The expression of the generalized inverse of the perturbed operator under Type I perturbation in Hilbert spaces

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Abstract

Let $H_1, H_2$ be two Hilbert spaces over the complex field $\mathbb{C}$ and let $T: H_1 \rightarrow H_2$ be a bounded linear operator with the generalized inverse $T^+$. Let $\widetilde{T} = T + \delta T$ be a bounded linear operator with $\|T^+\|\|\delta T\| < 1$. Suppose that $\dim \ker \widetilde{T} = \dim \ker T < \infty$ or $R(\widetilde{T}) \cap R(T)^\perp = 0$. Then $\widetilde{T}$ has the generalized inverse

$$
\widetilde{T}^+ = [I - (I + T^+ \delta T)^{-1}(I - T^+ T) - (I - T^+ T)(I + T^+ \delta T)^{-1}(I + \delta TT^+)^{-1}T^+ \\
	\times [(I + \delta TT^+)(I + \delta TT^+)^{-1} + (I + \delta TT^+)^{-1}TT^+(I + \delta TT^+)^* - I]^{-1}
$$

with

$$
\|\widetilde{T}^+\| \leq \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}.
$$

This result gives an analogue of Theorem 3.9 of M.Z. Nashed ("Generalized Inverses and Applications", Academic Press, New York, 1976) in Hilbert spaces. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let \((X_1, \| \cdot \|_1), (X_2, \| \cdot \|_2)\) be two Banach spaces over the complex field \(\mathbb{C}\) and let \(B(X_1, X_2)\) denote the Banach space of all bounded linear operators \(T : X_1 \to X_2\) with the norm

\[
\| T \| = \sup \{ \| Tx \|_2 \mid \| x \|_1 = 1, x \in X_1 \}.
\]

For \(T \in B(X_1, X_2)\), \(\ker T\) (resp. \(R(T)\)) denotes the null space (resp. range) of \(T\). Let \(T \in B(X_1, X_2)\) with \(R(T)\) closed. If there exist two idempotents \(P : X_1 \to \ker T, Q : X_2 \to R(T)\) and \(T^+ \in B(X_2, X_1)\) such that

\[
T^+ T T^+ = T^+, \quad TT^+ T = T, \quad T^+ T = I - P, \quad TT^+ = Q,
\]

then we say that \(T\) has uniquely the generalized inverse \(T^+ = T^+_p Q\) (with respect to \(P, Q\)). If \(X_1, X_2\) are Hilbert spaces, we require \((T^+ T)^* = T^+ T, (TT^+)^* = TT^+\) (see also [1]).

In [1], Nashed showed that for \(T \in B(X_1, X_2)\) with the generalized inverse \(T^+\) and \(\bar{T} = T + \delta T \in B(X_1, X_2)\) with \(\| T^+ \| \| \delta T \| < 1\), if

\[
(I + \delta TT^+)^{-1} \bar{T} \text{ maps } \ker T \text{ into } R(T),
\]

then \(\bar{T}^+\) exists and \(\bar{T}^+ = (I + T^+ \delta T)^{-1} T^+\) with

\[
\| \bar{T}^+ \| \leq \frac{\| T^+ \|}{1 - \| T^+ \| \| \delta T \|}.
\]

Here we need to mention that by Chen and Xue’s work (cf. [2]) the condition (1.2) can be replaced by the following condition:

\[
\dim \ker T = \dim \ker T < \infty \quad \text{or} \quad R(T) \cap \ker T^+ = 0.
\]

The Nashed’s result Eq. (1.3), however, has not been found by now in Hilbert spaces because \(\bar{T}^+ = (I + T^+ \delta T)^{-1} T^+\) is not the generalized inverse of \(\bar{T}\) in Hilbert spaces.

Recently there are some research works pertaining to the estimation of \(\| \bar{T}^+ \|\) such as [3,4] although these are not the best. So in this paper we will establish the result similar to [Na], Theorem 3.7 in Hilbert spaces. Using this result, we have improved a very important result of [3].

2. The main result

Throughout this section, we assume that \(H, H_1, H_2\) are all Hilbert spaces. Let \(S \in B(H, H)\) be an idempotent. We denote by \(O(S)\) the orthogonal projection of \(H\) onto \(R(S)\). Then it is easy to verify that

\[
O(S)S = S, \quad SO(S) = O(S),
\]
Moreover, we have the following deeper result spirited up by [5], Proposition 4.6.2.

