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# Multiple little $q$-Jacobi polynomials ☆

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## Abstract

We introduce two kinds of multiple little $q$-Jacobi polynomials $p_{\vec{n}}$ with multi-index $\vec{n} = (n_1, n_2, \ldots, n_r)$ and degree $|\vec{n}| = n_1 + n_2 + \cdots + n_r$ by imposing orthogonality conditions with respect to $r$ discrete little $q$-Jacobi measures on the exponential lattice $\{q^k, k = 0, 1, 2, 3, \ldots\}$, where $0 < q < 1$. We show that these multiple little $q$-Jacobi polynomials have useful $q$-difference properties, such as a Rodrigues formula (consisting of a product of $r$ difference operators). Some properties of the zeros of these polynomials and some asymptotic properties will be given as well.



*Keywords:* $q$-Jacobi polynomials; Basis hypergeometric polynomials; Multiple orthogonal polynomials

## 1. Little $q$-Jacobi polynomials

Little $q$-Jacobi polynomials are orthogonal polynomials on the exponential lattice $\{q^k, \ k=0, 1, 2, \ldots\}$, where $0 < q < 1$. In order to express the orthogonality relations, we will use the $q$-integral

$$\int_0^1 f(x)\, d_q x = (1-q) \sum_{k=0}^{\infty} q^k f(q^k) \tag{1.1}$$

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(see, e.g., [2, Section 10.1; 5, Section 1.11]) where $f$ is a function on [0, 1] which is continuous at 0. The orthogonality is given by

$$\int_0^1 p_n(x;\alpha,\beta\,|\,q)x^k w(x;\alpha,\beta\,|\,q)\,\mathrm{d}_q x = 0, \quad k = 0, 1, \ldots, n-1, \tag{1.2}$$

where

$$w(x;a,b\,|\,q) = \frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}\, x^\alpha. \tag{1.3}$$

We have used the notation

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

In order that the $q$-integral of $w$ is finite, we need to impose the restrictions $\alpha, \beta > -1$. The orthogonality conditions (1.2) determine the polynomials $p_n(x;\alpha,\beta\,|\,q)$ up to a multiplicative factor. In this paper, we will always use monic polynomials and these are uniquely determined by the orthogonality conditions. The $q$-binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|, |q| < 1 \tag{1.4}$$

(see, e.g., [2, Section 10.2; 5, Section 1.3]) implies that

$$\lim_{q\to 1} w(x;\alpha,\beta\,|\,q) = (1-x)^\beta x^\alpha, \quad 0 < x < 1,$$

so that $w(x;\alpha,\beta\,|\,q)$ is a $q$-analog of the beta density on [0, 1], and hence

$$\lim_{q\to 1} p_n(x;\alpha,\beta\,|\,q) = P_n^{(\alpha,\beta)}(x),$$

where $P_n^{(\alpha,\beta)}$ are the monic Jacobi polynomials on [0, 1]. Little $q$-Jacobi polynomials appear in representations of quantum SU(2) [9,10], and the special case of little $q$-Legendre polynomials was used to prove irrationality of a $q$-analog of the harmonic series and log 2 [14]. Their role in partitions was described in [1]. A detailed list of formulas for the little $q$-Jacobi polynomials can be found in [8, Section 3.12], but note that in that reference the polynomial $p_n(x;a,b\,|\,q)$ is not monic and that $a = q^\alpha$, $b = q^\beta$. Useful formulas are the *lowering operation*

$$\mathscr{D}_q p_n(x;\alpha,\beta\,|\,q) = \frac{1-q^n}{1-q}\, p_{n-1}(x;\alpha+1,\beta+1\,|\,q), \tag{1.5}$$

where $\mathscr{D}_q$ is the $q$-difference operator

$$\mathscr{D}_q f(x) = \begin{cases} \frac{f(x)-f(qx)}{(1-q)x} & \text{if } x \neq 0, \\ f'(0) & \text{if } x = 0 \end{cases} \tag{1.6}$$

and the *raising operation*

$$\mathscr{D}_p[w(x;\alpha,\beta\,|\,q)p_n(x;\alpha,\beta\,|\,q)] = -\frac{1-q^{n+\alpha+\beta}}{(1-q)q^{n+\alpha-1}}\, w(x;\alpha-1,\beta-1\,|\,q)p_{n+1}(x;\alpha-1,\beta-1\,|\,q), \tag{1.7}$$

where $p=1/q$. Repeated application of the raising operator gives the *Rodrigues formula*

$$w(x;\alpha,\beta\,|\,q)p_n(x;\alpha,\beta\,|\,q) = \frac{(-1)^n(1-q)^n q^{\alpha n+n(n-1)}}{(q^{\alpha+\beta+n+1};q)_n}\,\mathscr{D}_p^n w(x;\alpha+n,\beta+n\,|\,q). \tag{1.8}$$

A combination of the raising and the lowering operation gives a *second-order q-difference equation*. The Rodrigues formula enables us to give an explicit expression as a basic hypergeometric sum:

$$p_n(x;\alpha,\beta\,|\,q) = \frac{x^n q^{n(n+\alpha)}(q^{-n-\alpha};q)_n}{(q^{n+\alpha+\beta+1};q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-n-\alpha}, 1/x \\ q^{\beta+1}, 0 \end{matrix}\,\middle|\, q;q\right),$$

which by some elementary transformations can also be written as

$$\begin{aligned} p_n(x;\alpha,\beta\,|\,q) &= \frac{q^{(n+\alpha)n}(q^{-n-\alpha};q)_n}{(q^{n+\alpha+\beta+1};q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{\alpha+1} \end{matrix}\,\middle|\, q;qx\right) \\ &= \frac{q^{(n+\alpha)n}(q^{-n-\alpha};q)_n}{(q^{n+\alpha+\beta+1};q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{n+\alpha+\beta+1};q)_k}{(q^{\alpha+1};q)_k}\,\frac{q^k x^k}{(q;q)_k}. \end{aligned} \tag{1.9}$$

## 2. Multiple orthogonal polynomials

Multiple orthogonal polynomials (of type II) are polynomials satisfying orthogonality conditions with respect to $r \geqslant 1$ positive measures [3,4,11, Section 4.3; 15]. Let $\mu_1, \mu_2, \dots, \mu_r$ be $r$ positive measures on the real line and let $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$ be a multi-index of length $|\vec{n}| = n_1 + n_2 + \cdots + n_r$. The corresponding type II multiple orthogonal polynomial $p_{\vec{n}}$ is a polynomial of degree $\leqslant |\vec{n}|$ satisfying the orthogonality relations

$$\int p_{\vec{n}}(x)x^k\,\mathrm{d}\mu_j(x) = 0, \quad k = 0, 1, \dots, n_j - 1, \; j = 1, 2, \dots, r.$$

