Existence and Nonuniqueness of Solutions of a Singular Nonlinear Boundary-Layer Problem

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Sufficient conditions for existence and nonuniqueness of positive solutions of the singular boundary value problem \( g(x) g''(x) + h(x) = 0, \quad -k < x < 1, \quad k > 0, \)
\[ g'(-k) = C, \quad g(1) = 0 \]
are obtained. Also, it is proved that the solutions with \( g(-k) > -Ck \) (for \( C < 0 \)) and \( g(-k) > (k/2) \sqrt{-2h(-k)} \) (for \( C > 0 \)) are unique. Furthermore, it is shown numerically that for \( h(x) = x \) there are exactly two solutions for the problem.


1. Introduction


Using Crocco variable transformation [8], Callegari and Friedman [5] developed an analytical solution for the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence. Subsequently, Callegari and Nachman [6] obtained results for uniqueness and analyticity for boundary value problems corresponding to the flow behind weak expansion and shock waves and for the flow above a moving conveyor belt by standardizing the initial conditions. Recently, Vajravelu, Soewono, and Mohapatra [18] studied the above two problems (see [5, 6]) under the influence of suction and injection and developed an alternative procedure to obtain existence and uniqueness of solutions since the standardization technique failed to yield appropriate results.

Klemp and Acrivos [14] studied the Blasius problem when the plate moves in the direction opposite to that of the main stream. Subsequently

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Hussaini and Lakin [12] showed that the solutions of such boundary-layer problems exist only up to a certain critical value of the velocity ratio parameter. Analyticity of the solutions were studied by Hussaini, Lakin, and Nachman [13].

Since the effects of suction (or injection) on the boundary-layer flow is of interest in increasing (or reducing) the drag force and in controlling the boundary-layer separation (see [16]), we study the problem of Hussaini, Lakin, and Nachman [13] under the effect of suction/injection. Existence, nonuniqueness, and analyticity results are established for a class of boundary value problems from which results of [12, 13] can be obtained as a special case. Furthermore, the analytical results are compared with numerical solutions in Section 6. The numerical evidence reveals the existence of exactly two solutions.

2. FORMULATION OF THE PROBLEM

Consider a moving porous flat plate aligned with a uniform flow of constant speed $U_\infty$ and moving in the direction opposite to that of the stream. The boundary-layer equations for the flow are

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = v \frac{\partial^2 U}{\partial Y^2}, \quad (2.1)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \quad (2.2)$$

where $X$ and $Y$ axes are taken along and perpendicular to the plate and the other symbols have their usual meanings. The appropriate boundary conditions are

$$U = -U_w, \quad V = V_w(X) \quad \text{at} \quad Y = 0, \quad (2.3, 1, 2, 3)$$

$$U \to U_\infty \quad \text{as} \quad Y \to \infty.$$

In terms of the stream function $\psi$ defined by

$$U = \partial \psi / \partial Y, \quad V = -\partial \psi / \partial X,$$

Eq. (2.1) becomes

$$\frac{\partial \psi}{\partial Y} \frac{\partial^2 \psi}{\partial X \partial Y} - \frac{\partial \psi}{\partial Y} \frac{\partial^2 \psi}{\partial Y^2} = v \frac{\partial^3 \psi}{\partial Y^3}, \quad (2.4)$$

and Eq. (2.2) is satisfied automatically. Defining new variables,

$$\eta = Y(U_\infty/2vX)^{1/2} \quad \text{and} \quad \psi = (2vXU_\infty)^{1/2} f(\eta),$$
Eq. (2.4) and the boundary conditions (2.3, 1, 2, 3) can be written as

\[ f''' + ff'' = 0, \tag{2.5} \]

\[ f = -C, \quad f' = -k \quad \text{at} \quad \eta = 0, \tag{2.6, 1, 2, 3} \]

\[ f' \to 1 \quad \text{as} \quad \eta \to \infty, \]

where \( k = U_w/U_{\infty} \), the velocity ratio parameter and \( C = V_w(2X/vU_{\infty})^{1/2} \), the suction/injection parameter. Here \( C \) can be positive, negative, and zero. Physically \( C > 0 \) means the injection of fluid into the boundary layer and \( C < 0 \) implies the suction of the fluid from the boundary layer.

Using Crocco variable formulation, that is, in terms of shear stress \( g(=f'') \) as the dependent variable and tangential velocity \( x(=f') \) as the independent variable, Eq. (2.5) and the boundary conditions (2.6, 1, 2, 3) can be written as

\[ g(x) g''(x) + x = 0, \tag{2.7} \]

\[ g'(-k) = C \quad \text{and} \quad g(1) = 0, \quad k > 0, \tag{2.8, 1, 2} \]

A general form of (2.7 and 2.8, 1, 2) for the case \( k = 0 \) and \( C = 0 \) has been studied in [4, 9, 15] in the form of

\[ y'' + f(x, y) = 0, \]

\[ xy(0) - \beta y'(0) = 0, \]

\[ \gamma y(1) + \delta y'(1) = 0, \]

where \( f: (0, 1) \times (0, \infty) \to (0, \infty) \) is continuous and \( \lim_{y \to 0^+} f(x, y) = \infty \). It has been shown that this problem under appropriate conditions on \( f(x, y) \) has a unique positive solution.

