Discontinuous solutions of the compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum

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Abstract

In this paper, we study the evolutions of the interfaces between gas and the vacuum for one-dimensional viscous gas motions when the initial density connects to vacuum continuously. The degeneracy appears in the initial data and has effect on the viscosity coefficient because the coefficient is assumed to be a power function of the density. Using some new a priori estimates, we establish the new local (in time) existence and uniqueness results under minimal hypotheses on the initial density, so that the interval for the power of the density in the viscosity coefficient is enlarged to $(0, \gamma)$. In particular, we include the important case that the initial density could be piecewise smooth with arbitrarily large jump discontinuities, and could degenerate to zero.

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1. Introduction

We study the free boundary problem for the one-dimensional compressible Navier–Stokes equations with density-dependent viscosity, which can be written in Eulerian coordinates as

\[
\begin{cases}
\rho_\tau + (\rho u)_\xi = 0, \\
(\rho u)_\tau + (\rho u^2 + P(\rho))_\xi = (\mu(\rho)u_\xi)_\xi, \\
\end{cases}
\]

for \( \tau > 0 \), \( a(\tau) < \xi < b(\tau) \), \( \xi \in \mathbb{R} \) (1.1)

with initial data

\[
(\rho, \rho u)(\xi, 0) = (\rho_0, m_0)(\xi), \quad \xi \in [a, b],
\]

where \( \rho, u \) and \( P(\rho) \) are the density, the velocity and the pressure, respectively, \( \mu(\rho) \geq 0 \) is the viscosity coefficient with the property \( \mu(0) = 0 \). Here, \( a(\tau) \) and \( b(\tau) \) are the free boundaries, i.e., the interfaces of the gas and the vacuum, satisfying

\[
\begin{cases}
\frac{d}{d\tau} a(\tau) = u(a(\tau), \tau), \\
\frac{d}{d\tau} b(\tau) = u(b(\tau), \tau),
\end{cases}
\]

and the boundary condition

\[
\begin{cases}
(P(\rho) + \mu(\rho)u_\xi)(a(\tau), \tau) = 0, \\
(P(\rho) + \mu(\rho)u_\xi)(b(\tau), \tau) = 0.
\end{cases}
\]

It is interesting to study the case when \( \rho_0 \) is compactly supported and connects to vacuum continuously. In this case, the boundary conditions (1.4) should be replaced by

\[
\rho(a(\tau), \tau) = \rho(b(\tau), \tau) = 0, \quad \tau \geq 0.
\]

Within moderate ranges of temperature and density, a real gas is well approximated by an ideal gas (the heat conductivity \( \kappa \) and viscosity \( \mu \) are constant). At high temperatures and densities, however, the specific heat flow, the conductivity \( \kappa \) and viscosity \( \mu \) vary with the temperature and the density. We refer to [1,18] for the extensive discussions and experimental evidence in this direction. And in mathematics, the study in [6] shows that the continuous dependence on the initial data of the solutions to the Navier–Stokes equations with vacuum and constant viscosity coefficient fails. Considering modified Navier–Stokes system in which the viscosity coefficient depends on the density, Liu, Xin and Yang in [12] proved that the system is local well-posedness. As remarked in [12], the Navier–Stokes equations can be derived from the Boltzmann equation through the viscosity coefficient is a function of temperature. For isentropic flow, this dependence is translated into the dependence of the viscosity on the density. Here, for simplicity, we consider only the polytropic gas throughout the rest of this paper. That is, we assume \( P(\rho) = A\rho^\gamma \) and \( \mu(\rho) = c\rho^\theta \) with

\[
0 < \theta < \gamma, \quad \gamma > 1,
\]

and \( A > 0, c > 0 \) being constants. We normalized \( A = 1 \) and \( c = 1 \).

In particular, the viscosity of gas is proportional to the square root of the temperature for hard sphere collision (as pointed out in [13]), and the relation between \( \theta \) and \( \gamma \) is

\[
\theta = \frac{\gamma - 1}{2}.
\]
Our hypotheses (1.6) covers it.

Since the free boundaries \( x = a(\tau) \) and \( x = b(\tau) \) are unknown in Euler coordinates, we will convert them to fixed boundaries by using Lagrangian coordinates. We introduce the following coordinate transformation

\[
x = \int_{a(\tau)}^{\xi} \rho(y, \tau) \, dy, \quad t = \tau,
\]

then the free boundaries \( x = a(\tau) \) and \( x = b(\tau) \) become

\[
\tilde{a}(t) = 0 \quad \text{and} \quad \tilde{b}(t) = \int_{a(t)}^{b(t)} \rho(y, t) \, dy = \int_{a}^{b} \rho_{0}(y) \, dy,
\]

where \( \int_{a}^{b} \rho_{0}(y) \, dy \) is the total mass initially, and without loss of generality, we can normalize it to 1. So in terms of Lagrangian coordinates, the free boundaries become fixed.

Under the coordinate transformation, Eqs. (1.1) are transformed into

\[
\begin{align*}
\rho_t + \rho^2 u_x &= 0, & t > 0, \\
u_t + (\rho \nu)_x &= (\rho^{1+\theta} u_x)_x, & 0 < x < 1,
\end{align*}
\]

while the initial data and boundary conditions as

\[
(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in [0, 1],
\]

\[
\rho(0, t) = \rho(1, t) = 0, \quad t \geq 0.
\]

It is standard that if we can solve problem (1.7)–(1.9), then the Cauchy problem (1.1)–(1.2) and (1.5) has a solution.

We are interested in the case that the viscous gas is in contact with the vacuum when the viscosity depends on the density. There are many works ([11–13,15] et al.), based on the assumption that the gas connects to vacuum with jump discontinuities, and the density of the gas has compact support. Among of them, T.P. Liu, Z. Xin and T. Yang in [12] obtained the local existence of weak solutions to Navier–Stokes equations. If \( \mu = c \rho^\theta \), M. Okada, Š. Matušů-Nečasová and T. Makino in [13] obtained the global existence of weak solutions when \( 0 < \theta < 1/3 \) with the same property. This result was later generalized to the case when \( 0 < \theta < 1/2 \) and \( 0 < \theta < 1 \) in [15] and [11], respectively.

When the viscous gas is connected to vacuum states with continuous density, Yang and Zhao in [16] obtained the local existence of weak solutions when \( \frac{1}{2} < \theta \leq \gamma - \frac{1}{3} \). Then Yang and Zhu in [17] get the global existence of weak solutions when \( 0 < \theta < \frac{2}{5} \) and \( \frac{1}{3} < \theta < \frac{3}{7} \). This result was later generalized to the case when \( 0 < \theta < \frac{1}{2} \) and \( 0 < \theta < \frac{1}{3} \) in [4,14], respectively. In [14], authors obtained the uniqueness of weak solution when \( 0 < \theta < 1 - \frac{\sqrt{6}}{3} \). Then, in [5], we obtained the uniqueness of weak solution when \( 0 < \theta < \frac{1}{2} \).

The questions addressed here are motivated by the fact that discontinuous solutions are fundamental both in the physical theory of non-equilibrium thermodynamics as well as in the mathematical study of inviscid models for compressible flow. On the other hand, Hoff
and Serre in [6], used the existence and uniqueness results of (1.1) when \( \mu = \epsilon \) with discontinuous initial density. And in [4,5,11–17], the initial density \( \rho_0 \) must be a differentiable function in \( x \in (0, 1) \). It is natural, therefore, to seek a rigorous theory which accommodates these discontinuities. In this paper, we consider the case that the initial density \( \rho_0 \) could be discontinuous at \( 0 < x < 1 \).