These orthogonality relations give $|\vec{n}|$ homogeneous equations for the $|\vec{n}|+1$ unknown coefficients of $p_{\vec{n}}$. We say that $\vec{n}$ is a normal index if the orthogonality relations determine the polynomial $p_{\vec{n}}$ up to a multiplicative factor. Multiple orthogonal polynomials of type I (see, e.g., [3,11, Section 4.3; 4,15]) will not be considered in this paper. Multiple little $q$-Jacobi polynomials are multiple orthogonal polynomials, where the measures $\mu_1, \dots, \mu_r$ are supported on the exponential lattice $\{q^k,\; k = 0, 1, 2, \dots\}$ and are all of the form $\mathrm{d}\mu_i(x) = w(x;\alpha_i,\beta_i\,|\,q)\,\mathrm{d}_q x$, where $w(x;\alpha,\beta\,|\,q)\,\mathrm{d}_q x$ is the orthogonality measure for little $q$-Jacobi polynomials. It turns out that in order to have formulas and identities similar to those of the usual little $q$-Jacobi polynomials one needs to keep one of the parameters $\alpha_i$ or $\beta_i$ fixed and change the other parameters for the $r$ measures. This gives two kinds of multiple little $q$-Jacobi polynomials. Note that these multiple little $q$-Jacobi polynomials should not be confused with multivariable little $q$-Jacobi polynomials, introduced in [13]. In [12] the multiple little $q$-Jacobi polynomials of the first kind are used to prove some irrationality results for $\zeta_q(1)$ and $\zeta_q(2)$.

### *2.1. Multiple little q-Jacobi polynomials of the first kind*

Multiple little $q$-Jacobi polynomials of the first kind $p_{\vec{n}}(x;\vec{\alpha},\beta\,|\,q)$ are monic polynomials of degree $|\vec{n}|$ satisfying the orthogonality relations

$$\int_0^1 p_{\vec{n}}(x;\vec{\alpha},\beta\,|\,q)x^k w(x;\alpha_j,\beta\,|\,q)\,\mathrm{d}_q x = 0, \quad k=0,1,\ldots,n_j-1,\ \ j=1,2,\ldots,r, \tag{2.1}$$

where $\alpha_1,\ldots,\alpha_r,\beta>-1$. Observe that all the measures are orthogonality measures for little $q$-Jacobi polynomials with the same parameter $\beta$ but with different parameters $\alpha_j$. All the multi-indices will be normal when we impose the condition that $\alpha_i-\alpha_j\notin\mathbb{Z}$ whenever $i\neq j$, because then all the measures are absolutely continuous with respect to $w(x;0,\beta\,|\,q)\,\mathrm{d}_q x$ and the system of functions

$$x^{\alpha_1}, x^{\alpha_1+1},\ldots,x^{\alpha_1+n_1-1}, x^{\alpha_2}, x^{\alpha_2+1},\ldots,x^{\alpha_2+n_2-1},\ldots,x^{\alpha_r},x^{\alpha_r+1},\ldots,x^{\alpha_r+n_r-1}$$

is a Chebyshev system on $(0,1)$, so that the measures $(\mu_1,\ldots,\mu_r)$ form a so-called AT-system, which implies that all the multi-indices are normal [11, Theorem 4.3].

There are $r$ *raising operations* for these multiple orthogonal polynomials.

**Theorem 2.1.** *Suppose that* $\alpha_1,\ldots,\alpha_r,\beta>0$, *with* $\alpha_i-\alpha_j\notin\mathbb{Z}$ *whenever* $i\neq j$, *and put* $p=1/q$, *then*

$$\begin{aligned}&\mathscr{D}_p[w(x;\alpha_j,\beta\,|\,q)p_{\vec{n}}(x;\vec{\alpha},\beta\,|\,q)]\\&\quad=\frac{q^{\alpha_j+\beta+|\vec{n}|}-1}{(1-q)q^{\alpha_j+|\vec{n}|-1}}\,w(x;\alpha_j-1,\beta-1\,|\,q)p_{\vec{n}+\vec{e}_j}(x;\vec{\alpha}-\vec{e}_j,\beta-1\,|\,q),\end{aligned} \tag{2.2}$$

*for* $1\leqslant j\leqslant r$, *where* $\vec{e}_1=(1,0,0,\ldots,0),\ldots,\vec{e}_r=(0,\ldots,0,0,1)$ *are the standard unit vectors.*

Observe that these operations raise one of the indices in the multi-index and lower the parameter $\beta$ and one of the components of $\vec{\alpha}$.

**Proof.** First observe that

$$\begin{aligned}&\mathscr{D}_p[w(x;\alpha_j,\beta\,|\,q)p_{\vec{n}}(x;\vec{\alpha},\beta\,|\,q)]\\&\quad=w(x;\alpha_j-1,\beta-1\,|\,q)\,\frac{(1-q^\beta x)p_{\vec{n}}(x;\vec{\alpha},\beta\,|\,q)-p^{\alpha_j}(1-x)p_{\vec{n}}(px;\vec{\alpha},\beta\,|\,q)}{1-p},\end{aligned}$$

so that

$$\mathscr{D}_p[w(x;\alpha_j,\beta\,|\,q)p_{\vec{n}}(x;\vec{\alpha},\beta\,|\,q)]=-\frac{1-q^{\alpha_j+\beta+|\vec{n}|}}{(1-q)q^{\alpha_j+|\vec{n}|-1}}\,w(x;\alpha_j-1,\beta-1\,|\,q)Q_{|\vec{n}|+1}(x), \tag{2.3}$$

where $Q_{|\vec{n}|+1}$ is a monic polynomial of degree $|\vec{n}|+1$. We will show that this monic polynomial $Q_{|\vec{n}|+1}$ satisfies the multiple orthogonality conditions (2.1) of $p_{\vec{n}+\vec{e}_j}(x;\vec{\alpha}-\vec{e}_j,\beta-1\,|\,q)$ and hence, since all

$\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, the unicity of the multiple orthogonal polynomials implies that $Q_{|\vec{n}|+1}(x) = p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1 \,|\, q)$. Integration by parts for the $q$-integral is given by the rule

$$\int_0^1 f(x) \mathscr{D}_p g(x) \, \mathrm{d}_q x = -q \int_0^1 g(x) \mathscr{D}_q f(x) \, \mathrm{d}_q x \quad \text{if } g(p) = 0. \tag{2.4}$$

