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An Application of Discrete Inequality to Second Order Nonlinear Oscillation

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By using some simple discrete inequalities oscillation criteria are provided for the second order difference equations

$$\Delta^2 y_n + a_{n+1} f(y_{n+1}) = 0, \qquad n \in N,$$

where the operator Δ is defined by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}$ is a real sequence. The function f is such that $uf(u) > 0$ for $u \neq 0$ and $f(u) - f(v) = g(u, v)(u - v)$ for u, $v \neq 0$ for some nonnegative function g. © 1994 Academic Press, Inc.

1. INTRODUCTION

Consider the difference equation

$$\Delta^2 y_n + a_{n+1} f(y_{n+1}) = 0, \qquad n \in N, \tag{1}$$

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where $N = \{0, 1, 2, \cdots\}$, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}$ is a real sequence, and $f: R \to R$ is such that

$$uf(u) > 0 \quad \text{for } u \neq 0 \quad \text{and} \quad f(u) - f(v) = g(u, v)(u - v) \quad (2)$$

for $u, v \neq 0$ and for some nonnegative function g. We let $N_{n_0}^{\alpha} = \{n_0, n_0 + 1, \ldots, \alpha\}$, where $n_0 \in N$, $\alpha > n_0$ and when $\alpha = \infty$, $N_{n_0}^{\alpha}$ is denoted by N_{n_0}.

By a solution of (1) we always mean a nontrivial sequence $\{y_n\}$ which satisfies (1) for all $n \in N$. A solution $\{y_n\}$ of (1) is said to be oscillatory if for every $n_0 \in N$ there exists $n \geq n_0$ such that $y_n y_{n+1} < 0$; otherwise it is called nonoscillatory. We say that (1) is oscillatory if all nontrivial solutions are oscillatory.

In recent years there has been considerable interest in the study of oscillation and asymptotic behavior of solutions of difference equations; see for example [1, 4, 5, 7–10] and the references cited therein. In most of the results mentioned above it is assumed that $\{a_n\}$ is a nonnegative sequence. Our aim is to obtain sufficient conditions for all solutions of (1) to be oscillatory without any sign condition on $\{a_n\}$. This is done by using the theory of discrete inequalities.

2. Some Basic Lemmas

In this section we present two lemmas which are interesting in their own right and which will be used in the proofs of our main results given in Section 3.

Lemma 1. *Let $K(n, s, y)$ be defined on $N_{n_0} \times N_{n_0} \times R^+$ such that for fixed n and s, K is a nondecreasing function of y. Let $\{p_n\}$ be a given sequence and let $\{y_n\}$ and $\{z_n\}$ be defined on N_{n_0} satisfying for all $n \in N_{n_0}$*

$$y_n \geq p_n + \sum_{s=n_0}^{n-1} K(n, s, y_n) \quad (3)$$

and

$$z_n = p_n + \sum_{s=n_0}^{n-1} K(n, s, z_n).$$

respectively. Then $z_n \leq y_n$ for all $n \in N_{n_0}$.

Proof. Suppose there exists an integer $\ell \in N_{n_0}$ such that $z_{\ell+1} > y_{\ell+1}$ and $z_s \leq y_s$ for $s \leq \ell$. But then

$$z_{\ell+1} - y_{\ell+1} \leq \sum_{s=n_0}^{\ell} [K(\ell+1, s, z_s) - K(\ell+1, s, y_s)] \leq 0,$$

which is a contradiction.

Remark. The importance of the above lemma is that once the discrete inequality (3) is known then a lower bound of z_n can be found by replacing the inequality by an equation and solving the latter.

LEMMA 2. *Suppose the function f satisfies condition* (2). *Let $\{y_n\}$ be a positive (negative) solution of* (1) *for $n \in N^{\alpha}_{n_0}$ such that*

$$\frac{-\Delta y_{n_0}}{f(y_{n_0})} + \sum_{s=n_0}^{n-1} a_{s+1} + \sum_{s=n_0}^{n_1-1} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \geq m \tag{4}$$

for all $n \in N^{\alpha}_{n_1}$ and for some positive constant m; then

$$\Delta y_n \leq -mf(y_{n_1})(\Delta y_n \geq -mf(y_{n_1})) \qquad \text{for all n} \in N^{\alpha}_{n_1}.$$

