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On Identity Bases of Epigroup Varieties

K. Auinger

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

and

M. B. Szendrei*

Bolyai Institute, József Attila University, Aradi vértanúk tere 1, H-6720 Szeged, Hungary
E-mail: M.Szendrei@math.u-szeged.hu

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Let $C\mathscr{CR}_n$ and $C\mathscr{CS}_n$ be the varieties of all completely regular and of all completely simple semigroups, respectively, whose idempotent generated subsemigroups are periodic with period n. We use Ol'shanskiĭ's theory of geometric group presentations to show that for large odd n these varieties (and similarly defined varieties of epigroups) do not have finitely axiomatizable equational theories. © 1999 Academic Press

1. INTRODUCTION AND PRELIMINARIES

The class of all *completely regular semigroups*, that is, the class of all semigroups which are unions of groups, has been intensively studied and plays a significant role in semigroup theory (see, e.g., the book of Petrich and Reilly [10]). It has turned out to be natural and useful to endow a completely regular semigroup with the unary operation $^{-1}$ that assigns to

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each element x its inverse within the maximal subgroup containing x and to consider such a semigroup as an algebra of type $\langle 2, 1\rangle$. As such the class $\mathscr{CR}$ of all completely regular semigroups forms a variety.

A semigroup S is *group bound* or an *epigroup* if each element $a \in S$ has a power a^n which is in a subgroup of S. Each completely regular semigroup is therefore an epigroup. On any epigroup one can naturally define a unary operation which generalizes the inversion in a completely regular semigroup as follows. For an element a, denote by a^0 the (uniquely determined) identity element in the maximal subgroup of S containing all but finitely many powers of a, and put a^{-1} to be the inverse of aa^0 within this subgroup. This unary operation satisfies

$$aa^{-1} = a^{-1}a = a^0 \qquad \text{and} \qquad a^{-1}aa^{-1} = a^{-1}.$$

The element a^{-1} is actually the so-called *Drazin inverse* of a (see [2]). As in the case of completely regular semigroups, it is natural to consider any epigroup as a semigroup endowed with the unary operation $^{-1}$, that is, as an algebra of type $\langle 2, 1\rangle$. The class of all epigroups does not form a variety because it is not closed under the formation of direct products, yet the class is closed under forming substructures (also called subepigroups), morphic images, and finitary direct products. An epigroup S is of *index* t if for every $a \in S$, a^t, and therefore also a^n for all $n \geq t$ belongs to a subgroup of S. In contrast to the class of all epigroups, the class $\mathscr{P}_t$ of all epigroups of index t does form a variety of algebras of type $\langle 2, 1\rangle$, and a basis for its equational theory is

$$xx^{-1} \simeq x^{-1}x, \qquad x^{-1}xx^{-1} \simeq x^{-1}, \qquad x^{t+1}x^{-1} \simeq x^t$$

together with the associative law for the multiplication. We have the following hierarchy of varieties:

$$\mathscr{CR} = \mathscr{P}_1 \subseteq \mathscr{P}_2 \subseteq \cdots \subseteq \mathscr{P}_t \subseteq \mathscr{P}_{t+1} \subseteq \cdots.$$

The systematic study of varieties of epigroups has been proposed by Shevrin [11, 12]. The variety $\mathscr{P}_{t,n}$ of all periodic semigroups of index t and period n, that is, the (semigroup) variety determined by the law $x^t \simeq x^{t+n}$, can be regarded as a subclass of $\mathscr{P}_t$, and in $\mathscr{P}_{t,n}$ the unary operation of $\mathscr{P}_t$ can be expressed as a semigroup term: $x^{-1} \simeq x^{2tn-1}$.

For each semigroup S denote by $C(S)$ the *core* of S, that is, the subsemigroup of S generated by its idempotents (provided S has idempotents). It is known (see Volkov [15]) that the core of an epigroup forms a subepigroup, that is, $C(S)$ is an epigroup and its unary operation coincides with the restriction of the unary operation of S. Let $\mathscr{V}$ be a subvariety of $\mathscr{P}_t$ and put

$$C_t\mathscr{V} = \{S \in \mathscr{P}_t \mid C(S) \in \mathscr{V}\}.$$

It is easy to check that $C_t\mathcal{V}$ is also a subvariety of $\mathcal{P}_t$ (and clearly $\mathcal{V} \subseteq C_t\mathcal{V}$). For $t = 1$ we will write C instead of C_1. The operator $\mathcal{V} \mapsto C\mathcal{V}$ (which operates on the lattice of all completely regular semigroup varieties) has been intensively studied; see [10].