Here, we assume only that, \( \rho_0 \) is bounded above, and degenerate to zero at the vacuum boundaries. In particular, we allow piecewise smooth density with arbitrarily large jump discontinuities. Hence, the viscosity vanishes at vacuum, and how to overcome this degeneracy while density function has lower regularity is the main task of this paper. The main estimate is to obtain the lower bound of the density function. Since the density function vanishes at vacuum boundary, we can only obtain a lower bound in the form of a power function \((x(1-x))^\alpha\) determined by the initial density. The better lower bound implies large interval of \( \theta \) \((0 < \theta < \gamma)\) for local existence. And using weight function to control the upper bound of density function, we will obtain the uniqueness of weak solution when \( 0 < \theta < \gamma \). Then, some important physical consequences of our results are that neither vacuum states nor concentration states can occur in the interior of the gas, and the interface separating the gas and vacuum propagates with finite speed in local time, no matter how large the oscillation of the initial density. When viscosity is constant, D. Hoff (see [7–10] et al.), consider the case of discontinuous initial data. When the gas connects to vacuum with jump discontinuities, we in [2,3] consider the case of discontinuous initial density, and obtained the existence and uniqueness of the weak solution.

Our hypotheses on the initial data are then that

(A1) there exist two positive constants \( 0 < \alpha < \min\{\frac{1}{2\gamma}, \frac{1}{1+\theta}\} \) and \( B \geq A > 0 \) such that

\[
B(x(1-x))^\alpha \geq \rho_0(x) \geq A(x(1-x))^\alpha
\]

and

\[
\lim_{x \to 0^+} \rho_0(x) = \lim_{x \to 1^-} \rho_0(x) = 0.
\]

(A2) \( u_0(x) \in H^1([0, 1]), \rho_0(x) \in L^\infty([0, 1]) \).

We denote by \( \langle f \rangle_X^{a,b} \) the usual Hölder norm

\[
\sup \left\{ \frac{|f(x,t) - f(y,s)|}{|x-y|^a + |t-s|^b}, (x,t), (y,s) \in X, (x,t) \neq (y,s) \right\}.
\]

We now state the main results in this paper as follows.

**Theorem 1.1 (Existence).** Under the conditions (1.6), (A1) and (A2), there exists \( T_1 > 0 \) such that system (1.7)–(1.9) has a weak solution \((\rho, u)\) in the sense that for \( 0 \leq x \leq 1 \) and \( 0 \leq t \leq T_1 \),

\[
\frac{A}{2}(x(1-x))^\alpha \leq \rho(x,t) \leq C,
\]

\[
\rho(0,t) = \rho(1,t) = 0,
\]

\[
\rho, u \in C^{\frac{1}{2}}([0,T_1]; L^2([0,1])) \cap L^\infty([0, 1] \times [0, T_1]),
\]
\begin{align}
\|u\|_{L^\infty([0,T_1];L^2([0,1]))} + \|u_x\|_{L^\infty([0,T_1];L^\beta([0,1]))} + \|u_t\|_{L^2([0,1] \times [0,T_1])} & \leq C(T_1), \quad (1.15) \\
\sup_{t \in (0,T_1)} \left\{ t \int_0^{T_1} \left[ u_x^2 + u_t^2 \right] dt + \int_0^{T_1} \rho^{1+\theta} u_x^2 dt \right\} \\
+ \int_0^{T_1} \|\rho u_x\|_{L^\infty} dt \int_0^{T_1} t \rho^{1+\theta} u_x^2 dt + \int_0^{T_1} t \|u_x\|_{L^\beta} dt & \leq C(T_1), \quad (1.16) \\
\|u\|_{L^\infty([0,1] \times [0,T_1])} + \tau^{\theta} \|u\|_{L^\beta([0,1] \times [\tau,T_1])} & \leq C(T_1), \quad \forall 0 < \tau < T_1, \quad (1.17)
\end{align}

where \(1 < \beta < \frac{2}{1+\alpha+\alpha^2}\), and
\begin{equation}
\sup_{t \in (0,T_1)} \tau^\gamma \left\| (\rho^{1+\theta} u_x - \rho^\gamma) \right\|_{H^1} \leq C(T_1), \quad (1.18)
\end{equation}

for some positive constants \(C(T_1) = C(T_1, A, \alpha, \|\rho_0\|_{L^\infty}, \|u_0\|_{H^1})\) and \(C = C(\|\rho_0\|_{L^\infty}, \|u_0\|_{L^2})\), and the following equations hold:
\begin{equation}
\rho_t + \rho^2 u_x = 0, \quad \rho(x,0) = \rho_0(x) \quad \text{for a.e. } x \in (0,1) \text{ and } \forall 0 < t \leq T_1, \quad (1.19)
\end{equation}
\begin{equation}
\int_0^{T_1} \left\{ u \phi_t + \left( \rho^\gamma - \rho^{1+\theta} u_x \right) \phi_x \right\} dx dt + \int_0^1 u_0(x) \phi(x,0) dx = 0, \quad (1.20)
\end{equation}

for any test function \(\phi(x,t) \in C_0^\infty(Q)\) with \(Q = \{(x,t) \mid 0 \leq x \leq 1, 0 \leq t \leq T_1\}\).

**Theorem 1.2 (Uniqueness).** Under the conditions (1.6), (A1)–(A2), and \(\rho_0 \leq B(x(1-x))^{\alpha_1}\), for some constants \(\alpha \geq \alpha_1 \geq \max\left\{ \frac{\theta}{\alpha}, 2\theta \alpha^2 \right\}\) and \(B > 0\), let \((\rho_1, u_1)(x,t)\) and \((\rho_2, u_2)(x,t)\) be two weak solutions to the initial-boundary value problem (1.7)–(1.9) in \(0 \leq t \leq T_1\) as described in Theorem 1.1. Then \((\rho_1, u_1)(x,t) = (\rho_2, u_2)(x,t)\) a.e. in \((x,t) \in (0,1) \times [0,T_1]\).

The proof of the existence is given in Section 2 in a sequence of eight lemmas. The first of these gives some useful identities. The second gives a time-independent bound for \(\|u_t^2 + \frac{1}{\gamma-1} \rho^{\gamma-1} \|^1_{L^1}\), which represents the total energy in the fluid, and derive a priori upper bound for \(\rho\). This upper bound then enable us in Lemma 2.3 to derive the equicontinuity in time of \(u(\cdot, t)\) in \(L^2\)-norm at \(t = 0\). And using Lemma 2.3 and weight function \((x(1-x))^\alpha\), we can get the uniform lower bound for \(\rho\) in Lemma 2.4. These pointwise bounds for \(\rho\) then enable us in Lemmas 2.5–2.7 to exploit the parabolicity of the second equation in (1.7) to derive certain higher regularity estimates for \(u\). Of special interest here is the bound given above in (1.18) for the quantity \(\rho^{1+\theta} u_x - \rho^\gamma\). It is well known (see Hoff [8], for example) that initial discontinuities in \(\rho\) persist for all time, when \(\mu = \text{const.}\). And we prove that initial discontinuities in \(\rho\) persist for all time, when \(\mu = \rho^\theta\), in [2]. So that \(\rho(\cdot, t)\) may be integrable but not smooth. On the other hand, the fact that the difference \(\rho^{1+\theta} u_x - \rho^\gamma\) is in \(H^1_x \hookrightarrow C^{1/2}\) evidently implies a cancellation of singularities. In the present context, this cancellation of singularities can be shown to prevent oscillations.
in weakly converging sequences of solution of (1.7). These ideas are given in Lemma 2.8, where we complete the proof of the existence by obtaining the solution \((\rho, u)\) as the strong limit of approximate solutions corresponding to mollified initial data. The proof of the uniqueness is given in Section 3, applying energy method.

2. Proof of the existence

Let \((\rho_0, u_0)\) be as described in the theorem, for simplicity, we still let \((\rho_0, u_0)\) denote the extension of \((\rho_0, u_0)\) in \(\mathbb{R}\), i.e.,

\[
\rho_0(x) := \begin{cases} 
0, & x \in (1, \infty), \\
\rho_0(x), & x \in [0, 1], \\
0, & x \in (-\infty, 0),
\end{cases}
\]

and

\[
u_0(x) := \begin{cases} 
\nu_0(1), & x \in (1, \infty), \\
u_0(x), & x \in [0, 1], \\
u_0(0), & x \in (-\infty, 0),
\end{cases}
\]

We define the approximate initial data to \((\rho_\varepsilon^0, u_\varepsilon^0)\):

\[
\rho_\varepsilon^0(x) := (\rho_0 * j_\varepsilon)(x) + \varepsilon, \quad u_\varepsilon^0 = (u_0 * j_\varepsilon)(x), \quad x \in [0, 1],
\]

where \(j_\varepsilon(x) = \varepsilon^{-1} j(x/\varepsilon)\) is the standard mollifier.