In what follows we study the existence, uniqueness, and analyticity results for the system

\[ g(x) g''(x) + h(x) = 0, \quad -k < x < 1, \tag{2.9} \]

\[ g'(-k) = C \quad \text{and} \quad g(1) = 0, \quad k > 0 \tag{2.10, 1, 2} \]

(where \( h \) is a continuous and increasing function defined on \([-k, \infty)\)) satisfying

\[ M_2 |x| \leq |h(x)| \leq M_1 |x| \quad \text{for} \quad -k \leq x \leq 1 \]

and \( \lim_{x \to \infty}(h(x)/x) \geq M \) which arises in the study of the compressible laminar boundary layers over a semi-infinite flat plate with zero incidence in a uniform stream.
Note that $h(x)$ changes sign at $x = 0$. When $h(x) = x$ and $C = 0$, we get the results of Hussaini and Lakin [12] and Hussaini, Lakin, and Nachman [13]. In Sections 3 and 4, we prove the existence and nonuniqueness of the solutions. And in Sections 5 and 6 we present the numerical results.

3. **Main Results**

Consider a boundary value problem

$$g(x) - g''(x) + h(x) = 0, \quad -k < x < l, \quad (k > 0),$$

(3.1)

$$g'(-k) = C \quad \text{and} \quad g(1) = 0,$$

(3.2)

where $h$ is a continuous and increasing function defined on $[-k, \infty)$ satisfying

$$M_1 |x| \leq |h(x)| \leq M_2 |x| \quad \text{for} \quad -k \leq x \leq 1,$$

(3.3)

and

$$\lim_{x \to \infty} \frac{h(x)}{x} > M,$$

(3.4)

for some positive constants $M_1, M_2$, and $M$. The above growth restrictions on $h$ are crucial to the proof of Lemma 5. For a special case when $h(x) = x$, we have $M_1 = M_2 = M = 1$. The second inequality in (3.3) is naturally satisfied when $h$ is continuously differentiable, although this is not needed in the theorems. The first inequality in (3.3) may be improved by replacing $M_2 |x|$ by $M_2 |x|^n$ for some positive number $n$ without changing the idea of the proofs of the results.

We are interested in positive solutions $g$ in $[-k, 1)$; therefore (3.1) can be written as

$$g''(x) = -\frac{h(x)}{g(x)}, \quad -k < x < 1.$$  

(3.5)

Instead of solving (3.1) and (3.2) directly, we consider the initial value problem

$$g''(x) = -\frac{h(x)}{g(x)}, \quad -k < x$$

(3.6)

$$g(-k) = \alpha > 0, \quad g'(-k) = C.$$  

(3.7)

This formulation of the problem is an adaptation of the shooting technique (see the general idea in [2, Chap. 2]). The problem now is to
find \( z > 0 \) such that the solution \( g \) of (3.6) and (3.7) is defined and positive on \([-k, 1)\) and \( \lim_{x \to 1^-} g(x) = 0 \).

Concerning the existence and uniqueness of the positive solution \( g \) of (3.6) and (3.7), it can be remarked that by using standard techniques the following result (Theorem 1) can be established.

**Theorem 1.** For any \( C \) and \( z > 0 \), the initial value problem (3.6), (3.7) has a unique positive solution \( g \) on its maximal interval of existence \([-k, x^*_\infty)\). Moreover \( g(x^*_\infty) = 0 \).

In the next two theorems we show that the boundary value problem (3.1), (3.2) has at least two solutions. This shows a different behavior of the solutions from the case when \(-k \geq 0\) where the solution is unique (see [5, 6, 12, 13, 18]).

**Theorem 2.** If \( C < 0 \) and \( M_1, M_2, k \) satisfy

(i) \(-C \geq \sqrt{M_1} k\)

(ii) \( \int_{-k}^0 sh(s) \, ds \leq 2M_2/27\)

(iii) \( k > M_1/\pi M_2 \sqrt{3}\),

then the boundary value problem (3.1), (3.2) has a monotonic decreasing solution with \( z > -Ck \) and another solution with a negative local minimum point and a positive maximum point with \( z < -Ck \).

**Theorem 3.** If \( C \geq 0 \) and

\[
\begin{align*}
a_1 \sqrt{\frac{2}{M_2 b_1}} + a_2 \sqrt{\frac{2b_2}{M_2}} + b_2 &< 1, \\
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= -Ck + k \sqrt{2M_1} k, \\
a_2 &= C + \frac{M_1 \sqrt{k}}{\sqrt{2M_2}}, \\
b_1 &= \sqrt{\frac{Ck \sqrt{-2h(-k)} + M_2 k^2}{M_1}}, \\
b_2 &= \frac{1}{M_2} [a_2^2 + \sqrt{a_2^4 + 2M_2 a_1 a_2}],
\end{align*}
\]

then the boundary value problem (3.1), (3.2) has at least two solutions, one with \( z > (k/2) \sqrt{-2h(-k)} \) and the other with \( z < (k/2) \sqrt{-2h(-k)} \). Both
solutions are increasing from \(-k\) to its positive maximum point \(\beta\) and decreasing in \((\beta, 1)\).

**Remark.** The conditions in Theorem 3 are satisfied if \(C\) and \(k\) are sufficiently small. For \(C=0\) and \(h(x)=x\), the conditions are reduced to \(k<1/(4+\sqrt{2})\).

