Frenkel electron on an arbitrary electromagnetic background and magnetic *Zitterbewegung*

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Abstract

We present Lagrangian which implies both necessary constraints and dynamical equations for position and spin of relativistic spin one-half particle. The model is consistent for any value of magnetic moment μ and for arbitrary electromagnetic background. Our equations coincide with those of Frenkel in the approximation in which the latter have been obtained by Frenkel. Transition from approximate to exact equations yields two structural modifications of the theory. First, Frenkel condition on spin-tensor turns into the Pirani condition. Second, canonical momentum is no more proportional to velocity. Due to this, even when $\mu = 1$ (Frenkel case), the complete and approximate equations predict different behavior of a particle. The difference between momentum and velocity means extra contribution to spin–orbit interaction. To estimate the contribution, we found exact solution to complete equations for the case of uniform magnetic field. While Frenkel electron moves around the circle, our particle experiences magnetic *Zitterbewegung*, that is oscillates in the direction of magnetic field with amplitude of order of Compton wavelength for the fast particle. Besides, the particle has dipole electric moment.

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1. Introduction

Consistent and complete description of spin effects of the relativistic electron is achieved in QED on the base of Dirac equation. However, starting from the pioneer works [1–3] and up to

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present date, interpretation of final results in some cases is under permanent and controversial debates in various theoretical and experimental set-ups [4–16]. Understanding of spin precession in the case of arbitrary magnetic moment in an external electromagnetic field is important in the development of experimental technics for measurements of anomalous magnetic moment [17,18]. In accelerator physics [19] it is important to control resonances leading to depolarization of a beam. In the case of vertex electrons carrying arbitrary angular momentum, semiclassical description can also be useful [20]. So the relationship among classical and quantum descriptions remains an important step of analysis, providing the interpretation of results of QFT computations in usual terms: particles and their interactions. Hence an actual task is to develop, in a systematic form, the classical model of an electron [21–26,28,29,31–38,40] which would be as close as possible to the Dirac equation.

Maybe the best candidates for classical equations of relativistic electron are those of Frenkel [21,22] and Bargmann, Michel and Telegdi (BMT) [23]. They almost exactly reproduce spin dynamics of polarized beams in uniform fields, and thus might be proper classical analog for the Dirac theory. Integrability of BMT equations in the case of rather general electromagnetic backgrounds were studied in [50]. Solutions to BMT equations in a constant magnetic field can be associated with those of Dirac [49].

However, to be able to describe other spin effects, it is desirable to have systematically constructed Lagrangian and Hamiltonian formulations as well as proper quantization scheme for these equations (note that one needs a Hamiltonian to describe, for instance, Stark and Zeeman effects). Then it would be possible to use them as a semiclassical approximation of the QFT computations based on Dirac equation.

Non-relativistic spin operators are proportional to the Pauli matrices, so they form a simple algebra with respect to commutator

$$[\hat{S}_i, \hat{S}_j]_- = i\hbar\epsilon_{ijk}\hat{S}_k, \tag{1}$$

as well as with respect to anticommutator

$$[\hat{S}_i, \hat{S}_j]_+ = \frac{\hbar^2}{2}\delta_{ij}. \tag{2}$$

These equations prompt that spin-space in classical model can be described by either even or odd (Grassmann) variables. The pioneer model based on odd variables have been constructed by Berezin and Marinov [26]. This gives very economic and elegant scheme for semiclassical description of spin. For non-relativistic spin, the Lagrangian reads $\frac{m}{2}(\dot{x}_i)^2 + i\xi_i\dot{\xi}_i$, where the inner space of spin is constructed from vector-like Grassmann variables ξ_i, $\xi_i\xi_j = -\xi_j\xi_i$. Since the Lagrangian is linear on $\dot{\xi}_i$, their conjugate momenta coincide with ξ, $\pi_i = \frac{\partial L}{\partial \dot{\xi}_i} = i\xi_i$. The relations represent the Dirac second-class constraints and are taken into account by transition from the grassmannian Poisson bracket to the Dirac one. After that, the constraints can be used to exclude π_i. Dirac bracket of the remaining variables reads $\{\xi_i, \xi_j\}_{DB} = i\delta_{ij}$. Comparing this with Eq. (2), we quantize the model replacing $\xi_i \rightarrow \frac{\hbar}{2}\sigma_i$. Relativistic spin is described in a similar way [26–29]. The problem here is that Grassmann classical mechanics represents a rather formal mathematical construction. It leads to certain difficulties [26,29] in attempts to use it for description the spin effects on the semiclassical level, before the quantization. Besides, generalization of Grassmann mechanics to higher spins is not known [30]. Hence it would be interesting to describe spin on a base of usual variables, that is we intend to arrive at the commutator algebra (1) instead of (2).

Contrary to the models based on commuting spinors [32,33], in the Berezin–Marinov approach the σ_i (or γ^μ) matrices do not appear in classical theory but produced through the quantization process. The same turns out to be true in our model based on non-Grassmann vector for description of spin.

Very general approach to description of rotational degrees of freedom in relativistic theory has been developed by Hanson and Regge [25]. They suggested to represent a relativistic spherical top as a point on a world-line to which a body-fixed frame is attached. The frame is identified with the Lorentz-group element, so the trajectory $(x^\mu(\tau), \Lambda^{\mu\nu}(\tau))$ of the top represents a line of the Poincare-group manifold. The antisymmetric tensor $\Lambda^{-1}\dot\Lambda$ has been chosen as the basic quantity to describe the rotational degrees of freedom. They asked on the most general form of Lagrangian which yields only three physical rotational degrees of freedom. The Lagrangian gives generalized mass-shell constraint which relates mass with spin, so in quantum mechanics they obtained a string-like spectrum composed by a family of particles with varying mass and spin. They also analyzed whether their spin-tensor couples directly with electromagnetic fields, and concluded on impossibility to construct the interaction in a closed form. As we show below, this can be achieved in closed form for the vector that constitutes our spin-tensor.

Since the commutator (1) represents the angular-momentum algebra, it is natural to represent the spin in classical theory as the composed quantity, $\vec{S} = \vec{\omega} \times \vec{\pi}$, constructed from spacial components of some inner-space coordinate ω^μ and its conjugated momentum π^μ. As in the Hanson–Regge approach, the main problem here is to construct the Poincare-invariant Lagrangian which has the right number of degrees of freedom and admits an interaction in closed and relatively simple form. We need a variational problem which yields the appropriate constraints. In turn, this implies the use of Dirac's machinery for analysis of constrained systems. Though a number of models [31–35,39] with vector variables yield Frenkel and BMT equations, they also contain extra degrees of freedom. At the classical level one can simply ignore them. However, they should be taken into account during quantization procedure, this leads to quantum models essentially different from the Dirac electron.

In this work we continue detailed analysis of the Frenkel and BMT equations started in [41,42], and construct the Lagrangian which yields generalization of these equations to the case of arbitrary electromagnetic background. Even for non-interacting theory search for Lagrangian represents rather non-trivial problem [22,25]. In [42] we have solved this problem, considering spin as angular momentum of inner four-dimensional vector space attached to the point of a world-line.

In Hamiltonian formulation the model represents a non-trivial example of a constrained system. Phase space of the model turns out to be curved manifold equipped, in a natural way, with the structure of fiber bundle. Detailed analysis of the underlying geometry has been presented in [41]. This allowed us to develop the proper quantization scheme. In [42] we have performed both canonical (in physical-time parametrization) and manifestly covariant (in arbitrary parametrization) quantization of the free model, and established the relation with one-particle sector of Dirac equation as well as with quantum theory of two-component Klein–Gordon equation developed by Feynman and Gell-Mann [43]. It has been demonstrated that various known in the literature non-covariant, covariant and manifestly-covariant operators of position and spin acquire clear meaning and interpretation in the Lagrangian model of Frenkel electron. In particular, we have found the manifestly covariant form of position and spin operators in the space of positive-energy Dirac spinors.

In the Hamiltonian formulation two second-class constraints appeared which, at the end, supply the Frenkel condition on spin-tensor. They depend on both position and spin-sector variables.

This leads to new properties as compared with non-relativistic spin [41]. The constraints must be taken into account by transition from Poisson to Dirac bracket, this leads to non-vanishing classical brackets for the position variables. In the result, the position space is endowed, in a natural way, with non-commutative structure which originates from accounting of spin degrees of freedom. Our model represents an example of a situation, when physically interesting non-commutative relativistic particle emerges in a natural way. For the case, the "parameter of non-commutativity" is proportional to spin-tensor. As a consequence, operators corresponding to position of the electron are non-commutative (they can be identified [42] with Pryce (d) operators). This implies that effects of non-commutativity could be presented at the Compton wave length, in contrast to conventional expectations of non-commutativity at Planck length.

There are a lot of candidates for spin and position operators of the relativistic electron [1–3, 44,24]. Different position observables coincide when we consider standard quasi-classical limit. So, in absence of a systematically constructed classical model of an electron it is difficult to understand the difference between these operators. Our approach allows us to do this, after realizing all them at the classical level. Besides, all the candidates obey the same equations in free theory, so the question of which of them are the true position and spin is a matter of convention. The situation changes in the interacting theory considered below, where we can distinguish the variables according their classical dynamics in an external field.

In the present work we construct and study an interacting theory. In Section 2 we show that our Lagrangian admits interaction with an arbitrary electromagnetic background.[1] The model contains two coupling constants – charge e and the interaction constant μ of basic spin variables with $F_{\mu\nu}$. Provisionally, we call this magnetic moment. The theory is consistent for arbitrary values of μ. For the position variable we have the minimal interaction term $\frac{e}{c}A_\mu \dot{x}^\mu$. As for spin, when the particle has non-vanishing magnetic moment, this interacts with electromagnetic field in a highly nonlinear way. This turns out to be necessary for preservation of the number and algebraic structure of constraints in the passage from free to interacting theory. In Section 3 we present and analyze equations of motion in Hamiltonian and Lagrangian formulations. We show that they follow from simple and expected Hamiltonian (36), when we deal with the Dirac bracket. We compare our equations with those of Frenkel [21,22]. Frekel considered the case $\mu = 1$, and found his equations in the quadratic approximation on spin–tensor. We show that our exact equations coincide with those of Frenkel in these limits. Hence our Lagrangian gives complete Frenkel equations for arbitrary field and magnetic moment.

Frenkel tensor can be used to construct BMT-type four-vector. We write the corresponding equations of motion. While Hamiltonian equations can be rewritten in closed form in terms of BMT vector, see Eqs. (60)–(62), we do not achieved this for Lagrangian equations in our theory. In the Lagrangian form, the equation for BMT vector contains Frenkel tensor, see Eq. (63). It seems that Frenkel spin in our theory represents more fundamental object as compared with BMT spin.

While equations of motion have a rather complicated structure, in the case of uniform magnetic field there are a lot of symmetries and hence integrals of motions providing complete analytical solution. In Section 4 we find exact solution to our equations for this case. As compared with Frenkel and BMT equations, our model takes into account two effects. First, magnetic moment interacted with a magnetic field results in additional mass of electron. Second, in the case of anomalous magnetic moment the velocity and the momentum are not collinear, this modifies the

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[1] Interaction with an arbitrary curved background is presented in [45].