First, the existence of a weak solution \((\rho_\varepsilon, u_\varepsilon)\) of (1.7) with the initial data \((\rho_\varepsilon^0, u_\varepsilon^0)\) and the boundary conditions

\[
(\rho_\varepsilon u_\varepsilon^p + \rho_\varepsilon^{p+1} u_\varepsilon x)(d, t) = 0, \quad d = 0, 1,
\]

is proved in [12]. Here we need to get further a priori estimates on \((\rho_\varepsilon, u_\varepsilon)\), that the bounds of \((\rho_\varepsilon, u_\varepsilon)\) are independent of \(\varepsilon\). That is, we assume that \((\rho_\varepsilon, u_\varepsilon)\) is a solution of (1.7)–(1.8) and (2.1), which is as smooth as required for the arguments presented here. Alternatively, the reader may regard the present analysis as applying at the level of discrete approximations, where a priori regularity is not an issue; the continuous estimates in lemmas would then follow in the limit as the mesh size tends to zero. In the following lemmas we give various bounds for \((\rho_\varepsilon, u_\varepsilon)\) in terms of a constant \(C_\varepsilon\), which are generic positive constants depending only on the initial data, and are independent of \(\varepsilon\). For simplicity, we omit the superscripts \(\varepsilon\) in \((\rho_\varepsilon, u_\varepsilon)\) from now on till Lemma 2.7.

First we list some useful identities.

**Lemma 2.1 (Some useful identities).** Under the conditions of Theorem 1.1, we have for \(0 < x < 1, \ t > 0\) that

\[
\frac{d}{dt} \int_0^x u(y, t) \, dy = -\frac{d}{dt} \int_x^1 u(y, t) \, dy,
\]

\[
(\rho^{p+1} u_x)(x, t) = \rho^p(x, t) + \int_0^x u_t(y, t) \, dy = \rho^p(x, t) - \int_x^1 u_t(y, t) \, dy
\]

\[
\rho^p(x, t) + \theta \int_0^t \rho^p(x, s) \, ds = \rho^p_0(x) - \theta \int_0^t u_t(y, s) \, dy \, ds
\]
\[ \rho(x,t) = \rho_0^\theta(x) + \theta \int_0^t \int_0^1 u_t(y,s) \, dy \, ds. \] (2.4)

The proof of Lemma 2.1 is standard, and we omit the details.

The second lemma is concerned with the standard energy estimate, a uniform upper bound for the density function, whose proof can be found in [4].

**Lemma 2.2.** Under the conditions of Theorem 1.1, we can deduce for \( 0 < x < 1 \), and each fixed constant \( T > 0 \) that

\[
\int_0^1 \left( \frac{1}{2} u^2 + \frac{1}{\gamma - 1} \rho^{\gamma - 1} \right) \, dx + \int_0^t \int_0^1 \rho^{1+\theta} u_x^2 \, dx \, ds = \int_0^1 \left( \frac{1}{2} u_0^2 + \frac{1}{\gamma - 1} \rho_0^{\gamma - 1} \right) \, dx = C_0, \tag{2.5}
\]

\[
\rho(x,t) \leq C \rho_0 + C C_0^{1/2} \left( x(1-x) \right)^{1/2} \leq C_1, \tag{2.6}
\]

where \( 0 < t \leq T \), and \( C_i = C_i(\|\rho_0\|_{L^\infty}, \|u_0\|_{L^2}) \) (\( i = 0, 1 \)) be positive constants.

Next, in order to derive the uniform lower bound for \( \rho(x,t) \), we need to prove the equicontinuity in time of \( u(\cdot,t) \) in \( L^2 \)-norm at \( t = 0 \).

**Lemma 2.3.** There exists a positive constant \( C_2 = C_2(\|\rho_0\|_{L^\infty}, \|u_0\|_{H^1}) \), such that

\[
\int_0^1 (u(t,x) - u_0(x))^2 \, dx \leq C_2 t. \tag{2.7}
\]

**Proof.** We rewrite (1.7) as

\[
\frac{d}{dt}(u - u_0) + (\rho^\gamma)_x = (\rho^{1+\theta} u_x)_x. \tag{2.8}
\]

Multiplying both sides of (2.8) by \( (u - u_0) \), and integrating it with respect to \( x \) over \([0, 1]\), we have

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (u - u_0)^2 \, dx + \int_0^1 (u - u_0)(\rho^\gamma - \rho^{1+\theta} u_x)_x \, dx = 0.
\]

Using integration by parts, the boundary conditions (2.1) and Cauchy–Schwartz inequality, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (u - u_0)^2 \, dx + \int_0^1 \rho^{1+\theta} u_x^2 \, dx.
\]
$$\begin{align*}
&= \int_0^1 \left\{ \rho^\gamma u_x - \rho^\gamma u_{0x} + \rho^{1+\theta} u_x u_{0x} \right\} dx \\
&\leq \frac{1}{2} \int_0^1 \rho^{1+\theta} u_x^2 dx + \frac{3}{2} \int_0^1 \rho^{2\gamma-\theta-1} dx + \frac{3}{2} \int_0^1 \rho^{1+\theta} u_{0x}^2 dx.
\end{align*}$$

Consequently, we have
$$\frac{d}{dt} \int_0^1 (u - u_0)^2 dx + \int_0^1 \rho^{1+\theta} u_x^2 dx \leq 3 \int_0^1 \left[ \rho^{2\gamma-\theta-1} + \rho^{1+\theta} u_{0x}^2 \right] dx \leq C_2,$$
provided $\|u_{0x}\|_{L^2}$ is bounded, and we have used (2.6). Thus Lemma 2.3 is true. 

As a consequence of this lemma, we are going to derive the lower bound for $\rho(x,t)$ locally in time.

**Lemma 2.4.** Under the conditions in Theorem 1.1, there is a sufficiently small positive constant $T_1 = T_1(A, \alpha, \|\rho_0\|_{L^\infty}, \|u_0\|_{H^1})$, such that for $0 < x < 1$, $0 < t \leq T_1$,
$$\rho(x,t) \geq \frac{A}{2} \left( x(1-x) \right)^\alpha. \quad (2.9)$$

**Proof.** By (2.4), (2.7) and the fact that $\alpha \theta < \frac{1}{2}$, we have
$$\begin{align*}
\rho^\theta (x,t) + \theta \int_0^t \rho^\gamma (x,s) ds \\
&= \rho^\theta_0 (x) - \theta \int_0^t \int_0^x u_r(y,s) dy ds \\
&= \rho^\theta_0 (x) - \theta \int_0^x \left[ u(y,t) - u_0(y) \right] dy \\
&\geq A^\theta (x(1-x))^{\theta\alpha} - \theta \left( \int_0^1 |u(y,t) - u_0(y)|^2 dy \right)^{\frac{1}{2}} (x(1-x))^{\frac{1}{2}} \\
&\geq (x(1-x))^{\theta\alpha} \left[ A^{\theta} - \theta C_2 \frac{1}{2} t^{\frac{1}{2}} \right] \\
&\geq \left( \frac{3}{4} A \right)^\theta (x(1-x))^{\theta\alpha}, \quad (2.10)
\end{align*}$$
provided that
$$0 \leq t \leq t_2, \quad \text{where} \quad A^{\theta} - \theta C_2 \frac{1}{2} t^{\frac{1}{2}} \geq \left( \frac{3}{4} A \right)^\theta. \quad (2.11)$$
To get a lower bound on $\rho(x,t)$ from (2.10), we need to get an upper bound on the term $\int_0^t \rho \gamma(x,s) ds$ for sufficiently small $t$. For this purpose, we set

$$E(t) = \int_0^t \rho \gamma(x,s) ds,$$

(2.12)

and similar to that of (2.10), we can deduce that $E(t)$ satisfies the following differential inequality from (2.4):

$$\left[E'(t)\right]^\theta + \theta E(t) \leq \left(B^\theta + \theta C_2^\frac{1}{\gamma} t^{\frac{1}{\gamma}}\right)(x(1-x))^{\theta \alpha} \leq C_3(t_2)(x(1-x))^{\theta \alpha},$$

(2.13)

where $0 \leq t \leq t_2$.