Proof. Let $\{y_n\}$ be a solution of (1) satisfying the hypotheses of the lemma. Since

$$\frac{\Delta^2 y_n}{f(y_{n+1})} + a_{n+1} = 0,$$

on summing from n_0 to $n - 1$, and using (4) along with the formula

$$\sum_{s=n_0}^{n-1} z_{s+1}\Delta v_s = z_s v_s|_{s=n_0}^{n} - \sum_{s=n_0}^{n-1} v_s \Delta z_s$$

for summation by parts and the inequality (4) we have

$$\frac{-\Delta y_n}{f(y_n)} \geq m + \sum_{s=n_1}^{n-1} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})}, \qquad n \in N^{\alpha}_{n_1}. \tag{5}$$

Since the sum in (5) is nonnegative, in view of (2), we get

$$y_n \Delta y_n < 0 \qquad \text{for } n \in N^{\alpha}_{n_1}.$$

If $\{y_n\}$ is positive, then (5) becomes

$$\omega_n \geq mf(y_n) + \sum_{s=n_1}^{n-1} \frac{f(y_n)g(y_s, y_{s+1})(-\Delta y_s)}{f(y_s)f(y_{s+1})} \omega_s.$$

where $\omega_n = -\Delta y_n$. Define

$$K(n, s, z) = \frac{f(y_n)g(y_s, y_{s+1})(-\Delta y_s)}{f(y_s)f(y_{s+1})} z$$

for $n \in N_{n_1}^{\alpha}$ and $z \in R^+$. We note that $K(n, s, z)$ is nondecreasing in z for each fixed n and s. So we can use Lemma 1 with $p_n = mf(y_n)$ to obtain $\omega_n \geq \nu_n$, where ν_n satisfies

$$\nu_n = mf(y_n) + \sum_{s=n_1}^{n-1} \frac{f(y_n)g(y_s, y_{s+1})(-\Delta y_s)}{f(y_s)f(y_{s+1})} \nu_s \tag{6}$$

provided that $\nu_s \in R^+$ for all $s \in N_{n_1}^{\alpha}$. First dividing (6) by $f(y_n)$ and then applying operator Δ one can easily verify that $\Delta\nu_v \equiv 0$. Thus

$$\nu_n = \nu_1 = mf(y_{n_1}) \qquad \text{for } n \in N_{n_1}^{\alpha}.$$

The proof for the case when $\{y_n\}$ is negative follows from a similar argument by taking $\omega_n = \Delta y_n$ and $p_n = -mf(y_n)$.

3. Main Result

We begin with the following theorem which is the discrete analogue of a theorem of Waltman [11].

THEOREM 1. *In addition to condition* (2), *assume that*

$$\lim_{n\to\infty} \sum_{s=0}^{n-1} a_{s+1} = \infty; \tag{A}$$

then (1) *is oscillatory.*

Proof. Suppose the contrary and let $\{y_n\}$ be a nonoscillatory solution of (1), which we may (and do) assume to be positive on N_{n_0}. In view of (A), (4) is satisfied on N_{n_1} for some sufficiently large n_1. Thus, from Lemma 2

$$\Delta y_n \leq -mf(y_{n_1}) \qquad \text{for all } n \in N_{n_1},$$

which, after summing yields

$$y_n \leq y_{n_1} - mf(y_{n_1})(n - n_1).$$

Hence $y_n \to -\infty$ as $n \to \infty$. This contradicts the fact that $y_n > 0$ for all $n \in N_{n_0}$. The proof for the case when $\{y_n\}$ is negative is similar and hence is omitted.

The next result is concerned with the situation when the solutions of (1) are bounded away from zero.

THEOREM 2. *In addition to condition* (2), *suppose that*

$$\sum_{s=n_0}^{\infty} a_{s+1} \qquad \textit{converges} \tag{7}$$

and

$$g(u, v) \geq \lambda > 0 \qquad \textit{for all } u, v \neq 0. \tag{8}$$

If $\{y_n\}$ *is a solution of* (1) *such that* $\liminf_{n\to\infty} |y_n| > 0$, *then*

$$\sum_{s=n_0}^{\infty} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} < \infty, \tag{9}$$

$$\frac{\Delta y_n}{f(y_n)} \to 0 \qquad \text{as } n \to \infty, \tag{10}$$

and

$$\frac{\Delta y_n}{f(y_n)} = \sum_{s=n}^{\infty} a_{s+1} + \sum_{s=n}^{\infty} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \tag{11}$$

for all sufficiently large n.

