Normal families and uniqueness theorems for entire functions

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Abstract

There exists a set $S$ with 3 elements such that if $f$ is a non-constant entire function satisfying $E(S, f) = E(S, f')$, then $f \equiv f'$. The number 3 is best possible. The proof uses the theory of normal families in an essential way.

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1. Introduction

Let $f$ be a non-constant meromorphic function in the complex plane and let $S$ be a set of complex numbers. Put

$$E(S, f) = \bigcup_{a \in S} \{ z : f(z) - a = 0 \},$$

where a zero of multiplicity $m$ is counted $m$ times in the set.

Answering a question of Gross [2], Yi [14] proved the following result.

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**Theorem A.** There exists a finite set $S$ containing 7 elements such that if $f$ and $g$ are two non-constant entire functions and $E(S, f) = E(S, g)$, then $f \equiv g$.

Earlier, Rubel and Yang [10] had shown

**Theorem B.** Let $a$ and $b$ be distinct complex numbers, and let $f$ be a non-constant entire function. If $E(a, f) = E(a, f')$ and $E(b, f) = E(b, f')$, then $f \equiv f'$.

In this paper, we use the theory of normal families to prove

**Theorem 1.** There exists a set $S$ with 3 elements such that if a non-constant entire function $f$ and its derivative $f'$ satisfy $E(S, f) = E(S, f')$, then $f \equiv f'$.

Let $S = \{a, b\}$, where $a$ and $b$ are any two distinct complex numbers. Let $f(z) = e^{-z} + a + b$; then $f'(z) = -e^{-z}$. Obviously, $E(S, f) = E(S, f')$, but $f \not\equiv f'$. This shows that the number 3 in Theorem 1 is best possible.

Jank et al. [6] proved

**Theorem C.** Let $f$ be a non-constant entire function, and let $a$ be a non-zero constant. If $E(a, f) = E(a, f')$ and $f''(z) = a$ whenever $f'(z) = a$, then $f \equiv f'$.

Again, using the theory of normal families, we prove

**Theorem 2.** Let $f$ be a non-constant entire function and $k \geq 2$ a positive integer. Let $a$ and $b$ be complex numbers such that $b \neq 0$. If $E(a, f) = E(a, f')$ and $f^{(k)}(z) = b$ whenever $f'(z) = b$, then

$$f(z) = de^{cz} + \frac{c-1}{c}a,$$

where $c$ and $d$ are two non-zero constants with $c^{k-1} = 1$. In particular, $f \equiv f'$ for $k = 2$.

From Theorem 2 we obtain the following result.

**Theorem D** [7, Theorem 2]. Let $f$ be a non-constant entire function and $k \geq 2$ a positive integer. Let $a$ be a non-zero constant. If $E(a, f) = E(a, f') = E(a, f^{(k)})$, then

$$f(z) = de^{cz} + \frac{c-1}{c}a,$$

where $c$, $d$ are two non-zero constants with $c^{k-1} = 1$.

It does not seem that Theorem 2 can be proved by using the methods in [6] and [7].


**Theorem E.** Let $a$ be a non-zero complex number and $k$ a positive integer. Let $f$ be a non-constant entire function. If $f(z) f^{(k)}(z) \neq 0$ and $f(z) = a$ if and only if $f^{(k)}(z) = a$, then $f(z) = e^{Az+B}$, where $A \neq 0$ and $B$ are constants satisfying $A^k = 1$. 
As an application of the theory of normal families, we improve Theorem E as follows.

**Theorem 3.** Let \( a \) and \( b \) be distinct non-zero complex numbers and \( k \) a positive integer. Let \( f \) be a non-constant entire function. If \( f(z) \neq 0 \) and \( f^{(k)}(z) = b \) whenever \( f(z) = a \), then \( f(z) = e^{Az+B} \), where \( A \neq 0 \) and \( B \) are constants satisfying \( Ak = b/a \).

**Corollary 4.** Let \( a \) be a non-zero complex number and \( k \) a positive integer. Let \( f \) be a non-constant entire function. If \( f(z) \neq 0 \) and \( f^{(k)}(z) = a \) whenever \( f(z) = a \), then \( f(z) = e^{Az+B} \), where \( A \neq 0 \) and \( B \) are constants satisfying \( Ak = 1 \).