If we apply this, then

$$\frac{1 - q^{\alpha_j+\beta+|\vec{n}|}}{(1-q)q^{\alpha_j+|\vec{n}|-1}} \int_0^1 x^k w(x; \alpha_j - 1, \beta - 1 \,|\, q) Q_{|\vec{n}|+1}(x) \, \mathrm{d}_q x$$
$$= -q \int_0^1 w(x; \alpha_j, \beta \,|\, q) p_{\vec{n}}(x; \vec{\alpha}, \beta \,|\, q) \mathscr{D}_q x^k \, \mathrm{d}_q x,$$

and since

$$\mathscr{D}_q x^k = \begin{cases} \frac{1-q^k}{1-q} x^{k-1} & \text{if } k \geqslant 1, \\ 0 & \text{if } k = 0, \end{cases}$$

we find that

$$\int_0^1 x^k w(x; \alpha_j - 1, \beta - 1 \,|\, q) Q_{|\vec{n}|+1}(x) \, \mathrm{d}_q x = 0, \quad k = 0, 1, \ldots, n_j.$$

For the other components $\alpha_i$ $(i \neq j)$ of $\vec{\alpha}$ we have

$$\frac{1 - q^{\alpha_j+\beta+|\vec{n}|}}{(1-q)q^{\alpha_j+|\vec{n}|-1}} \int_0^1 x^k w(x; \alpha_i, \beta - 1 \,|\, q) Q_{|\vec{n}|+1}(x) \, \mathrm{d}_q x$$
$$= \frac{1 - q^{\alpha_j+\beta+|\vec{n}|}}{(1-q)q^{\alpha_j+|\vec{n}|-1}} \int_0^1 x^{k+\alpha_i-\alpha_j+1} w(x; \alpha_j - 1, \beta - 1 \,|\, q) Q_{|\vec{n}|+1}(x) \, \mathrm{d}_q x$$
$$= -q \int_0^1 w(x; \alpha_j, \beta \,|\, q) p_{\vec{n}}(x; \vec{\alpha}, \beta \,|\, q) \mathscr{D}_q x^{k+\alpha_i-\alpha_j+1} \, \mathrm{d}_q x,$$

and since $\alpha_i - \alpha_j \notin \mathbb{Z}$ we have

$$\mathscr{D}_q x^{k+\alpha_i-\alpha_j+1} = \frac{1 - q^{k+\alpha_i-\alpha_j}}{1-q} x^{k+\alpha_i-\alpha_j},$$

hence

$$\int_0^1 x^k w(x; \alpha_i, \beta - 1 \,|\, q) Q_{|\vec{n}|+1}(x) \, \mathrm{d}_q x = 0, \quad k = 0, 1, \ldots, n_i - 1.$$

Hence all the orthogonality conditions for $p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1 \,|\, q)$ are indeed satisfied. □

As a consequence we find a *Rodrigues formula*:

**Theorem 2.2.** *The multiple little q-Jacobi polynomials of the first kind are given by*

$$p_{\vec{n}}(x; \vec{\alpha}, \beta \,|\, q) = C(\vec{n}, \vec{\alpha}, \beta) \frac{(q^{\beta+1}x; q)_\infty}{(qx; q)_\infty} \prod_{j=1}^{r} \left(x^{-\alpha_j} \mathscr{D}_p^{n_j} x^{\alpha_j+n_j}\right) \frac{(qx; q)_\infty}{(q^{\beta+|\vec{n}|+1}x; q)_\infty}, \tag{2.5}$$

*where the product of the difference operators can be taken in any order and*

$$C(\vec{n},\vec{\alpha},\beta)=(-1)^{|\vec{n}|}\frac{(1-q)^{|\vec{n}|}q^{\sum_{j=1}^{r}(\alpha_j-1)n_j+\sum_{1\leqslant j\leqslant k\leqslant r}n_jn_k}}{\prod_{j=1}^{r}(q^{\alpha_j+\beta+|\vec{n}|+1};q)_{n_j}}.$$

**Proof.** If we apply the raising operator for $\alpha_j$ recursively $n_j$ times, then

$$\begin{aligned}&\mathscr{D}_p^{n_j}w(x;\alpha_j,\beta\,|\,q)p_{\vec{m}}(x;\vec{\alpha},\beta\,|\,q)\\&\quad=(-1)^{n_j}\frac{(q^{\alpha_j+\beta+|\vec{m}|-n_j+1};q)_{n_j}}{(1-q)^{n_j}q^{(\alpha_j+|\vec{m}|-1)n_j}}\\&\qquad\times w(x;\alpha_j-n_j,\beta-n_j\,|\,q)p_{\vec{m}+n_j\vec{e}_j}(x;\vec{\alpha}-n_j\vec{e}_j,\beta-n_j\,|\,q).\end{aligned}\tag{2.6}$$

Use this expression with $\vec{m}=\vec{0}$ and $j=1$, then

$$\mathscr{D}_p^{n_1}w(x;\alpha_1,\beta\,|\,q)=(-1)^{n_1}\frac{(q^{\alpha_1+\beta-n_1+1};q)_{n_1}}{(1-q)^{n_1}q^{(\alpha_1-1)n_1}}\times w(x;\alpha_1-n_1,\beta-n_1\,|\,q)p_{n_1\vec{e}_1}(x;\vec{\alpha}-n_1\vec{e}_1,\beta-n_1\,|\,q).$$

Multiply both sides by $w(x;\alpha_2,\beta-n_1\,|\,q)$ and divide by $w(x;\alpha_1-n_1,\beta-n_1\,|\,q)$, then

$$x^{n_1+\alpha_2-\alpha_1}\mathscr{D}_p^{n_1}w(x;\alpha_1,\beta\,|\,q)=(-1)^{n_1}\frac{(q^{\alpha_1+\beta-n_1+1};q)_{n_1}}{(1-q)^{n_1}q^{(\alpha_1-1)n_1}}\times w(x;\alpha_2,\beta-n_1\,|\,q)p_{n_1\vec{e}_1}(x;\vec{\alpha}-n_1\vec{e}_1,\beta-n_1\,|\,q).$$

Apply (2.6) with $j=2$, then

$$\begin{aligned}&\mathscr{D}_p^{n_2}x^{n_1+\alpha_2-\alpha_1}\mathscr{D}_p^{n_1}w(x;\alpha_1,\beta\,|\,q)\\&\quad=(-1)^{n_1+n_2}\frac{(q^{\alpha_1+\beta-n_1+1};q)_{n_1}(q^{\alpha_2+\beta-n_2+1};q)_{n_2}}{(1-q)^{n_1+n_2}q^{(\alpha_1-1)n_1+(\alpha_2-1+n_1)n_2}}\\&\qquad\times w(x;\alpha_2-n_2,\beta-n_1-n_2\,|\,q)p_{n_1\vec{e}_1+n_2\vec{e}_2}(x;\vec{\alpha}-n_1\vec{e}_1-n_2\vec{e}_2,\beta-n_1-n_2\,|\,q).\end{aligned}$$