Lorentz force. Our model naturally incorporates both these effects and leads to small corrections of the trajectory and spin precession.

2. Lagrangian and Hamiltonian of interacting theory

To start with, we shortly describe the structure of free theory [41,42]. Configuration space of the model consist of the position $x^\mu(\tau) = (ct, \boldsymbol{x})$ and the vector-like variable of spin $\omega^\mu(\tau) = (\omega^0, \boldsymbol{\omega})$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ in an arbitrary parametrization τ. p^μ and π^μ are conjugate momenta for x^μ and ω^μ. The variables in the free theory are subject to the constraints (we set $a_3 = \frac{3\hbar^2}{4a_4}$)

$$T_1 = p^2 + (mc)^2 = 0, \tag{3}$$

$$T_3 = \pi^2 - a_3 = 0, \qquad T_4 = \omega^2 - a_4 = 0, \qquad T_5 = \omega\pi = 0, \tag{4}$$

$$T_6 = p\omega = 0, \qquad T_7 = p\pi = 0, \tag{5}$$

where $\omega\pi = -\omega^0\pi^0 + \boldsymbol{\omega}\boldsymbol{\pi}$ and so on. As the Hamiltonian action functional, we simply take $L_H = p\dot{x} + \pi\dot{\omega} - H$, with the Hamiltonian $H = g_i T_i$ in the form of linear combination the constraints T_i multiplied by auxiliary variables g_i, $i = 1, 3, 4, 5, 6, 7$. The constraint T_3 belongs to first-class and is related with local spin-plane symmetry presented in the theory [41]. The basic spin-sector variables change under the symmetry, so they do not represent an observables quantities. As the observable quantity we take the Frenkel spin-tensor $J^{\mu\nu}$

$$J^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu). \tag{6}$$

The constraints (4) and (5) imply the following restrictions (in the free theory the conjugated momentum is proportional to four-velocity, $p^\mu \sim u^\mu$)

$$J^{\mu\nu}p_\nu = 0, \qquad J^2 = 6\hbar^2. \tag{7}$$

Spacial components of the Frenkel tensor can be used to construct the quantity

$$S^i = \frac{1}{4}\epsilon^{ijk}J_{jk}, \tag{8}$$

which we identify with non-relativistic spin of Pauli theory. In the interacting theory p^μ turns into canonical momentum $\mathcal{P}^\mu$, which for non-uniform fields or/and $\mu \neq 1$ does not proportional to four-velocity. Hence in this case the Frenkel condition $J^{\mu\nu}u_\nu = 0$ turns into the Pirani condition [46,48,47] $J^{\mu\nu}\mathcal{P}_\nu = 0$.

Frenkel tensor can be used to construct four-vector

$$s^\mu(\tau) \equiv \frac{1}{4\sqrt{-p^2}}\epsilon^{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}, \quad \text{then} \quad s^\mu p_\mu = 0, \quad s^2 = \frac{3\hbar^2}{4}. \tag{9}$$

In our theory, even in the case of interaction, the condition $s^\mu\mathcal{P}_\mu = 0$ implies $s^\mu u_\mu = 0$, see Eq. (58) below. So we can identify s^μ with BMT vector [23]. In the rest frame, spacial components s^i of BMT vector coincide with S^i. In an arbitrary frame, they are related as follows:

$$S^i = \frac{p^0}{\sqrt{-p^2}}\left(\delta_{ij} - \frac{p_i p_j}{(p^0)^2}\right)s^j. \tag{10}$$

In the free theory Eq. (9) can be inverted, $J^{\mu\nu} = -\frac{2}{\sqrt{-p^2}}\epsilon^{\mu\nu\alpha\beta}p_\alpha s_\beta$, so the two quantities are mathematically equivalent. In the interacting theory, we have $p \to \mathcal{P} = \mathcal{P}(u, J, F)$, see Eqs. (30) and (31), so (9) becomes non-linear equation.

In [42] we developed Lagrangian formulation of the theory. Excluding conjugate momenta from L_H, we obtained the Lagrangian action. Further, excluding the auxiliary variables, one after another, we obtained various equivalent formulations of the model. In the end, we got the "minimal" formulation without auxiliary variables. This reads[2]

$$S = \int d\tau - mc\sqrt{-\dot{x}N\dot{x}} + \sqrt{a_3}\sqrt{\dot{\omega}N\dot{\omega}} - \frac{1}{2}g_4(\omega^2 - a_4), \tag{11}$$

where $N^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{\omega^\mu\omega^\nu}{\omega^2}$ is projector on the plane transverse to the direction of ω^μ. The action is written in a parametrization τ which obeys $\frac{dt}{d\tau} > 0$.

Interaction with an external background should not spoil the number and algebraic properties of constraints (3)–(5). We do not know how to achieve this for the minimal action.[3] For instance, the natural reparametrization-invariant interaction $\frac{e}{c}A_\mu\dot{x}^\mu + \mu F_{\mu\nu}\omega^\mu\dot{\omega}^\nu$, even for vanishing magnetic moment, leads to the theory with the number and algebraic structure of constraints different from those of free theory. So we start with the equivalent Lagrangian with four auxiliary variables, g_1, g_3, g_4 and g_7, this turns out to be appropriate to our aims. Of course, the auxiliary variables will be excluded from final equations of motion, see Eqs. (46)–(52).

To introduce coupling of the position variable with an electro-magnetic field, we add the minimal interaction term $A_\mu\dot{x}^\mu$. As for spin, we propose to modify derivative of ω as follows

$$\dot{\omega}^\mu \rightarrow D\omega^\mu = \dot{\omega}^\mu - g_1\frac{e\mu}{c}(F\omega)^\mu. \tag{12}$$

This is the only term which we have found to be consistent with the constraints T_i. Lagrangian reads

$$L = \frac{1}{2\det\tilde{G}}\Big[g_3(\dot{x}N\dot{x}) - 2g_7(\dot{x}ND\omega) + g_1(D\omega ND\omega)\Big]$$
$$+ \frac{e}{c}A_\mu\dot{x}^\mu - \frac{g_4}{2}(\omega^2 - a_4) - \frac{g_1}{2}m^2c^2 + \frac{g_3}{2}a_3, \tag{13}$$

where $\det\tilde{G} = g_1g_3 - g_7^2$. Since L contains auxiliary variables, even for $\mu = 0$ we have highly nonlinear interaction. As a consequence, motion of spin influences motion of the particle and *vice versa*.

We first establish whether our Lagrangian gives the desired constraints. The momenta read

$$p^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{\det\tilde{G}}\big(g_3N\dot{x}^\mu - g_7ND\omega^\mu\big) + \frac{e}{c}A^\mu,$$

$$\pi^\mu = \frac{\partial L}{\partial \dot{\omega}^\mu} = \frac{1}{\det\tilde{G}}\big(-g_7N\dot{x}^\mu + g_1ND\omega^\mu\big), \tag{14}$$

$$\pi_{g_i} = \frac{\partial L}{\partial \dot{g}_i} = 0. \tag{15}$$

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[2] The last term in (11) represents kinematic (velocity-independent) constraint which is well known from classical mechanics. So, we might follow the classical-mechanics prescription to exclude g_4 as well. But this would lead to lose of manifest covariance of the formalism.

[3] Hanson and Regge [25] have found highly non-linear interaction for the case of their relativistic top. It would be interesting to apply their formalism to our minimal action.

According to (15), the momenta π_{gi} represent primary constraints, $\pi_{gi} = 0$. Using the property $N\omega = 0$ of the projector N, from Eq. (14) more primary constraints follow, $T_5 = \pi\omega = 0$ and $T_6 = \mathcal{P}\omega = 0$. It has been denoted

$$\mathcal{P}^\mu = p^\mu - \frac{e}{c}A^\mu. \tag{16}$$

To write Hamiltonian, we solve the system (14) with respect to projected velocities

$$N\dot{x}^\mu = g_1\mathcal{P}^\mu + g_7\pi^\mu, \qquad ND\omega^\mu = g_3\pi^\mu + g_7\mathcal{P}^\mu. \tag{17}$$

Using these expressions as well as the identities $\mathcal{P}\dot{x} = \mathcal{P}N\dot{x}$, $\pi\dot{\omega} = \pi N\dot{\omega}$ we obtain Hamiltonian $H = p\dot{x} + \pi\dot{\omega} - L + \lambda_a\Phi_a$ in the form

$$H = \frac{g_1}{2}\left(\mathcal{P}^2 - \frac{\mu e}{2c}(JF) + m^2c^2\right) + \frac{g_3}{2}\left(\pi^2 - a_3\right) + \frac{g_4}{2}\left(\omega^2 - a_4\right)$$
$$+ \lambda_5(\omega\pi) + \lambda_6(\mathcal{P}\omega) + g_7(\mathcal{P}\pi) + \lambda_{gi}\pi_{gi}, \tag{18}$$

where λ_5 and λ_6 appear as Lagrangian multipliers for primary constraints T_5 and T_6. We denote $(JF) = J^{\mu\nu}F_{\mu\nu}$ and so on. From (18) we conclude that T_1, T_3, T_4 and T_7 appear as secondary constraints when we impose the compatibility conditions $\dot{\pi}_{gi} = \{\pi_{gi}, H\} = 0$. The second, third and fourth stages of the Dirac–Bergmann algorithm can be resumed as follows

$$T_1 = 0 \quad \Rightarrow \quad \lambda_6 C + g_7 D = 0, \tag{19}$$
$$T_3 = 0 \quad \Rightarrow \quad \lambda_5 = 0, \tag{20}$$
$$T_4 = 0 \quad \Rightarrow \quad \lambda_5 = 0, \tag{21}$$
$$T_5 = 0 \quad \Rightarrow \quad g_4 = \frac{a_3}{a_4}g_3, \qquad \Rightarrow \quad \lambda_{g_4} = \frac{a_3}{a_4}\lambda_{g_3}, \tag{22}$$
$$T_6 = 0 \quad \Rightarrow \quad g_1 C - g_7 M^2 c^2 = 0, \qquad \Rightarrow \quad \lambda_{g_7} = f(\lambda_{g_1}), \tag{23}$$
$$T_7 = 0 \quad \Rightarrow \quad g_1 D + \lambda_6 M^2 c^2 = 0. \tag{24}$$

We have denoted

$$M^2 = m^2 - \frac{e(2\mu + 1)}{4c^3}F_{\mu\nu}J^{\mu\nu}, \tag{25}$$
$$C = -\frac{e}{c}(\mu - 1)(\omega F\mathcal{P}) + \frac{e\mu}{4c}(\omega\partial)(JF),$$
$$D = -\frac{e}{c}(\mu - 1)(\pi F\mathcal{P}) + \frac{e\mu}{4c}(\pi\partial)(JF). \tag{26}$$

Eq. (19) turns out to be a consequence of (23) and (24), $\lambda_6(23) + g_7(24) = g_1(19)$, and can be omitted. Eq. (23) determines $g_7 = \frac{C}{M^2c^2}g_1$ while (24) gives the lagrangian multiplier $\lambda_6 = -\frac{D}{M^2c^2}g_1$. The Dirac–Bergmann algorithm stops at the fourth stage. This yields all the desired constraints T_a, $a = 1, 3, 4, 5, 6, 7$. Two auxiliary variables, g_1 and g_3, and the corresponding Lagrange multipliers λ_{g_1}, λ_{g_3} have not been determined.