**Theorem 5.** Let \( a \) and \( b \) be distinct non-zero complex numbers and \( k \) a positive integer. Let \( f \) be a non-constant entire function. If \( f(z) \neq 0 \) and \( f^{(k)}(z) = b \) whenever \( f(z) = a \), then \( f(z) = e^{Az+B} \), where \( A \neq 0 \) and \( B \) are constants satisfying \( Ak = b/a \).

**Corollary 6.** Let \( a \) be a non-zero complex number and \( k \) a positive integer. Let \( f \) be a non-constant entire function. If \( f(z) \neq 0 \) and \( f(z) = a \) whenever \( f^{(k)}(z) = a \), then \( f(z) = e^{Az+B} \), where \( A \neq 0 \) and \( B \) are constants satisfying \( Ak = 1 \).

Throughout this paper, we use the standard notation of Nevanlinna theory (cf. [5,12]). In particular, \( S(r,f) \) denotes any function satisfying

\[
S(r,f) = O(\log T(r,f)) + O(\log r)
\]
as \( r \to +\infty \), possibly outside of a set of positive measure, where \( T(r,f) \) is Nevanlinna’s characteristic function. In fact, the functions for which we use this notation are all of finite order, so the exceptional set does not occur. For such functions, we have \( S(r,f) = o(T(r,f)) \) (cf. [5, p. 41]).

### 2. Some lemmas

For the proof of our results, we need the following lemmas.

**Lemma 1** ([1], cf. [8]). Let \( f \) be an entire function, \( M \) a positive number. If \( f^k(z) \leq M \) for any \( z \in \mathbb{C} \), then \( f \) is of exponential type.

Here, as usual, \( f^k(z) = |f'(z)|/(1 + |f(z)|^2) \) is the spherical derivative.

**Lemma 2** [4, Theorem 1]. Let \( f \) be a non-constant entire function with finite order, and let \( a \) be a finite value. If \( E(a,f) = E(a,f') \), then

\[
f'(z) - a = A[f(z) - a],
\]

where \( A \) is a non-zero constant.

**Lemma 3** [9, Lemma 2]. Let \( \mathcal{F} \) be a family of functions holomorphic on the unit disc, all of whose zeros have multiplicity at least \( k \). Suppose that there exists \( A \geq 1 \) such that \( |f^{(k)}(z)| \leq A \) whenever \( f(z) = 0 \). If \( \mathcal{F} \) is not normal, there exist, for each \( 0 \leq \alpha \leq k \),
(a) a number $0 < r < 1$,
(b) points $z_n$ with $|z_n| < r$,
(c) functions $f_n \in \mathcal{F}$, and
(d) positive numbers $\rho_n \to 0$

such that $\rho_n^{-a} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where $g$ is a non-constant entire function, all of whose zeros have multiplicity at least $k$, such that $g^a(\xi) \leq g^a(0) = kA + 1$. In particular, $g$ is of exponential type.

For $0 \leq \alpha < k$, the hypothesis on $f^{(\alpha)}(z)$ can be dropped, and $kA + 1$ can be replaced by an arbitrary positive constant.

**Lemma 4.** Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D$. Let $k$ be a positive integer. Let $a$, $b$, and $c$ be three distinct finite complex numbers and $M$ a positive number. If, for any $f \in \mathcal{F}$, the zeros of $f$ are of multiplicity $\geq k$ and $|f^{(k)}(z)| \leq M$ whenever $f(z) = a, b, or c$, then $\mathcal{F}$ is normal in $D$.

**Proof.** Suppose that $\mathcal{F}$ is not normal on $D$. By Lemma 3, there exist points $z_n \in D$, positive numbers $\rho_n \to 0^+$, and functions $f_n \in \mathcal{F}$ such that $g_n(\xi) = f_n(z_n + \rho_n \xi)$ converges locally uniformly to a non-constant entire function $g$, whose zeros have multiplicity $\geq k$.

Obviously, $g^{(k)}(\xi) \not\equiv 0$, for otherwise $g$ would be a polynomial of degree less than $k$, and so could not have zeros of multiplicity at least $k$.