Noticing $E(0) = 0$ and $0 < \theta < \gamma$, we finally deduce from (2.13) that

$$-\theta E(t) \geq C_3(t_2)(x(1-x))^{\theta \alpha} \left[1 + t(\gamma - \theta)(C_3(t_2)(x(1-x))^{\theta \alpha})^{\frac{\gamma - \theta}{\gamma}}\right]^{\theta - \gamma} - 1.$$

(2.14)

By choosing $t_3 \leq t_2$ sufficiently small such that

$$C_3(t_2)\left[1 - \left[1 + t_3(\gamma - \theta)(C_3(t_2)(x(1-x))^{\theta \alpha})^{\frac{\gamma - \theta}{\gamma}}\right]^{\theta - \gamma}\right] \leq \left((\frac{3}{4})^{\theta} - \left(\frac{1}{2}\right)^{\theta}\right)A^\theta,$$

(2.15)

we can get from (2.14)–(2.15) that

$$-\theta \int_0^t \rho \gamma(x,s) ds \geq -\left((\frac{3}{4})^{\theta} - \left(\frac{1}{2}\right)^{\theta}\right)A^\theta(x(1-x))^{\theta \alpha},$$

(2.16)

where $0 \leq t \leq t_3$.

Inserting (2.16) into (2.10), we can arrive at

$$\rho^\theta(x,t) \geq \left(\frac{3}{4}A\right)^\theta(x(1-x))^{\theta \alpha} - \theta \int_0^t \rho \gamma(x,s) ds \geq \left(\frac{A}{2}\right)^\theta(x(1-x))^{\theta \alpha},$$

(2.17)

provided $0 \leq t \leq t_3$. Thus we choose $T_1 = t_3$, and complete the proof of Lemma 2.4.

We can now exploit the parabolicity of the second equation in (1.7) to obtain certain higher order regularity estimates for $u$.

**Lemma 2.5.** There exist two positive constants $C_i = C_i(T_1, A, \alpha, \|\rho_0\|_{L^\infty}, \|u_0\|_{H^1})$ ($i = 4, 5$), such that for $0 \leq t \leq T_1$,
\[
\int_0^1 (\rho^{1+\theta} u_x^2)(x,t) \, dx + \int_0^t \int_0^1 u_t^2 \, dx \, ds \leq C_4, \quad (2.18)
\]

\[
\int_0^1 \| (\rho u_x)(\cdot,t) \|^2_{L^\infty} \, dt \leq C_5. \quad (2.19)
\]

**Proof.** Multiplying \((1.7)\_2\) by \(u_t(x,s)\), and integrating it with respect to \(x\) and \(s\) over \([0, 1] \times [0, t]\), we have

\[
\int_0^t \int_0^1 u_t^2 \, dx \, ds = \int_0^t \int_0^1 u_t [\rho^{1+\theta} u_x - \rho^{\gamma}] \, dx \, ds.
\]

Using integration by parts, \((1.7)\_1\) and boundary conditions \((2.1)\), we obtain

\[
\int_0^t \int_0^1 u_t [\rho^{1+\theta} u_x - \rho^{\gamma}] \, dx \, ds
\]

\[
= \int_0^t \int_0^1 u_t [\rho^{\gamma} - \rho^{1+\theta} u_x] \, dx \, ds
\]

\[
= \int_0^t \int_0^1 u_t \rho^{\gamma} - \rho^{1+\theta} \left( \frac{1}{2} u_x^2 \right)_t \, dx \, ds
\]

\[
= \int_0^1 \left\{ u_x \left[ \rho^{\gamma} - \frac{1}{2} \rho^{1+\theta} u_x \right] - u_{0x} \left[ \rho^{\gamma}_0 - \frac{1}{2} \rho^{1+\theta}_0 u_{0x} \right] \right\} \, dx
\]

\[
+ \int_0^t \int_0^1 \left\{ \gamma u_x^2 \rho^{1+\gamma} - \frac{1 + \theta}{2} u_x^2 \rho^{2+\theta} \right\} \, dx \, ds. \quad (2.20)
\]

Thus

\[
\int_0^t \int_0^1 u_t^2 \, dx \, ds + \frac{1}{2} \int_0^1 \rho^{1+\theta} u_x^2 \, dx
\]

\[
= \int_0^1 \left\{ \rho^{\gamma} u_x - u_{0x} \left[ \rho^{\gamma}_0 - \frac{1}{2} \rho^{1+\theta}_0 u_{0x} \right] \right\} \, dx
\]

\[
+ \int_0^t \int_0^1 \left\{ \gamma u_x^2 \rho^{1+\gamma} - \frac{1 + \theta}{2} u_x^2 \rho^{2+\theta} \right\} \, dx \, ds
\]
\[
\begin{align*}
\int_0^1 \left\{ \frac{1}{4} \rho^{1+\theta} u_x^2 + \rho^{2\gamma-1-\theta} + \frac{1}{2} u_0^2 + \frac{1}{2} \rho_0^{2\gamma} + \frac{1}{2} \rho_0^{1+\theta} u_{0x}^2 \right\} dx \\
+ \int_0^t \int_0^1 \left\{ \gamma \rho^{\gamma+1} u_x^2 + \frac{1}{2} \rho^{2+\theta} |u_x|^3 \right\} dx ds.
\end{align*}
\]

Consequently, using (1.6), (2.5), (2.6) and (2.21), we get
\[
\begin{align*}
\int_0^1 \rho^{1+\theta} u_x^2 ds + \int_0^1 u_t^2 dx ds & \leq C_7 + \frac{1+\theta}{2} \int_0^1 \rho^{2+\theta} |u_x|^3 dx ds. \quad (2.22)
\end{align*}
\]

To estimate the last integral on the right-hand side of the inequality (2.22), we estimate \( \sup_{x \in [0,1]} |\rho u_x|(x,s) \) first. From (1.6), (2.3), (2.6), (2.9) and the fact that \( \theta \alpha < \frac{1}{2} \), we have
\[
|\rho u_x|(x,s) = \left| \rho^{\gamma-\theta}(x,s) + \rho^{-\theta}(x,s) \int_0^x u_t(y,s) dy \right|
\]
\[
\leq C_1^{\gamma-\theta} + \left( \frac{A}{2} \right)^{-\theta} (x(1-x))^{-\theta\alpha} \left( \int_0^1 u_t^2(y,s) dy \right)^{\frac{1}{2}} (x(1-x))^{\frac{1}{2}}
\]
\[
\leq C_8 + C_9 \| u_t(\cdot,s) \|_{L^2}. \quad (2.23)
\]

Therefore,
\[
\int_0^t \int_0^1 \rho^{2+\theta} |u_x|^3 dx ds
\]
\[
\leq \int_0^t \sup_{x \in [0,1]} |\rho u_x| \int_0^1 u_x^2 dx ds
\]
\[
\leq C_8 \int_0^t \int_0^1 \rho^{1+\theta} u_x^2 ds ds + C_9 \int_0^t \| u_t(\cdot,s) \|_{L^2} \int_0^1 \rho^{1+\theta} u_x^2 dx ds
\]
\[
\leq C_8 C_0 + \frac{1}{1+\theta} \int_0^t \int_0^1 u_t^2 dx ds + C_10 \int_0^t \| \rho^{1+\theta} u_x^2(\cdot,s) \|_{L^1} \int_0^1 \rho^{1+\theta} u_x^2 dx ds. \quad (2.24)
\]

It follows from (2.22) and (2.24) that
\[
\int_0^1 \rho^{1+\theta} u_x^2 dx + \int_0^1 u_t^2 dx ds
\]
\[
\leq C(T_1) + C(T_1) \int_0^t \left\| \rho^{1+\theta} u_x^2 (\cdot, s) \right\|_{L^1} \int_0^1 \rho^{1+\theta} u_x^2 \, dx \, ds,
\]

using Gronwall’s inequality and (2.5), we complete the proof of (2.18). From (2.18) and (2.23), we obtain (2.19) immediately.