**Lemma 1.** Let \( S, O(S) \) be as above. Then
1. \( I - S - S^* \) is invertible in \( B(H, H) \);
2. \( S(I - S - S^*)^{-1} = (I - S - S^*)^{-1}S^*, S^*(I - S - S^*)^{-1} = (I - S - S^*)^{-1}S \);
3. \( O(S) = -S(I - S - S^*)^{-1} \).

**Proof.** (1) Since \( a = Z + (S - S^*)(S - S^*)^* \) is strictly positive and \( S^2 = S, (S^*)^2 = S^* \), we have
\[
a = I + SS^* + S^*S - S - S^* = (I - S - S^*)^2
\]
is invertible. So is the \( I - S - S^* \).

(2) The assertion comes from the identities:
\[
S(I - S - S^*) = -SS^* = (I - S - S^*)s^*; \quad (2.2)
S^*(I - S - S^*) = -S'S = (I - S - S^*)S \quad (2.3)
\]
and (1).

(3) Let \( r = -S(I - S - S^*)^{-1} \). We will prove that \( r \) is an orthogonal projection of \( H \) onto \( R(S) \) so that \( O(S) = r \).

By (2) and Eq. (2.3), we have
\[
r^* = -(Z - S - S^*)^{-1}r = -S(Z - S - S^*)^{-1} = r
\]
and
\[
r^2 = S(I - S - S^*)^{-1}S(I - S - S^*)^{-1}
= (I - S - S^*)^{-1}S^*S(I - S - S^*)^{-1}
= -(I - S - S^*)^{-1}S^* = -S(I - S - S^*)^{-1} = r.
\]
Now from \( r = -S(I - S - S^*)^{-1} \), we have \( R(r) \subset R(S) \). On the other hand, by (2) and Eq. (2.2), we get that
\[
rS = -S(I - S - S^*)^{-1}S = -SS^*(I - S - S^*)^{-1} = S.
\]
This means that \( R(S) \subset R(r) \). Therefore \( R(r) = R(S) \). \( \Box \)

The next lemma concerns about the construction of the generalized inverses of the operators in Hilbert spaces.

**Lemma 2.** Let \( T \in B(H_1, H_2) \) with \( R(T) \) closed. Assume that there are an idempotent \( P : H_1 \rightarrow \text{Ker} \ T \) and an \( A \in B(H_2, H_1) \) such that \( AT = I - P \). Then \( T^+ = [I - O(P)]AO(TA) \).
Proof. Put $Q = TA$. Then $Q^2 = TATA = T(I - P)A = TA - Q$. Now from $Q = TA$ and $QT = T(I - P) = T$, we get that $R(T) = R(Q)$. Put $B = [I - O(P)]AO(Q)$. Then

$$TB = T[I - O(P)]AO(Q) = TAO(Q) = QO(Q) = O(Q),$$

$$BT = [I - O(P)]AO(Q)T = [I - O(P)]AT = [I - O(P)](I - P) = I - O(P).$$

The above indicates that $T^+ = B$ is the generalized inverse of $T$ in $B(H_2, H_1)$. □

Now we present our main result as follows.

Theorem 1. Let $T \in B(H_1, H_2)$ with the generalized inverse $T^+$ and let $\overline{T} = T + \delta T \in B(H_1, H_2)$ with $\|T^+\| \|\delta T\| < 1$. Suppose that $\dim \ker \overline{T} = \dim \ker T < \infty$ or $\overline{T}$ is a Type I perturbation of $T$ (i.e., $R(\overline{T}) \cap R(T)^\perp = 0$). Then $\overline{T}$ has the generalized inverse $\overline{T}^+$ admitting the form

$$\overline{T}^+ = [I - (I + T^+\delta T)^{-1}(I - T^+\delta T)]$$

$$- (I - T^+\delta T)(I + T^+\delta T)^{-1}
\times [(I + \delta TT^+)(I + \delta TT^+)^{-1}
+ (I + \delta TT^+)^{-1} TT^+(I + \delta TT^+)^* - I]^{-1},$$

(2.4)

and

$$\|\overline{T}^+\| \leq \frac{\|T^+\|}{1 - \|T^+\| \|\delta T\|}.$$

Proof. Noting that $\ker T^+ = R(T - TT^+) = R(T)^\perp$ in $H_2$, we obtain that by [2], Propositions 3.1, 3.2, and Corollary 3.1, $S = (I + T^+\delta T)^{-1}(I - T^+\delta T)$ is an idempotent with $R(S) = \ker(\overline{T})$ and $\overline{T}$ has the generalized inverse $A = (I + T^+\delta T)^{-1} T^+ \in B(H_2, H_1)$ in the sense of Eq. (1.1).