Continuing this way we arrive at

$$\begin{aligned}&(\mathscr{D}_p^{n_r}x^{\alpha_r})(x^{n_{r-1}-\alpha_{r-1}}\mathscr{D}_p^{n_{r-1}}x^{\alpha_{r-1}})\cdots(x^{n_1-\alpha_1}\mathscr{D}_p^{n_1})w(x;\alpha_1,\beta\,|\,q)\\&\quad=\frac{(-1)^{|\vec{n}|}\prod_{j=1}^{r}(q^{\alpha_j+\beta-n_j+1};q)_{n_j}}{(1-q)^{|\vec{n}|}q^{\sum_{j=1}^{r}(\alpha_j-1)n_j+\sum_{1\leqslant j<k\leqslant r}n_jn_k}}w(x;\alpha_r-n_r,\beta-|\vec{n}|\,|\,q)p_{\vec{n}}(x;\vec{\alpha}-\vec{n},\beta-|\vec{n}|\,|\,q).\end{aligned}$$

Now replace each $\alpha_j$ by $\alpha_j+n_j$ and $\beta$ by $\beta+|\vec{n}|$, then the required expression follows. The order in which we took the raising operators is irrelevant. □

We can obtain an explicit expression of the multiple little $q$-Jacobi polynomials of the first kind using this Rodrigues formula. Indeed, if we use the $q$-binomial theorem, then

$$\frac{(qx;q)_\infty}{(q^{\beta+|\vec{n}|+1}x;q)_\infty}=\sum_{k=0}^{\infty}\frac{(q^{-\beta-|\vec{n}|};q)_k}{(q;q)_k}q^{(\beta+|\vec{n}|+1)k}x^k.$$

Use this in (2.5), together with

$$x^{-\alpha}\mathscr{D}_p^n x^{\alpha+n+k} = \frac{(q^{\alpha+1};q)_n}{(1-q)^n}\,\frac{(q^{\alpha+n+1};q)_k}{(q^{\alpha+1};q)_k}\,q^{-n(k+\alpha)-n(n-1)/2}x^k,$$

then this gives

$$p_{\vec{n}}(x;\vec{\alpha},\beta\,|\,q) = C(\vec{n},\vec{\alpha},\beta)\,\frac{\prod_{j=1}^r (q^{\alpha_j+1};q)_{n_j}}{(1-q)^{|\vec{n}|}}\,q^{-\sum_{j=1}^r \alpha_j n_j - \sum_{j=1}^r \binom{n_j}{2}} \times \frac{(q^{\beta+1}x;q)_\infty}{(qx;q)_\infty}\,{}_{r+1}\phi_r\left(\begin{matrix} q^{-\beta-|\vec{n}|}, q^{\alpha_1+n_1+1},\ldots,q^{\alpha_r+n_r+1} \\ q^{\alpha_1+1},\ldots,q^{\alpha_r+1}\end{matrix}\,\middle|\, q; q^{\beta+1}x\right). \qquad (2.7)$$

This explicit expression uses a nonterminating basic hypergeometric series, except when $\beta$ is an integer. Another representation, using only finite sums, can be obtained by using the Rodrigues formula (1.8) $r$ times. For $r=2$ this gives,

**Theorem 2.3.** *The multiple little q-Jacobi polynomials of the first kind* (*for* $r=2$) *are given by*

$$p_{n,m}(x;(\alpha_1,\alpha_2),\beta\,|\,q) = \frac{q^{nm+m^2+n^2+\alpha_1 n+\alpha_2 m}(q^{-\alpha_1-n};q)_n(q^{-\alpha_2-m};q)_m}{(q^{\alpha_1+\beta+n+m+1};q)_n(q^{\alpha_2+\beta+n+m+1};q)_m} \times \sum_{\ell=0}^{n}\sum_{k=0}^{m} \frac{(q^{-n};q)_\ell(q^{-m};q)_k(q^{\alpha_2+\beta+m+n+1};q)_k(q^{\alpha_1+\beta+n+1};q)_{k+\ell}(q^{\alpha_1+n+1};q)_k}{(q^{\alpha_2+1};q)_k(q^{\alpha_1+1};q)_{k+\ell}(q^{\alpha_1+\beta+n+1};q)_k} \times \frac{q^{k+\ell}x^{k+\ell}}{q^{kn}(q;q)_k(q;q)_\ell}. \qquad (2.8)$$

**Proof.** For $r=2$ the Rodrigues formula (2.5) is

$$p_{n,m}(x;(\alpha_1,\alpha_2),\beta\,|\,q) = \frac{(-1)^{n+m}(1-q)^{n+m}q^{\alpha_1 n+\alpha_2 m-n-m+nm+n^2+m^2}}{(q^{\alpha_1+\beta+n+m+1};q)_n(q^{\alpha_2+\beta+n+m+1};q)_m} \times \frac{(q^{\beta+1}x;q)_\infty}{(qx;q)_\infty}x^{-\alpha_1}\mathscr{D}_p^n x^{\alpha_1+n-\alpha_2}\mathscr{D}_p^m x^{\alpha_2+m}\,\frac{(qx;q)_\infty}{(q^{\beta+n+m+1}x;q)_\infty}.$$

Observe that by the Rodrigues formula (1.8) for the little $q$-Jacobi polynomials

$$\mathscr{D}_p^m x^{\alpha_2+m}\,\frac{(qx;q)_\infty}{(q^{\beta+n+m+1}x;q)_\infty} = \frac{(-1)^m(q^{\alpha_2+\beta+n+m+1};q)_m}{(1-q)^m q^{\alpha_2 m+m^2-m}}\,\frac{(qx;q)_\infty}{(q^{\beta+n+1}x;q)_\infty}\,x^{\alpha_2}p_m(x;\alpha_2,\beta+n\,|\,q),$$

and hence

$$p_{n,m}(x;(\alpha_1,\alpha_2),\beta\,|\,q) = \frac{(-1)^n(1-q)^n q^{\alpha_1 n-n+nm+n^2}}{(q^{\alpha_1+\beta+n+m+1};q)_n}\,\frac{(q^{\beta+1}x;q)_\infty}{(qx;q)_\infty}\,x^{-\alpha_1} \times \mathscr{D}_p^n x^{\alpha_1+n}\,\frac{(qx;q)_\infty}{(q^{\beta+n+1}x;q)_\infty}\,p_m(x;\alpha_2,\beta+n\,|\,q).$$