It is useful to summarize the algebra of Poisson brackets between constraints in a compact form, see Table 1. We note that Poisson brackets of T_1 and $\tilde{T}_3 = T_3 + \frac{a_3}{a_4}T_4$ vanish on the constraint surface, so they form the first-class subset. The presence of two first-class constraints is in a correspondence with the fact that two lagrangian multipliers remain undetermined within the Dirac procedure. Matrix of Poisson brackets of the remaining constraints, T_4, T_5, T_6 and T_7, is non-degenerate, so this is a set of second-class constraints. All this is in correspondence with free theory [42].

In resume, the interaction does not spoil the structure and algebraic properties of Hamiltonian constraints of the free theory.

Table 1
Algebra of constraints.

	T_1	T_3	T_4	T_5	T_6	T_7
$T_1 = \mathcal{P}^2 - \frac{\mu e}{2c} F^{\mu\nu} J_{\mu\nu}$ $+ m^2 c^2$	0	0	0	0	$-2C$	$-2D$
$T_3 = \pi^2 - a_3$	0	0	$-4T_5$	$-2(a_3 + T_3)$	$-2T_7$	0
$T_4 = \omega^2 - a_4$	0	$4T_5$	0	$2(T_4 + a_4)$	0	$2T_6$
$T_5 = \omega\pi$	0	$2(T_3 + a_3)$	$-2(a_4 + T_4)$	0	$-T_6$	T_7
$T_6 = \mathcal{P}\omega$	$2C$	$2T_7$	0	T_6	0	$T_1 - M^2 c^2$
$T_7 = \mathcal{P}\pi$	$2D$	0	$-2T_6$	$-T_7$	$-T_1 + M^2 c^2$	0

3. Exact Frenkel equations on arbitrary background

3.1. Hamiltonian equations of motion

The Hamiltonian (18) determines evolution of the basic variables through the Poisson bracket $\dot{q} = \{q, H\}$. Equivalently, we can pass from Poisson to Dirac bracket constructed on the base of the second-class constraints T_4, T_5, T_6, T_7. The list of Dirac brackets is presented in Appendix A. After that, our highly nonlinear interaction turns out to be hidden in the Dirac bracket: the constraints can be used in the Hamiltonian (18), this gives the expression

$$H_1 = \frac{g_1}{2}\left(\mathcal{P}^2 - \frac{\mu e}{2c}(JF) + m^2 c^2\right) + \frac{g_3}{2}\left(J^2 - 8a_3 a_4\right). \tag{27}$$

Equations of motion now can be obtained with help of H_1 and the Dirac bracket, $\dot{q} = \{q, H_1\}_{DB}$. They read

$$\dot{x}^\mu = g_1 u^\mu, \qquad \dot{\mathcal{P}}^\mu = g_1 \frac{e}{c}(Fu)^\mu + g_1 \frac{\mu e}{4c}\partial^\mu(JF), \tag{28}$$

$$\dot{\omega}^\mu = g_1 \frac{e\mu}{c}(F\omega)^\mu + g_3 \pi^\mu + g_7 \mathcal{P}^\mu,$$

$$\dot{\pi}^\mu = g_1 \frac{e\mu}{c}(F\pi)^\mu - \frac{a_3}{a_4}g_3 \omega^\mu - g_6 \mathcal{P}^\mu, \tag{29}$$

where $\partial^\mu(JF) = J^{\alpha\beta}\partial^\mu F_{\alpha\beta}$. According to (23) and (24), the four-velocity u^μ is not proportional to canonical momentum $\mathcal{P}^\mu$

$$u^\mu = \mathcal{P}^\mu + \frac{g_7}{g_1}\pi^\mu + \frac{\lambda_6}{g_1}\omega^\mu = T^\mu{}_\nu \mathcal{P}^\nu + Y^\mu. \tag{30}$$

We have denoted

$$T^{\mu\nu} = \eta^{\mu\nu} - (\mu - 1)a(JF)^{\mu\nu}, \qquad Y^\mu = \frac{\mu a}{4}J^{\mu\alpha}\partial_\alpha(JF),$$

$$a = -\frac{e}{2M^2 c^3} \equiv \frac{-2e}{4m^2 c^3 - e(2\mu + 1)(JF)}. \tag{31}$$

Matrix T is invertible, the inverse matrix $\tilde{T}$ has the same structure (we used the identity $(JFJ)^{\mu\nu} = -\frac{1}{2}(JF)J^{\mu\nu}$ which implied by (6))

$$\tilde{T}^{\mu\nu} = \eta^{\mu\nu} + (\mu - 1)b(JF)^{\mu\nu},$$

$$b = \frac{2a}{2 + (\mu - 1)a(JF)} \equiv \frac{-2e}{4m^2c^3 - 3e\mu(JF)}. \tag{32}$$

All the basic variables have ambiguous evolution. x^μ and $\mathcal{P}^\mu$ have one-parametric ambiguity due to g_1 (they change under reparametrizations) while ω and π have two-parametric ambiguity due to g_1 and g_3 (they change under reparametrizations and spin-plane symmetry). The quantities x^μ, $\mathcal{P}^\mu$ and the spin-tensor $J^{\mu\nu}$ are spin-plane invariants. Their equations of motion form a closed system

$$\dot{x}^\mu = g_1 \left[\mathcal{P}^\mu - a J^{\mu\alpha} \left((\mu - 1)(F\mathcal{P})_\alpha - \frac{\mu}{4} \partial_\alpha (JF) \right) \right], \tag{33}$$

$$\dot{\mathcal{P}}^\mu = \frac{e}{c} (F\dot{x})^\mu + g_1 \frac{\mu e}{4c} \partial^\mu (JF), \tag{34}$$

$$j^{\mu\nu} = g_1 \left[\frac{e\mu}{c} F^{[\mu}{}_\alpha J^{\alpha\nu]} - 2a\mathcal{P}^{[\mu} J^{\nu]\alpha} \left((\mu - 1)(F\mathcal{P})_\alpha - \frac{\mu}{4} \partial_\alpha (JF) \right) \right]. \tag{35}$$

The last term in (27) does not contributes to the equations of motion for $x, \mathcal{P}$ and J, and can be omitted. Then the Hamiltonian for these variables acquires a simple and expected form

$$H = \frac{g_1}{2} \left(\mathcal{P}^2 - \frac{\mu e}{2c} (JF) + m^2 c^2 \right). \tag{36}$$

The interaction yields two essential structural modifications of the theory. Free theory implies the Frenkel condition, $J^{\mu\nu} \dot{x}_\nu = 0$, and $p^\mu \sim \dot{x}^\mu$. Interaction modifies not only dynamical equations but also the Frenkel condition, the latter necessarily turns into the Pirani condition

$$J^{\mu\nu} \mathcal{P}_\nu = 0, \tag{37}$$

where, due to (30), $\mathcal{P}^\mu$ is not proportional to $\dot{x}^\mu$. Then Eqs. (33) and (34) imply that the interaction leads to a modification of the Lorentz-force equation even for uniform fields. Only for the non-anomalous value of magnetic moment, $\mu = 1$, and uniform electromagnetic field Eqs. (23) and (24) would be the same as in free theory, $\lambda_6 = g_7 = 0$. Then $T^{\mu\nu} = \eta^{\mu\nu}$, $Y^\mu = 0$, and four-velocity becomes proportional to $\mathcal{P}^\mu$. Contribution of anomalous magnetic moment $\mu \neq 1$ to the difference between u and $\mathcal{P}$ is proportional to $\frac{J}{c^3} \sim \frac{\hbar}{c^3}$, while the term with a gradient of field is proportional to $\frac{J^2}{c^3} \sim \frac{\hbar^2}{c^3}$.

The remaining ambiguity due to g_1 in Eqs. (33)–(35) reflects the reparametrization symmetry of the theory. Assuming that the functions $x^\mu(\tau)$, $p^\mu(\tau)$ and $J^{\mu\nu}(\tau)$ represent the physical variables $x^i(t)$, $p^\mu(t)$ and $J^{\mu\nu}(t)$ in the parametric form, their equations read

$$\frac{dx^i}{dt} = c \frac{u^i}{u^0}, \qquad \frac{dx^0}{dt} = c, \tag{38}$$

$$\frac{d\mathcal{P}^\mu}{dt} = \frac{e}{u^0} F^{\mu\nu} u_\nu + \frac{\mu e}{4u^0} \partial^\mu (JF), \tag{39}$$

$$\frac{dJ^{\mu\nu}}{dt} = \frac{c}{g_1 u^0} j^{\mu\nu}. \tag{40}$$

As it should be, they have unambiguous dynamics. Eqs. (33)–(35) are written in an arbitrary parametrization of the world-line. In the next subsection we exclude $\mathcal{P}^\mu$ and g_1, and then analyze the resulting equations in the proper-time parameterizations. This allow us to compare them with original Frenkel equations.

3.2. Lagrangian form of equations

Hamiltonian equations from the previous section can be rewritten in the Lagrangian form for the set x, J. Let us analyze the relation between velocity and momentum given by the Hamiltonian equation (33). This can be written in the form

$$\dot{x}^\mu = g_1\left(T^\mu{}_\nu \mathcal{P}^\nu + Y^\mu\right). \tag{41}$$

From this equation we express $\mathcal{P}$ through $\dot{x}$

$$\mathcal{P}^\mu = \frac{1}{g_1}\tilde{T}^\mu{}_\nu \dot{x}^\nu - \tilde{T}^\mu{}_\nu Y^\nu. \tag{42}$$

We can find g_1 calculating square of the following expression

$$\mathcal{P}^\mu + \tilde{T}^\mu{}_\nu Y^\nu = \frac{1}{g_1}\tilde{T}^\mu{}_\nu \dot{x}^\nu,$$

which yields

$$\mathcal{P}^2 + (\tilde{T}Y)^\mu(\tilde{T}Y)_\mu = \frac{1}{g_1^2}(\tilde{T}\dot{x})^\mu(\tilde{T}\dot{x})_\mu.$$