Since $T^+\overline{T} = T^+T + T^+\delta T$, it follows that

$$AT = (I + T^+\delta T)^{-1}T^+ = I - S.$$

Therefore by Lemma 1 and Lemma 2, $\overline{T}$ has the generalized inverse $\overline{T}^+$ of the form

$$\overline{T}^+ = [I - O(S)](I + T^+\delta T)^{-1}T^+O(Q),$$

(2.5)

where

$$Q = \overline{T}A = (I + \delta T)(I + T^+\delta T)^{-1}T^+$$

$$= (T + \delta T)T^+(I + \delta TT^+)^{-1} = (I + \delta TT^+)(I + \delta TT^+)^{-1}.$$
Since we have by Lemma 1,
\[ I - O(S) = I + S(I - S - S^*)^{-1} = (I - S^*)(I - S - S^*)^{-1} \]
\[ = (I - S - S^*)^{-1}(I - S) \]
\[ = [I - (I + T^+\delta T)^{-1}(I - T^+T) - (I - T^+T)] \times (I + T^+\delta T)^{-1}T^+T \]

and
\[ O(Q) = - (I + \delta TT^+)(I + \delta TT^+)^{-1}[I - (I + \delta TT^+)TT^+(I + \delta TT^+)^{-1} \]
\[ - (I + \delta TT^+)^{-1}TT^+(I + \delta TT^+)^{-1}T^+T^+T \]

it follows from the simple computation that \( T^+ \) has the form (2.4).

Finally, by Eq. (2.5) we have
\[ \|T^+\| \leq \|I - O(S)\|(I + T^+\delta T)^{-1}T^+\|O(Q)\| \]
\[ \leq \frac{\|T^+\|}{1 - \|T^+\|\|\delta T\|}. \quad \Box \]

In order to estimate \( \|T^+ - T^+\| \) when \( \dim \text{Ker} T = \dim \text{Ker} T < \infty \) or \( T \) is a Type I perturbation of \( T \), we need following lemma which comes from the combination of [6] Theorem 1.6.34 and [3], Lemma 2.1.

**Lemma 3.** Let \( S_1, S_2 \) be to idempotents in \( B(H, H) \). Then \( \|O(S_1) - O(S_2)\| \)
\[ \leq \|S_1 - S_2\|. \]

The following theorem shows what the \( \|T^+ - T^+\| \) is.

**Theorem 2.** Let \( T, T \) satisfy the conditions of Theorem 1. Then \( T^+ \) exits and
\[ \frac{\|T^+ - T^+\|}{\|T^+\|} \leq \frac{3\|T^+\|\|\delta T\|}{1 - \|T^+\|\|\delta T\|}. \]

**Proof.** According to Theorem 1, we have
\[ T^+ - T^+ = [I - O(S)][(I + T^+\delta T)^{-1}T^+ - T^+]O(Q) \]
\[ + [I - O(S) - T^+T] \times T^+O(Q) + T^+[O(Q) - TT^+], \quad (2.6) \]

where \( S, Q \) are given in the proof of Theorem 1. Thus applying Lemma 3 to
Eq. (2.6), we obtain that
\[
\|T^- - T^+\| \leq \|(I + T^+ \delta T)^{-1} T^- - T^+\|
\]
\[
+ \|I - T^+ T - O(S)\| \|T^+\| + \|T^+\| \|O(S) - TT^+\|
\]
\[
\leq \|(I + T^+ \delta T)^{-1} - I\| \|T^+\| + \|I -(I + T^+ \delta T)^{-1}\|
\]
\[
(I - T^+ T) \|T^+\| + \|T^+\| \|(I + \delta TT^+) TT^+\|
\]
\[
(I + \delta TT^+)^{-1} - TT^+ \leq \frac{2\|T^+\|^2 \|\delta T\|}{1 - \|T^+\| \|\delta T\|}
\]
\[
+ \|T^+\| \|(I - TT^+) \delta TT^+ (I + T^+ \delta T)^{-1}\|
\]
and consequently,
\[
\frac{\|T^- - T^+\|}{\|T^+\|} \leq \frac{3\|T^+\| \|\delta T\|}{1 - \|T^+\| \|\delta T\|}.
\]

**Remark.** Comparing this paper with [3], we have seen that on the same assumptions, Theorem 1 improves the result of [3], Theorems 3.2 and Theorem 2 gives a much better improvement of the result of [3], Theorem 4.1.

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**References**