Now use the explicit expression (1.9) to find

$$
\begin{aligned}
p_{n,m}(x;(\alpha_1,\alpha_2),\beta\,|\,q) &= \frac{(-1)^n(1-q)^n q^{\alpha_1 n+\alpha_2 m-n+nm+n^2+m^2}(q^{-m-\alpha_2};q)_m}{(q^{\alpha_1+\beta+n+m+1};q)_n(q^{\alpha_2+\beta+n+m+1};q)_m} \\
&\quad\times \frac{(q^{\beta+1}x;q)_\infty}{(qx;q)_\infty}\, x^{-\alpha_1} \sum_{k=0}^{m} \frac{(q^{-m};q)_k(q^{\alpha_2+\beta+n+m+1};q)_k q^k}{(q^{\alpha_2+1};q)_k(q;q)_k} \\
&\quad\times \mathscr{D}_p^n x^{\alpha_1+n+k}\,\frac{(qx;q)_\infty}{(q^{\beta+n+1}x;q)_\infty}.
\end{aligned}
$$

In this expression we recognize

$$
\begin{aligned}
&\mathscr{D}_p^n x^{\alpha_1+n+k}\,\frac{(qx;q)_\infty}{(q^{\beta+n+1}x;q)_\infty} \\
&\quad= \frac{(-1)^n(q^{\alpha_1+\beta+k+n+1};q)_n}{(1-q)^n q^{\alpha_1 n+kn+n^2-n}}\, x^{\alpha_1+k}\,\frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}\, p_n(x;\alpha_1+k,\beta\,|\,q),
\end{aligned}
$$

hence

$$
\begin{aligned}
&p_{n,m}(x;(\alpha_1,\alpha_2),\beta\,|\,q) \\
&\quad= \frac{q^{\alpha_2 m+nm+m^2}(q^{-m-\alpha_2};q)_m}{(q^{\alpha_1+\beta+n+m+1};q)_n(q^{\alpha_2+\beta+n+m+1};q)_m} \\
&\qquad\times \sum_{k=0}^{m} \frac{(q^{-m};q)_k(q^{\alpha_2+\beta+n+m+1};q)_k(q^{\alpha_1+\beta+k+n+1};q)_n q^k}{(q^{\alpha_2+1};q)_k(q;q)_k q^{kn}}\, x^k p_n(x;\alpha_1+k,\beta\,|\,q).
\end{aligned}
$$

If we use the explicit expression (1.9) for the little $q$-Jacobi polynomials once more, then after some simplifications we finally arrive at (2.8). □

### 2.2. Multiple little q-Jacobi polynomials of the second kind

Multiple little $q$-Jacobi polynomials of the second kind $p_{\vec{n}}(x;\alpha,\vec{\beta}\,|\,q)$ are monic polynomials of degree $|\vec{n}|$ satisfying the orthogonality relations

$$
\int_0^1 p_{\vec{n}}(x;\alpha,\vec{\beta}\,|\,q)x^k w(x;\alpha,\beta_j\,|\,q)\,\mathrm{d}_q x = 0, \quad k=0,1,\ldots,n_j-1,\ \ j=1,2,\ldots,r, \tag{2.9}
$$

where $\alpha,\beta_1,\ldots,\beta_r > -1$. Observe that all the measures are orthogonality measures for little $q$-Jacobi polynomials with the same parameter $\alpha$ but with different parameters $\beta_j$. All the multi-indices will be normal when we impose the condition that $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, because then all the measures are absolutely continuous with respect to $(qx;q)_\infty w(x;\alpha,0\,|\,q)\,\mathrm{d}_q x$ and the system of functions

$$\frac{1}{(q^{\beta_1+1}x;q)_\infty}, \frac{x}{(q^{\beta_1+1}x;q)_\infty}, \dots, \frac{x^{n_1-1}}{(q^{\beta_1+1}x;q)_\infty}, \frac{1}{(q^{\beta_2+1}x;q)_\infty}, \frac{x}{(q^{\beta_2+1}x;q)_\infty}, \dots, \frac{x^{n_2-1}}{(q^{\beta_2+1}x;q)_\infty}, \dots, \frac{1}{(q^{\beta_r+1}x;q)_\infty}, \frac{x}{(q^{\beta_r+1}x;q)_\infty}, \dots, \frac{x^{n_r-1}}{(q^{\beta_r+1}x;q)_\infty}$$

is a Chebyshev system[1] on [0, 1], so that the vector of measures $(\mu_1,\dots,\mu_r)$ forms an AT-system, which implies that all the multi-indices are normal [11, Theorem 4.3].

Again there are *r raising operations*

**Theorem 2.4.** *Suppose that* $\alpha,\beta_1,\dots,\beta_r>0$, *with* $\beta_i-\beta_j\notin\mathbb{Z}$ *when* $i\neq j$, *and put* $p=1/q$, *then*

$$\mathscr{D}_p[w(x;\alpha,\beta_j\,|\,q)p_{\vec n}(x;\alpha,\vec\beta\,|\,q)] = \frac{q^{\alpha+\beta_j+|\vec n|}-1}{(1-q)q^{\alpha+|\vec n|-1}}\, w(x;\alpha-1,\beta_j-1\,|\,q)\,p_{\vec n+\vec e_j}(x;\alpha-1,\vec\beta-\vec e_j\,|\,q), \tag{2.10}$$

*for* $1\leqslant j\leqslant r$, *where* $\vec e_1=(1,0,0,\dots,0),\dots,\vec e_r=(0,\dots,0,0,1)$ *are the standard unit vectors.*

Observe that these operations raise one of the indices in the multi-index and lower the parameter $\alpha$ and one of the components of $\vec\beta$.

**Proof.** Again we see that

$$\mathscr{D}_p[w(x;\alpha,\beta_j\,|\,q)p_{\vec n}(x;\alpha,\vec\beta\,|\,q)] = \frac{q^{\alpha+\beta_j+|\vec n|}-1}{(1-q)q^{\alpha+|\vec n|-1}}\, w(x;\alpha-1,\beta_j-1\,|\,q)\,Q_{|\vec n|+1}(x), \tag{2.11}$$

where $Q_{|\vec n|+1}$ is a monic polynomial of degree $|\vec n|+1$. We will show that this monic polynomial $Q_{|\vec n|+1}$ satisfies the multiple orthogonality conditions (2.9) of $p_{\vec n+\vec e_j}(x;\alpha-1,\vec\beta-\vec e_j\,|\,q)$ and hence, since all $\beta_i-\beta_j\notin\mathbb{Z}$ whenever $i\neq j$, the unicity of the multiple orthogonal polynomials implies that $Q_{|\vec n|+1}(x)=p_{\vec n+\vec e_j}(x;\alpha-1,\vec\beta-\vec e_j\,|\,q)$. Integration by parts gives

$$\frac{1-q^{\alpha+\beta_j+|\vec n|}}{(1-q)q^{\alpha+|\vec n|-1}}\int_0^1 x^k w(x;\alpha-1,\beta_j-1\,|\,q)\,Q_{|\vec n|+1}(x)\,\mathrm{d}_q x = -q\int_0^1 w(x;\alpha,\beta_j\,|\,q)\,p_{\vec n}(x;\alpha,\vec\beta\,|\,q)\,\mathscr{D}_q x^k\,\mathrm{d}_q x,$$

so that

$$\int_0^1 x^k w(x;\alpha-1,\beta_j-1\,|\,q)\,Q_{|\vec n|+1}(x)\,\mathrm{d}_q x=0,\qquad k=0,1,\dots,n_j.$$

[1] The fact that this system is a Chebyshev system is not obvious but is left as an advanced problem for the reader.