**Theorem 4.** If \(C<0\) and

1. \(-C \geq \sqrt{M_1(k+(1/2)k^2)}\)
2. \(\int_{-k}^{0} sh(s)\,ds \leq 2M_1/27\)
3. \(k > M_1/\pi M_2 \sqrt{3}\)
4. \(2k^2 + (4/3)k^3 < 2M_1/M_2\),

then the boundary value problem \((3.1), (3.2)\) has exactly one monotonic decreasing solution with \(x > -Ck\).

**Theorem 5.** If \(C>0\) and in addition to the conditions in Theorem 3, \(C\) and \(k\) satisfy

1. \(d_1 \geq k \sqrt{M_1(1/2 + (1/3)k)}\)
2. \(d_2 \leq (k/2) \sqrt{-2h(-k)}\),

where
\[
d_{1,2} = \frac{1}{2} \left[ \sqrt{M_2/6 - Ck - C \pm \sqrt{(\sqrt{M_2/6 - Ck - C)^2 - 2M_1(k^2 + k^3)}}} \right]
\]

then the boundary value problem \((3.1), (3.2)\) has exactly one solution with \(x > (k/2) \sqrt{-2h(-k)}\).

**Remark.** The conditions in Theorem 5 are satisfied if \(C\) and \(k\) are relatively small. Numerical evidence shows that the second solution is also unique for \(C \geq 0\) and \(C < 0\) although we are not able to prove this conjecture.

4. **Proofs of the Results**

Since \(\alpha > 0\), the standard existence theorem (see [7]) gives a unique positive solution of \((3.6), (3.7)\) on its maximal interval of existence \([-k, x^*_\alpha]\). In this interval, \((3.6), (3.7)\) is equivalent to

\[
g'(x) = C - \int_{-k}^{x} \frac{h(s)}{g(s)}\,ds, \quad (4.1)
\]
\[
g(-k) = \alpha. \quad (4.2)
\]

The following lemma is needed to prove our results.
**Lemma 1.** For any \( C \) and any \( \alpha > 0 \), we have \( x_\alpha^* > 0 \). If \( C \geq 0 \), the strict inequality \( x_\alpha^* > 0 \) occurs.

**Proof.** For \( C \geq 0 \), \( g' \) is positive, increasing, and bounded from above by \( C - (1/\alpha) \int_{-k}^{0} h(s) \, ds \) in \([-k, x_0)\), where \( x_0 = \min(0, x_\alpha^*) \). Since \( g''(x) = -h(x)/g(x) \) is decreasing in \([-k, x_0)\), \( g(x) \) is bounded by \( - (h(-k)/2\alpha) (x + k)^2 + C(x + k) + \alpha \), for \( -k \leq x < x_0 \). By definition of maximal interval of existence we conclude that \( x_\alpha^* > 0 \). For \( C < 0 \), suppose \( x_\alpha^* < 0 \). We have from (4.1), \( g'(x) \) is initially negative and increasing. If \( \lim_{x \to -x_\alpha^*} g'(x) > 0 \), then the argument in the case of \( C \geq 0 \) implies \( x_\alpha^* > 0 \). However, \( \lim_{x \to -x_\alpha^*} g'(x) \leq 0 \) implies \( g(x_\alpha^*) = 0 \). Using the fact that \( g''(x) > 0 \) for \( -k < x < x_\alpha^* \), we have

\[
\frac{\alpha(x_\alpha^* - x)}{x_\alpha^* + k} = f(x) \quad \text{for} \quad -k \leq x \leq x_\alpha^*.
\]

But this implies

\[
g'(x) = C - \int_{-k}^{x} \frac{h(s)}{g(s)} \, ds \geq C - \int_{-k}^{x} \frac{h(s)}{f(s)} \, ds \to \infty
\]
as \( x \to x_\alpha^* \) which contradicts \( g'(x) < 0 \) for \( -k \leq x < x_\alpha^* \).

**Lemma 2.** For any fixed \( C \), \( x_\alpha \) is a continuous function of \( \alpha \).

Proof of this lemma is similar to that of Lemma 2 in [18].

**Lemma 3.** For fixed \( C \), \( x_\alpha^* \to \infty \) as \( \alpha \to \infty \).

**Proof.** For \( C \geq 0 \), \( g(0) > g(-k) = \alpha \) and for \( C < 0 \) and \( \alpha \) large, \( g(0) > \alpha + Ck \). Therefore, if \( \alpha \) is large, the solution \( g \) of (3.6), (3.7) will intersect the line \( y = 1 \) at \( x_{1,\alpha} > 0 \) and \( g(x) \geq 1 \) for \( -k \leq x \leq x_{1,\alpha} \). Then we can follow the proof of Lemma 3 in [18]. Integrate (4.1) over \([-k, x_{1,\alpha}]\),

\[
1 = \alpha + C(x_1 + k) - \int_{-k}^{x_{1,\alpha}} \frac{(x_{1,\alpha} - s) h(s)}{g(s)} \, ds.
\]

Since \( g(x) \geq 1 \) on \([-k, x_{1,\alpha}]\) we have from the last equation

\[
C(x_{1,\alpha} + k) + \alpha - 1 \leq \int_{-k}^{x_{1,\alpha}} (x_{1,\alpha} - s) h(s) \, ds,
\]

which implies that \( x_{1,\alpha} \to \infty \) as \( \alpha \to \infty \) and consequently \( x_\alpha^* \to \infty \) as \( \alpha \to \infty \); otherwise the right side of the above inequality is bounded while the left side is unbounded.
Lemma 4. If \( C < 0, \alpha = -Ck \) and