Proof. Let $\{y_n\}$ be a nonoscillatory solution of (1) such that $\liminf_{n\to\infty} |y_n| > 0$. Then there exists $m_1, m_2 > 0$, and $n_1 > n_0$ such that $|y_n| \geq m_1$ and $|f(y_n)| \geq m_2$ for $n \in N_{n_1}$. Suppose that (9) does not hold; then it follows from (7) that (4) is satisfied on N_{n_1} for some sufficiently large n_1. For the case when $\{y_n\}$ is positive on N_{n_1}, it follows from Lemma 2 that

$$\Delta y_n \leq -mf(y_{n_1}).$$

After summing we have

$$y_n \leq y_{n_1} - mf(y_{n_1})(n - n_1) \to -\infty \qquad \text{as } n \to \infty,$$

which contradicts the fact that $y_n > 0$ on N_{n_1}. The proof for the case when

$\{y_n\}$ is negative on N_{n_1} is similar and is omitted. This completes the proof for (9). Next, summing the equation (1) once, we have for $n \in N_{n_0}$

$$\frac{\Delta y_n}{f(y_n)} = \beta + \sum_{s=n}^{\infty} a_{s+1} + \sum_{s=n}^{\infty} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})}, \tag{12}$$

where β is a constant such that

$$\beta = \frac{\Delta y_{n_0}}{f(y_{n_0})} - \lim_{n\to\infty} \sum_{s=n_0}^{n-1} a_{s+1} - \sum_{s=n_0}^{\infty} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})}.$$

To prove that (10) and (11) hold it suffices to show that $\beta = 0$. First suppose that $y_n > 0$ for all $n \in N_{n_0}$. If $\beta < 0$, then (7) and (9) imply that there exists $n_1 > n_0$ such that

$$\left|\sum_{s=n_1}^{n-1} a_{s+1}\right| \leq -\frac{\beta}{4} \quad \text{and} \quad \left|\sum_{s=n_1}^{n-1} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})}\right| < -\frac{\beta}{4}$$

for all $n \in N_{n_1}$. Now (12) implies that (4) holds on N_{n_1}. But then, by the argument given above, Lemma 2 and its proof lead to a contradiction of the fact that $y_n > 0$. If $\beta > 0$ then, since, $\lim_{n\to\infty} \Delta y_n/f(y_n) = \beta$, we have $\Delta y_n/f(y_n) \geq \beta/2$ for all $n \in N_{n_1}$ with $n_1 \geq n_0$. We use (2) and (8) to get

$$\frac{g(y_n, y_{n+1})\Delta y_n}{f(y_{n+1})} \geq \frac{\lambda\beta}{2 + \lambda\beta}.$$

Thus

$$\sum_{s=n_1}^{\infty} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \geq \lim_{n\to\infty} \sum_{s=n_1}^{n} \frac{\beta}{2}\left(\frac{\lambda\beta}{2 + \lambda\beta}\right) \to \infty,$$

which contradicts (9). This completes the proof that $\beta = 0$ for the case when $y_n > 0$. The proof for the case when $y_n < 0$ is similar.

THEOREM 3. *Suppose that* (2), (7), *and* (8) *hold. Further, assume that*

$$\liminf_{n\to\infty} A(n) > \frac{-1}{\lambda} \quad \textit{where } A(n) = \sum_{s=n}^{\infty} a_{s+1}. \tag{13}$$

If

$$\sum_{n_0}^{\infty} \frac{A_+^2(n)}{1 + \lambda A(n)} = \infty, \tag{14}$$

where $A_+(n) = \max\{A(n), 0\}$, *then* (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $\{y_n\}$ such that $|y_n| > 0$ for $n \in N_{n_1}$. Since (8) implies that $f(u)$ is strictly increasing for $u \neq 0$, we have $|f(y_n)| > 0$ for $n \in N_{n_1}$. And hence by Theorem 2 we have

$$\frac{\Delta y_n}{f(y_n)} \geq A(n) \tag{15}$$

and the inequality (9). Use (2) and (8) in (15) to get

$$\frac{g(y_n, y_{n+1})\Delta y_n}{f(y_{n+1})} \geq \frac{\lambda A(n)}{1 + \lambda A(n)}. \tag{16}$$

Now, from (15) and (16), we obtain

$$\sum_{s=n_1}^{\infty} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \geq \lambda \sum_{n_1}^{\infty} \frac{A_+^2(n)}{1 + \lambda A(n)},$$

which contradicts (9). This completes the proof of the theorem.