For the other components $\beta_i$ $(i \neq j)$ of $\vec{\beta}$ we have

$$\frac{1-q^{\alpha+\beta_j+|\vec{n}|}}{(1-q)q^{\alpha+|\vec{n}|-1}} \int_0^1 x^k w(x;\alpha-1,\beta_i\,|\,q) Q_{|\vec{n}|+1}(x)\,\mathrm{d}_q x$$
$$= \frac{1-q^{\alpha+\beta_j+|\vec{n}|}}{(1-q)q^{\alpha+|\vec{n}|-1}} \int_0^1 x^k \frac{(q^{\beta_j}x;q)_\infty}{(q^{\beta_i+1}x;q)_\infty} w(x;\alpha-1,\beta_j-1\,|\,q) Q_{|\vec{n}|+1}(x)\,\mathrm{d}_q x$$
$$= -q \int_0^1 w(x;\alpha,\beta_j\,|\,q) p_{\vec{n}}(x;\alpha,\vec{\beta}\,|\,q) \mathscr{D}_q \left( x^k \frac{(q^{\beta_j}x;q)_\infty}{(q^{\beta_i+1}x;q)_\infty} \right) \mathrm{d}_q x,$$

and since $\beta_i - \beta_j \notin \mathbb{Z}$ we have

$$\mathscr{D}_q \left( x^k \frac{(q^{\beta_j}x;q)_\infty}{(q^{\beta_i+1}x;q)_\infty} \right) = x^{k-1} \frac{(q^{\beta_j+1}x;q)_\infty}{(q^{\beta_i+1}x;q)_\infty} a_k(x),$$

where each $a_k$ is a polynomial of degree exactly 1 and $a_0(0)=0$. Therefore

$$\int_0^1 x^k w(x;\alpha-1,\beta_i\,|\,q) Q_{|\vec{n}|+1}(x)\,\mathrm{d}_q x = 0, \quad k=0,1,\ldots,n_i-1.$$

Hence all the orthogonality conditions for $p_{\vec{n}+\vec{e}_j}(x;\alpha-1,\vec{\beta}-\vec{e}_j\,|\,q)$ are indeed satisfied. □

As a consequence we again find a *Rodrigues formula*:

**Theorem 2.5.** *The multiple little q-Jacobi polynomials of the second kind are given by*

$$p_{\vec{n}}(x;\alpha,\vec{\beta}\,|\,q) = \frac{C(\vec{n},\alpha,\vec{\beta})}{(qx;q)_\infty x^\alpha} \prod_{j=1}^r \left( (q^{\beta_j+1}x;q)_\infty \mathscr{D}_p^{n_j} \frac{1}{(q^{\beta_j+n_j+1}x;q)_\infty} \right) (qx;q)_\infty x^{\alpha+|\vec{n}|}, \quad (2.12)$$

*where the product of the difference operators can be taken in any order and*

$$C(\vec{n},\alpha,\vec{\beta}) = (-1)^{|\vec{n}|} \frac{(1-q)^{|\vec{n}|} q^{(\alpha+|\vec{n}|-1)|\vec{n}|}}{\prod_{j=1}^r (q^{\alpha+\beta_j+|\vec{n}|+1};q)_{n_j}}.$$

**Proof.** The proof can be given in a similar way as in the case of multiple little $q$-Jacobi polynomials of the first kind by repeated application of the raising operators. Alternatively one can use induction on $r$. For $r=1$ the Rodrigues formula is the same as (1.8). Suppose that the Rodrigues formula (2.12) holds for $r-1$. Observe that the multiple orthogonal polynomials with multi-index $(n_1,\ldots,n_{r-1})$ for $r-1$ measures $(\mu_1,\ldots,\mu_{r-1})$ coincide with the multiple orthogonal polynomials with multi-index $(n_1,n_2,\ldots,n_{r-1},0)$ for $r$ measures $(\mu_1,\ldots,\mu_r)$ for any measure $\mu_r$. Use the Rodrigues formula for $r-1$ for the polynomial

$p_{\vec{n}-n_r\vec{e}_r}(x;\alpha+n_r,\vec{\beta}+n_r\vec{e}_r\,|\,q)$ to find

$$\begin{aligned} &w(x;\alpha+n_r,\beta_r+n_r\,|\,q)\,p_{\vec{n}-n_r\vec{e}_r}(x;\alpha+n_r,\vec{\beta}+n_r\vec{e}_r\,|\,q)\\ &\quad = C(\vec{n}-n_r\vec{e}_r,\alpha+n_r,\vec{\beta})\,\frac{1}{(q^{\beta_r+n_r+1}x;q)_\infty}\\ &\qquad \times \prod_{j=1}^{r-1}\left((q^{\beta_j+1}x;q)_\infty \mathscr{D}_p^{n_j}\frac{1}{(q^{\beta_j+n_j+1}x;q)_\infty}\right)(qx;q)_\infty x^{\alpha+|\vec{n}|}. \end{aligned}$$

Now apply the raising operation (2.10) for $\beta_r$ to this expression $n_r$ times to find the required expression. □

In a similar way, as for the first kind multiple little $q$-Jacobi polynomials, we can find an explicit formula with finite sums using the Rodrigues formula for little $q$-Jacobi polynomials $r$ times. For $r=2$ this gives the following:

**Theorem 2.6.** *The multiple little q-Jacobi polynomials of the second kind* (for $r=2$) *are explicitly given by*

$$\begin{aligned} p_{n,m}(x;\alpha,(\beta_1,\beta_2)\,|\,q) &= \frac{q^{\alpha(n+m)+n^2+m^2+nm}(q^{-m-\alpha};q)_m(q^{-n-\alpha};q)_n(q^{\alpha+1};q)_{m+n}}{(q^{\alpha+\beta_1+n+m+1};q)_n(q^{\alpha+\beta_2+n+m+1};q)_m(q^{\alpha+1};q)_n(q^{\alpha+1};q)_m}\\ &\quad\times\sum_{\ell=0}^{n}\sum_{k=0}^{m}\frac{(q^{-n};q)_\ell(q^{-m};q)_k(q^{\alpha+\beta_2+n+m+1};q)_k(q^{\alpha+\beta_1+n+1};q)_{k+\ell}}{(q^{\alpha+1};q)_{k+\ell}(q^{\alpha+\beta_1+n+1};q)_k}\\ &\quad\times\frac{q^{k+\ell}x^{k+\ell}}{q^{nk}(q;q)_k(q;q)_\ell}. \end{aligned} \tag{2.13}$$