(i) \( -C > \sqrt{M_1k} \)

(ii) \( \int_{-k}^{0} sh(s) \, ds \leq 2M_2/27, \)

then \( x_2^* < 1. \)

Proof. For \( \alpha \geq -Ck, \) since \( g''(x) > 0 \) for \( -k \leq x < 0, \) we have

\[
g(x) \geq \alpha + C(x + k) \quad \text{for} \quad -k \leq x \leq 0.
\]

Therefore

\[
g'(0) = C - \int_{-k}^{0} \frac{h(s)}{g(s)} \, ds \leq C - \int_{-k}^{0} \frac{h(s)}{\alpha + C(s + k)} \, ds.
\]

The assumption on \( h \) in (3.3) implies

\[
g'(0) \leq \frac{C^2 - M_1k}{C} \leq 0 \quad \text{if} \quad -C \geq \sqrt{M_1k}.
\]

This implies \( g'(x) \leq 0 \) for \( -k \leq x \leq 0. \) For \( \alpha = -Ck, \) integrate (4.1) over \( [-k, 0]; \) we have

\[
0 < g(0) = -Ck + Ck + \int_{-k}^{0} \frac{sh(s)}{g(s)} \, ds \leq \frac{1}{g(0)} \int_{-k}^{0} sh(s) \, ds \quad (4.3)
\]

or

\[
0 < g(0) \leq \left[ \int_{-k}^{0} sh(s) \, ds \right]^{1/2} = a. \quad (4.4)
\]

Next we find a bound for \( x_2^* \) for \( \alpha = -Ck. \) Since \( g''(0) = 0, \) we obtain a majorant \( f \) of \( g \) given by

\[
g(x) \leq f(x) = \begin{cases} \alpha - \frac{M_2 \varepsilon}{2a} (\varepsilon - x)^2, & \varepsilon < x < x_2^* \\ \alpha, & 0 \leq x \leq \varepsilon, \end{cases}
\]

where \( \varepsilon = (a^2/2M_2)^{1/3}. \) The function \( f \) intersects the \( x \)-axis at

\[
x = \varepsilon + a \sqrt{\frac{2}{M_2 \varepsilon}} = 3 \left( \frac{a^2}{2M_2} \right)^{1/3} \leq 1
\]

if \( a^2 \leq 2M_2/27. \) This completes the proof of Lemma 4.
Remark. From Lemmas 2, 3, and 4, we have the existence of a solution of (3.1), (3.2) with \( \alpha > -Ck \), if \( C \) and \( k \) satisfy the conditions in Lemma 4.

The following lemma is needed for the existence of a second solution for \( C < 0 \).

**Lemma 5.** Suppose \( C \) and \( k \) satisfy the conditions in Lemma 4. If \( k > M_1 / \pi M_2 \sqrt{3} \), then a second solution \( g \) of (3.1), (3.2) exists with \( 0 < \alpha < -Ck \). Furthermore, \( g \) has a negative local minimum point and a positive maximum point.

**Proof.** If \( \alpha \) is small enough, since \( C \) is negative, \( g \) attains its local minimum at \( \gamma, -k < \gamma < 0 \). This follows immediately from Lemma 1 and (4.1). Consider now the solution \( g \) of (3.6), (3.7) in \([-k, \gamma]\) with \( \alpha \) small enough. Since \( g(x) \leq \alpha \) for \(-k < x \leq \gamma\), we have from (4.1)

\[
-C = - \int_{-k}^{\gamma} \frac{h(s)}{g(s)} \, ds \geq - \frac{1}{\alpha} \int_{-k}^{\gamma} h(s) \, ds.
\]

This implies \( \gamma \to -k \) and therefore \( g(\gamma) \to 0 \) as \( \alpha \to 0 \). Consider now the solution \( g \) of (3.6), (3.7) for small \( \alpha \) in \([\gamma, 0]\). We have from (4.1)

\[
g'(0) = - \int_{\gamma}^{0} \frac{h(s)}{g(s)} \, ds.
\]

The solution \( g \) is increasing in \([\gamma, 0]\) and

\[
g(x) \leq f(x) = - \frac{M_1 \gamma}{2g(\gamma)} \left\{ (x - \gamma)^2 - \frac{2g^2(\gamma)}{\gamma M_1} \right\}.
\]

Therefore (4.5) yields

\[
g'(0) \geq \frac{2g(\gamma)}{M_1 \gamma} \int_{\gamma}^{0} \frac{h(s)}{(s - \gamma)^2 - 2g^2(\gamma) / \gamma M_1} \, ds
\]

\[
\geq \frac{2M_2}{M_1 \gamma} \left\{ \frac{\gamma \sqrt{-\gamma M_1}}{g(\gamma) \sqrt{2}} \arctan \frac{-\gamma \sqrt{-\gamma M_1}}{g(\gamma) \sqrt{2}} \right. \\
\left. + \frac{1}{2} \ln \left( \gamma^2 - \frac{2g^2(\gamma)}{\gamma M_1} \right) \left( \frac{2g^2(\gamma)}{\gamma M_1} \right) \right\}
\]

\[
= \frac{M_2}{M_1} \left\{ \sqrt{-2\gamma M_1} \arctan \frac{-\gamma \sqrt{-\gamma M_1}}{g(\gamma) \sqrt{2}} + \frac{g(\gamma)}{\gamma} \ln \left( 1 - \frac{\gamma^2 M_1}{2g^2(\gamma)} \right) \right\}.
\]