The purpose of this note is to study the finite basis problem for certain varieties of the form $C_t\mathcal{P}_{s,n}$ and for similarly defined varieties. It is obvious that an infinite basis for $C_t\mathcal{P}_{s,n}$ is given by the system

$$(x_1x_1^{-1}\cdots x_kx_k^{-1})^{n+s} \simeq (x_1x_1^{-1}\cdots x_kx_k^{-1})^{s} \qquad \text{for } k = 1, 2, \ldots \tag{1}$$

together with an identity basis for $\mathcal{P}_t$. Yet it can happen that in certain contexts such a system of identities is equivalent to a finite one. For example, from Theorem VIII.6.15 in [10] it follows that the variety of all completely simple semigroups whose idempotent generated members are periodic with period 2 is finitely based (of course, here it is essential that groups of exponent 2 are abelian). The intention of this paper is to show that for large n this can never happen. That is, for large n, varieties of the form $C_t\mathcal{P}_{t,n}$, $C_t\mathcal{P}_{1,n}$, $C\mathcal{P}_{1,n}$, etc., are never finitely based. In the main result (Theorem 3.1) we will actually show a more general result which holds in the case n is large and odd.

We close this section by recalling all definitions and results of semigroup theory which are needed to understand the paper. A general reference for semigroup theory is the book of Howie [4]. As already mentioned, the core $C(S)$ of an epigroup [completely regular semigroup] is again an epigroup [completely regular semigroup]. By the *self-conjugate core* $C^*(S)$ of a completely regular semigroup S we mean the least subsemigroup of S which contains all idempotents of S and which is closed under conjugation, that is, $xax' \in C^*(S)$ for all $a \in C^*(S)$ and $x, x' \in S^1$ such that x and x' are mutually inverse. The definition of the self-conjugate core in [10] or [16] is slightly different but turns out to be equivalent to the given one. This one applies to the larger class of all regular semigroups. One can define an analogous concept for epigroups as well but we will not need it here. (This analogue would, for finite semigroups coincide with the well-known type-II subsemigroup.) For any variety $\mathcal{V}$ of completely regular semigroups put

$$C^*\mathcal{V} = \{S \in \mathcal{CR} \mid C^*(S) \in \mathcal{V}\}.$$

It is known (e.g., [16]) that $C^*\mathcal{V}$ is again a variety of completely regular semigroups. Since $C(S) \subseteq C^*(S)$, we have $\mathcal{V} \subseteq C^*\mathcal{V} \subseteq C\mathcal{V}$ for all completely regular semigroup varieties $\mathcal{V}$. A semigroup is *completely simple* if it is completely regular and simple. The class $\mathcal{CS}$ of all completely simple semigroups forms a subvariety of $\mathcal{CR}$. Completely simple semigroups admit a nice structural description via Rees matrix semigroups: let I, Λ be non-empty sets, let G be a group, and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G. Let $\mathscr{M}(I, G, \Lambda, P)$ be the set $I \times G \times \Lambda$ endowed with the multiplication $(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu)$; then $\mathscr{M}(I, G, \Lambda, P)$ is a completely simple semigroup, and each completely simple semigroup can be so constructed (up to isomorphism). For more details see [4]. The following varieties of epigroups will play some role in the paper:

$\mathscr{P}_t$	epigroups of index t
$\mathscr{CR} = \mathscr{P}_1$	completely regular semigroups
$\mathscr{CS}$	completely simple semigroups = completely regular semigroups satisfying $(xyx)^{-1}xyx \simeq x^{-1}x$
$\mathscr{P}_{t,n}$	epigroups of index t with periodic subgroups of exponent n = semigroups satisfying $x^{t+n} \simeq x^t$
$\mathscr{CR}_n = \mathscr{P}_{1,n}$	periodic completely regular semigroups with period n = semigroups satisfying $x^{n+1} \simeq x$
$\mathscr{CS}_n$	periodic completely simple semigroups with period n = semigroups satisfying $(xyx)^n x \simeq x$
$C_t\mathscr{P}_{s,n}$	epigroups of index t whose core has index s and is periodic with period n
$C\mathscr{CR}_n$	completely regular semigroups whose core is periodic with period n
$C^*\mathscr{CR}_n$	completely regular semigroups whose self-conjugate core is periodic with period n
$C\mathscr{CS}_n$	completely simple semigroups whose core is periodic with period n
$C_{\mathscr{CS}_k}\mathscr{CS}_n$	periodic completely simple semigroups with period k whose core has period n
$C^*\mathscr{CS}_n$	completely simple semigroups whose self-conjugate core is periodic with period n
$C^*_{\mathscr{CS}_k}\mathscr{CS}_n$	periodic completely simple semigroups with period k whose self-conjugate core has period n

