ELSEVIER Linear Algebra and its Applications 285 (1998) 1–6

LINEAR ALGEBRA AND ITS APPLICATIONS

The expression of the generalized inverse of the perturbed operator under Type I perturbation in Hilbert spaces

Guoliang Chen [a,*], Yifeng Xue [b]

[a] *Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China*

[b] *Institute of Fundamental Education, East China University of Science and Technology, Shanghai 200237, People's Republic of China*

Received 4 December 1997; received in revised form 17 March 1998; accepted 9 April 1998

Submitted by R. Bhatia

Abstract

Let H_1,H_2 be two Hilbert spaces over the complex field **C** and let $T{:}H_1 \to H_2$ be a bounded linear operator with the generalized inverse T^+. Let $\overline{T} = T + \delta T$ be a bounded linear operator with $\|T^+\|\|\delta T\| < 1$. Suppose that $\dim\ker \overline{T} = \dim\ker T < \infty$ or $\mathrm{R}(\overline{T}) \cap \mathrm{R}(T)^{\perp} = 0$. Then $\overline{T}$ has the generalized inverse

$$\begin{aligned}\overline{T}^{+} = {} & [I - (I + T^+\delta T)^{-1}(I - T^+T) \\ & - (I - T^+T)(I + T^+\delta T)^{*-1}]^{-1}(I + \delta TT^+)^{-1}T^+ \\ & \times [(I + \delta TT^+)TT^+(I + \delta TT^+)^{-1} + (I + \delta TT^+)^{*-1}TT^+(I + \delta TT^+)^* - I]^{-1}\end{aligned}$$

with

$$\|\overline{T}^{+}\| \leqslant \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}.$$

This result gives an analogue of Theorem 3.9 of M.Z. Nashed ("Generalized Inverses and Applications", Academic Press, New York, 1976) in Hilbert spaces. © 1998 Elsevier Science Inc. All rights reserved.

AMS classification: 65H05

Keywords: Generalized inverses; Type I perturbation of operators; Orthogonal projections

* Corresponding author.

PII: S0024-3795(98)10066-6

1. Introduction

Let $(X_1, \|\cdot\|_1), (X_2, \|\cdot\|_2)$ be two Banach spaces over the complex field **C** and let $B(X_1,X_2)$ denote the Banach space of all bounded linear operators $T : X_1 \to X_2$ with the norm

$$\|T\| = \sup\{\|Tx\|_2 \mid \|x\|_1 = 1, x \in X_1\}.$$

For $T \in B(X_1,X_2)$, Ker T (resp. R(T)) denotes the null space (resp. range) of T. Let $T \in B(X_1,X_2)$ with R(T) closed. If there exist two idempotents $P : X_1 \to \operatorname{Ker} T$, $Q: X_2 \to \mathrm{R}(T)$ and $T^+ \in B(X_2,X_1)$ such that

$$T^+TT^+ = T^+, \quad TT^+T = T, \quad T^+T = I - P, \quad TT^+ = Q, \tag{1.1}$$

then we say that T has uniquely the generalized inverse $T^+ = T^+_{P,Q}$ (with respect to P, Q). If X_1, X_2 are Hilbert spaces, we require $(T^+T)^* = T^+T$, $(TT^+)^* = TT^+$ (see also [1]).

In [1], Nashed showed that for $T \in B(X_1,X_2)$ with the generalized inverse T^+ and $\overline{T} = T + \delta T \in B(X_1,X_2)$ with $\|T^+\|\|\delta T\| < 1$, if

$$(I + \delta TT^+)^{-1}\overline{T} \text{ maps Ker } T \text{ into R}(T), \tag{1.2}$$

then $\overline{T}^+$ exists and $\overline{T}^+ = (I + T^+\delta T)^{-1}T^+$ with

$$\|\overline{T}^+\| \leqslant \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}. \tag{1.3}$$

Here we need to mention that by Chen and Xue's work (cf. [2]) the condition (1.2) can be replaced by the following condition:

$$\dim \operatorname{Ker} \overline{T} = \dim \operatorname{Ker} T < \infty \quad \text{or} \quad \mathrm{R}(\overline{T}) \cap \operatorname{Ker} T^+ = 0.$$

The Nashed's result Eq. (1.3), however, has not been found by now in Hilbert spaces because $\overline{T}^+ = (I + T^+\delta T)^{-1}T^+$ is not the generalized inverse of $\overline{T}$ in Hilbert spaces.