We used that $\mathcal{P}_\mu \tilde{T}^\mu{}_\nu = \mathcal{P}_\mu$ and $\mathcal{P}_\mu Y^\mu = 0$. Using the last equation and T_1-constraint we find g_1

$$g_1 = \sqrt{\frac{(\tilde{T}\dot{x})^2}{(\tilde{T}Y)^2 - m^2c^2 + \frac{\mu e(JF)}{2c}}} \equiv \frac{\sqrt{-g\dot{x}\dot{x}}}{m_r c}, \tag{43}$$

where we have introduced the symmetric matrix

$$g_{\mu\nu} = \left(\tilde{T}^T \tilde{T}\right)_{\mu\nu}, \tag{44}$$

and the radiation mass

$$m_r^2 = m^2 - \frac{\mu e}{2c^3}(JF) - \frac{gYY}{c^2}. \tag{45}$$

In the natural parametrization $\sqrt{-g\dot{x}\dot{x}} = c$, we have $g_1 = m_r^{-1}$, that is the auxiliary variable, which appeared in front of mass-shell constraint $T_1 = 0$, is the inverse radiation mass. Due to the identity $(\tilde{T}Y)^\mu = \frac{b}{a}Y^\mu$ we also can write $gYY = \frac{b^2}{a^2}Y^2$. Using (42) and (43) in (34) and (35) we write closed system of equations for x^μ and $J^{\mu\nu}$ in the form

$$\frac{d}{d\tau}\left[m_r c\frac{(\tilde{T}\dot{x})^\mu}{\sqrt{-g\dot{x}\dot{x}}} - (\tilde{T}Y)^\mu\right] = \frac{e}{c}(F\dot{x})^\mu + \frac{\mu e\sqrt{-g\dot{x}\dot{x}}}{4m_r c^2}\partial^\mu(JF), \tag{46}$$

$$j^{\mu\nu} = \frac{e\mu}{m_r c^2}\sqrt{-g\dot{x}\dot{x}}F^{[\mu}{}_\alpha J^{\alpha\nu]} - \frac{2b(\mu-1)m_r c}{\sqrt{-g\dot{x}\dot{x}}}\dot{x}^{[\mu}(JF\dot{x})^{\nu]} + \frac{2b}{a}\dot{x}^{[\mu}Y^{\nu]}, \tag{47}$$

$$J^{\mu\nu}\tilde{T}_{\nu\alpha}\left(m_r c\dot{x}^\alpha - \sqrt{-g\dot{x}\dot{x}}Y^\alpha\right) = 0. \tag{48}$$

Let us compare them with Frenkel equations. Frenkel found equations of motion consistent with the condition $J^{\mu\nu}u_\nu = 0$ up to order $O^3(J, F, \partial F)$. Besides, he considered the case $\mu = 1$. Taking these approximations in our equations in the proper-time parametrization $\sqrt{-(\dot{x})^2} = c$, we arrive at those of Frenkel (our J is $\frac{2mc}{e}$ of Frenkel J)

$$\frac{d}{d\tau}\left[\left(m - \frac{e}{4mc^3}(JF)\right)\dot{x}^\mu + \frac{e}{8m^2c^3}J^{\mu\alpha}\partial_\alpha(JF)\right] = \frac{e}{c}(F\dot{x})^\mu + \frac{e}{4mc}\partial^\mu(JF), \tag{49}$$

$$j^{\mu\nu} = \frac{e}{mc}\left[F^{[\mu}{}_\alpha J^{\alpha\nu]} - \frac{1}{4mc^2}\dot{x}^{[\mu}J^{\nu]\alpha}\partial_\alpha(JF)\right], \qquad J^{\mu\nu}\dot{x}_\nu = 0. \tag{50}$$

In general case, our equations (46)–(48) involve two types of corrections as compared with those of Frenkel. First, the energy of magnetic moment in non-uniform field leads to the contribution $-\frac{gYY}{c^2}$ into the Frenkel radiation mass, see (44). Second, when $\mu \neq 0$, a contribution arises because the Frenkel condition which has been satisfied for the free particle, turns into Pirani condition in the interacting theory. Its Lagrangian form is written in (48). In the result, the components J^{0i} vanish in the frame $\mathcal{P}^\mu = (\mathcal{P}^0, \vec{0})$ instead of the rest frame. Hence our model predicts small dipole electric moment of the particle.

The structure of our equations simplified significantly for the stationary homogeneous field $\partial_\alpha F^{\mu\nu} = 0$. In this case (46) and (47) read

$$\left[\frac{m_r(\tilde{T}\dot{x})^\mu}{\sqrt{-(g\dot{x}\dot{x})}}\right]^\cdot = \frac{e}{c^2}(F\dot{x})^\mu, \tag{51}$$

$$j^{\mu\nu} = \frac{e\mu}{m_r c^2}\sqrt{-g\dot{x}\dot{x}}F^{[\mu}{}_\alpha J^{\alpha\nu]} - 2b\frac{(\mu-1)m_r c}{\sqrt{-g\dot{x}\dot{x}}}\dot{x}^{[\mu}(JF\dot{x})^{\nu]}, \tag{52}$$

$$(J\tilde{T}\dot{x})^\mu = 0. \tag{53}$$

Eq. (52) implies $\dot{j}^{\mu\nu}F_{\mu\nu} = 0$. Hence JF and m_r are conserved quantities. Then $T_1 = 0$ implies that $\mathcal{P}^2$ is a conserved quantity as well. The equation $\dot{m}_r = 0$ can also be obtained contracting (51) with $(\tilde{T}\dot{x})_\mu$ and using the identity $v_\mu[\frac{v^\mu}{\sqrt{-v^2}}]^\cdot \equiv 0$.

Contracting (51) with T we can further simplify this equation

$$\frac{d}{d\tau}\left[\frac{m_r \dot{x}^\mu}{\sqrt{-g\dot{x}\dot{x}}}\right] = \frac{e}{c^2}(F'\dot{x})^\mu, \qquad F' = TF - \frac{m_r c^2}{e\sqrt{-g\dot{x}\dot{x}}}T\dot{\tilde{T}}. \tag{54}$$

Let us choose a parametrization which implies

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2. \tag{55}$$

Since $g\dot{x}\dot{x} = \dot{x}^2 + O(J^2)$, in the linear approximation on J this is just the proper-time parametrization. Then Eqs. (52) and (54) read

$$\frac{d(\tilde{T}\dot{x})^\mu}{d\tau} = \frac{e}{m_r c}(F\dot{x})^\mu, \quad \text{or,} \quad \ddot{x}^\mu = \frac{e}{m_r c}(F'\dot{x})^\mu, \qquad F' = TF - \frac{m_r c}{e}T\dot{\tilde{T}}, \tag{56}$$

$$j^{\mu\nu} = \frac{e\mu}{m_r c}F^{[\mu}{}_\alpha J^{\alpha\nu]} - 2b(\mu-1)m_r \dot{x}^{[\mu}(JF\dot{x})^{\nu]}. \tag{57}$$

So, when $\mu \neq 0$, the exact equations differ from the approximate equations (49) and (50) even for uniform fields.

3.3. BMT vector in Frenkel theory

Since $J^{\mu\nu}\mathcal{P}_\nu = 0$, the spin-tensor is equivalent to the four-vector (9) where we replace $p^\mu \to \mathcal{P}^\mu$. Then $s^\mu\mathcal{P}_\mu = 0$. Due to Eqs. (30) and (31) together with (6), s^μ also obeys the condition

$$s^\mu u_\mu = s^\mu \dot{x}_\mu = 0. \tag{58}$$

The physical dynamics can be described using s^μ instead of $J^{\mu\nu}$. Eq. (58) suggests that s^μ could be candidate for BMT-vector in our model. Using the identities

$$J^{\mu\nu} = \frac{2}{\sqrt{-\mathcal{P}^2}}\epsilon^{\mu\nu\alpha\beta}s_\alpha \mathcal{P}_\beta, \qquad \epsilon^{\mu\nu\alpha\beta}J_{\alpha\beta} = \frac{4}{\sqrt{-\mathcal{P}^2}}\mathcal{P}^{[\mu}s^{\nu]}, \tag{59}$$

to represent $J^{\mu\nu}$ through s^μ in Eqs. (33)–(35), we obtain the closed system of equations for spin-plane invariant quantities

$$\dot{x}^\mu = g_1\left[\mathcal{P}^\mu - \frac{2(\mu-1)a}{\sqrt{-\mathcal{P}^2}}\epsilon^{\mu\nu\alpha\beta}(F\mathcal{P})_\nu s_\alpha \mathcal{P}_\beta - \frac{\mu a}{\mathcal{P}^2}\epsilon^{\mu\alpha\gamma\delta}\epsilon^{\rho\beta\gamma'\delta'}s_\gamma \mathcal{P}_\delta s_{\gamma'}\mathcal{P}_{\delta'}\partial_\alpha F_{\rho\beta}\right], \tag{60}$$

$$\dot{\mathcal{P}}^\mu = \frac{e}{c}(F\dot{x})^\mu + g_1\frac{\mu e}{2c\sqrt{-\mathcal{P}^2}}\epsilon_{\alpha\beta\gamma\rho}s^\gamma \mathcal{P}^\rho \partial^\mu F^{\alpha\beta}, \tag{61}$$

$$\dot{s}^\mu = g_1\frac{e\mu}{c}\left[(Fs)^\mu + \frac{1}{\mathcal{P}^2}(sF\mathcal{P})\mathcal{P}^\mu\right] - \frac{1}{\mathcal{P}^2}(\dot{\mathcal{P}}s)\mathcal{P}^\mu. \tag{62}$$

These equations valid for arbitrary electro-magnetic fields. Let us consider the case of uniform field discussed by Bargmann Michel and Telegdi. Then we can compare these equations with BMT equations. First, we should exclude $\mathcal{P}$ and g_1 from Eqs. (61) and (62) using (60). In contrast to (41), where $\dot{x}^\mu$ is a linear function of $\mathcal{P}^\mu$ and $J^{\mu\nu}$, in (60) $\dot{x}^\mu$ is a non-linear function of $\mathcal{P}^\mu$ and s^μ. Inverse function which express $\mathcal{P}^\mu$ as a function of $\dot{x}^\mu$, s^μ exists, though we can't find its explicit form even in the case of uniform fields. Formally using (42) and (43) in the case of uniform fields, $\partial_\alpha F^{\mu\nu} = 0$, we get

$$\dot{s}^\mu = \sqrt{-g\dot{x}\dot{x}}\,\frac{e\mu}{m_r c^2}(Fs)^\mu$$
$$- \frac{e}{m_r c^2 \sqrt{-g\dot{x}\dot{x}}}\left[(\mu-1)(sF\dot{x}) + \mu b(sFJF\dot{x})\right](\tilde{T}\dot{x})^\mu, \quad s\dot{x} = 0. \tag{63}$$

Eq. (63) contains J but for weak fields the corresponding contribution can be neglected. In the uniform field and in the parametrization (55) we have Eq. (56) for x and

$$\dot{s}^\mu = \frac{e\mu}{m_r c}(Fs)^\mu - \frac{e}{m_r c^3}\left[(\mu-1)(sF\dot{x}) + \mu b(sFJF\dot{x})\right](\tilde{T}\dot{x})^\mu. \tag{64}$$

This can be compared with BMT-equations

$$\ddot{x}^\mu = \frac{e}{mc}(F\dot{x})^\mu, \tag{65}$$

$$\dot{s}^\mu = \frac{e\mu}{mc}(Fs)^\mu - \frac{e}{mc^3}(\mu-1)(sF\dot{x})\dot{x}^\mu. \tag{66}$$

We can also introduce BMT-tensor dual to s^μ

$$J^{\mu\nu}_{BMT} = \frac{2}{c}\epsilon^{\mu\nu\alpha\beta}s_\alpha \dot{x}_\beta.$$

Due to (66) this obeys the equation

$$\dot{J}^{\mu\nu}_{BMT} = \frac{e}{mc}\left[\mu F^{[\mu}{}_\alpha J^{\alpha\nu]}_{BMT} - \frac{(\mu-1)}{c^2}(J_{BMT}F\dot{x})^{[\mu}\dot{x}^{\nu]}\right]. \tag{67}$$

This can be compared with (57).