For any n, q, l the following inclusions are obvious: $C^*_{\mathscr{CS}_{nq}}\mathscr{CS}_n \subseteq C\mathscr{CS}_n \subseteq C\mathscr{CR}_n \subseteq C_t\mathscr{P}_{t,nl}$. The main theorem in Section 3 states that for suitable n, q, l no variety in the interval $[C^*_{\mathscr{CS}_{nq}}\mathscr{CS}_n, C_t\mathscr{P}_{t,nl}]$ has a finite identity basis.

2. GROUPS

The finite basis problem for $C\mathscr{CR}_n$, $C\mathscr{CS}_n$, and similarly defined varieties can be reduced to the problem of whether finitely generated groups having certain properties do exist. We shall use the theory developed by Ol'shanskiĭ [8] in the context of free Burnside groups to show that indeed such groups exist. In particular, we shall refer to and use the main definitions and statements in Sections 18, 19, 25, and 26 of [8].

In the following let n be any odd integer greater than 10^{10}; let m be any fixed positive integer. Denote by $\mathscr{A} = \{x_1, \ldots, x_m\}$ an alphabet of m variables, let $\mathscr{A}^{-1} = \{x_1^{-1}, \ldots, x_m^{-1}\}$ and $\bar{\mathscr{A}} = \mathscr{A} \cup \mathscr{A}^{-1}$. As usual, for any non-empty set X, let X^* be the free monoid on X, that is, the set of all finite words in the elements of X. For each $\lambda \in \{1, \ldots, m\}$ and $W \in \bar{\mathscr{A}}^*$ put

$$\sigma_\lambda(W) = \text{the sum of the exponents of } x_\lambda \text{ in } W.$$

Throughout this section, each presentation $\langle \mathscr{A} \mid \cdots \rangle$ is understood as a presentation of a group. The next lemma is obvious.

LEMMA 2.1. *Let $\mathscr{B} \subseteq \bar{\mathscr{A}}^*$ and $G = \langle \mathscr{A} \mid W^n = 1, W \in \mathscr{B} \rangle$ and $A, B \in \bar{\mathscr{A}}^*$; if $A = B$ in G then for each λ, $\sigma_\lambda(A) \equiv \sigma_\lambda(B) \pmod{n}$.*

In the following we shall define a sequence of sets of relators $\mathscr{R}_i$ and a sequence of groups $G_i = \langle \mathscr{A} \mid R = 1, R \in \mathscr{R}_i \rangle$ in a way very similar to the definition in Section 18.1 (on p. 197) in [8]. For $i = 0, 1, \ldots, m-1$ the definition is precisely the same as on page 197 in [8]. The only modification applies to $i = m$.

First we note that the word $x_1 \cdots x_m$ is simple of rank $m-1$: otherwise this word would be conjugate in rank $m-1$ to a power of some shorter word (including periods of length $\leq m-1$) . That is, in G_{m-1} we would have $x_1 \cdots x_m = Z^{-1} C^t Z$ for some word C of length less than m and for some integer t. This implies $\sigma_\lambda(C) = 0$ for some λ and so $\sigma_\lambda(Z^{-1} C^t Z) = 0$. Since $\sigma_\lambda(x_1 \cdots x_m) = 1 \not\equiv 0 \pmod{n}$, this contradicts Lemma 2.1. Now we choose $\mathscr{X}_m$ to be a set of words of length m such that:

- $x_1 \cdots x_m \in \mathscr{X}_m$
- each word in $\mathscr{X}_m$ is simple of rank $m-1$
- $\mathscr{X}_m$ is *maximal* with respect to the property on page 197 in [8]: if A and B are distinct words in $\mathscr{X}_m$ then A is not conjugate in G_{m-1} to B or B^{-1}.

Let $q > 1$ be a divisor of n; the set $\mathscr{S}_m$ of relators is defined as

$$\mathscr{S}_m = \left\{(x_1 \cdots x_m)^{nq}\right\} \cup \left\{W^n \mid W \in \mathscr{X}_m \setminus \{x_1 \cdots x_m\}\right\}$$

and $\mathscr{R}_m = \mathscr{R}_{m-1} \cup \mathscr{S}_m$. For each $i > m$ the sets $\mathscr{X}_i$, $\mathscr{S}_i$, and $\mathscr{R}_i$ are again defined precisely as on page 197 in [8].