**Lemma 2.6.** There exists a positive constant \( C_{11} = C_{11}(T_1, A, \alpha, \|\rho_0\|_{L^\infty}, \|u_0\|_{H^1}) \) such that

\[
\sup_{0 < t \leq T_1} t \int_0^1 u_t^2 \, dx + \int_0^{T_1} t \int_0^1 \rho^{1+\theta} u_x^2 \, dx \, dt \leq C_{11}.
\]

**(2.25)**

**Proof.** By differentiating (1.7)_2 with respect to the time \( t \), then integrating it after multiplying \( s u_t(x, s) \) with respect to \( x \) and \( s \) over \([0, 1] \times [0, t]\), we deduce

\[
\frac{1}{2} t \int_0^1 u_t^2 \, dx + \int_0^1 s (\rho^{1+\theta})_{x_t} u_t \, dx \, ds
\]

\[
= \frac{1}{2} \int_0^1 u_t^2 \, dx + \int_0^1 s (\rho^{1+\theta} u_x)^{x_t} u_t \, dx \, ds.
\]

**(2.26)**

Using integration by parts, we have from (1.7)_1 and the boundary conditions (2.1) that

\[
\int_0^1 s (\rho^{1+\theta} u_x - \rho^\gamma)_{x_t} u_t \, dx \, ds
\]

\[
= - \int_0^1 s (\rho^{1+\theta} u_x - \rho^\gamma) u_{tx} \, dx \, ds
\]

\[
= - \int_0^1 \left[ s \rho^{1+\theta} u_x^2_{xt} - (1 + \theta) s \rho^{2+\theta} u_x^2_{tx} + \gamma s \rho^{1+\gamma} u_x u_{xt} \right] dx \, ds.
\]

**(2.27)**

Substituting (2.27) into (2.26), using Cauchy–Schwartz inequality, (2.5)–(2.6) and (2.18), we have

\[
\frac{1}{2} t \int_0^1 u_t^2 \, dx + \int_0^1 s \rho^{1+\theta} u_x^2 \, dx \, ds
\]

\[
= \frac{1}{2} \int_0^1 u_t^2 \, dx - \int_0^1 \left[ s \rho^{1+\gamma} u_x u_{tx} - (1 + \theta) s \rho^{2+\theta} u_x^2 \right] dx \, ds
\]
\[ \leq \frac{1}{2} C_4 + \frac{1}{2} \int_0^1 \int_0^1 s \rho^{1+\theta} u_{tx}^2 \, dx \, ds + C \int_0^1 \int_0^1 s \rho^{2\gamma+1-\theta} u_x^2 \, dx \, ds \\
+ C \int_0^1 \int_0^1 s \rho^{\theta+3} u_x^4 \, dx \, ds \]
\[ \leq \frac{1}{2} C_4 + \frac{1}{2} \int_0^1 \int_0^1 s \rho^{1+\theta} u_{tx}^2 + C C_0 C_1^{2\gamma-2\theta} t + C t \int_0^1 \int_0^1 \rho^{\theta+3} u_x^4 \, dx \, ds. \] (2.28)

By (2.18)–(2.19) and (2.28), we get
\[ t \int_0^1 u_t^2 \, dx + \int_0^1 s \rho^{1+\theta} u_{xt}^2 \, dx \, ds \]
\[ \leq C_{13} + C T_1 \int_0^1 \rho^{3+\theta} u_x^4 \, dx \, ds \]
\[ \leq C_{13} + C T_1 \int_0^1 \| \rho u_x (\cdot, s) \|_{L^\infty}^2 \, ds \sup_{s \in [0, T_1]} \int_0^1 \rho^{1+\theta} u_x^2 (x, s) \, dx \]
\[ \leq C_{13} + C C_4 C_5 T_1 \leq C_{11}. \]

This complete the proof of Lemma 2.6. \( \square \)

**Lemma 2.7.** There exist five positive constants \( C_i = C_i (T_1, A, \alpha, \| \rho_0 \|_{L^\infty}, \| u_0 \|_{H^1}) \) \((i = 14, 15, 16, 17, 18), such that for \( 0 < x < 1 \) and \( 0 \leq t \leq T_1, \)
\[ \int_0^1 |u_x|^\beta (x, t) \, dx \leq C_{14}, \] (2.29)
\[ |u(x, t)| \leq C_{15}, \] (2.30)
\[ \sup_{t \in (0, T_1]} t \int_0^1 |u_x|^2 (x, t) \, dx \leq C_{16}, \] (2.31)
\[ \int_0^{T_1} \int_0^1 t |u_{xt}|^\beta \, dx \, dt \leq C_{17}, \] (2.32)
\[ \langle u \rangle_{[0,1] \times [\tau, T_1]} \leq C_{18} \tau^{-\frac{1}{\beta}}, \forall \tau \in (0, T_1), \] (2.33)

where \( 1 < \beta < \frac{2}{1+\alpha+\theta \alpha}. \)
Proof. From (2.9) and (2.18), using Young’s inequality and the fact that \( 1 < \beta < \frac{2}{1 + \alpha + \theta \alpha} \), we have

\[
\int_0^1 |u_x|^{\beta} \, dx \leq C \int_0^1 \rho^{1 + \theta} u_x^2 \, dx + C \int_0^1 \rho^{-(1 + \theta) - \frac{\theta \alpha}{2 - \beta}} \, dx
\]

\[
\leq CC_4 + C \left( \frac{A}{2} \right)^{-(1 + \theta) - \frac{\theta \alpha}{2 - \beta}} \int_0^1 (x(1-x))^{-(1 + \theta) - \frac{\theta \alpha}{2 - \beta}} \, dx
\]

\[
\leq C_{14}.
\]

This complete the proof of (2.29). And from (2.5) and (2.29), we get (2.30) immediately.

In fact, from (2.3) and (2.9), we get

\[
|u_x|(x, t) \leq \rho^{1 - \theta} - \frac{1}{\theta} \int_0^x u_t(y, t) \, dy + \rho^{1 - \theta} (x, t)
\]

\[
\leq \left( \frac{A}{2} \right)^{1 - \theta} (x(1-x))^{1 - \theta} - \frac{1}{\theta} \left( \int_0^1 u_t^2 \, dx \right)^{\frac{1}{2}}
\]

\[
+C_{1}^{1 - \theta} \left( \frac{A}{2} \right)^{1 - \theta} (x(1-x))^{1 - \theta} - \frac{\alpha}{\theta}
\]

and from (2.25), we have

\[
\sup_{t \in [0, T_1]} \int_0^1 |u_x|^2(x, t) \leq C_{19} \int_0^1 \left[ (x(1-x))^{1 - 2\alpha(1 + \theta)} + (x(1-x))^{2\theta \alpha} \right] \, dx
\]

\[
\leq C_{16},
\]

since \( \alpha < \min\left\{ \frac{1}{2\eta}, \frac{1}{1 + \theta} \right\} \). This complete the proof of (2.31).