We claim that $g^{(k)}(\xi) = 0$ whenever $g(\xi) = a, b, or c$.

Using standard results of Nevanlinna theory, we have

$$2T(r, g) \leq N\left(r, \frac{1}{g-a}\right) + N\left(r, \frac{1}{g-b}\right) + N\left(r, \frac{1}{g-c}\right) + S(r, g)$$

$$\leq N\left(r, \frac{1}{g^{(k)}}\right) + S(r, g) \leq T\left(r, \frac{1}{g^{(k)}}\right) + S(r, g)$$

$$\leq T(r, g^{(k)}) + S(r, g) \leq T(r, g) + S(r, g).$$

Note that we have used the fact that $g$ is entire in both the first and last inequalities above.

Thus we get a contradiction: $T(r, g) = o(T(r, g))$. Hence $\mathcal{F}$ is normal in $D$. This completes the proof of the lemma. \qed

**Example.** Let $S = \{1, -1\}$. Set

$$\mathcal{F} = \{ f_n(z): n = 2, 3, 4, \ldots \},$$

where

$$f_n(z) = \frac{n+1}{2n} e^{nz} + \frac{n-1}{2n} e^{-nz}, \quad D = \{z: |z| < 1\}.$$
Then, for any $f_n \in \mathcal{F}$, we have
$$n^2 \left[ f_n^2(z) - 1 \right] = \left[ f_n(z) \right]^2 - 1.$$ 
Thus $E(S, f) = E(S, f')$, but $\mathcal{F}$ is not normal in $D$.

**Lemma 5.** Let $\mathcal{F}$ be a family of holomorphic functions in a domain $D$ and $k \geq 2$ a positive integer. Let $a$, $b$, and $c$ be three complex numbers such that $b \neq 0$. If, for any $f \in \mathcal{F}$, $E(0, f) = E(a, f')$ and $f^{(k)}(z) = c$ whenever $f'(z) = b$, then $\mathcal{F}$ is normal in $D$.

**Proof.** Suppose that $\mathcal{F}$ is not normal on $D$. By Lemma 3, there exist sequences $z_n \in D$, $\rho_n \to 0^+$, and $f_n \in \mathcal{F}$ such that $g_n(\xi) = \rho_n^{-1} f_n(z_n + \rho_n \xi)$ converges locally uniformly to a non-constant entire function $g$ of exponential type.

We consider two cases.

**Case 1:** $a \neq 0$. Suppose that $g(\xi_0) = 0$. Then by Hurwitz’s theorem, there exists a sequence $\{\xi_n\}$ with $\xi_n \to \xi_0$ such that (for $n$ sufficiently large)
$$g_n(\xi_n) = \rho_n^{-1} f_n(z_n + \rho_n \xi_n) = 0.$$ 
Thus $f_n(z_n + \rho_n \xi_n) = 0$. Since $E(0, f_n) = E(a, f_n')$, we have
$$g_n'(\xi_n) = f_n'(z_n + \rho_n \xi_n) = a.$$ 
Hence $g'(\xi_0) = \lim_{n \to \infty} g_n'(\xi_n) = a$. Thus $g'(\xi) = a$ whenever $g(\xi) = 0$.

Now suppose that $g'(\xi_0) = a$. We claim that $g'(\xi) \neq a$, for otherwise $g(\xi) = a(\xi - \xi_1)$. A simple calculation then shows that
$$g^k(0) \leq \begin{cases} 1 & \text{if } |\xi_1| \geq 1, \\ |a| & \text{if } |\xi_1| < 1. \end{cases}$$
Hence we have $g^k(0) < (|a| + 1) + 1$, which is a contradiction. Since $g'(\xi_0) = a$ but $g'(\xi) \neq a$, there exist $\xi_n$, $\xi_n \to \xi_0$, such that (for $n$ large) $f_n'(z_n + \rho_n \xi_n) = g_n'(\xi_n) = a$.

It follows that $f_n(z_n + \rho_n \xi_n) = 0$, so that $g_n(\xi_n) = f_n(z_n + \rho_n \xi_n)/\rho_n = 0$. Since $g(\xi_0) = \lim_{n \to \infty} g_n(\xi_n) = 0$, we have shown that $g(\xi) = 0$ whenever $g'(\xi) = a$.