Notice that the sequence of groups $G_i = \langle \mathscr{A} \mid R = 1, R \in \mathscr{R}_i \rangle$ and the sequence $\mathscr{R}_i$ of sets of relators satisfy the requirements of Section 25.1 (all periods and relators are of the first type). In particular, Lemma 25.2, Theorem 26.2, and Theorem 26.4 of [8] apply. By Theorem 26.4(1), the order of $x_1 \cdots x_m$ in $G = G_\infty$ is nq whence $(x_1 \cdots x_m)^n \neq 1$ in G. In the following, for words $A, B \in \bar{\mathscr{A}}^*$ we write $A \equiv B$ to indicate that A and B are graphically equal, that is, equal as words. By Theorem 26.2 in [8], the order of any element of G is equal to the order of some relator. Consequently, G satisfies the law $x^{nq} \simeq 1$.

LEMMA 2.2. *Let $X \in \bar{\mathscr{A}}^*$ be a word such that $\sigma_\lambda(X) \equiv 0 \pmod{n}$ for some $\lambda \in \{1, \ldots, m\}$. Then $X^n = 1$ in G.*

Proof. Let q be the divisor of n which is used in the definition of $\mathscr{S}_m$ above. The proof of Theorem 26.2 in [8], in fact, shows that each word $A \in \bar{\mathscr{A}}^*$, in particular also X, is conjugate in G to a power C^t of some period $C \in \bigcup \mathscr{X}_j$. (Take any word A and denote its length by l. Either A is conjugate in rank $l-1$ to a power of a shorter word (including periods of rank less than l)—then the assertion follows from the induction hypothesis; otherwise A is simple of rank $l-1$—then the assertion follows from the maximality assumption on $\mathscr{X}_l$.) If $C \not\equiv x_1 \cdots x_m$ then we are done because in this case $C^n = 1$ in G (by 26.1(1)). Suppose that $C \equiv x_1 \cdots x_m$ and in G, $X = Z^{-1}C^tZ$ for some integer t. By assumption, $\sigma_\lambda(X) \equiv 0 \pmod{n}$ for some λ. By Lemma 2.1 we have $t = \sigma_\lambda(Z^{-1}C^tZ) \equiv \sigma_\lambda(X) \pmod{n}$, hence $t \equiv 0 \pmod{n}$, that is, $t = nk$ for some k. But then, in G we have $X^n = (Z^{-1}C^{nk}Z)^n = Z^{-1}C^{n^2k}Z = Z^{-1}C^{nq(n/q)k}Z = 1$, as required. ∎

Summing up, we have shown the existence of a group $G = \langle x_1, \ldots, x_m \rangle$ enjoying the properties we need.

COROLLARY 2.3. *Let n be odd and greater than 10^{10} and let q be a divisor of n, $q \neq 1$. Then for each $m > 1$ there exists a group $G = \langle x_1, \ldots, x_m \rangle$ such that in G*

1. $(x_1 \cdots x_m)^n \neq 1$,
2. *$W^n = 1$ for each word $W \in \{x_1^{\pm 1}, \ldots, x_m^{\pm 1}\}^*$ for which $\sigma_\lambda(W) = 0$ for some $\lambda \in \{1, \ldots, m\}$,*
3. *G satisfies the law $x^{nq} \simeq 1$.*

The authors are indebted to V. Guba for drawing their attention to the work of Ol'shanskiĭ and for sketching a proof of Corollary 2.3. The authors are also grateful to S. Ivanov for providing valuable information about the case n being even. Indeed, recently he developed, for large even exponents, a theory of group presentations which, to a certain extent, is analogous to that in [8, Sects. 25, 26] (see [6]). However, not all results can be carried over from odd to even exponents. For example, it is no longer true that each element is conjugate to a power of some period. And this fact has essentially been used in the construction of the groups we need. Therefore, it is not clear if an analogue of Corollary 2.3 holds for even exponents. Yet a much weaker statement follows immediately from Ivanov's presentation of free Burnside groups of even exponents [5]:

RESULT 2.4. *Let n be a positive integer such that* $2^9 \mid n$ *and* $n > 2^{45}$. *Then for each m there exists a group* $G = \langle x_1, \ldots, x_m \rangle$ *such that in G*:

1. $(x_1 \cdots x_m)^n \neq 1$,
2. $W^n = 1$ *for each word W with length less than m*.

3. NON-EXISTENCE OF FINITE BASES

In this section we prove the main theorem of the paper.