From (2.9) and (2.25), using similar proof of (2.29), we obtain

\[
\int_0^{T_1} \int_0^1 t |ux_t|^{\beta} \, dx \, dt \leq C_{17}, \quad 1 < \beta < \frac{2}{1 + \alpha + \theta \alpha},
\]

and using (2.18) and Gagliardo–Nirenberg’s inequality (\( \| f \|_{L^\infty} \leq C \| f \|_{L^2}^\eta \| f_x \|_{L^\beta}^{1-\eta} \), where \( \frac{1}{\eta} + (\frac{1}{\beta} - 1)(1 - \eta) = 0 \)), we have

\[
\int_\tau^{T_1} |u_t|^{\beta}(x, s) \, ds \leq C_{20} \tau^{-1}, \quad \forall x \in [0, 1], \quad \tau \in (0, T_1).
\]
From (2.35), and using Young’s inequality, we get for \( y \in [0, 1] \) and \( t, s \in [\tau, T_1] \)

\[
\frac{|u(y, t) - u(y, s)|}{|t - s|^{\frac{\beta - 1}{\beta}}} = \frac{|\int_s^t u_y(y', t') \, dt'|}{|t - s|^{\frac{\beta - 1}{\beta}}} \leq C \left( \int_{\tau}^{T_1} |u_t(y, t)| \, dt \right)^{\frac{1}{\beta}} \\
\leq CC_{20} \tau^{-\frac{1}{\beta}}.
\]

From (2.36), using Young’s inequality, we get for \( \tau \leq t \leq T_1 \),

\[
\frac{|u(x, t) - u(y, t)|}{|x - y|^{\frac{2}{3}}} = \frac{\int_x^y u_x(z, t) \, dz}{|x - y|^{\frac{2}{3}}} \leq C \frac{|x - y|^{\frac{1}{2}} (\int_0^1 u_x^2 \, dx)^{\frac{1}{2}}}{|x - y|^{\frac{1}{3}}} \leq CC_{16} \tau^{-\frac{1}{2}}.
\]

From (2.36)–(2.37), we get

\[
\langle u \rangle_{[0,1] \times [\tau, T_1]} = \sup_{(x,t)(y,s) \in [0,1] \times [\tau, T_1]} \frac{|u(x, t) - u(y, s)|}{|x - y|^{\frac{1}{2}} + |t - s|^{\frac{\beta - 1}{\beta}}},
\]

\[
\leq \sup_{x,y \in [0,1]} \frac{|u(x, t) - u(y, t)|}{|x - y|^{\frac{1}{2}}} + \sup_{t,s \in [\tau, T_1]} \frac{|u(y, t) - u(y, s)|}{|t - s|^{\frac{\beta - 1}{\beta}}} \leq C_{18} \tau^{-\frac{1}{2}}.
\]

This complete the proof of Lemma 2.7. \( \square \)

**Proof of the existence.** Similar to the arguments in [12], the estimates of Lemmas 2.1–2.7 then apply to show that the corresponding weak solutions \((\rho^\varepsilon, u^\varepsilon)\) of (1.7) with initial data \((\rho_0^\varepsilon, u_0^\varepsilon)\) exists for all \( t \in [0, T_1] \) and satisfy all the estimates of Lemmas 2.1–2.7 with constants \( C(T_1) \) and \( C \) which are independent of \( \varepsilon \). That means

\[
\frac{A}{2} (x(1 - x))^a \leq \rho^\varepsilon \leq C \rho_0^\varepsilon + CC_{20} (x(1 - x))^{\frac{1}{3\beta}} \leq C,
\]

\[
\rho^\varepsilon, u^\varepsilon \in C^{2}([0, T_1]; L^2([0, 1])) \cap L^\infty([0, 1] \times [0, T_1]),
\]

\[
\| u_{t} \|_{L^{\infty}(0,T_{1});L^{2}(0,1)} + \| u_{x} \|_{L^{\infty}(0,T_{1});L^{\beta}(0,1)} + \| u_{y} \|_{L^{2}(0,1)} \leq C(T_{1}),
\]

\[
\sup_{t \in (0,T_{1})} \left( \int_{0}^{T_{1}} \left[ (u_{x}^{\varepsilon})^{2} + (u_{y}^{\varepsilon})^{2} \right] \, dx + \int_{0}^{1} \left( \rho_{t}^{\varepsilon} \right)^{1+\theta} (u_{x}^{\varepsilon})^{2} \, dx \right) \right) \\
+ \int_{0}^{T_{1}} \left( \rho_{t}^{\varepsilon} u_{x}^{\varepsilon} \right) \, dt \int_{0}^{T_{1}} \left( u_{x}^{\varepsilon} \right)^{2} \, dx \, dt + \int_{0}^{T_{1}} \int_{0}^{T_{1}} \left( u_{x}^{\varepsilon} \right)^{2} \, dx \, dt \\
\leq C(T_{1}),
\]

\[
\| u_{t} \|_{L^{\infty}(0,1 \times [0,T_{1}])} + \tau^{\frac{1}{2}} \| u_{x}^{\varepsilon} \|_{L^{\infty}(0,1 \times [\tau,T_{1}])} \leq C(T_{1}), \quad \forall 0 < \tau < T_{1},
\]

where \( 1 < \beta < \frac{2}{1 + \alpha + \alpha \sigma} \), and the following equations hold:
\[ \rho_\varepsilon^e + (\rho_\varepsilon^e)^2 u_\varepsilon^e_x = 0, \quad \rho_\varepsilon^e(x, 0) = \rho_0^e(x) \quad \text{for a.e. } x \in (0, 1) \text{ and } \forall 0 < t \leq T_1. \]  
\[ \int_0^1 \int 0^1 \left\{ u_\varepsilon^e \phi_t + \left( (\rho_\varepsilon^e)^\gamma - (\rho_\varepsilon^e)^{1+\theta} u_\varepsilon^e \right) \phi_x \right\} \, dx \, dt + \int_0^1 u_0^e(x) \phi(x, 0) \, dx = 0, \]  
for any test function \( \phi(x, t) \in C_0^\infty(Q) \) with \( Q = \{(x, t) \mid 0 \leq x \leq 1, 0 \leq t \leq T_1\} \).

By Arzela–Ascoli’s theorem, we obtain that there is a subsequence \( \varepsilon \to 0 \) for which
\[ u_\varepsilon^e \to u \text{ uniformly on compact sets in } [0, 1] \times (0, T_1), \]  
and we have
\[ u_\varepsilon^e \rightharpoonup u \quad \text{strongly in } C([0, 1]; L^q([0, 1])), \quad \forall q \in [1, \infty), \]  
\[ u_\varepsilon^e_x \rightharpoonup u_x \quad \text{weak-* in } L^\infty([0, T_1]; L^\beta([0, 1])), \]  
\[ u_\varepsilon^e \rightharpoonup u \quad \text{weakly in } L^2([0, 1] \times [0, T_1]), \]  
\[ t^{\frac{2}{p}} u_\varepsilon^e \rightharpoonup t^{\frac{1}{p}} u_t \quad \text{weak-* in } L^\infty([0, T_1]; L^2([0, 1])), \]  
\[ t^{\frac{2}{p}} u_\varepsilon^e_x \rightharpoonup t^{\frac{1}{p}} u_{tx} \quad \text{weakly in } L^\beta([0, 1] \times [0, T_1]). \]  

And using the arguments in Lemma 2.7, we obtain \( \langle u^{\frac{1}{2}, \frac{\beta-1}{p}} \rangle_{\tau,T_1} \leq C(T_1)^{\tau^{-\frac{1}{p}}} \), \( \forall \tau \in (0, T_1) \).

We could also obtain a subsequence for which \( \rho_\varepsilon^e \) converges weakly, say to a function \( \rho \), and
\[ \rho(0, t) = \rho(1, t) = 0, \quad \forall t \in [0, T_1]. \]  
This would not guarantee, however, that \( (\rho_\varepsilon^e)^\gamma \) (or \( (\rho_\varepsilon^e)^\theta \)) converges, even weakly, to \( \rho^\gamma \) (or \( \rho^\theta \)). We therefore could not conclude that limiting pair \( (\rho, u) \) is a weak solution of the second equation in (1.7). We shall have to obtain the density \( \rho \) as a strong limit of the \( \rho_\varepsilon^e \), and this will require a somewhat deeper analysis.

By the Lemma 2.8, it is a simple matter to check that the limiting pair \( (\rho, u) \) is indeed a weak solution of (1.7) with initial data \( (\rho_0, u_0) \), and take the regularity results (1.12)–(1.18) hold. This will then complete the proof of the Theorem 1.1.