Obtaining their equation (66) in uniform field, Bargmann, Michel and Telegdi supposed that the motion of particle (65) is independent from the motion of spin. Besides they looked for the

equation for s^μ linear on s and F. Obtaining Eqs. (56) and (64) we have not made any supposition of such a kind. Our approach is based on the variational formulation which satisfies all the necessary symmetries. The exact equations (56) and (64) involve two types of essential corrections as compared with BMT equations. First, an energy of magnetic moment in electromagnetic fields leads to the radiation mass m_r. Second, anomalous magnetic moment affects trajectory of a particle.

Neglecting non-linear on F and s terms in our equations (56), (64), we obtain those of Bargmann, Michel and Telegdi. The same holds if we take the proper-time parametrization, $\dot{x}^\mu \dot{x}_\mu = -c^2$, instead of (55).

4. Exact solution in uniform magnetic field

We take BMT vector of our model as the basic quantity for description of spin, and compare dynamics of our and BMT models in the case of uniform magnetic fields. We already established that $\mathcal{P}^2$ and (FJ) are integrals of motion for uniform fields. The quantity a given in Eq. (31) is also a constant, which practically (for the magnetic fields smaller than Schwinger field) can be taken as $a \approx -\frac{e}{2m^2c^3}$. In the case of uniform magnetic field we have

$$(FJ) = 4\gamma\big[(\mathbf{Bs}) - (\mathbf{B}\boldsymbol{\beta})(\boldsymbol{\beta}\mathbf{s})\big].$$

Here and through the rest of this section we use following notations

$$\gamma = \frac{\mathcal{P}^0}{\sqrt{-\mathcal{P}^2}}, \qquad \boldsymbol{\beta} = \frac{\mathcal{P}}{\mathcal{P}^0},$$

in accordance with our construction of Lorentz invariant $SO(3)$ spin fiber bundle [41], where γ plays a role of relativistic factor which in the limit of free electron reads $\gamma = (1 - \mathbf{v}^2/c^2)^{-1/2}$. Denote by β module of vector $\boldsymbol{\beta}$.

Consider a particle with initial momentum $\mathcal{P}(0)$ moving in the uniform magnetic field directed along z-axis, $\mathbf{B} = B\mathbf{e}_z$, of a laboratory Cartesian coordinate system defined by an orthonormal basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. Let $s_y(0) = \gamma^{-1}s^{(0)}\cos\phi$ and $s_x(0) = s^{(0)}\cos\phi$ be initial y and x components of BMT spin s^μ, respectively. From the system of the following algebraic equations $\mathcal{P}^2 - \frac{\mu e}{2c}(JF) + m^2c^2 = 0$, $s^2 = 3\hbar^2/4$, $\mathcal{P}s = 0$, one can define initial values of $\mathcal{P}^0(0)$, $s_z(0)$, $s_0(0)$.

The Hamiltonian equations of motions (60)–(62) written in the parametrization of physical time read

$$\frac{d\mathbf{x}}{dt} = \frac{c\boldsymbol{\beta}[1 + 2a(\mu-1)\gamma((\mathbf{Bs}) - (\mathbf{B}\boldsymbol{\beta})(\boldsymbol{\beta}\mathbf{s}))] - \frac{2ac(\mu-1)}{\gamma}\mathbf{B}(\boldsymbol{\beta}\mathbf{s})}{[1 + 2a(\mu-1)\gamma(\boldsymbol{\beta}^2(\mathbf{Bs}) - (\mathbf{B}\boldsymbol{\beta})(\boldsymbol{\beta}\mathbf{s}))]}, \tag{68}$$

$$\frac{d\mathcal{P}}{dt} = \frac{e(1 + 2a(\mu-1)\gamma((\mathbf{Bs}) - (\mathbf{B}\boldsymbol{\beta})(\boldsymbol{\beta}\mathbf{s})))[\boldsymbol{\beta}, \mathbf{B}]}{c[1 + 2a(\mu-1)\gamma(\boldsymbol{\beta}^2(\mathbf{Bs}) - (\mathbf{B}\boldsymbol{\beta})(\boldsymbol{\beta}\mathbf{s}))]}, \qquad \frac{d\mathcal{P}^0}{dt} = 0, \tag{69}$$

$$\frac{d\mathbf{s}}{dt} = \frac{e\mu[\mathbf{s}, \mathbf{B}]}{\mathcal{P}^0[1 + 2a(\mu-1)\gamma(\boldsymbol{\beta}^2(\mathbf{Bs}) - (\mathbf{B}\boldsymbol{\beta})(\boldsymbol{\beta}\mathbf{s}))]}$$
$$+ \frac{e(\mu-1)(\mathbf{s}, \mathcal{P}, \mathbf{B})\mathcal{P}[1 - 2a\gamma((\mathbf{Bs}) - (\mathbf{B}\boldsymbol{\beta})(\boldsymbol{\beta}\mathbf{s}))]}{\mathcal{P}^2\mathcal{P}^0[1 + 2a(\mu-1)\gamma(\boldsymbol{\beta}^2(\mathbf{Bs}) - (\mathbf{B}\boldsymbol{\beta})(\boldsymbol{\beta}\mathbf{s}))]}, \tag{70}$$

where $[\mathbf{s}, \mathbf{B}]$ and $(\mathbf{s}, \mathcal{P}, \mathbf{B})$ mean the vector and mixed product of 3-dimensional vectors.

From (69) follows the following set of integrals of motion: $\mathcal{P}^0 = \text{const}$, $\mathcal{P}^2 = \text{const}$, $\gamma = \text{const}$, $\boldsymbol{\beta}^2 = \text{const}$ and $(\mathcal{P}\mathbf{B}) = \text{const}$. To simplify our calculations we assume without loosing
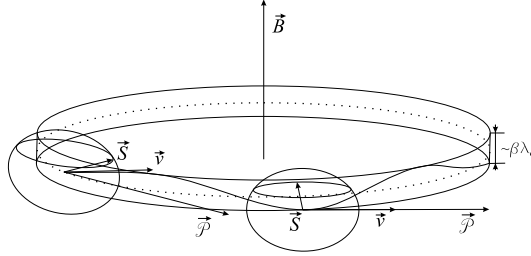
Fig. 1. Momentum, velocity, spin and trajectory of a charged spinning particle in the uniform magnetic field.

generality that the initial vector of momentum is orthogonal to magnetic field, $(\mathcal{P}\mathbf{B}) = 0$. Indeed, other values of $(\mathcal{P}\mathbf{B})$ can be obtained by boosts along $\mathbf{B}$ which do not modify electromagnetic tensor ($\mathbf{B}' = \mathbf{B}, \mathbf{E}' = \mathbf{E} = 0$). Choice $(\mathcal{P}\mathbf{B}) = 0$ implies $(\mathbf{B}\mathbf{s}) = \text{const}$. It is convenient to introduce the following constants (frequencies)

$$\Omega_p = \frac{eB(1 + 2a(\mu - 1)\gamma(\mathbf{B}\mathbf{s}))}{\mathcal{P}^0(1 + 2a(\mu - 1)\gamma\beta^2(\mathbf{B}\mathbf{s}))}, \tag{71}$$

$$\Omega_s' = \frac{eB(\mu - 1)\mathcal{P}^0(1 - 2a\gamma(\mathbf{B}\mathbf{s}))}{\mathcal{P}^2(1 + 2a(\mu - 1)\gamma\beta^2(\mathbf{B}\mathbf{s}))}. \tag{72}$$

Then, solution of equations of motion reads

$$\mathcal{P}(t) = |\mathcal{P}(0)|(\mathbf{e}_x \cos(\Omega_p t) + \mathbf{e}_y \sin(\Omega_p t)), \tag{73}$$

$$\mathbf{s}(t) = \mathbf{e}_x s^{(0)}\left[\cos\left(\frac{\Omega_s'}{\gamma}t + \phi\right)\cos(\Omega_p t) - \frac{1}{\gamma}\sin\left(\frac{\Omega_s'}{\gamma}t + \phi\right)\sin(\Omega_p t)\right]$$

$$+ \mathbf{e}_y s^{(0)}\left[\cos\left(\frac{\Omega_s'}{\gamma}t + \phi\right)\sin(\Omega_p t) + \frac{1}{\gamma}\sin\left(\frac{\Omega_s'}{\gamma}t + \phi\right)\cos(\Omega_p t)\right]$$

$$+ \mathbf{e}_z s_z(0), \tag{74}$$

$$\mathbf{x}(t) = \mathbf{x}_c + \frac{c\mathcal{P}(t)}{eB} - \frac{2ac|\mathcal{P}(0)|s^{(0)}}{e\gamma^2(1 - 2a\gamma(\mathbf{B}\mathbf{s}))}\mathbf{e}_z \sin\left(\frac{\Omega_s'}{\gamma}t + \phi\right), \tag{75}$$

where vector $\mathbf{x}_c$ defines the center of circle. Note that the angular velocity of precession of vector $\mathbf{s}$ around $\mathbf{B}$ is time-dependent. Nevertheless, the helicity $(\mathcal{P}\mathbf{s})$ changes with the constant rate, Ω_s'/γ. Indeed, $(\mathcal{P}\mathbf{s}) = s^{(0)}|\mathcal{P}^{(0)}|\cos(\frac{\Omega_s'}{\gamma}t + \phi)$.

Trajectory $\mathbf{x}(t)$ represents sum of two motions: circular motion in the plane orthogonal to $\mathbf{B}$ and oscillations along $\mathbf{B}$. These oscillations accompany variations of the helicity. The amplitude of oscillations along $\mathbf{B}$, $\Delta z = |\mathcal{P}(0)|s^{(0)}/[e\gamma^2(1 - 2a\gamma(\mathbf{B}\mathbf{s}))] \lesssim \beta\lambda_C$, less than the Compton wave-length. The trajectory of the particle is shown schematically in Fig. 1. In the case of usual magnetic moment $\mu = 1$, helicity is an integral of motion, additional oscillations vanish and the particle moves along circular trajectory in the plane orthogonal to the magnetic field (dotted line in Fig. 1).

Oscillations of the trajectory along $\mathbf{B}$ with the amplitude of Compton wavelength, appearing in our model in the case of anomalous magnetic moment, $\mu \neq 1$, can be called magnetic *Zitter-bewegung*. When $\mu \neq 1$, the velocity and canonical momentum are non-collinear [24]. As we have started from Lagrangian variational problem, we have explicit relation between velocity and canonical momentum. This allows us to exclude the canonical momentum from our equations, see Eq. (56). The result is an additional spin–orbit interaction, $F'\dot{x}$, instead of $F\dot{x}$. Hence

the magnetic *Zitterbewegung* appears due to the modification of Lorentz force for the spinning particle.