**Proof.** The Rodrigues formula (2.12) for $r=2$ becomes

$$\begin{aligned} p_{n,m}(x;\alpha,(\beta_1,\beta_2)\,|\,q) &= \frac{(-1)^{n+m}(1-q)^{n+m}q^{(\alpha+n+m-1)(n+m)}}{(q^{\alpha+\beta_1+n+m+1};q)_n(q^{\alpha+\beta_2+n+m+1};q)_m}\\ &\quad\times x^{-\alpha}\frac{(q^{\beta_1+1}x;q)_\infty}{(qx;q)_\infty}\mathscr{D}_p^n\frac{(q^{\beta_2+1}x;q)_\infty}{(q^{\beta_1+n+1}x;q)_\infty}\mathscr{D}_p^m\frac{(qx;q)_\infty}{(q^{\beta_2+m+1}x;q)_\infty}x^{\alpha+n+m}. \end{aligned}$$

The Rodrigues formula (1.8) for little $q$-Jacobi polynomials gives

$$\begin{aligned} &\mathscr{D}_p^m\frac{(qx;q)_\infty}{(q^{\beta_2+m+1}x;q)_\infty}x^{\alpha+n+m}\\ &\quad=\frac{(-1)^m(q^{\alpha+\beta_2+n+m+1};q)_m}{(1-q)^m q^{\alpha m+m^2-m+nm}}\,x^{\alpha+n}\,\frac{(qx;q)_\infty}{(q^{\beta_2+1}x;q)_\infty}\,p_m(x;\alpha+n,\beta_2\,|\,q), \end{aligned}$$

hence

$$\begin{aligned} p_{n,m}(x;\alpha,(\beta_1,\beta_2)\,|\,q) &= \frac{(-1)^n(1-q)^n q^{\alpha n+n^2+nm-n}}{(q^{\alpha+\beta_1+n+m+1};q)_n}\,x^{-\alpha}\,\frac{(q^{\beta_1+1}x;q)_\infty}{(qx;q)_\infty}\\ &\quad\times\mathscr{D}_p^n x^{\alpha+n}\,\frac{(qx;q)_\infty}{(q^{\beta_1+n+1}x;q)_\infty}\,p_m(x;\alpha+n,\beta_2\,|\,q). \end{aligned}$$

Now use the explicit expression (1.9) for the little $q$-Jacobi polynomials to find

$$\begin{aligned} p_{n,m}(x;\alpha,(\beta_1,\beta_2)\,|\,q) &= \frac{(-1)^n(1-q)^n q^{\alpha(n+m)+n^2+m^2+2nm-n}(q^{-m-n-\alpha};q)_m}{(q^{\alpha+\beta_1+n+m+1};q)_n(q^{\alpha+\beta_2+n+m+1};q)_m} \\ &\quad\times \frac{(q^{\beta_1+1}x;q)_\infty}{x^\alpha(qx;q)_\infty}\sum_{k=0}^{m}\frac{(q^{-m};q)_k(q^{\alpha+\beta_2+n+m+1};q)_k q^k}{(q^{\alpha+n+1};q)_k(q;q)_k} \\ &\quad\times \mathscr{D}_p^n x^{\alpha+n+k}\frac{(qx;q)_\infty}{(q^{\beta_1+n+1}x;q)_\infty}. \end{aligned}$$

Again we recognize a little $q$-Jacobi polynomial

$$\begin{aligned} &\mathscr{D}_p^n x^{\alpha+n+k}\frac{(qx;q)_\infty}{(q^{\beta_1+n+1}x;q)_\infty} \\ &\quad= \frac{(-1)^n(q^{\alpha+\beta_1+k+n+1};q)_n}{(1-q)^n q^{\alpha n+kn+n^2-n}}\,x^{\alpha+k}\frac{(qx;q)_\infty}{(q^{\beta_1+1}x;q)_\infty}\,p_n(x;\alpha+k,\beta_1\,|\,q), \end{aligned}$$

and if we use the explicit expression (1.9) for this little $q$-Jacobi polynomial, then we find (2.13) after some simplifications. □

## 3. Zeros

The zeros of the multiple little $q$-Jacobi polynomials (first and second kind) are all real, simple and in the interval $(0,1)$. This is a consequence of the fact that $\mu_1,\ldots,\mu_r$ form an AT-system [11, first Corollary on p. 141]. For the usual orthogonal polynomials with positive orthogonality measure $\mu$ we know that an interval $[c,d]$ for which the orthogonality measure has no mass, i.e., $\mu([c,d])=0$, can have at most one zero of each orthogonal polynomial $p_n$. In particular this means that each orthogonal polynomial $p_n$ on the exponential lattice $\{q^k,\ k=0,1,2,\ldots\}$ can have at most one zero between two points $q^{k+1}$ and $q^k$ of the lattice. A similar result holds for multiple orthogonal polynomials if we impose some conditions on the measures $\mu_i$.

**Theorem 3.1.** *Suppose $\mu_1,\ldots,\mu_r$ are positive measures on $[a,b]$ with infinitely many points in their support, which form an AT-system, i.e., $\mu_k$ is absolutely continuous with respect to $\mu_1$ for $2\leqslant k\leqslant r$ with*

$$\frac{\mathrm{d}\mu_k(x)}{\mathrm{d}\mu_1(x)} = w_k(x),$$

*and*

$$1,x,\ldots,x^{n_1-1},w_2(x),xw_2(x),\ldots,x^{n_2-1}w_2(x),\ldots,w_r(x),xw_r(x),\ldots,x^{n_r-1}w_r(x)$$

*are a Chebyshev system on $[a,b]$ for every multi-index $\vec{n}$. If $[c,d]$ is an interval such that $\mu_1([c,d])=0$, then each multiple orthogonal polynomial $p_{\vec{n}}$ has at most one zero in $[c,d]$.*

**Proof.** Suppose that $p_{\vec{n}}$ is a multiple orthogonal polynomial with two zeros $x_1$ and $x_2$ in $[c, d]$. We can then write it as $p_{\vec{n}}(x) = (x - x_1)(x - x_2)q_{|\vec{n}|-2}(x)$, where $q_{|\vec{n}|-2}$ is a polynomial of degree $|\vec{n}| - 2$. Consider a function $A(x) = \sum_{j=1}^{r} A_j(x)w_j(x)$, where $w_1 = 1$ and each $A_j$ is a polynomial of degree $m_j - 1 \leqslant n_j - 1$, with $|\vec{m}| = |\vec{n}| - 1$. Since we are dealing with a Chebyshev system, there is a unique function $A$ satisfying the interpolation conditions

$$A(y) = \begin{cases} 0 & \text{if } y \text{ is a zero of } q_{|\vec{n}|-2}, \\ 1 & \text{if } y = x_1. \end{cases}$$