Remark. If $\sum_{s=n_0}^{\infty} a_{s+1}$ converges absolutely then $A(n) \to 0$ as $n \to \infty$ and condition (13) is automatically satisfied. We can replace (14) by $\sum_{n_0}^{\infty} A_+^2(n) = \infty$. The authors have not been able to improve condition (14). It would be better if one could impose a condition on $A(n)$ which is independent of λ.

Before formulating the next theorem we point out that if $A(n)$ converges then $h_0(n) = \sum_{s=n}^{\infty} a_{s+1}$ is well defined for all $n \in N_{n_0}$ and we assume that $h_0(n)$ is positive for all sufficiently large values of n. We define

$$h_1(n) = \sum_{s=n}^{\infty} \frac{[h_0(s)_+]^2}{1 + \lambda h_0(s)}$$

and

$$h_{m+1}(n) = \sum_{s=n}^{\infty} \frac{[h_0(s)_+ + \lambda h_m(s)_+]^2}{1 + \lambda(h_0(s) + \lambda h_m(s))}$$

for $m = 1, 2, \cdots$ and $h_0(n)_+ = \max\{h_0(n), 0\}$.

In our next theorem we need the following condition

There exists a positive integer M such that h_m exists for $m = 1, 2, \ldots, M - 1$ and h_M does not exist. (B)

THEOREM 4. *Suppose that* (2), (7), (8), (13), *and* (*B*) *hold. Then* (1) *is oscillatory.*

Proof. From the proof of Theorem 2 it follows that if $\{y_n\}$ is a nonoscillatory solution of (1) with $|y_n| > 0$ for $n \in N_{n_1}$ then

$$\frac{\Delta y_n}{f(y_n)} = h_0(n) + \sum_{s=n}^{\infty} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \tag{17}$$

and

$$\sum_{s=n}^{\infty} \frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} < \infty. \tag{18}$$

From (17) we get $\Delta y_n/f(y_n) \geq h_0(n)$, which implies that

$$\frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \geq \frac{\lambda[h_0(n)_+]^2}{1 + \lambda h_0(n)}. \tag{19}$$

If $M = 1$, then (18) and (19) imply that

$$h_1(n) = \sum_{s=n}^{\infty} \frac{[h_0(s)_+]^2 \cdot}{1 + \lambda h_0(s)} < \infty,$$

which contradicts the non-existence of $h_M(n) = h_1(n)$. If $M = 2$, then from (17) and (19) we have

$$\frac{\Delta y_n}{f(y_n)} \geq h_0(n) + \lambda \sum_{s=n}^{\infty} \frac{[h_0(s)_+]^2}{1 + \lambda h_0(s)} = h_0(n) + \lambda h_1(n),$$

from which it follows that

$$\frac{g(y_s, y_{s+1})(\Delta y_s)^2}{f(y_s)f(y_{s+1})} \geq h_0(n) + \frac{\lambda[h_0(n) + \lambda h_1(n))_+]^2}{1 + \lambda(h_0(n) + \lambda h_1(n))}.$$

Then, in view of (18), a summation of the last inequality leads to the contradiction of the non-existence of $h_M \equiv h_2$. A similar argument provides a contradiction for any integer $M > 2$.

EXAMPLE 1. Consider the equation

$$\Delta^2 y_n + \frac{(n+1)^2 2(2n^2+4n+1)}{n(n+2)} y_{n+1}^3 = 0, \qquad n = 1, 2, \ldots \qquad (E_1)$$

The hypotheses of Theorem 1 are satisfied and hence (E_1) is oscillatory. One such solution is $y_n = (-1)^n/n$.

EXAMPLE 2. Consider the difference equation

$$\Delta^2 y_n - \frac{1}{2^{2n+3}} y_{n+1}^3 = 0, \qquad n = 0, 1, \ldots \qquad (E_2)$$

It is easy to verify that all conditions of Theorem 2 are satisfied for (E_2). Hence every nonoscillatory solution of (E_2) satisfies (9)–(11). One such solution is $y_n = 2^n$ with $\liminf_{n\to\infty} |y_n| > 0$.

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