Using the result that \( \gamma \to -k \) and \( g(\gamma) \to 0 \) as \( \alpha \to 0 \), we have from the last inequality

\[
\lim_{\alpha \to 0} g'(0) \geq \frac{M_2 \pi \sqrt{k}}{\sqrt{M_1} \sqrt{2}}.
\]
Similarly, integrating (4.1) over \([\gamma, 0]\), we have
\[
g(0) = g(\gamma) + \int_{\gamma}^{0} \frac{sh(s)}{g(s)} \, ds
\]
\[
\geq g(\gamma) - \frac{2M_2 g(\gamma)}{M_1 \gamma} \int_{\gamma}^{0} \frac{s^2 \, ds}{(s - \gamma)^2 - 2g^2(\gamma) / \gamma M_1}
\]
\[
= g(\gamma) - 2 \frac{M_2 g(\gamma)}{M_1 \gamma} \left[ -\gamma + \gamma \ln \left( 1 - \frac{\gamma^2 M_1}{2g^2(\gamma)} \right) \right.
\]
\[
\times \sqrt{\frac{2 M_1 \gamma^3 + g^2(\gamma)}{g(\gamma) \sqrt{-2 \gamma M_1}}} \arctan \frac{-\gamma \sqrt{-\gamma M_1}}{g(\gamma) \sqrt{2}} \right].
\]

Then we have
\[
\lim_{x \to 0} g(0) \geq \frac{M_2 \pi k \sqrt{k}}{\sqrt{2 M_1}}. \tag{4.7}
\]

For small \(x > 0\), we have from (4.6) \(g'(0) > 0\). Following the proof of Theorem 1, \(g\) attains its maximum at \(\beta > 0\) and intersects the x-axis at \(x^*_x\). Let \(x_1 > 0\) be such that \(g(x_1) = g(0)\). Then integrate (4.1) over \([0, x_1]\) to get
\[
g(x_1) = g(0) + g'(0) x_1 - \int_{0}^{x_1} \frac{(x_1 - s) h(s)}{g(s)} \, ds,
\]
or
\[
g'(0) x_1 = \int_{0}^{x_1} \frac{(x_1 - s) h(s)}{g(s)} \, ds
\]
\[
\leq \frac{1}{g(0)} \int_{0}^{x_1} (x_1 - s) h(s) \, ds \leq \frac{M_1 x_1^3}{6g(0)}.
\]

Therefore
\[
x_1 \geq \sqrt{6g(0) g'(0)/M_1}. \tag{4.8}
\]

We conclude from (4.6), (4.7), and (4.8)
\[
\lim_{x \to 0} x^*_x \geq \lim_{x \to 0} x_1 \geq \frac{M_2}{M_1} k \sqrt{3} > 1, \tag{4.9}
\]
if \(k > M_1 / \pi M_2 \sqrt{3}\).

The conclusion of Lemma 5 follows immediately from Lemma 2, Lemma 4, and (4.9).
Proof of Theorem 2. It follows immediately from Lemmas 1, 2, 3, 4, and 5. The following lemmas are needed to prove Theorem 3.

**Lemma 6.** Let \( C > 0 \) and \( g \) be the solution of (3.6), (3.7) with \( x = (k/2) \sqrt{-2h(-k)}. \) If

\[
\sqrt{\frac{2}{M_2 b_1}} + a_2 \sqrt{\frac{2b_2}{M_2}} + b_2 < 1,
\]

where

\[
a_1 = Ck + k \sqrt{2M_1 k}
\]

\[
a_2 = C + \frac{M_1 \sqrt{k}}{\sqrt{2M_2}}
\]

\[
b_1 = \frac{Ck \sqrt{-2h(-k) + M_2 k^2}}{M_1}
\]

and \( b_2 = (1/M_2)[a_2^2 + \sqrt{a_2^4 + 2M_2 a_1 a_2}], \) then \( x_x^* < 1. \)

**Proof.** We first derive upper bounds for \( g(0) \) and \( g'(0). \) We have for \(-k \leq x \leq 0\)

\[
g(x) \leq f(x) = -\frac{h(-k)}{2a} (x + k)^2 + C(x + k) + x.
\]

By use of (3.3) and the choice of \( \alpha, \) we have from the last inequality

\[
g(0) \leq x + Ck - \frac{k^2 h(-k)}{2a} \leq Ck + k \sqrt{2M_1 k} = a_1.
\]

(4.10)

Again use (3.3) and \( g(x) \geq g(-k) \) for \(-k \leq x \leq 0\) in (4.1) to obtain

\[
g'(0) \leq C - \frac{2}{k \sqrt{-2h(-k) - k}} \int_{-k}^{0} h(s) \, ds \leq C + \frac{M_1 \sqrt{k}}{\sqrt{2M_2}} = a_2.
\]

(4.11)

Next, the bounds for \( \beta \) and \( g(\beta), \) where \( g'(\beta) = 0, \) are derived. Integrate (3.5) over \([0, \beta]\) and use the inequality \( g(0) \leq g(x) \leq g(\beta) \) for \( 0 \leq x \leq \beta \) to obtain

\[
\frac{1}{g(\beta)} \int_{0}^{\beta} h(s) \, ds \leq g'(0) = \int_{0}^{\beta} \frac{h(s)}{g(s)} \, ds \leq \frac{1}{g(0)} \int_{0}^{\beta} h(s) \, ds
\]