Recently there are some research works pertaining to the estimation of $\|\overline{T}^+\|$ such as [3,4] although these are not the best. So in this paper we will establish the result similar to [Na], Theorem 3.7 in Hilbert spaces. Using this result, we have improved a very important result of [3].

2. The main result

Throughout this section, we assume that H, H_1, H_2 are all Hilbert spaces. Let $S \in B(H,H)$ be an idempotent. We denote by $O(S)$ the orthogonal projection of H onto R(S). Then it is easy to verify that

$$O(S)S = S, \qquad SO(S) = O(S),$$

$$(I - O(S))(I - S) = I - O(S). \tag{2.1}$$

Moreover, we have the following deeper result spirited up by [5], Proposition 4.6.2.

Lemma 1. *Let $S, O(S)$ be as above. Then*
1. *$I - S - S^*$ is invertible in $B(H, H)$;*
2. *$S(I - S - S^*)^{-1} = (I - S - S^*)^{-1}S^*$, $S^*(I - S - S^*)^{-1} = (I - S - S^*)^{-1}S$;*
3. *$O(S) = -S(I - S - S^*)^{-1}$.*

Proof. (1) Since $a = I + (S - S^*)(S - S^*)^*$ is strictly positive and $S^2 = S, (S^*)^2 = S^*$, we have

$$a = I + SS^* + S^*S - S - S^* = (I - S - S^*)^2$$

is invertible. So is the $I - S - S^*$.

(2) The assertion comes from the identities:

$$S(I - S - S^*) = -SS^* = (I - S - S^*)S^*; \tag{2.2}$$

$$S^*(I - S - S^*) = -S^*S = (I - S - S^*)S \tag{2.3}$$

and (1).

(3) Let $r = -S(I - S - S^*)^{-1}$. We will prove that r is an orthogonal projection of H onto $\mathrm{R}(S)$ so that $O(S) = r$.

By (2) and Eq. (2.3), we have

$$r^* = -(I - S - S^*)^{-1}S^* = -S(I - S - S^*)^{-1} = r$$

and

$$\begin{aligned} r^2 &= S(I - S - S^*)^{-1}S(I - S - S^*)^{-1} \\ &= (I - S - S^*)^{-1}S^*S(I - S - S^*)^{-1} \\ &= -(I - S - S^*)^{-1}S^* = -S(I - S - S^*)^{-1} = r. \end{aligned}$$

Now from $r = -S(I - S - S^*)^{-1}$, we have $\mathrm{R}(r) \subset \mathrm{R}(S)$. On the other hand, by (2) and Eq. (2.2), we get that

$$rS = -S(I - S - S^*)^{-1}S = -SS^*(I - S - S^*)^{-1} = S.$$

This means that $\mathrm{R}(S) \subset \mathrm{R}(r)$. Therefore $\mathrm{R}(r) = \mathrm{R}(S)$. □

The next lemma concerns about the construction of the generalized inverses of the operators in Hilbert spaces.

Lemma 2. *Let $T \in B(H_1, H_2)$ with $\mathrm{R}(T)$ closed. Assume that there are an idempotent $P : H_1 \to \operatorname{Ker} T$ and an $A \in B(H_2, H_1)$ such that $AT = I - P$. Then $T^+ = [I - O(P)]AO(TA)$.*

Proof. Put $Q = TA$. Then $Q^2 = TATA = T(I-P)A = TA = Q$. Now from $Q = TA$ and $QT = T(I-P) = T$, we get that $\mathrm{R}(T) = \mathrm{R}(Q)$. Put $B = [I - O(P)]AO(Q)$. Then

$$\begin{aligned} TB &= T[I - O(P)]AO(Q) = TAO(Q) = QO(Q) = O(Q), \\ BT &= [I - O(P)]AO(Q)T = [I - O(P)]AT \\ &= [I - O(P)](I - P) = I - O(P). \end{aligned}$$

The above indicates that $T^+ = B$ is the generalized inverse of T in $B(H_2, H_1)$. □

Now we present our main result as follows.