Thus $g(\xi) = 0$ if and only if $g'(\xi) = a$.

Let $\xi_0$ be a zero of $g'(\xi) - a$ with multiplicity $\geq 1$. Then $g(\xi_0) = 0$, and there exists a positive number $\delta > 0$ such that for $0 \leq |\xi - \xi_0| < \delta$
$$g'(\xi) \neq 0.$$ 
(2.1)

By Hurwitz’s theorem, there exist $m$ sequences $\{\xi_{in}\}$ $(i = 1, 2, \ldots, m)$ such that $\lim_{n \to \infty} \xi_{in} = \xi_0$, and (for large $n$)
$$g_n(\xi_{in}) = a, \quad i = 1, 2, \ldots, m.$$ 
(2.2)

Thus
$$f_n'(z_n + \rho_n \xi_{in}) = a, \quad i = 1, 2, \ldots, m.$$ 
Hence, by $E(0, f_n) = E(a, f_n')$ and $a \neq 0$, we have $f_n(z_n + \rho_n \xi_{in}) = 0$ for $i = 1, 2, \ldots, m$, and each $\xi_{in}$ is a simple zero of $g_n$. Thus
$$\xi_{in} \neq \xi_{jn}, \quad 1 \leq i < j \leq m.$$ 
(2.3)
By Rouché’s theorem, the order of the zero of \( g \) at \( \xi_0 \) is \( m \). This implies that \( E(0, g) = E(a, g') \). (In fact, \( m = 1 \).

If \( g' \neq b \), then since \( g \) is of order at most one, there exist non-zero constants \( A \) and \( B \) such that

\[
g'(\xi) = Be^{A\xi} + b. \tag{2.4}
\]

Thus we have

\[
g(\xi) = \frac{B}{A}e^{A\xi} + b\xi + C. \tag{2.5}
\]

Obviously, \( g(\xi) = 0 \) has infinitely many solutions. Suppose \( g(\xi_0) = 0 \). Then by (2.4), (2.5), and \( E(0, g) = E(a, g') \), we get \( \xi_0 = (b - CA - a)/bA \), which is a contradiction.

Thus there exists \( \xi_0 \) such that \( g'(\xi_0) = b \). Clearly, \( g'(\xi) \neq b \), for otherwise \( g(\xi) = b\xi + C \), which contradicts \( E(0, g) = E(a, g') \). Hence there exist \( \xi_n, \xi_n \to \xi_0 \), such that \( (n \text{ large}) \ g_n'(\xi_n) = b \). Thus, \( f_n'(\xi_n + \rho_n\xi) = b \). Since \( f_n'' = c \) whenever \( f_n' = b \), we have

\[
f_n''(\xi_n + \rho_n\xi) = c \quad \text{and} \quad g_n''(\xi_n) = \rho_n^{-1}f_n''(\xi_n + \rho_n\xi) = \rho_n^{-1}c \to 0
\]
as \( n \to \infty \). Thus \( g_n''(\xi_n) = \lim_{n \to \infty} g_n''(\xi_n) = 0 \). By Lemma 2, we have

\[
g'(\xi) - a = Ag(\xi),
\]

where \( A \) is a non-zero constant.

Thus we have

\[
g(\xi) = Be^{A\xi} - \frac{a}{A}, \tag{2.6}
\]

\[
g^{(k)}(\xi) = A^kBe^{A\xi}, \tag{2.7}
\]

where \( B \) is a non-zero constant.

By \( g^{(k)}(\xi_0) = 0 \), (2.7), and \( AB \neq 0 \), we have a contradiction.

Case 2: \( a = 0 \). In this case, it is clear that \( g(\xi) \neq 0 \). Thus

\[
g(\xi) = Be^{A\xi}, \tag{2.8}
\]

where \( A, B \) are non-zero constants. Clearly, there exists \( \xi_0 \) such that \( g'(\xi_0) = b \). Using the same argument as in Case 1, we obtain \( g^{(k)}(\xi_0) = 0 \), which contradicts (2.8).