**Lemma 2.8.** Let \( \rho_\varepsilon^e, u_\varepsilon^e \) and \( u \) be as above. Then there is a further subsequence \( \varepsilon \to 0 \) and a limiting density \( \rho \) such that
\[ \rho_\varepsilon^e \rightharpoonup \rho \quad \text{weak-* in } L^\infty([0, 1] \times [0, T_1]), \]  
\[ \rho_\varepsilon^e \to \rho \quad \text{strongly in } L^p([0, 1] \times [0, T_1]), \quad \forall p \in [1, \infty), \]  
\[ (\rho_\varepsilon^e)^2 u_\varepsilon^e_x \rightharpoonup \rho^2 u_x \quad \text{weakly in } L^2([0, 1] \times [0, T_1]), \]  
\[ (\rho_\varepsilon^e)^{1+\theta} u_\varepsilon^e - (\rho_\varepsilon^e)^\gamma \rightharpoonup \rho^{1+\theta} u_x - \rho^\gamma \quad \text{strongly in } L^2([0, 1] \times [0, T_1]). \]

**Proof.** We define
\[ F^e(x, t) = [(\rho_\varepsilon^e)^{1+\theta} u_\varepsilon^e - (\rho_\varepsilon^e)^\gamma](x, t), \]
and
\[ L^\epsilon(x, t) = \theta^{-1}(\rho^\epsilon)^\theta(x, t). \]

We claim that there is a further subsequence \( \epsilon \to 0 \) for which
\[ F^\epsilon \to F \text{ strongly in } L^2([\tau, T_1]; L^2([0, 1])), \quad \forall 0 < \tau < T_1, \]
\[ L^\epsilon(\cdot, t) \to L(\cdot, t) \text{ strongly in } L^2([0, 1]), \text{ uniformly for } t \in [0, T_1], \]
for appropriate functions \( F \) and \( L \). To prove (2.56) we simply compute from (2.43) that
\[ F^\epsilon_t = u^\epsilon_t, \text{ in } \mathcal{D}', \]
which by (2.40) is bounded in \( L^2([0, 1] \times [0, T_1]), \text{ uniformly in } \epsilon. \) Also,
\[
F^\epsilon_t = (1 + \theta)(\rho^\epsilon)^\theta \rho^\epsilon u^\epsilon_t + (\rho^\epsilon)^{1+\theta} u^\epsilon_{tx} - \gamma (\rho^\epsilon)^{-1} \rho^\epsilon_t = -(1 + \theta)(\rho^\epsilon)^\theta (\rho^\epsilon u^\epsilon_x)^2 + (\rho^\epsilon)^{1+\theta} u^\epsilon_{tx} + \gamma (\rho^\epsilon)^{1+\gamma} u^\epsilon_x,
\]
which by (2.38) and (2.41) is bounded in \( L^2([0, 1] \times [\tau, T_1]), \forall \tau \in (0, T_1), \text{ uniformly in } \epsilon. \) These estimates prove (2.56).

To prove (2.57), we compute from (2.43) that
\[ L^\epsilon_t = (\rho^\epsilon)^{-1} \rho^\epsilon_t = -(\rho^\epsilon)^{1+\theta} u^\epsilon_x = -\left( F^\epsilon + (\rho^\epsilon)^\gamma \right). \]
Fixing \( \epsilon_1 \) and \( \epsilon_2 \), and defining \( \alpha = ((\rho^\epsilon)^\gamma - (\rho^\epsilon_0)^\gamma)/(L^{\epsilon_2} - L^{\epsilon_1}). \) We then obtain that
\[
\frac{\partial}{\partial t}\left( L^{\epsilon_2} - L^{\epsilon_1} \right) + \alpha \left( L^{\epsilon_2} - L^{\epsilon_1} \right) = -(F^{\epsilon_2} - F^{\epsilon_1}).
\]

By (1.6) and (2.38), we have\[
\alpha \geq 0.
\]

It follows easily that
\[
\left\| L^{\epsilon_2}(\cdot, t) - L^{\epsilon_1}(\cdot, t) \right\|^2_{L^2} \\
\leq C(T_1) \left[ \left\| L^{\epsilon_2}(\cdot, 0) - L^{\epsilon_1}(\cdot, 0) \right\|^2_{L^2} + \int_0^{T_1} \left\| F^{\epsilon_2}(\cdot, s) - F^{\epsilon_1}(\cdot, s) \right\|^2_{L^2} ds \right].
\]

This together with (2.56) proves (2.57). (\( \forall \epsilon > 0 \); first, we could choose \( \tau \in (0, T_1) \) independent of \( t \), such that \( C(T_1) \int_0^T \left\| F^{\epsilon_2}(\cdot, s) - F^{\epsilon_1}(\cdot, s) \right\|^2_{L^2} ds < \frac{\epsilon}{4} \); then, by (2.56), we can choose \( \epsilon_0 > 0 \) independent of \( t \), such that \( \forall \epsilon_1, \epsilon_2 < \epsilon_0 \) satisfy \( C(T_1) \int_\tau^{T_1} \left\| F^{\epsilon_2}(\cdot, s) - F^{\epsilon_1}(\cdot, s) \right\|^2_{L^2} ds < \frac{\epsilon}{4} \) and \( C(T_1) \left\| L^{\epsilon_2}(\cdot, 0) - L^{\epsilon_1}(\cdot, 0) \right\|^2_{L^2} < \frac{\epsilon}{4} \), then \( \left\| L^{\epsilon_2}(\cdot, t) - L^{\epsilon_1}(\cdot, t) \right\|^2_{L^2} < \epsilon. \)

From (2.38) and (2.57), we can obtain (2.52)–(2.53) immediately.

Finally, from (2.38), (2.41), (2.47) and (2.53), we have
\[
(\rho^\epsilon)^2 u^\epsilon_x \to \rho^2 u_x \text{ weakly in } L^2([0, 1] \times [0, T_1]),
\]
(2.58)
and
\[(\rho^\varepsilon)^{1+\theta} u_x^\varepsilon \rightharpoonup \rho^{1+\theta} u_x \text{ weakly in } L^2([0, 1] \times [0, T_1]).\] (2.59)

From (2.53), (2.56) and (2.59), we have
\[(\rho^\varepsilon)^{1+\theta} u_x^\varepsilon - (\rho^\varepsilon)^\gamma \rightharpoonup \rho^{1+\theta} u_x - \rho^\gamma \text{ strongly in } L^2([0, 1] \times [\tau, T_1]).\] (2.60)

for all \(\tau \in (0, T_1).\) Since \(\rho^{1+\theta} u_x - \rho^\gamma\) bounded in \(L^2([0, 1] \times [0, T_1]),\) we have
\[(\rho^\varepsilon)^{1+\theta} u_x^\varepsilon - (\rho^\varepsilon)^\gamma \rightharpoonup \rho^{1+\theta} u_x - \rho^\gamma \text{ strongly in } L^2([0, 1] \times [0, T_1]).\] (2.61)

Then completes the proof of Lemma 2.8. □

3. Uniqueness of weak solution

In this section, we will prove the uniqueness of the weak solution constructed in Section 2, applying energy method. For simplicity, we may assume that \((\rho, u)(x, t), (\rho_1, u_1)(x, t)\) and \((\rho_2, u_2)(x, t)\) are suitably smooth, since we can get the following estimates by using the Friedrichs mollifier.

First, we give the fine upper bound of density function \(\rho(x, t)\) in following lemma.

**Lemma 3.1.** Under the condition of Theorem 1.2, we have
\[\rho \leq C (x(1-x))^{\alpha_1}.\] (3.1)

**Proof.** From (2.4)–(2.5), using the assumption \(\rho_0 \leq B(x(1-x))^{\alpha_1}\) and the fact that \(\alpha_1 \leq \alpha < \frac{1}{2\theta},\) we have for \(0 < t \leq T_1,\)
\[\rho^\theta (x, t) \leq \rho_0^\theta (x) - \theta \int_0^x u(y, t) \, dy + \theta \int_0^x u_0 \, dy\]
\[\leq \rho_0^\theta (x) + C \left( \int_0^1 u^2(y, t) \, dy \right)^{\frac{1}{2}} (x(1-x))^{\frac{1}{2}} + C(x(1-x))\]
\[\leq B^\theta (x(1-x))^{\alpha_1 \theta} + C(x(1-x))^{\frac{1}{2}}\]
and Lemma 3.1 follows. □

We give the bound for function \(\rho |u_x|\) in Theorem 1.1, and could reduce the index of \(\rho\) in following lemma.