(4.12)
and
\[ g(\beta) \leq g(0) + \beta g'(0). \]  \hspace{1cm} (4.13)

From (4.1) and (3.3) we have
\[ g'(0) = C - \int_{-k}^{0} \frac{h(s)}{g(s)} ds > C + \frac{M_2 k^2}{2g(0)}, \]
which gives
\[ g(0) g'(0) > Cx + \frac{1}{2} M_2 k^2. \]  \hspace{1cm} (4.14)

Combine (4.12), (4.14), and (3.3) to get
\[ \frac{1}{2} M_1 \beta^2 \geq g(0) g'(0) > Cx + \frac{1}{2} M_2 k^2, \]

or
\[ \beta \geq \sqrt{\frac{Ck + 2h(-k) + M_2 k^2}{M_1}} = b_1. \]  \hspace{1cm} (4.15)

Combine (4.12), (4.13), and (3.3) to get
\[ \frac{1}{2} M_2 \beta^2 \leq g'(0) g(\beta) \leq g(0) g'(0) + \beta g'(0)^2 \]

or
\[ \beta \leq \frac{1}{M_2} \left[ g'(0)^2 + \sqrt{g'(0)^4 + 2M_2 g(0) g'(0)} \right]. \]

By use of (4.10) and (4.11), the last inequality is reduced to
\[ \beta \leq \frac{1}{M_2} \left[ a_2^2 + \sqrt{a_2^4 + 2M_2 a_2 a_2} \right] = b_2. \]  \hspace{1cm} (4.16)

The last step is to find an upper bound for \( x_* \). Since \( g \) is dominated by
\[ f(x) = g(\beta) - \frac{h(\beta)}{2g(\beta)} (x - \beta)^2 \quad \text{for} \quad \beta \leq x \leq x_*^{\pm}, \]

we have
\[ x_*^{\pm} \leq \beta + \frac{g(\beta) \sqrt{2}}{\sqrt{h(\beta)}}. \]
Use (4.10), (4.11), (4.13), (4.15), (4.16), and (3.3) to conclude

\[ x^* \approx \beta + g(0) \sqrt{\frac{2}{M_2 \beta}} + g'(0) \sqrt{\frac{2\beta}{M_2}} \]
\[ \leq b_2 + a_1 \sqrt{\frac{2}{M_2 b_1}} + a_2 \sqrt{\frac{2b_2}{M_2}} < 1. \]

**Lemma 7.** Let \( C \geq 0 \) and \( g \) be the solution of (3.6), (3.7). Then \( g(0) \to \infty \) as \( \alpha \to 0. \)

**Proof.** We have from (3.3) and integrating (4.1) over \([-k, 0]\)

\[ g(0) = \alpha + Ck + \int_{-k}^{0} \frac{sh(s)}{g(s)} ds \]
\[ \geq \alpha + Ck + \frac{M_2 k^3}{3g(0)} \]

or

\[ g(0)^2 - (\alpha + Ck) g(0) - \frac{M_2 k^3}{3} \geq 0, \]

which implies

\[ g(0) \geq \frac{1}{2} \left( \alpha + Ck \right) + \frac{1}{2} \sqrt{\left( \alpha + Ck \right)^2 + \frac{4M_2 k^3}{3}}. \]

Therefore

\[ \lim_{\alpha \to 0} g(0) \geq \frac{1}{2} Ck + \frac{1}{2} \sqrt{C^2 k^2 + \frac{4M_2 k^3}{3}}. \quad (4.17) \]

Also we have from integrating (4.1) over \([-k, 0]\)

\[ g(0) = \alpha + Ck + \int_{-k}^{0} \frac{sh(s)}{g(s)} ds \]
\[ \geq \alpha + Ck + M_2 \int_{-k}^{0} \frac{s^2}{g(0) + \left( \frac{(g(0) - \alpha)}{k} \right)_s} ds \]
\[ = \alpha + Ck + \frac{M_2 k}{g(0) - \alpha} \left[ - \frac{1}{2} k^2 - \frac{g(0)k^2}{g(0) - \alpha} \right. \]
\[ + \left. \left( \frac{g(0)k}{g(0) - \alpha} \right)^2 \ln \frac{g(0)}{\alpha} \right]. \]
Suppose \( g(0) \) is bounded as \( x \to 0 \), using (4.17), then the right-hand side of the above inequality is unbounded as \( x \to 0 \), while the left-hand side is bounded. Therefore \( g(0) \to \infty \) as \( x \to 0 \).

**Remark.** Lemma 7 implies that \( x_k^* \to \infty \) as \( x \to 0 \).

**Proof of Theorem 3.** This follows immediately from Lemmas 3, 6, and 7.

In the next two lemmas we show the uniqueness of the first solutions of (3.1), (3.2).

**Lemma 8.** In addition to conditions (i), (ii), and (iii) in Theorem 2, suppose \( C \) and \( k \) satisfy

(i) \(-C \geq \sqrt{M_1(k + (1/2)k^2)}\) and

(ii) \(2k^2 + (4/3)k^3 < 2M_2/3M_1\),

then the boundary value problem (3.1), (3.2) has at most one decreasing solution.