Hence \( F \) is normal in \( D \). This completes the proof of the lemma. \( \square \)

**Lemma 6.** Let \( F \) be a family of holomorphic functions in a domain \( D \) and \( k \) a positive integer. Let \( a \) and \( b \) be distinct non-zero complex numbers. If, for any \( f \in F \), \( f \neq 0 \) and \( f^{(k)}(z) = b \) whenever \( f(z) = a \), then \( F \) is normal in \( D \).

**Proof.** Suppose that \( F \) is not normal on \( D \). By Lemma 3, there exist points \( z_n \in D \), numbers \( \rho_n \to 0^+ \), and functions \( f_n \in F \) such that \( g_n(\xi) = f_n(z_n + \rho_n\xi) \) converges locally uniformly to a non-constant entire function \( g \). Moreover, \( g \) has no zeros and is of exponential type. It follows that \( g(\xi) = e^{A\xi + B} \), where \( A \neq 0 \) and \( B \) are constants. Suppose
that \( g(\xi_0) = a \). Then by Hurwitz’s theorem, there exist \( \xi_n, \xi_n \to \xi_0 \), such that (for \( n \) large) \( a = g_n(\xi_n) = f_n(z_n + \rho_n\xi_n) \). Hence \( f^{(k)}(z_n + \rho_n\xi_n) = b \), so that

\[
\begin{align*}
g^{(k)}(\xi_0) &= \lim_{n \to \infty} g_n^{(k)}(\xi_n) = \lim_{n \to \infty} \rho_n^k f_n^{(k)}(z_n + \rho_n\xi_n) = \lim_{n \to \infty} \rho_n^k b = 0.
\end{align*}
\]

This is a contradiction, since \( g^{(k)}(\xi_0) = \Lambda e^{A\xi_0 + B} \neq 0 \). The proof of the lemma is completed. \( \square \)

Using the same argument as in the proof of Lemma 6, we can prove the following lemma. We omit the details here.

**Lemma 7.** Let \( \mathcal{F} \) be a family of holomorphic functions in a domain \( D \), let \( k \) be a positive integer, and let \( a, b \) be two non-zero finite complex numbers. If, for any \( f \in \mathcal{F} \), \( f \neq 0 \) and \( f(z) = a \) whenever \( f^{(k)}(z) = b \), then \( \mathcal{F} \) is normal in \( D \).

Finally, we recall Marty’s well-known characterization of normal families.

**Lemma 8** [11, p. 75]. Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( D \). Then \( \mathcal{F} \) is normal in \( D \) if and only if the spherical derivatives of functions \( f \in \mathcal{F} \) are uniformly bounded on compact subsets of \( D \).

### 3. Proof of Theorem 1

Set \( S = \{0, a, b\} \), where \( a, b \) are two non-zero distinct finite complex numbers satisfying \( a^2 \neq b^2 \), \( a \neq 2b \), \( b \neq 2a \), \( a^2 - ab + b^2 \neq 0 \). Suppose that \( E(S, f) = E(S, f') \). Set

\[
\phi(z) = \frac{f'(z)[f'(z) - a][f'(z) - b]}{[f(z) - a][f(z) - b]}.
\]  

(3.1)

Then by \( E(S, f) = E(S, f') \), there exists an entire function \( h \) satisfying

\[
\phi(z) = \frac{f'(z)[f'(z) - a][f'(z) - b]}{[f(z) - a][f(z) - b]} = e^{h(z)}.
\]  

(3.2)

Standard computations involving the lemma on the logarithmic derivative (see [6, pp. 32, 34, 55]) show that

\[
m(r, \phi) = S(r, f),
\]  

(3.3)

and hence

\[
T(r, \phi) = S(r, f).
\]  

(3.4)