**Lemma 3.2.** For \(0 < \theta < \gamma,\) let
\[\lambda = \max \left\{ 1 + \theta - \gamma, 1 + \theta - \frac{1}{2\alpha} \right\} < 1.\] (3.2)
Then there exists a constant $C(T_1)$ such that
\[ \| (\rho^\lambda u_x) (x, t) \|_{L^2(0, T_1; L^\infty[0, 1])} \leq C(T_1). \]  

**Proof.** From (1.12), (2.3) and (3.2), using Hölder’s inequality, we have for $0 < t \leq T_1$,
\[
\rho^\lambda u_x = \rho^{\gamma+\lambda-1-\theta} + \rho^{\lambda-1-\theta} \int_0^x u_t(y, t) \, dy \\
\leq C + \rho^{\lambda-1-\theta} \left( \int_0^1 u_t^2(x, t) \, dx \right)^{\frac{1}{2}} (x(1-x))^{\frac{1}{2}} \\
\leq C + C(x(1-x))^{\frac{1}{2}+\alpha(\lambda-1-\theta)} \left( \int_0^1 u_t^2(x, t) \, dx \right)^{\frac{1}{2}} \\
\leq C + C \left( \int_0^1 u_t^2(x, t) \, dx \right)^{\frac{1}{2}}.
\]

This and (1.15) show (3.3) and completes the proof of Lemma 3.2. \qed

**Proof of Theorem 1.2.** Let $(\rho_1, u_1)(x, t)$ and $(\rho_2, u_2)(x, t)$ be two weak solutions in Theorem 1.2, in $0 \leq t \leq T_1$. Then from Lemmas 3.1–3.2, we have
\[ \rho_i(x, t) \leq C(x(1-x))^{\alpha_i}, \quad i = 1, 2, \]  

\[ \| (\rho_i^\lambda \partial_x u_i)(\cdot, t) \|_{L^\infty} \leq H(t), \quad \int_0^{T_1} H^2(t) \, dt \leq C(T_1). \]  

Let
\[
\begin{cases}
\varrho(x, t) = (\rho_1 - \rho_2)(x, t), \\
\omega(x, t) = \int_0^1 (u_1 - u_2)(y, t) \, dy,
\end{cases}
\]
for $0 \leq x \leq 1$ and $0 \leq t \leq T_1$.

By the boundary condition (1.9) and (2.2), we have
\[ \varrho(0, t) = \varrho(1, t) = \omega(0, t) = \omega(1, t) = 0, \]
for $0 \leq t \leq T_1$.

The equations change to
\[
\left( \frac{\varrho}{\rho_1 \rho_2} \right)_t + \omega_{xx} = 0, \]  

\[ \omega_t + \frac{\rho_1^{\gamma} - \rho_2^{\gamma}}{\rho_1 - \rho_2} \varrho = \rho_1^{1+\theta} \omega_{xx} + \frac{\rho_1^{1+\theta} - \rho_2^{1+\theta}}{\rho_1 - \rho_2} \varrho u_{2x}. \]
Multiplying (3.8) by $\rho_1^\theta \rho_2^{-1} \varrho$, we get
\[
(\rho_1^{-1+\theta} \rho_2^{-2} \varrho^2)_{x} + (1 + \theta) \rho_1^\theta \rho_2^{-2} \varrho^2 u_{1x} + 2 \rho_1^\theta \rho_2^{-1} \varrho \omega_{xx} = 0.
\] (3.10)

Integrating (3.10) in $x$ over $[0,1]$ and using Cauchy–Schwartz inequality, we have
\[
\frac{d}{dt} \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \varrho^2 \, dx \\
\leq C \int_0^1 \rho_1^{\theta-\lambda} \rho_2^{-2} \varrho^2 |\rho_1^\lambda u_{1x}| \, dx + \frac{1}{4} \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx + C \int_0^1 \rho_1^{\theta-1} \rho_2^{-2} \varrho^2 \, dx \\
\leq CH(t) \int_0^1 \rho_1^{\theta-\lambda} \rho_2^{-2} \varrho^2 \, dx + C \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx.
\] (3.11)

From (3.2), we have
\[
\theta - \lambda > -1 + \theta.
\]

Therefore,
\[
\frac{d}{dt} \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \varrho^2 \, dx \leq C (1 + H(t)) \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \varrho^2 \, dx + \frac{1}{4} \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx.
\] (3.12)

Multiplying (3.9) by $\omega_{xx}$, we have
\[
\left(\frac{1}{2} \omega_x^2\right)_t + \rho_1^{1+\theta} \omega_{xx}^2 = \frac{\rho_1^\gamma - \rho_2^\gamma}{\rho_1 - \rho_2} Q \omega_{xx} - \frac{\rho_1^{1+\theta} - \rho_2^{1+\theta}}{\rho_1 - \rho_2} Q u_{2x} \omega_{xx} + (\omega_t \omega_x)_x.
\] (3.13)

Integrating (3.13) in $x$ over $[0,1]$, using Cauchy–Schwartz inequality and (3.4)–(3.5), we get
\[
\frac{d}{dt} \int_0^1 \frac{1}{2} \omega_x^2 \, dx + \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx \\
= \int_0^1 \rho_1^\gamma - \rho_2^\gamma \omega_{xx} \, dx - \int_0^1 \rho_1^{1+\theta} - \rho_2^{1+\theta} Q u_{2x} \omega_{xx} \, dx \\
\leq \int_0^1 (\rho_1^{\gamma-1} + \rho_2^{\gamma-1})|Q \omega_{xx}| \, dx + \int_0^1 (\rho_1^\theta + \rho_2^\theta)|Q u_{2x} \omega_{xx}| \, dx \\
\leq \frac{1}{4} \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx + C \int_0^1 (\rho_1^{2\gamma-2} + \rho_2^{2\gamma-2}) \rho_1^{-1-\theta} \varrho^2 \, dx.
\[ + C \int_0^1 \left( \rho_1^{2\theta} + \rho_2^{2\theta} \right) \rho_1^{1-\theta} \rho_2^2 u_{2x}^2 \, dx \]
\[ \leq \frac{1}{4} \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx + C \max_{x \in [0,1]} \left[ \rho_1^{-2\theta} \rho_2^2 \left( \rho_1^{2\gamma-2} + \rho_2^{2\gamma-2} \right) \right] \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \rho^2 \, dx \]
\[ + C \max_{x \in [0,1]} \left[ (\rho_1^{2\theta} + \rho_2^{2\theta}) (\rho_1^{-2\theta} \rho_2^{2-2\lambda}) \right] \max_{x \in [0,1]} \left( \rho_2^5 u_{2x} \right)^2 \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \rho^2 \, dx \]
\[ \leq \frac{1}{4} \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx + C \max_{x \in [0,1]} (x(1-x))^{2\gamma \alpha_1 - 2\theta \alpha} \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \rho^2 \, dx \]
\[ + CH^2(t) \max_{x \in [0,1]} (x(1-x))^{(2+2\theta - 2\lambda) \alpha_1 - 2\theta \alpha} \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \rho^2 \, dx. \]

From (3.2) and the fact that \( \alpha \geq \alpha_1 \geq \max\{ \frac{\theta}{\gamma} \alpha, 2\theta \alpha^2 \} \), we obtain
\[ 2\gamma \alpha_1 - 2\theta \alpha \geq 0, \quad (2+2\theta - 2\lambda) \alpha_1 - 2\theta \alpha \geq 0. \]

Therefore,
\[ \frac{d}{dt} \int_0^1 \frac{1}{2} \omega_x^2 \, dx + \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx \leq C \left( 1 + H^2(t) \right) \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \rho^2 \, dx. \]  
(3.14)

From (3.12) and (3.14), we have
\[ \frac{d}{dt} \left( \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \rho^2 \, dx + \int_0^1 \frac{1}{2} \omega_x^2 \, dx \right) + \frac{1}{2} \int_0^1 \rho_1^{1+\theta} \omega_{xx}^2 \, dx \]
\[ \leq C \left( 1 + H^2(t) \right) \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \rho^2 \, dx. \]

Using Gronwall’s inequality and (3.5), we have for any \( 0 < t \leq T_1 \),
\[ \int_0^1 \rho_1^{-1+\theta} \rho_2^{-2} \rho^2 \, dx + \int_0^1 \omega_x^2 = 0. \]  
(3.15)

Those prove Theorem 1.2. \( \square \)

References