THEOREM 3.1. *Let t be any positive integer, let n be odd and greater than* 10^{10}; *moreover, let* $q > 1$ *be a divisor of n and let l be any positive integer which is not a multiple of q. Then no variety* $\mathscr{V}$ *between* $C^*_{\mathscr{CS}_{nq}}\mathscr{CS}_n$ *and* $C_t\mathscr{P}_{t,\,nl}$ *has a finite basis for its identities*.

Proof. Let $\mathscr{V}$ be a variety between $C^*_{\mathscr{CS}_{nq}}\mathscr{CS}_n$ and $C_t\mathscr{P}_{t,\,nl}$. We argue by a standard argument: for each positive integer k we construct a (completely simple) semigroup S_k such that $S_k \notin \mathscr{V}$ but each completely simple subsemigroup (that is, each subepigroup) of S_k which is generated by less than k elements does belong to $\mathscr{V}$.

Choose a group $G = \langle x_1, \ldots, x_{2k} \rangle$ as in Corollary 2.3 (for $m = 2k$) and let $S_k = \mathscr{M}(2, G, 2k, P)$ be the $2 \times 2k$ Rees matrix semigroup over G with the $2k \times 2$ sandwich matrix

$$P = \begin{pmatrix} 1 & 1 \\ 1 & x_2^{-1}x_1^{-1} \\ 1 & x_3x_1^{-1} \\ \vdots & \vdots \\ 1 & x_{2k-1}x_1^{-1} \\ 1 & x_{2k}^{-1}x_1^{-1} \end{pmatrix}.$$

In S_k, the $\mathscr{R}$-classes are $R_i = \{i\} \times G \times \{1, 2, \ldots, 2k\}$, $i = 1, 2$, and the $\mathscr{L}$-classes are $L_j = \{1, 2\} \times G \times \{j\}$, $j = 1, 2, \ldots, 2k$. Denote by f_1, $f_2, \ldots, f_{2k}$ and $g_1, g_2, \ldots, g_{2k}$ the idempotents in R_1 and R_2, respectively, such that f_j, g_j belong to L_j, $j = 1, 2, \ldots, 2k$. Thus S_k looks like

f_1	f_2	f_3	$\cdots$	f_{2k}
g_1	g_2	g_3	$\cdots$	g_{2k}

We consider the product of idempotents

$$\begin{aligned} & f_1 g_2 f_3 \cdots f_{2k-1} g_{2k} \\ & \quad = (1,1,1)\big(2,(x_2^{-1}x_1^{-1})^{-1},2\big)(1,1,3)\cdots(1,1,2k-1)\big(2,(x_{2k}^{-1}x_1^{-1})^{-1},2k\big) \\ & \quad = \big(1,(x_1x_2)(x_3x_1^{-1})(x_1x_4)\cdots(x_{2k-1}x_1^{-1})(x_1x_{2k}),2k\big) \\ & \quad = (1, x_1x_2\cdots x_{2k}, 2k), \end{aligned}$$

whence, for any positive integer s,

$$\big(f_1 g_2 \cdots f_{2k-1} g_{2k}\big)^s = \big(1,(x_1x_2\cdots x_{2k})^s,2k\big).$$

By our assumption on G, the order of $(x_1 \cdots x_{2k})$ is nq. Since $nq \nmid nl$ we have $(x_1 \cdots x_{2k})^{nl} \neq 1$ hence

$$\big(f_1 g_2 \cdots f_{2k-1} g_{2k}\big)^{t+nl} \neq \big(f_1 g_2 \cdots f_{2k-1} g_{2k}\big)^t.$$

The element $f_1 g_2 \cdots f_{2k-1} g_{2k}$ surely belongs to the core of S_k. Therefore S_k does not belong to $C_t\mathscr{P}_{t,nl}$ (though it clearly belongs to $\mathscr{P}_t$). Consequently, S_k does not belong to $\mathscr{V}$.

Now let T be any completely simple subsemigroup of S_k which is generated by less than k (even less than $2k$) elements. Then for some $\lambda \in \{1,2,\ldots,2k\}$, $T \subseteq T_\lambda := S_k \setminus L_\lambda$, where (as above) $L_\lambda = \{1,2\} \times G \times \{\lambda\}$. Clearly, since G satisfies $x^{nq} \simeq 1$, T_λ (and therefore also T) belongs to $\mathscr{CS}_{nq}$. In order to show that T_λ, and thus T, is contained in $\mathscr{V}$, it suffices to show that $C^*(T_\lambda) \in \mathscr{CS}_n$. T_λ is a completely simple semigroup isomorphic to $\mathscr{M}(2, G, 2k-1, P_\lambda)$ with a normalized matrix P_λ, where P_λ is obtained as follows. If $\lambda > 1$ then simply delete the row λ in P to get P_λ. For $\lambda = 1$, T_1 is, first of all, isomorphic to $\mathscr{M}(2, G, 2k-1, \overline{P_1})$, where $\overline{P_1}$ is obtained from P by deletion of the first row:

$$\overline{P_1} = \begin{pmatrix} 1 & x_2^{-1}x_1^{-1} \\ 1 & x_3x_1^{-1} \\ \vdots & \vdots \\ 1 & x_{2k-1}x_1^{-1} \\ 1 & x_{2k}^{-1}x_1^{-1} \end{pmatrix}.$$

However, this matrix is not normalized. Normalization of $\overline{P_1}$ then gives (see, e.g., [10, Lemma III.3.6])

$$P_1 = \begin{pmatrix} 1 & 1 \\ 1 & x_3 x_2 \\ 1 & x_4^{-1} x_2 \\ \vdots & \vdots \\ 1 & x_{2k}^{-1} x_2 \end{pmatrix}$$

and $T_1 \cong \mathscr{M}(2, G, 2k-1, P_1)$. Notice that, for each λ, the entries in P_λ do not contain the element x_λ. Let N_λ be the normal subgroup of G generated by the entries of P_λ. Each element in N_λ can be expressed as a word $W \in \{x_1^{\pm 1}, \ldots, x_{2k}^{\pm 1}\}^*$ such that $\sigma_\lambda(W) = 0$. (The element x_λ does not occur in the entries of P_λ and therefore it can come into play only by conjugation.) From the choice of G it follows that N_λ satisfies the identity $x^n \simeq 1$. From Lemma 2.3 in [16] we have that $C^*(T_\lambda) \cong \mathscr{M}(2, N_\lambda, 2k-1, P_\lambda)$. In other words, the subgroups of $C^*(T_\lambda)$ satisfy $x^n \simeq 1$ whence $T_\lambda \in C^*_{\mathscr{CS}_{nq}} \mathscr{CS}_n \subseteq \mathscr{V}$. ∎

Immediately we have the following.

COROLLARY 3.2. *Let $t \geq 1$ and $m \geq 0$ be integers; let n be odd and greater than 10^{10} and let $q \neq 1$ be any divisor of n. Then the varieties $C_t \mathscr{P}_{t, n2^m}$, $C_t \mathscr{CR}_{n2^m}$, $C\mathscr{CR}_{n2^m}$, $C^* \mathscr{CR}_{n2^m}$, $C\mathscr{CS}_{n2^m}$, $C_{\mathscr{CS}_{nq}} \mathscr{CS}_n$, $C^* \mathscr{CS}_{n2^m}$, $C^*_{\mathscr{CS}_{nq}} \mathscr{CS}_n$, etc., are not finitely based.*

For an even number n such that $2^9 \mid n$ and $n > 2^{45}$ the sequence S_k, $k = 1, 2, \ldots$, based on groups as in Result 2.4, shows that the system of identities (1) is not equivalent to any finite subsystem. From the Compactness Theorem of Equational Logic we get the following.

COROLLARY 3.3. *For each integer n such that $2^9 \mid n$ and $n > 2^{45}$ and for any s and t the varieties $C_t \mathscr{P}_{s,n}$—in particular, $C_t \mathscr{P}_{t,n}$, $C_t \mathscr{CR}_n$, $C\mathscr{CR}_n$—and, moreover, the variety $C\mathscr{CS}_n$ are not finitely based.*

It should be noted that Theorem 3.1 is, in fact, new only for a particular subinterval of $I := [C^*_{\mathscr{CS}_{nq}} \mathscr{CS}_n, C_t \mathscr{P}_{t, nl}]$. Namely, let $\mathscr{DS}_t$ denote the class of all members of $\mathscr{P}_t$ all of whose regular $\mathscr{D}$-classes are (completely simple) subsemigroups. Then $\mathscr{DS}_t$ is a subvariety of $\mathscr{P}_t$ and can be defined by the law

$$\left((xy)^0(yx)^0(xy)^0\right)^0 \simeq (xy)^0,$$

where, for a term w, w^0 stands for $w^{-1}w$ (see, for example, Pastijn [9]). Furthermore, let $\mathcal{DS}_{t,\,nl}$ be the subvariety of $\mathcal{DS}_t$ consisting of all periodic members with period nl. It is well known and follows from the ideas of Volkov presented in [13] and [14] that all members of I which are not contained in $[C^*_{\mathcal{CS}_{nq}}\mathcal{CS}_n, C_t\mathcal{DS}_{t,\,nl}]$ (that is, all varieties in I which contain the five element combinatorial non-orthodox completely 0-simple semigroup C_2) are not finitely based. In this latter case, a much more general result holds: the group variety defined by the law $x^{nl} \simeq 1$ may be replaced by any non-trivial group variety.