Let us now show that \( f \) is of exponential type. Set \( \mathcal{F} = \{f(z + w) : w \in \mathbb{C}\} \). Then \( \mathcal{F} \) is a family of holomorphic functions on the unit disc \( \Delta \). By the assumption, for any function \( g(z) = f(z + w) \), we have \( |g'(z)| \leq \max(|a|, |b|) \) whenever \( g(z) = 0, a, b \). Hence by Lemma 4, \( \mathcal{F} \) is normal in \( \Delta \). Thus by Lemma 8, there exists \( M > 0 \) satisfying \( f^{(k)}(z) \leq M \) for all \( z \in \mathbb{C} \). By Lemma 1, \( f \) is of exponential type.
Therefore, $T(r, f) = O(r)$, whence $S(r, f) = O(\log r)$. It then follows from (3.4) that $\phi$ is a polynomial, so by (3.2) $\phi$ must be a non-zero constant $A$. Hence

$$f'(z)[f'(z) - a][f'(z) - b] = A,$$

that is,

$$f'(z)[f'(z) - a][f'(z) - b] = Af(z)[f(z) - a][f(z) - b]. \quad (3.5)$$

Differentiating the two sides of (3.5), we obtain

$$[3(f')^2 - 2(a + b)f' + ab]f'' = A[3f^2 - 2(a + b)f + ab]f'.$$

We claim $f' \neq 0$. Indeed, suppose that $f'(z_0) = 0$ and

$$f(z) = f(z_0) + A_n(z - z_0)^n + \cdots,$$

where $A_n \neq 0$, $n \geq 2$. Then the left-hand side of (3.6) vanishes at $z_0$ to order $n - 2$, while the right-hand side vanishes to the order at least $n - 1$, a contradiction. Hence

$$f'(z) = BCe^{Cz}, \quad (3.7)$$

and

$$f(z) = D + Be^{Cz}, \quad (3.8)$$

where $B \neq 0$, $C \neq 0$, and $D$ are constants.

If $D \neq 0$, $a$, $b$, then by Nevanlinna’s second fundamental theorem,

$$T(r, f) \leq N\left(r, \frac{1}{f} \right) + N\left(r, \frac{1}{f - D} \right) + S(r, f) = N\left(r, \frac{1}{f} \right) + S(r, f),$$

that is,

$$N\left(r, \frac{1}{f} \right) = T(r, f) + S(r, f). \quad (3.9)$$

Similarly, we have

$$N\left(r, \frac{1}{f - a} \right) = T(r, f) + S(r, f),$$

$$N\left(r, \frac{1}{f - b} \right) = T(r, f) + S(r, f). \quad (3.10)$$

By (3.9), (3.10), $E(S, f) = E(S, f')$, and Nevanlinna’s first fundamental theorem, we have

$$3T(r, f) = N\left(r, \frac{1}{f} \right) + N\left(r, \frac{1}{f - a} \right) + N\left(r, \frac{1}{f - b} \right) + S(r, f)$$

$$\leq N\left(r, \frac{1}{f' - a} \right) + N\left(r, \frac{1}{f' - b} \right) + S(r, f)$$

$$\leq 2T(r, f') + S(r, f) \leq 2T(r, f) + S(r, f).$$

Hence we obtain $T(r, f) = S(r, f)$, which contradicts (3.8). Thus $D \in \{0, a, b\}$. 

Now we consider the following three cases.

Case 1: $D = 0$. By (3.7) and (3.8), we have

$$f(z) = Be^{Cz}, \quad f'(z) = BCe^{Cz}. \quad (3.11)$$

Suppose $f(z_1) = a$. Then since $E(S, f) = E(S, f')$, we have either $f'(z_1) = a$ or $f'(z_1) = b$. If $f'(z_1) = a$, then by (3.11), $C = 1$, so $f \equiv f'$. If $f'(z_1) = b$, then by (3.11),

$$C = \frac{b}{a}. \quad (3.12)$$

Similarly, if $f(z_2) = b$, then either $f'(z_2) = a$ or $f'(z_2) = b$. If $f'(z_2) = b$, then $C = 1$, so that $f \equiv f'$. If $f'(z_2) = a$, then by (3.11),

$$C = \frac{a}{b}. \quad (3.13)$$

Thus either $f \equiv f'$ or, by (3.12) and (3.13), $a^2 = b^2$. However, this last relation is ruled out by our choice of $a$ and $b$. It follows that if $D = 0$, then $f \equiv f'$.