Theorem 1. *Let $T \in B(H_1, H_2)$ with the generalized inverse T^+ and let $\overline{T} = T + \delta T \in B(H_1, H_2)$ with $\|T^+\|\|\delta T\| < 1$. Suppose that* $\dim \operatorname{Ker} \overline{T} = \dim \operatorname{Ker} T < \infty$ *or $\overline{T}$ is a Type I perturbation of T (i.e., $\mathrm{R}(\overline{T}) \cap \mathrm{R}(T)^{\perp} = 0$). Then $\overline{T}$ has the generalized inverse $\overline{T}^+$ admitting the form*

$$\begin{aligned} \overline{T}^+ = {} & [I - (I + T^+\delta T)^{-1}(I - T^+T) \\ & - (I - T^+T)(I + T^+\delta T)^{*-1}]^{-1}(I + T^+\delta T)^{-1}T^+ \\ & \times [(I + \delta TT^+)TT^+(I + \delta TT^+)^{-1} \\ & + (I + \delta TT^+)^{*-1}TT^+(I + \delta TT^+)^* - I]^{-1}, \end{aligned} \tag{2.4}$$

with

$$\|\overline{T}^+\| \leqslant \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}.$$

Proof. Noting that $\operatorname{Ker} T^+ = \mathrm{R}(I - TT^+) = \mathrm{R}(T)^{\perp}$ in H_2, we obtain that by [2], Propositions 3.1, 3.2, and Corollary 3.1, $S = (I + T^+\delta T)^{-1}(I - T^+T)$ is an idempotent with $\mathrm{R}(S) = \operatorname{Ker}(\overline{T})$ and $\overline{T}$ has the generalized inverse $A = (I + T^+\delta T)^{-1}T^+ \in B(H_2, H_1)$ in the sense of Eq. (1.1).

Since $T^+\overline{T} = T^+T + T^+\delta T$, it follows that

$$A\overline{T} = (I + T^+\delta T)^{-1}T^+\overline{T} = I - S.$$

Therefore by Lemma 1 and Lemma 2, $\overline{T}$ has the generalized inverse $\overline{T}^+$ of the form

$$\overline{T}^+ = [I - O(S)](I + T^+\delta T)^{-1}T^+O(Q), \tag{2.5}$$

where

$$\begin{aligned} Q = \overline{T}A &= (T + \delta T)(I + T^+\delta T)^{-1}T^+ \\ &= (T + \delta T)T^+(I + \delta TT^+)^{-1} = (I + \delta TT^+)TT^+(I + \delta TT^+)^{-1}. \end{aligned}$$

Since we have by Lemma 1,

$$\begin{aligned} I - O(S) &= I + S(I - S - S^*)^{-1} = (I - S^*)(I - S - S^*)^{-1} \\ &= (I - S - S^*)^{-1}(I - S) \\ &= [I - (I + T^+\delta T)^{-1}(I - T^+T) - (I - T^+T) \\ &\quad \times (I + T^+\delta T)^{*-1}]^{-1}(I + T^+\delta T)^{-1}T^+\overline{T} \end{aligned}$$

and

$$\begin{aligned} O(Q) = &- (I + \delta TT^+)TT^+(I + \delta TT^+)^{-1}[I - (I + \delta TT^+)TT^+(I + \delta TT^+)^{-1} \\ &- (I + \delta TT^+)^{*-1}TT^+(I + \delta TT^+)^*]^{-1}, \end{aligned}$$

it follows from the simple computation that $\overline{T}^+$ has the form (2.4).