Furthermore, $A$ has $|\vec{n}| - 2$ zeros in $[a, b]$ and these are the only sign changes on $[a, b]$. Hence

$$\int_a^b p_{\vec{n}}(x)A(x)\,\mathrm{d}\mu_1(x) = \int_{[a,b]\backslash[c,d]} (x - x_1)(x - x_2)q_{|\vec{n}|-2}(x)A(x)\,\mathrm{d}\mu_1(x) \neq 0,$$

since the integrand does not change sign on $[a, b]\backslash[c, d]$. On the other hand,

$$\int_a^b p_{\vec{n}}(x)A(x)\,\mathrm{d}\mu_1(x) = \sum_{j=1}^{r} \int_a^b p_{\vec{n}}(x)A_j(x)\,\mathrm{d}\mu_j(x) = 0,$$

since every term in the sum vanishes because of the orthogonality conditions. This contradiction implies that $p_{\vec{n}}$ can't have two zeros in $[c, d]$. □

In particular, this theorem tells us that the zeros of the multiple little $q$-Jacobi polynomials are always separated by the points $q^k$ and that between two points $q^{k+1}$ and $q^k$ there can be at most one zero of a multiple little $q$-Jacobi polynomial. Note that the points $q^k$ have one accumulation point at 0, hence as a consequence the zeros of the multiple little $q$-Jacobi polynomials (first and second kind) accumulate at the origin.

## 4. Asymptotic behavior

The asymptotic behavior of little $q$-Jacobi polynomials was given in [7] and an asymptotic expansion was given in [6]. In this section, we give the asymptotic behavior of the multiple little $q$-Jacobi polynomials which extends the result of Ismail and Wilson.

**Theorem 4.1.** *For the multiple little q-Jacobi polynomials of the first kind we have*

$$\lim_{n,m\to\infty} x^{n+m} p_{n,m}(1/x; (\alpha_1, \alpha_2), \beta \,|\, q) = (x; q)_\infty. \tag{4.1}$$

*The order in which the limits for n and m are taken is irrelevant.*

**Proof.** If we use (2.8) and reverse the order of summation (i.e., change variables $m-k=j$ and $n-\ell=i$), then

$$\begin{aligned}
&x^{n+m} p_{n,m}(1/x;(\alpha_1,\alpha_2),\beta\,|\,q)\\
&\quad=\frac{q^{nm+m^2+n^2+\alpha_1 n+\alpha_2 m}(q^{-\alpha_1-n};q)_n(q^{-\alpha_2-m};q)_m}{(q^{\alpha_1+\beta+n+m+1};q)_n(q^{\alpha_2+\beta+n+m+1};q)_m}\\
&\qquad\times\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{(q^{-n};q)_{n-i}(q^{-m};q)_{m-j}(q^{\alpha_2+\beta+m+n+1};q)_{m-j}(q^{\alpha_1+\beta+n+1};q)_{m+n-i-j}(q^{\alpha_1+n+1};q)_{m-j}}{(q^{\alpha_2+1};q)_{m-j}(q^{\alpha_1+1};q)_{m+n-i-j}(q^{\alpha_1+\beta+n+1};q)_{m-j}}\\
&\qquad\times\frac{q^{m+n-i-j}x^{i+j}}{q^{(m-j)n}(q;q)_{m-j}(q;q)_{n-i}}.
\end{aligned}$$

Now observe that

$$(q^{-m};q)_{m-j}=(-1)^{m-j}q^{-\frac{m(m+1)}{2}+\frac{j(j+1)}{2}}\frac{(q;q)_m}{(q;q)_j},$$

$$(q^{-m-\alpha};q)_m=(-1)^m q^{-m(m+1)/2}q^{-m\alpha}(q^{\alpha+1};q)_m,$$

$$(q^{c+n};q)_m=\frac{(q^c;q)_{n+m}}{(q^c;q)_n},$$

therefore we find

$$\begin{aligned}
&x^{n+m} p_{n,m}(1/x;(\alpha_1,\alpha_2),\beta\,|\,q)\\
&\quad=\frac{(q^{\alpha_2+1};q)_m(q^{\alpha_1+\beta+1};q)_{n+m}(q;q)_m(q;q)_n}{(q^{\alpha_1+\beta+1};q)_{2n+m}(q^{\alpha_2+\beta+1};q)_{n+2m}}\\
&\qquad\times\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{(q^{\alpha_2+\beta+1};q)_{n+2m-j}(q^{\alpha_1+\beta+1};q)_{2n+m-i-j}(q^{\alpha_1+1};q)_{n+m-j}}{(q^{\alpha_2+1};q)_{m-j}(q^{\alpha_1+1};q)_{m+n-i-j}(q^{\alpha_1+\beta+1};q)_{m+n-j}(q;q)_{n-i}(q;q)_{m-j}}\\
&\qquad\times(-1)^{i+j}q^{\binom{i}{2}+\binom{j}{2}}\frac{q^{nj}x^{i+j}}{(q;q)_i(q;q)_j}.
\end{aligned}$$

If we use Lebesgue's dominated convergence theorem, then we take $n,m\to\infty$ in each term of the sum. The factor $q^{nj}$ tends to zero whenever $j>0$, hence the only contributions come from $j=0$, and we find

$$\lim_{n,m\to\infty}x^{n+m}p_{n,m}(1/x;(\alpha_1,\alpha_2),\beta\,|\,q)=\sum_{i=0}^{\infty}q^{\binom{i}{2}}\frac{(-x)^i}{(q;q)_i}.$$

The right-hand side is the $q$-exponential function

$$E_q(-x)=(x,q)_\infty,$$

[5, (II.2) in Appendix II], which gives the required result. □

**Theorem 4.2.** *For the multiple little q-Jacobi polynomials of the second kind we have*

$$\lim_{n,m\to\infty} x^{n+m} p_{n,m}(1/x; \alpha, (\beta_1, \beta_2) \mid q) = (x; q)_\infty. \tag{4.2}$$

*The order in which the limits for n and m are taken is irrelevant.*

**Proof.** The proof is similar to the case of the first kind multiple little $q$-Jacobi polynomials, except that now we use expression (2.13). □

As a consequence (using Hurwitz' theorem) we see that every zero of $(1/x; q)_\infty$, i.e., each number $q^k$, $k = 0, 1, 2, \ldots$, is an accumulation point of zeros of the multiple little $q$-Jacobi polynomial $p_{n,m}$ of the first and of the second kind.

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