Magnetic *Zitterbewegung* leads also to the corrections of the angular velocity of orbital motion Ω_p given by

$$\Omega_p \approx \frac{eB}{\gamma mc}\left(1 + \frac{e\gamma(\mu-1)}{m^2c^3}(\mathbf{sB}) + o(\hbar, \mu-1, B)\right), \tag{76}$$

where $eB/(\gamma mc)$ is the angular velocity of spinless or BMT particle. Frequency of helicity variations also corrected by high-order terms

$$\frac{\Omega'_s}{\gamma} = (\mu-1)\frac{eB}{mc}\left(1 - \frac{e\gamma}{m^2c^3}\big(1 + (\mu-1)\beta^2\big)(\mathbf{sB}) + o(\hbar, \mu-1, B)\right), \tag{77}$$

from the value $(\mu-1)\frac{eB}{mc} = (\frac{g}{2}-1)\frac{eB}{mc}$ computed by Bargmann, Michel and Telegdi [23]. The corrections are small, and for the experiments discussed by them, our equations give practically the same results. Therefore our model is compatible with these experiments. Probably, other artificial physical situations may be realized, where the corrections could become notable. For instance, this may be the case of quasiparticles with large magnetic moment [20].

5. Conclusions

In this work we have presented solution to the problem which has been posed by Frenkel in 1926. He noticed that search for variational formulation which takes into account the spin-tensor constraint $J^{\mu\nu}\dot{x}_\nu = 0$ represents rather non-trivial problem. He found equations of motion consistent with this condition in the approximation $O^3(J, F, \partial F)$, and when anomalous magnetic moment vanishes, $\mu = 1$. We have found Lagrangian action (13) for charged spinning particle which implies all the desired constraints and equations of motion without approximations. They remain consistent for any value of magnetic moment and for an arbitrary electromagnetic background. Besides, due to the constraints (3)–(5), our action guarantees the right number of both spacial and spin degrees of freedom. In the above mentioned approximations, our equations coincide with those of Frenkel. In the recent work [45], we also demonstrated that the classical spinning particle has an expected behavior in arbitrary curved background.

With the Lagrangian and Hamiltonian formulations at hands, we can unambiguously construct quantum mechanics of the spinning particle and establish its relation with the Dirac equation. For the free theory, this has been done in the work [42]. We showed that this gives one-particle sector of the Dirac equation. Due to the second-class constraints (5), the positions x_i obey to classical brackets with non-vanishing right hand side, see (A.1). So, in the Dirac theory they realized by non-commutative operators which we identified with Pryce (d) center-of-mass [1]. Since namely x_i has an expected behavior (56) as the position of spinning particle in classical interacting theory, our model argue in favor of covariant Pryce (d) operator as the position operator of Dirac theory.

In resume, we have constructed variational formulation for relativistic spin one-half particle which is self consistent and has reasonable behavior on both classical and quantum level.

As we have seen, interaction necessarily modifies some basic relations of the model. In the free theory the conjugated momentum is proportional to velocity, $p^\mu \sim \dot{x}^\mu$ and the Frenkel condition holds. This is no more true in interacting theory. The Frenkel condition turns into the Pirani condition, $J^{\mu\nu}\mathcal{P}_\nu = 0$, where the canonical momentum is not collinear to velocity. The advantage

of Lagrangian formulation is that this gives exact relation between them (see also Eqs. (41) and (43))

$$\dot{x}^\mu = g_1\big(T^\mu{}_\nu \mathcal{P}^\nu + Y^\mu\big), \qquad T^{\mu\nu} = \eta^{\mu\nu} + O(\mu - 1, J), \qquad Y^\mu = O(\partial F, J). \qquad (78)$$

Only when $\mu = 1$ and $\partial_\alpha F_{\mu\nu} = 0$, the interacting and free theory have the same structure. To resume, what happens in general case, let us consider our Hamiltonian equations with $Y^\mu = 0$: $\dot{x}^\mu = g_1 T^\mu{}_\nu \mathcal{P}^\nu$, $\dot{\mathcal{P}}^\mu = \frac{g_1 e}{c}(F\mathcal{P})^\mu$, and compare them with the standard expressions $\dot{x}^\mu = g_1 \mathcal{P}^\mu$, $\dot{\mathcal{P}}^\mu = \frac{g_1 e}{c}(F\mathcal{P})^\mu$. Due to $T^\mu{}_\nu$, excluding $\mathcal{P}$ from our equations, we obtain extra contributions to the standard expression for the Lorentz force, $\ddot{x}^\mu = \frac{g_1 e}{c}(F\dot{x})^\mu + O(J)$. So the modification (78) mean that complete theory yields an extra spin–orbit interaction as compared with the approximate Frenkel and BMT equations.

We studied possible effects of this spin–orbit interaction in the case of uniform magnetic fields. The exact analytical solution was obtained. We have found 4 independent integrals of motion $\mathcal{P}^0$, $\mathcal{P}^2$, $(\mathcal{P}\mathbf{B})$ s_z. These integrals commute with respect to Dirac brackets, and hence, form a complete set of integrals in involution, providing integrability of our Hamiltonian system with 8 degrees of freedom. Analytical solution shows that besides oscillations of the helicity first calculated by Bargmann, Michel and Telegdi, the particle with anomalous magnetic moment experiences an effect of magnetic *Zitterbewegung* of the trajectory. Usual circular motion in the plane orthogonal to **B** is perturbed by slow oscillations along **B** with the amplitude of order of Compton wavelength. The Larmor frequency (76) and the frequency of helicity oscillations (77) are also shifted by small corrections. It would be interesting to construct an experiment which could detect these possible corrections, for instance due to resonance effects. Another possibility is an artificial simulation of a point-like system with spin and a large anomalous magnetic moment. This could be inspired by simulations of *Zitterbewegung* itself with a trapped ions [15].

Acknowledgements

Appendix A. Dirac brackets

We construct Dirac brackets that take into account the second-class pairs T_3, T_4, T_5 and T_6. We will calculate them iteratively in the case of arbitrary electromagnetic background. Then Dirac brackets of the free theory can be obtained by substitution $F^{\mu\nu} = 0$. We start from the pair of second class constraints is T_6 and T_7,

$$\triangle_{67} = \{T_6, T_7\} = \mathcal{P}^2 + \frac{e}{4c} F_{\mu\nu} J^{\mu\nu}.$$

At the constraint surface $\triangle_{67} = -M^2 c^2$. The Poisson brackets of initial variables with constraints T_6 and T_7 are given in Table 2. Using Table 2 we calculate the Dirac brackets of basic variables with respect to constraints T_6 and T_7

$$\{Q_1, Q_2\}_{67} = \{Q_1, Q_2\} + \frac{1}{\triangle_{67}}\big(\{Q_1, T_6\}\{T_7, Q_2\} - \{Q_1, T_7\}\{T_6, Q_2\}\big).$$

Table 2
Constraints vs. variables.

$\{,\}$	x^μ	$\mathcal{P}^\mu$	π^μ	ω^μ	$J^{\mu\nu}$
$T_6 = \mathcal{P}\omega$	$-\omega^\mu$	$-\frac{e}{c} F^{\mu\nu}\omega_\nu$	$\mathcal{P}^\mu$	0	$2(\omega^\mu\mathcal{P}^\nu - \omega^\nu\mathcal{P}^\mu)$
$T_7 = \mathcal{P}\pi$	$-\pi^\mu$	$-\frac{e}{c} F^{\mu\nu}\pi_\nu$	0	$-\mathcal{P}^\mu$	$2(\pi^\mu\mathcal{P}^\nu - \pi^\nu\mathcal{P}^\mu)$

The brackets read

$$\left\{x^\mu, x^\nu\right\}_{67} = \frac{-J^{\mu\nu}}{2\triangle_{67}}, \tag{A.1}$$

$$\left\{x^\mu, \mathcal{P}^\nu\right\}_{67} = \eta^{\mu\nu} + \frac{e}{2c\triangle_{67}} J^{\mu\alpha} F_\alpha{}^\nu \equiv T_{(0)}^{\mu\nu}, \tag{A.2}$$

$$\left\{x^\mu, \omega^\nu\right\}_{67} = \frac{-\omega^\mu\mathcal{P}^\nu}{\triangle_{67}}, \tag{A.3}$$

$$\left\{x^\mu, \pi^\nu\right\}_{67} = \frac{-\pi^\mu\mathcal{P}^\nu}{\triangle_{67}}, \tag{A.4}$$

$$\left\{x^\mu, J^{\alpha\beta}\right\}_{67} = \frac{1}{\triangle_{67}}\left(J^{\mu\alpha}\mathcal{P}^\beta - J^{\mu\beta}\mathcal{P}^\alpha\right), \tag{A.5}$$

$$\left\{\omega^\mu, \omega^\nu\right\}_{67} = 0, \tag{A.6}$$

$$\left\{\pi^\mu, \pi^\nu\right\}_{67} = 0, \tag{A.7}$$

$$\left\{\omega^\mu, \pi^\nu\right\}_{67} = \eta^{\mu\nu} - \frac{\mathcal{P}^\mu\mathcal{P}^\nu}{\triangle_{67}} \equiv G^{\mu\nu}, \tag{A.8}$$

$$\left\{\omega^\mu, J^{\alpha\beta}\right\}_{67} = 2\left(\omega^\alpha G^{\mu\beta} - \omega^\beta G^{\mu\alpha}\right), \tag{A.9}$$

$$\left\{\pi^\mu, J^{\alpha\beta}\right\}_{67} = 2\left(\pi^\alpha G^{\mu\beta} - \pi^\beta G^{\mu\alpha}\right), \tag{A.10}$$

$$\left\{J^{\mu\nu}, J^{\alpha\beta}\right\}_{67} = 2\left(G^{\mu\alpha} J^{\nu\beta} - G^{\mu\beta} J^{\nu\alpha} - G^{\nu\alpha} J^{\mu\beta} + G^{\nu\beta} J^{\mu\alpha}\right), \tag{A.11}$$

$$\left\{\mathcal{P}^\mu, \mathcal{P}^\nu\right\}_{67} = \frac{e}{c} F^{\mu\nu} + \frac{e^2}{2\triangle_{67}c^2}(FJF)^{\mu\nu} = \frac{e}{c} F^\mu{}_\alpha T_{(0)}^{\alpha\nu}, \tag{A.12}$$

$$\left\{\mathcal{P}^\mu, \omega^\nu\right\}_{67} = -\frac{e}{\triangle_{67}c} F^{\mu\alpha}\omega_\alpha\mathcal{P}^\nu, \tag{A.13}$$

$$\left\{\mathcal{P}^\mu, \pi^\nu\right\}_{67} = -\frac{e}{\triangle_{67}c} F^{\mu\alpha}\pi_\alpha\mathcal{P}^\nu, \tag{A.14}$$

$$\left\{\mathcal{P}^\mu, J^{\alpha\beta}\right\}_{67} = -\frac{e}{\triangle_{67}c} F^{\mu\nu}\left(\mathcal{P}^\alpha J_\nu{}^\beta - \mathcal{P}^\beta J_\nu{}^\alpha\right). \tag{A.15}$$

We have defined define tensor $G^{\mu\nu}$ as the Dirac bracket of spin variables ω^μ and π^ν. Besides, $T_{(0)}^{\mu\nu} = T^{\mu\nu}(\mu = 0)$, where $T^{\mu\nu} = \eta^{\mu\nu} - \frac{e(\mu-1)}{2c\triangle_{67}}(JF)^{\mu\nu}$.