Finally, by Eq. (2.5) we have

$$\begin{aligned} \|\overline{T}^+\| &\leqslant \|I - O(S)\|\|(I + T^+\delta T)^{-1}T^+\|\|O(Q)\| \\ &\leqslant \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}. \quad \square \end{aligned}$$

In order to estimate $\|\overline{T}^+ - T^+\|$ when dim Ker $\overline{T}$ = dim Ker $T < \infty$ or $\overline{T}$ is a Type I perturbation of T, we need following lemma which comes from the combination of [6] Theorem I.6.34 and [3], Lemma 2.1.

Lemma 3. *Let S_1, S_2 be to idempotents in $B(H, H)$. Then* $\|O(S_1) - O(S_2)\| \leqslant \|S_1 - S_2\|$.

The following theorem shows what the $\|\overline{T}^+ - T^+\|$ is.

Theorem 2. *Let $T, \overline{T}$ satisfy the conditions of Theorem 1. Then $\overline{T}^+$ exits and*

$$\frac{\|\overline{T}^+ - T^+\|}{\|T^+\|} \leqslant \frac{3\|T^+\|\|\delta T\|}{1 - \|T^+\|\|\delta T\|}.$$

Proof. According to Theorem 1, we have

$$\begin{aligned} \overline{T}^+ - T^+ = &[I - O(S)][(I + T^+\delta T)^{-1}T^+ - T^+]O(Q) \\ &+ [I - O(S) - T^+T] \\ &\times T^+O(Q) + T^+[O(Q) - TT^+], \end{aligned} \tag{2.6}$$

where S, Q are given in the proof of Theorem 1. Thus applying Lemma 3 to Eq. (2.6), we obtain that

$$\begin{aligned}\|\overline{T}^+ - T^+\| &\leqslant \|(I + T^+\delta T)^{-1}T^+ - T^+\| \\ &+ \|I - T^+T - O(S)\|\|T^+\| + \|T^+\|\|O(Q) - TT^+\| \\ &\leqslant \|(I + T^+\delta T)^{-1} - I\|\|T^+\| + \|[I - (I + T^+\delta T)^{-1}] \\ &(I - T^+T)\|\|T^+\| + \|T^+\|\|(I + \delta TT^+)TT^+ \\ &(I + \delta TT^+)^{-1} - TT^+\| \leqslant \frac{2\|T^+\|^2\|\delta T\|}{1 - \|T^+\|\|\delta T\|} \\ &+ \|T^+\|\|(I - TT^+)\delta TT^+(I + T^+\delta T)^{-1}\|\end{aligned}$$

and consequently,

$$\frac{\|\overline{T}^+ - T^+\|}{\|T^+\|} \leqslant \frac{3\|T^+\|\|\delta T\|}{1 - \|T^+\|\|\delta T\|}. \quad \square$$

Remark. Comparing this paper with [3], we have seen that on the same assumptions, Theorem 1 improves the result of [3], Theorems 3.2 and Theorem 2 gives a much better improvement of the result of [3], Theorem 4.1.

Acknowledgements

The authors wish to thank the referee for his helpful comments and suggestions.

References

[1] M.Z. Nashed, in: M.Z. Nashed (Ed.), Perturbations and approximations for generalized inverses and linear operator equations Generalized Inverses and Applications, Academic Press, New York, 1976.

[2] G. Chen, Y. Xue, Perturbation analysis for the operator equation $Tx = b$ in Banach spaces, J. Math. Anal. Appl. 212 (1997) 107–125.

[3] G. Chen, M. Wei, Y. Xue, Perturbation analysis of the least square solution in Hilbert spaces, Linear Algebra Appl. 244 (1996) 69–80.

[4] J. Ding, L.J. Huang, On the continuity of generalized inverses of linear operators in Hilbert spaces, Linear Algebra Appl. 262 (1997) 229–242.

[5] B. Blackadar, *K*-Theory for Operator Algebras, Springer, New York, 1986.

[6] T. Kato, Perturbation Theory for Linear Operators, Springer, New York, 1984.