**Proof.** We first derive \( \alpha \), a lower bound of \( \alpha \) of any decreasing solution \( g \) of (3.1), (3.2). Integrate (4.1) over \([−k, 1]\) to get

\[
g(1) = 0 = \alpha + C(1 + k) - \int_{−k}^{1} \frac{(1 - s) h(s)}{g(s)} \, ds.
\]

By use of (3.3) we have

\[
\alpha + C(1 + k) = \left[ \int_{−k}^{0} + \int_{0}^{1} \right] \frac{(1 - s) h(s)}{g(s)} \, ds
\]

\[
\geq \frac{M_1}{g(0)} \left( \frac{1}{2} k^2 - \frac{1}{3} k^3 \right) + \frac{M_2}{6g(0)}
\]

\[
\geq \frac{1}{\alpha} \left( \frac{1}{6} M_2 - \left( \frac{1}{2} k^2 + \frac{1}{3} k^3 \right) M_1 \right).
\]

From the last inequality and condition (v), we conclude

\[
\alpha > -C(1 + k).
\]

Suppose \( g_1 \) and \( g_2 \) are decreasing solutions of (3.1), (3.2) corresponding to \( \alpha_1 \) and \( \alpha_2 \), \( \alpha_1 > \alpha_2 > -Ck \). In any subinterval \([−k, a]\) of \([−k, 0]\), where \( g_1(x) \geq g_2(x) \), we have from integrating (4.1) over \([−k, a]\)

\[
g_1(a) - g_2(a) - \alpha_1 - \alpha_2 + \int_{−k}^{a} \frac{g_1(s) - g_2(s)}{g_1(s) g_2(s)} (a - s) h(s) \, ds.
\]
Since \( g_1(x) - g_2(x) \) decreases in \([-k, a]\) and \( g_1(x) = Cx \) in \([-k, a]\), \( i = 1, 2 \), we have from the last equation

\[
g_1(a) - g_2(a) > (\alpha_1 - \alpha_2) \left[ 1 + \frac{1}{C^2} \left\{ M_1 a \ln \frac{|a|}{k} M_1 (a + k) \right\} \right] > (\alpha_1 - \alpha_2) \left( 1 - \frac{M_1 k}{C^2} \right).
\]

By use of condition (iv) we conclude

\[
g_1(x) - g_2(x) > (\alpha_1 - \alpha_2) \left( 1 - \frac{M_1 k}{C^2} \right) \text{ for } -k \leq x \leq 0. \tag{4.19}
\]

This also implies that \( g_1(x) - g_2(x) > 0 \) for \(-k \leq x < 0\). Now, we derive an upper bound for \(|g'_1(0) - g'_2(0)|\). We have from (4.1) and (4.18)

\[
|g'_1(0) - g'_2(0)| = \left| \int_{-k}^{0} h(s) \frac{g_1(s) - g_2(s)}{g_1(s) g_2(s)} ds \right| \leq \frac{M_1 k^2 (\alpha_1 - \alpha_2)}{2(\alpha_2 + Ck)^2} \leq \frac{M_1 k^2}{2C^2} (\alpha_1 - \alpha_2). \tag{4.20}
\]

From (4.1) we have

\[
g'(x) = g'(0) - \int_{-k}^{x} h(s) ds.
\]

We integrate the last equation over \([0, 1]\) for \( g_1 \) and \( g_2 \) and use (4.19), (4.20) and condition (iv) to obtain

\[
0 > \int_{0}^{1} (1 - s) h(s) \left( \frac{1}{g_2(s)} - \frac{1}{g_1(s)} \right) ds = g_1(0) g_2(0) + g_1'(0) g_2'(0) \geq (\alpha_1 - \alpha_2) \left( 1 - \frac{M_1 k}{C^2} - \frac{M_1 k^2}{2C^2} \right) \geq 0,
\]

which is a contradiction.

**Proof of Theorem 4.** This follows immediately from Theorem 2 and Lemma 8.

**Lemma 9.** Let \( C > 0 \) and

\[
\begin{align*}
(i) & \quad d_1 \geq k \sqrt{M_1(1/2 + (1/3)k)} \\
(ii) & \quad d_2 \leq (k/2) \sqrt{-2h(-k)},
\end{align*}
\]
where
\[ d_{1,2} = \frac{1}{2} \left[ \sqrt{M_2/6 - Ck - C} \pm \sqrt{(\sqrt{M_2/6 - Ck - C})^2 - 2M_1(k^2 + k^3)} \right]; \]
then the boundary value problem (3.1), (3.2) has at most one solution with \( \alpha > (k/2) \sqrt{-2h(-k)}. \)

**Proof.** Let \( g \) be a solution of (3.1), (3.2) corresponding to \( \alpha > (k/2) \sqrt{-2h(-k)}. \) It is first proven that \( \alpha > d_1. \) Integrate (4.1) over \([0, 1]\) for \( g \) to get
\[ 0 = g(1) = g(0) + g'(0) - \int_0^1 \frac{(1 - s)h(s)}{g(s)} ds. \]

Since \( g(x) \leq g(0) + g'(0) \) for \( 0 \leq x \leq 1, \) we have from the last equation
\[ g(0) + g'(0) > M_2 \int_0^1 \frac{(1 - s)s}{g(0) + g'(0)} ds, \]
which implies
\[ g(0) + g'(0) > \sqrt{M_2/6}. \quad (4.21) \]