On the next step we calculate Dirac brackets for the pair $\{T_4, T_5\}_{67} = 2(T_4 + a_4)$,

$$\{Q_1, Q_2\}_{4567} = \{Q_1, Q_2\}_{67} + \frac{1}{2\omega^2}\left(\{Q_1, T_4\}_{67}\{T_5, Q_2\}_{67} - \{Q_1, T_5\}_{67}\{T_4, Q_2\}_{67}\right).$$

The Dirac brackets $\{,\}_{67}$ of initial variables with T_4 and T_5 are given in Table 3.

From Table 3 it is seen that variables x^μ, $\mathcal{P}^\mu$, $J^{\mu\nu}$ have vanishing Dirac brackets $\{,\}_{67}$ with constraints T_4 and T_5. Therefore, new Dirac brackets $\{,\}_{4567}$ coincide with old Dirac brackets

Table 3
Constraints vs. variables.

$\{,\}_{67}$	x^μ	$\mathcal{P}^\mu$	π^μ	ω^μ	$J^{\mu\nu}$
$T_4 = \omega^2 - a_4$	0	0	$2\omega^\mu$	0	0
$T_5 = \omega\pi$	0	0	π^μ	$-\omega^\mu$	0

$\{,\}_{67}$ when at least one of arguments is a function $Z(x^\mu, \mathcal{P}^\mu, J^{\mu\nu})$ of variables x^μ, $\mathcal{P}^\mu$ and $J^{\mu\nu}$ only,

$$\{Z, Q\}_{4567} = \{Z, Q\}_{67}, \qquad Z = Z(x^\mu, \mathcal{P}^\mu, J^{\mu\nu}). \tag{A.16}$$

We omit subscripts of brackets, so that $\{,\}$ means $\{,\}_{4567}$. The only modification in the Dirac brackets comes from the basic variables in the spin sector

$$\{\omega^\mu, \omega^\nu\} = 0, \tag{A.17}$$

$$\{\omega^\mu, \pi^\nu\} = \eta^{\mu\nu} - \frac{\mathcal{P}^\mu \mathcal{P}^\nu}{\triangle_{67}} - \frac{\omega^\mu \omega^\nu}{\omega^2}, \tag{A.18}$$

$$\{\pi^\mu, \pi^\nu\} = \frac{-J^{\mu\nu}}{2\omega^2}. \tag{A.19}$$

Thus the complete list of Dirac brackets $\{,\}$ consist of expressions (A.1)–(A.5), (A.17)–(A.19) and (A.9)–(A.15).

Now the second class constraints can be put equal to zero, therefore we can rewrite the Hamiltonian as follows

$$H = \frac{g_1}{2}\left(\mathcal{P}^2 - \frac{e\mu}{2c}(FJ) + m^2 c^2\right). \tag{A.20}$$

The constraint T_3 can also be excluded since it has zero Dirac brackets with T_1 and with all spin-plane invariant variables of the theory. This Hamiltonian generates evolution

$$\dot{x}^\mu = \{x^\mu, H\}, \qquad \dot{\mathcal{P}}^\mu = \{\mathcal{P}^\mu, H\}, \qquad \dot{J}^{\mu\nu} = \{J^{\mu\nu}, H\}.$$

To check consistency of our calculations, let us obtain equations of motion using the Dirac brackets $\{,\}$ (and taking into account that at the constrained surface $\triangle_{67} = -M^2 c^2 = -m^2 c^2 - \frac{e(2\mu+1)}{4c^3}(FJ))$.

Equation for coordinate reads

$$\dot{x}^\mu = \frac{g_1}{2}\left\{x^\mu, \mathcal{P}^2 - \frac{e\mu}{2c}(FJ)\right\}$$

$$= g_1\{x^\mu, \mathcal{P}^\nu\}\mathcal{P}_\nu - g_1 \frac{e\mu}{4c}\{x^\mu, J^{\alpha\beta}\}F_{\alpha\beta} - g_1 \frac{e\mu}{4c}\{x^\mu, F^{\alpha\beta}\}J_{\alpha\beta}$$

$$= g_1\left(\eta^{\mu\nu} - \frac{e}{2M^2 c^3}J^{\mu\alpha}F_\alpha{}^\nu\right)\mathcal{P}_\nu - \frac{g_1 e\mu}{4M^2 c^3}(\mathcal{P}^\alpha J^{\mu\beta} - \mathcal{P}^\beta J^{\mu\alpha})F_{\alpha\beta}$$

$$\quad - \frac{g_1 e\mu}{8M^2 c^3}J^{\mu\rho}\partial_\rho(FJ)$$

$$= g_1\left(\eta^{\mu\nu} - \frac{e}{2M^2 c^3}J^{\mu\alpha}F_\alpha{}^\nu\right)\mathcal{P}_\nu + g_1 \frac{e\mu}{2M^2 c^3}J^{\mu\beta}F_{\beta\alpha}\mathcal{P}^\alpha - g_1 \frac{e\mu}{8M^2 c^3}J^{\mu\rho}\partial_\rho(FJ)$$

$$= g_1\left(\eta^{\mu\nu} + \frac{e(\mu-1)}{2M^2 c^3}J^{\mu\alpha}F_\alpha{}^\nu\right)\mathcal{P}_\nu - g_1 \frac{e\mu}{8M^2 c^3}J^{\mu\rho}\partial_\rho(FJ) = g_1 u^\mu.$$

Equation for momentum reads

$$
\begin{aligned}
\dot{\mathcal{P}}^\mu &= \frac{g_1}{2}\left\{\mathcal{P}^\mu, \mathcal{P}^2 - \frac{e\mu}{2c}(FJ)\right\} \\
&= g_1\{\mathcal{P}^\mu, \mathcal{P}^\nu\}\mathcal{P}_\nu - g_1\frac{e\mu}{4c}\{\mathcal{P}^\mu, J^{\alpha\beta}\}F_{\alpha\beta} - g_1\frac{e\mu}{4c}\{\mathcal{P}^\mu, x^\rho\}\partial_\rho F^{\alpha\beta}J_{\alpha\beta} \\
&= g_1\left(\frac{e}{c}F^{\mu\nu} + \frac{e^2}{2M^2c^4}F^{\mu\alpha}F^{\nu\beta}J_{\alpha\beta}\right)\mathcal{P}_\nu + g_1\frac{e^2\mu}{2M^2c^4}F^{\mu\nu}\mathcal{P}^\alpha J^\beta{}_\nu F_{\alpha\beta} \\
&\quad + g_1\frac{e\mu}{4c}\left(\eta^{\mu\rho} - \frac{e}{2M^2c^3}J^{\rho\alpha}F_\alpha{}^\mu\right)\partial_\rho(FJ) \\
&= g_1\frac{e}{c}F^{\mu\alpha}\left(\left(\eta_\alpha{}^\nu + \frac{e(\mu-1)}{2M^2c^3}J_{\alpha\beta}F^{\beta\nu}\right)\mathcal{P}_\nu - \frac{e\mu}{8M^2c^3}J_{\alpha\rho}\partial^\rho(FJ)\right) + g_1\frac{e\mu}{4c}\partial^\mu(FJ) \\
&= g_1\frac{e}{c}F^{\mu\alpha}u_\alpha + g_1\frac{e\mu}{4c}\partial^\mu(FJ).
\end{aligned}
$$

Equation for spin-tensor reads

$$
\begin{aligned}
j^{\alpha\beta} &= \frac{g_1}{2}\left\{J^{\alpha\beta}, \mathcal{P}^2 - \frac{e\mu}{2c}(FJ)\right\} \\
&= g_1\{J^{\alpha\beta}, \mathcal{P}^\mu\}\mathcal{P}_\mu - g_1\frac{e\mu}{4c}\{J^{\alpha\beta}, J^{\mu\nu}\}F_{\mu\nu} - g_1\frac{e\mu}{4c}\{J^{\alpha\beta}, x^\mu\}\partial_\mu(FJ) \\
&= g_1\frac{e}{M^2c^3}F^{\mu\nu}\left(\mathcal{P}^\alpha J^\beta{}_\nu - \mathcal{P}^\beta J^\alpha{}_\nu\right)\mathcal{P}_\mu + g_1\frac{e\mu}{c}\left(G^{\mu\alpha}J^{\nu\beta} - G^{\mu\beta}J^{\nu\alpha}\right)F_{\mu\nu} \\
&\quad + g_1\frac{e\mu}{4c}\frac{1}{M^2c^2}\left(\mathcal{P}^\alpha J^{\mu\beta} - \mathcal{P}^\beta J^{\mu\alpha}\right)\partial_\mu(FJ) \\
&= -g_1\frac{e}{M^2c^3}\mathcal{P}^{[\alpha}J^{\beta]}{}_\nu F^{\nu\mu}\mathcal{P}_\mu + g_1\frac{e\mu}{c}\left(F^\alpha{}_\nu J^{\nu\beta} - F^\beta{}_\nu J^{\nu\alpha}\right) \\
&\quad + g_1\frac{e\mu}{M^2c^3}\mathcal{P}^{[\alpha}J^{\beta]}{}_\nu F^{\nu\mu}\mathcal{P}_\mu + g_1\frac{e\mu}{4c}\frac{1}{M^2c^2}\left(\mathcal{P}^\alpha J^{\mu\beta} - \mathcal{P}^\beta J^{\mu\alpha}\right)\partial_\mu(FJ) \\
&= g_1\frac{e}{c}\Bigg(\mu\left(F^\alpha{}_\nu J^{\nu\beta} - F^\beta{}_\nu J^{\nu\alpha}\right) + \frac{(\mu-1)}{M^2c^2}\mathcal{P}^{[\alpha}J^{\beta]}{}_\nu F^{\nu\mu}\mathcal{P}_\mu \\
&\quad - \frac{1}{4M^2c^2}\mathcal{P}^{[\alpha}J^{\beta]}{}_\mu\partial^\mu(FJ)\Bigg).
\end{aligned}
$$

The Hamiltonian is proportional to the first class constraint T_1. Therefore equations of motion contain arbitrary function $g_1(\tau)$ which is related with reparametrization invariance of the model. To obtain unambiguous equations of evolution we can impose the gauge $x^0 = c\tau$. The gauge is often called canonical gauge. Constraint T_1 together with this condition form a pair of second class constraints. We have

$$
\{x^0 - c\tau, T_1\} = 2u^0,
$$

and Dirac brackets in the canonical gauge read

$$
\{Q_1, Q_2\}_\tau = \{Q_1, Q_2\} + \frac{1}{2u^0}\big(\{Q_1, G_1\}\{T_1, Q_2\} - \{Q_1, T_1\}\{G_1, Q_2\}\big).
$$

The Dirac brackets of constraint T_1 and canonical gauge G_1 with physical variables are given in Table 4. There compact notations

Table 4
Constraints vs. variables (canonical gauge).