Case 2: $D = a$. By (3.7) and (3.8), we have

$$f(z) = a + Be^{Cz}, \quad f'(z) = BCe^{Cz}. \quad (3.14)$$

Let $f(z_3) = 0$. Then since $E(S, f) = E(S, f')$, either $f'(z_3) = a$ or $f'(z_3) = b$.

Assume first that $f'(z_3) = a$. Then by (3.14), $C = -1$. Thus

$$f(z) = a + Be^{-z}, \quad f'(z) = -Be^{-z}. \quad (3.15)$$

Let $f(z_4) = b$. Then since $E(S, f) = E(S, f')$, either $f'(z_4) = a$ or $f'(z_4) = b$. If $f'(z_4) = a$, (3.15) gives $b = 0$, which contradicts our choice of $b$. If $f'(z_4) = b$, we obtain $a = 2b$, which also contradicts our choice of $a$ and $b$.

A similar argument applies in case $f'(z_3) = b$. In that case, $C = -b/a$ and

$$f(z) = a + Be^{-(b/a)z}, \quad f'(z) = -\frac{b}{a}Be^{-(b/a)z}. \quad (3.16)$$

Choosing $z_4$ so that $f(z_4) = b$, we have either $f'(z_4) = a$ or $f'(z_4) = b$. If $f'(z_4) = a$, (3.16) yields $a^2 - ab + b^2 = 0$, which contradicts our choice of $a$ and $b$. Similarly, $f'(z_4) = b$ leads to $b = 0$, which is also ruled out.

It follows that Case 2 cannot occur.

Case 3: $D = b$. This case is symmetric to Case 2 and can be eliminated by the same arguments.

In the above discussion we have shown that $f \equiv f'$. This completes the proof of the theorem. $\square$

4. Proof of Theorem 2

First, we prove that the order of $f$ is at most 1. Set $\mathcal{F} = \{f(z + w) - a : w \in \mathbb{C}\}$. Then $\mathcal{F}$ is a family of holomorphic functions on the unit disc $\Delta$. By assumption, for any function $g(z) = f(z + w) - a$, we have that $E(0, g) = E(a, g')$ and $g'(z) = b$ whenever $g(z) = b$.

Hence by Lemma 5, $\mathcal{F}$ is normal in $\Delta$. Thus by Lemma 8, there exists $M > 0$
satisfying \( f'(z) \leq M \) for all \( z \in \mathbb{C} \). By Lemma 1, \( f \) is of exponential type and hence of order at most one. Thus, by Lemma 2, we have

\[
f'(z) - a = c\left[f(z) - a\right],
\]

where \( c \) is a non-zero constant.

Hence

\[
f(z) = de^{cz} + \frac{c-1}{c}a,
\]

\[
f^{(k)}(z) = d^k d^{cz},
\]

where \( d \) is a non-zero constant. Clearly, there exists \( z_0 \) such that \( f'(z_0) = b \). Then

\[
f^{(k)}(z_0) = b, \text{ so by (4.3) } c^{k-1} = 1. \]

This completes the proof of Theorem 2.  

5. Proofs of Theorems 3 and 5

Because the proofs of Theorems 3 and 5 are similar, we give only the proof of Theorem 3.

First, we prove that \( f \) is of exponential type. Set

\[ \mathcal{F} = \{ f(z+w) : w \in \mathbb{C}, \quad z \in D = \{ z : |z| < 1 \} \}. \]

Then \( \mathcal{F} \) is a family of holomorphic functions in \( D \). By assumption, for any function \( g(z) = f(z+w) \), \( g(z) \neq 0 \) and \( g^{(k)}(z) = b \) whenever \( g(z) = a \). Hence by Lemma 6, \( \mathcal{F} \) is normal in \( D \). Thus by Lemma 8, there exists \( M > 0 \) satisfying \( f'(z) \leq M \) for all \( z \in \mathbb{C} \). By Lemma 1, \( f \) is of exponential type.

Since \( f \neq 0 \) and \( f \) is non-constant, \( f(z) = e^{Az+B} \), where \( A \neq 0 \), \( B \) are constants. From \( f^{(k)}(z) = b \) whenever \( f(z) = a \), we obtain \( A^k = b/a \). This concludes the proof of Theorem 3.  

References