From (4.1) we have
\[ g'(0) = C - \int_{-k}^0 \frac{h(s)}{g(s)} ds \leq C + \frac{M_1k^2}{2\alpha}. \quad (4.22) \]
Since \( g(x) \leq (-h(-k)/\alpha)(x + k)^2 + C(x + k) + \alpha \) for \(-k \leq x \leq 0, \) we have
\[ g(0) \leq \frac{M_1k^3}{2\alpha} + Ck + \alpha. \quad (4.23) \]

Combine (4.21), (4.22), and (4.23) to get
\[ 2\alpha^2 - 2\alpha\left(\sqrt{M_2/6 - Ck - C} + M_1(k^2 + k^3)\right) > 0. \]

By use of condition (ii) and the last inequality we conclude \( \alpha > d_1. \) Let \( g_1 \) and \( g_2 \) be solutions of (3.1), (3.2) corresponding to \( \alpha_1 \) and \( \alpha_2 \) respectively, where \( \alpha_1 > \alpha_2 > (k/2) \sqrt{-2h(-k)}. \) This implies \( \alpha_1 > \alpha_2 > d_1. \) Using the same argument as in the derivation of (4.19) we get
\[ g_1(0) - g_2(0) = \alpha_2 - \alpha_2 - \int_{-k}^0 \frac{g_1(s) - g_2(s)}{g_1(s) g_2(s)} sh(s) ds \]
\[ > \left( 1 - \frac{M_1k^3}{3d_1^2} \right) (\alpha_1 - \alpha_2). \quad (4.24) \]
Also using the same argument as in the derivation of (4.20) we have

\[ |g'_1(0) - g'_2(0)| \leq \left| \int_{-k}^{0} h(s) \frac{g_1(s) - g_2(s)}{g_1(s) g_2(s)} \, ds \right| \leq \frac{M_1 k^2}{2d_1^2} (x_1 - x_2). \quad (4.25) \]

Combine (4.24), (4.25), and condition (i) to conclude

\[ g_1(0) - g_2(0) + g'_1(0) - g'_2(0) \geq \left( 1 - \frac{M_1 k^2}{2d_1^2} - \frac{M_1 k^3}{3d_1^2} \right) (x_1 - x_2) \geq 0. \]

The conclusion of the lemma follows similarly to the last proof of Lemma 8.

**Proof of Theorem 5.** It follows immediately from Lemma 9 and Theorem 3.

5. **Method of Numerical Solutions**

The nonlinear differential equation (2.5), subject to conditions (2.6), is now to be solved for the dependent variable \( f \) as a function of \( \eta \). Let the range of numerical integration be restricted to finite dimensions (that is, \( \eta_{\text{max}} = 20 \) is considered). Numerical integration of (2.5) cannot be started at \( \eta = 0 \) because \( f'' \) is not known there. Boundary conditions (2.6) provide only two of the three values, \( f \) and \( f' \), that are required at \( \eta = 0 \), but provide another value of \( f' \) at infinity. At the beginning of numerical computation a value \( f'_0 \) is arbitrarily guessed and a positive increment \( \Delta f''_0 \) is picked. By letting \( f = -C, f' = -k \) and \( f'' = f''_0 \) at \( \eta = 0 \) Eq. (2.5) is integrated using fourth-order Runge-Kutta formulas and a shooting technique. The shooting technique takes care of the infinity condition on \( f' \) at \( \eta_{\text{max}} \); if not, the initial guess for \( f'' \) at \( \eta = 0 \) will be replaced by \( f''_0 + \Delta f''_0 \) and this process will be repeated until the infinity condition is satisfied.

It is observed that whatever value we may pick for \( f'' \) at \( \eta = 0 \), that value converges to one of the two values, say \( x_1 \) and \( x_2 \), to satisfy the infinity condition on \( f' \). This strongly suggests that there are only two solutions for Eq. (2.9) with conditions (2.10) when \( h(x) = x \) (i.e., \( f'' \)). Let the solutions corresponding to \( x_1 \) and \( x_2 \) be called \( g_1 \) (i.e., \( f''_1 \)) and \( g_2 \) (i.e., \( f''_2 \)) respectively. The solutions \( g_1 \) and \( g_2 \) are calculated for several sets of values of \( x_1, x_2, k, \) and \( C \). Some of the interesting results thus obtained are presented in Figs. 1 and 2.
6. DISCUSSION OF THE RESULTS

In Fig. 1, numerical results for $g_1(x)$ and $g_2(x)$ are presented when $C = -0.1, 0, 0.1,$ and $k = 0.1$. From Fig. 1, it is evident that both $g_1(x)$ and $g_2(x)$ decrease with an increasing $C$. Physical meaning of this is that the values of $g_1$ and $g_2$ decrease with injection and increase with suction. This qualitatively agrees with the expectations. Also, $g_1(x)$ and $g_2(x)$ are concave up in the interval $(-k, 0)$. However for a range of values of $C$ used in the computation $g_1(x)$ is higher than $g_2(x)$.

Figure 2 describes the behavior of the skin friction coefficients $\alpha_1$ and $\alpha_2$ with changes in the parameters $k$ and $C$. From this figure we note that the effects of the parameters $k$ and $C$ are to decrease $\alpha_1$. However, this phenomenon is quite opposite in the case of $\alpha_2$. As in Fig. 1, here for all
$C$, the value of $\alpha_1$ is higher than the value of $\alpha_2$. Further the numerical result of Hussaini, Lakin, and Nachman [13] is a special case of this study when $C = 0$. Finally it may be said that these numerical results qualitatively agree very well with the analytical predictions.

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