The sequence S_k, $k = 1, 2, \ldots,$ constructed in the proof of Theorem 3.1 has applications also to e-varieties of regular semigroups. A class of regular semigroups is termed an *e-variety* if it is closed under forming direct products, regular subsemigroups, and homomorphic images (see Hall [3] and also Kaďourek and Szendrei [7]). Broeksteeg [1] has shown that for each positive integer n, the class $\mathcal{ES}_n$ of all regular semigroups S satisfying $(ef)^{n+1} = ef$ for all idempotents $e, f \in S$ is an e-variety of E-solid semigroups, and moreover, the sub-e-varieties of $\mathcal{ES}_n$ are precisely the subclasses which are definable by bi-identities of the form introduced in [7]. Without giving the precise definition, we mention that such a bi-identity is an equation of the form $w \simeq v$, w and v being semigroup words on a doubled alphabet $X \cup X'$, where $X' = \{x' \mid x \in X\}$ is a disjoint copy of X. When evaluating such words in a regular semigroup, the pairs of variables x and x' have to be substituted by mutually inverse elements. For $k \geq 2$ define $\mathcal{ES}_{n,\,k}$ to be the class of all regular semigroups S such that $(e_1 \cdots e_k)^{n+1} = e_1 \cdots e_k$ for each k-tuple $e_1, \ldots, e_k$ of idempotents of S. Then $\mathcal{ES}_{n,\,k}$ is an e-variety and $\mathcal{ES}_n = \mathcal{ES}_{n,\,2} \supseteq \mathcal{ES}_{n,\,3} \supseteq \cdots \supseteq \mathcal{ES}_{n,\,k} \supseteq \mathcal{ES}_{n,\,k+1} \supseteq \cdots$. Moreover, $\bigcap_k \mathcal{ES}_{n,\,k}$ is the e-variety $C_{\mathcal{ES}}\mathcal{CR}_n$ of all E-solid semigroups S whose core $C(S)$ is periodic with period n. Since in each E-solid semigroup, not only the core but also the self-conjugate core is completely regular, another natural sub-e-variety in $\mathcal{ES}_n$ is $C^*_{\mathcal{ES}}\mathcal{CR}_n$, the class of all E-solid semigroups S whose self-conjugate core $C^*(S)$ is periodic with period n. Moreover, note that each variety of completely regular semigroups is in particular an e-variety of E-solid semigroups.

Now let n be odd and greater than 10^{10}. The sequence S_k, $k = 1, 2, \ldots,$ proves that, for each k, the e-variety $\mathcal{ES}_{n,\,2k}$ is properly contained in $\mathcal{ES}_{n,\,2k-1}$. The fact that $\mathcal{ES}_{n,\,2k}$ properly contains $\mathcal{ES}_{n,\,2k+1}$ for every k, can be proved in a similar way by means of a family U_k, $k = 1, 2, \ldots,$ of completely simple semigroups constructed as follows. Let $G = \langle x_1, \ldots, x_{2k+1}\rangle$ be a group as in Corollary 2.3 (for $m = 2k + 1$) and let $U_k =$

$\mathscr{M}(3, G, 2k+1, Q)$ be the $3 \times (2k+1)$ Rees matrix semigroup over G with the $(2k+1) \times 3$ sandwich matrix

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & x_2^{-1}x_1^{-1} & 1 \\ 1 & x_3x_1^{-1} & 1 \\ \vdots & \vdots & \vdots \\ 1 & x_{2k-1}x_1^{-1} & 1 \\ 1 & x_{2k}^{-1}x_1^{-1} & 1 \\ 1 & x_{2k+1}x_1^{-1} & x_{2k+1}^{-1} \end{pmatrix}.$$

Using an analogous notation for the idempotents introduced above for S_k, U_k looks like

f_1	f_2	f_3	$\cdots$	f_{2k}	f_{2k+1}
g_1	g_2	g_3	$\cdots$	g_{2k}	g_{2k+1}
h_1	h_2	h_3	$\cdots$	h_{2k}	h_{2k+1}

.

Then we have

$$\left(f_1g_2\cdots f_{2k-1}g_{2k}h_{2k+1}\right)^{n+1} \neq f_1g_2\cdots f_{2k-1}g_{2k}h_{2k+1}$$

for the idempotents $f_1, g_2, f_3, \ldots, f_{2k-1}, g_{2k}, h_{2k+1}$, but the self-conjugate core of each subsemigroup $U_k \setminus L_\lambda$ ($\lambda \in \Lambda$) is periodic with period n. In addition, each of the sequences (S_k) and (U_k) proves that no e-variety between $C^*_{\mathscr{CS}_{nq}}\mathscr{CS}_n$ and $C_{\mathscr{ES}}\mathscr{CR}_n$ is finitely based in the above-mentioned bi-identity signature. In particular, the e-variety $C^*_{\mathscr{ES}}\mathscr{CR}_n$ has no finite basis for its bi-identities. Moreover, taking into account that a suitable analogue of the Compactness Theorem also holds for the bi-identity signature (see [7]) we also obtain, by a reasoning similar to that prior to Corollary 3.3, that $C_{\mathscr{ES}}\mathscr{CR}_n$ is not finitely based provided $2^9 \mid n$ and $n > 2^{45}$.

Finally, it seems to be worth mentioning that a finitary analogue of Corollary 2.3, that is, a proof of the existence of *finite* groups enjoying the properties of this corollary, is highly desirable. A positive answer of whether such groups exist would solve the analogous finite basis problem in the context of pseudovarieties. The problem seems to be even more natural in this finitary context because there is no need to specify a fixed index t there. In addition, a positive solution would provide a large new class of non-finitely based pseudovarieties which are decidable in polynomial time (see [14]). But also the varietal question would profit from a finitary analogue of Corollary 2.3: the lower bound of the interval I could be pushed down to the variety of all completely simple semigroups S whose subgroups are *locally finite* and of exponent, say, nq and whose self-conjugate core $C^*(S)$ has period n.

REFERENCES

1. R. Broeksteeg, E-solid e-varieties of regular semigroups which are bi-equational, *Internat. J. Algebra Comput.* **6** (1996), 277–290.
2. M. P. Drazin, Pseudo-inverses in associative rings and semigroups, *Amer. Math. Monthly* **65** (1958), 506–514.
3. T. E. Hall, Identities for existence varieties of regular semigroups, *Bull. Austral. Math. Soc.* **40** (1989), 59–77.
4. J. M. Howie, "Fundamentals of Semigroup Theory," Clarendon, Oxford, 1995.
5. S. Ivanov, The free Burnside groups of sufficiently large exponents, *Internat. J. Algebra Comput.* **4** (1994), 1–308.
6. S. Ivanov, "On Finitely Presented Groups Given by Periodic Relators," preprint.
7. J. Kaďourek and M. B. Szendrei, A new approach in the theory of orthodox semigroups, *Semigroup Forum* **40** (1990), 257–297.
8. A. Yu. Ol'shanskiĭ, "Geometry of Defining Relations in Groups," Mathematics and Its Applications (Soviet Series), Kluwer, Dordrecht, 1991.
9. F. Pastijn, The idempotents in a periodic semigroup, *Internat. J. Algebra Comput.* **6** (1996), 511–540.
10. M. Petrich and N. R. Reilly, "Completely Regular Semigroups I," Wiley, New York, 1999.
11. L. N. Shevrin, On the theory of epigroups I, *Mat. Sb.* **185**, No. (8) (1994), 129–160 (in Russian); English transl., *Russian Acad. Sci. Sb. Math.* **82**, No. 2 (1995), 485–512.
12. L. N. Shevrin, On the theory of epigroups II, *Mat. Sb.* **185**, No. (9) (1994), 153–176 (in Russian); English transl. *Russian Acad. Sci. Sb. Math.* **83**, No. 1, (1995), 133–154.
13. M. V. Volkov, On finite basedness of semigroup varieties, *Mat. Zametki* **45** (1989), 12–23 (in Russian); English transl. in *Math. Notes* **45** (1989), 187–194.
14. M. V. Volkov, On a class of semigroup pseudovarieties without finite pseudoidentity basis, *Internat. J. Algebra Comput.* **5** (1995), 127–135.
15. M. V. Volkov, Covers in the lattices of semigroup varieties and pseudovarieties, *in* "Semigroups, Automata and Languages," (J. Almeida, G. M. S. Gomes, and P. V. Silva, Eds.), pp. 263–280, World Scientific, Singapore, 1996.
16. S. Zhang, Completely regular semigroup varieties generated by Mal'cev products with groups, *Semigroup Forum* **48** (1994), 180–192.