$\{,\}_{4567}$	x^μ	$\mathcal{P}^\mu$	ω^μ	π^μ	$J^{\mu\nu}$
$T_1 = \mathcal{P}^2 - \frac{e\mu}{2c}(FJ)$	$-2u^\mu$	$\frac{-2}{g_1}\dot{\mathcal{P}}^\mu$	$\frac{2C\mathcal{P}^\mu}{\triangle_{67}} - \frac{2e\mu}{c}(F\omega)^\mu$	$\frac{2D\mathcal{P}^\mu}{\triangle_{67}} - \frac{2e\mu}{c}(F\pi)^\mu$	$\frac{-2}{g_1}j^{\mu\nu}$
$G_1 = x^0 - c\tau$	$\frac{-1}{2\triangle_{67}}J^{0\mu}$	$T^{0\mu}_{(0)}$	$\frac{-\omega^0\mathcal{P}^\mu}{\triangle_{67}}$	$\frac{-\pi^0\mathcal{P}^\mu}{\triangle_{67}}$	$\frac{-1}{\triangle_{67}}J^{0[\nu}\mathcal{P}^{\mu]}$

$$\dot{\mathcal{P}}^\mu \equiv \frac{g_1}{2}\{\mathcal{P}^\mu, T_1\},$$

$$j^{\mu\nu} \equiv \frac{g_1}{2}\{J^{\mu\nu}, T_1\},$$

$$u^\mu \equiv \left(\eta^{\mu\nu} + \frac{e(\mu-1)}{2M^2c^3}J^{\mu\alpha}F_\alpha{}^\nu\right)\mathcal{P}_\nu - \frac{e\mu}{8M^2c^3}J^{\mu\rho}\partial_\rho(FJ).$$

$$C = -\frac{e}{c}(\mu-1)(\omega F\mathcal{P}) + \frac{e\mu}{4c}(\omega\partial)(JF),$$

$$D = -\frac{e}{c}(\mu-1)(\pi F\mathcal{P}) + \frac{e\mu}{4c}(\pi\partial)(JF),$$

were used.

The Dirac brackets which take into account the canonical gauge are as follow.
Spacial sector:

$$\{x^\mu, x^\nu\}_\tau = \frac{-1}{2u^0\triangle_{67}}\left(u^0 J^{\mu\nu} - u^\mu J^{0\nu} + u^\nu J^{0\mu}\right),$$

$$\{x^\mu, \mathcal{P}^\nu\}_\tau = \eta^{\mu\nu} - \frac{u^\mu}{u^0}\eta^{0\nu} + \frac{e}{2u^0c\triangle_{67}}\left(u^0 J^{\mu\alpha} - u^\mu J^{0\alpha} + u^\alpha J^{0\mu}\right)F_\alpha{}^\nu$$
$$\qquad - \frac{e\mu}{8u^0c\triangle_{67}}J^{0\mu}\partial^\nu(FJ),$$

$$\{\mathcal{P}^\mu, \mathcal{P}^\nu\}_\tau = \frac{e}{u^0c}\left(u^0 F^\mu{}_\alpha T^{\alpha\nu}_{(0)} - \frac{c}{eg_1}\dot{\mathcal{P}}^\mu T^{0\nu}_{(0)} + \frac{c}{eg_1}\dot{\mathcal{P}}^\mu T^{0\nu}_{(0)}\right). \tag{A.21}$$

Frenkel sector:

$$\{J^{\mu\nu}, J^{\alpha\beta}\}_\tau = \{J^{\mu\nu}, J^{\alpha\beta}\} - \frac{1}{g_1 u^0 \triangle_{67}}\left(j^{\mu\nu}J^{0[\beta}\mathcal{P}^{\alpha]} - j^{\alpha\beta}J^{0[\nu}\mathcal{P}^{\mu]}\right),$$

$$\{x^\mu, J^{\alpha\beta}\}_\tau = \frac{-1}{u^0\triangle_{67}}\left(u^\mu J^{0[\alpha}\mathcal{P}^{\beta]} - u^0 J^{\mu[\alpha}\mathcal{P}^{\beta]}\right) - \frac{1}{2u^0\triangle_{67}g_1}J^{0\mu}j^{\alpha\beta},$$

$$\{\mathcal{P}^\mu, J^{\alpha\beta}\}_\tau = \frac{e}{u^0c\triangle_{67}}F^\mu{}_\nu\left(u^0\mathcal{P}^{[\alpha}J^{\beta]\nu} - u^\nu\mathcal{P}^{[\beta}J^{\alpha]0}\right) - \frac{1}{u^0g_1}T^{0\mu}_{(0)}j^{\alpha\beta}$$
$$\qquad - \frac{e\mu}{4u^0c\triangle_{67}}\partial^\mu(FJ)\mathcal{P}^{[\beta}J^{\alpha]0}. \tag{A.22}$$

Basic spin variables:

$$\{\omega^\mu, \omega^\nu\}_\tau = -\frac{e\mu\omega^0}{2u^0c\triangle_{67}}\left(F^{\mu\alpha}\mathcal{P}^\nu - F^{\nu\alpha}\mathcal{P}^\mu\right)\omega_\alpha,$$

$$\{\omega^\mu, \pi^\nu\}_\tau = \eta^{\mu\nu} - \frac{\mathcal{P}^\mu\mathcal{P}^\nu}{\triangle_{67}}\left(1 + \frac{(\pi^0 C + \omega^0 D)}{u^0c\triangle_{67}}\right) - \frac{\omega^\mu\omega^\nu}{\omega^2}$$

$$- \frac{e\mu\omega^0}{u^0 c \triangle_{67}} \left(\pi^0 F^{\mu\alpha} \omega_\alpha \mathcal{P}^\nu - \omega^0 F^{\nu\alpha} \pi_\alpha \mathcal{P}^\mu \right),$$

$$\left\{ \pi^\mu, \pi^\nu \right\}_\tau = \frac{-J^{\mu\nu}}{2\omega^2} - \frac{e\mu\pi^0}{2u^0 c \triangle_{67}} \left(F^{\mu\alpha} \mathcal{P}^\nu - F^{\nu\alpha} \mathcal{P}^\mu \right) \pi_\alpha. \tag{A.23}$$

Other mixed brackets:

$$\left\{ x^\mu, \omega^\nu \right\}_\tau = \frac{-\omega^\mu \mathcal{P}^\nu}{\triangle_{67}} + \frac{1}{u^0} \left(\frac{J^{0\mu}}{2\triangle_{67}} \left(\frac{C\mathcal{P}^\nu}{\triangle_{67}} - \frac{e\mu}{c} (F\omega)^\nu \right) + \frac{u^\mu \omega^0 \mathcal{P}^\nu}{\triangle_{67}} \right),$$

$$\left\{ x^\mu, \pi^\nu \right\}_\tau = \frac{-\pi^\mu \mathcal{P}^\nu}{\triangle_{67}} + \frac{1}{u^0} \left(\frac{J^{0\mu}}{2\triangle_{67}} \left(\frac{D\mathcal{P}^\nu}{\triangle_{67}} - \frac{e\mu}{c} (F\pi)^\nu \right) + \frac{u^\mu \pi^0 \mathcal{P}^\nu}{\triangle_{67}} \right),$$

$$\left\{ \omega^\mu, J^{\alpha\beta} \right\}_\tau = 2 \left(\omega^\alpha G^{\mu\beta} - \omega^\beta G^{\mu\alpha} \right)$$
$$- \frac{1}{u^0 \triangle_{67}} \left(\omega^0 \mathcal{P}^\mu \frac{j^{\alpha\beta}}{g_1} + \left(\frac{C\mathcal{P}^\mu}{\triangle_{67}} - \frac{e\mu}{c} (F\omega)^\mu \right) J^{0[\beta} \mathcal{P}^{\alpha]} \right),$$

$$\left\{ \pi^\mu, J^{\alpha\beta} \right\}_\tau = 2 \left(\pi^\alpha G^{\mu\beta} - \pi^\beta G^{\mu\alpha} \right)$$
$$- \frac{1}{u^0 \triangle_{67}} \left(\pi^0 \mathcal{P}^\mu \frac{j^{\alpha\beta}}{g_1} + \left(\frac{D\mathcal{P}^\mu}{\triangle_{67}} - \frac{e\mu}{c} (F\pi)^\mu \right) J^{0[\beta} \mathcal{P}^{\alpha]} \right),$$

$$\left\{ \mathcal{P}^\mu, \omega^\nu \right\}_\tau = - \frac{e}{\triangle_{67} c} F^{\mu\alpha} \omega_\alpha \mathcal{P}^\nu - \frac{1}{u^0} \left(T^{0\mu}_{(0)} \left(\frac{C\mathcal{P}^\nu}{\triangle_{67}} - \frac{e\mu}{c} (F\omega)^\nu \right) - \frac{\dot{\mathcal{P}}^\mu}{g_1} \frac{\omega^0 \mathcal{P}^\nu}{\triangle_{67}} \right),$$

$$\left\{ \mathcal{P}^\mu, \pi^\nu \right\}_\tau = - \frac{e}{\triangle_{67} c} F^{\mu\alpha} \pi_\alpha \mathcal{P}^\nu$$
$$- \frac{1}{u^0} \left(T^{0\mu}_{(0)} \left(\frac{D\mathcal{P}^\nu}{\triangle_{67}} - \frac{e\mu}{c} (F\pi)^\nu \right) - \frac{\dot{\mathcal{P}}^\mu}{g_1} \frac{\pi^0 \mathcal{P}^\nu}{\triangle_{67}} \right). \tag{A.24}$$

In the free theory the algebra of Dirac brackets simplifies significantly. In this case $\mathcal{P}^\mu = p^\mu$, $u^\mu = p^\mu$, $\dot{j}^{\mu\nu} = \dot{\mathcal{P}}^\mu = 0$, $\triangle_{67} = p^2$, and in an arbitrary parametrization τ, we have the following brackets:
Basic variables of spin:

$$\left\{ \omega^\mu, \omega^\nu \right\} = 0, \qquad \left\{ \omega^\mu, \pi^\nu \right\} = g^{\mu\nu} - \frac{\omega^\mu \omega^\nu}{\omega^2}, \qquad \left\{ \pi^\mu, \pi^\nu \right\} = - \frac{1}{2\omega^2} J^{\mu\nu}; \tag{A.25}$$

$$\left\{ x^\mu, \omega^\nu \right\} = - \frac{\omega^\mu p^\nu}{p^2}, \qquad \left\{ x^\mu, \pi^\nu \right\} = - \frac{\pi^\mu p^\nu}{p^2}; \tag{A.26}$$

Spacial sector:

$$\left\{ x^\mu, x^\nu \right\} = - \frac{1}{2p^2} J^{\mu\nu}, \qquad \left\{ x^\mu, p^\nu \right\} = \eta^{\mu\nu}, \qquad \left\{ p^\mu, p^\nu \right\} = 0. \tag{A.27}$$

Frenkel sector:

$$\left\{ J^{\mu\nu}, J^{\alpha\beta} \right\} = 2 \left(g^{\mu\alpha} J^{\nu\beta} - g^{\mu\beta} J^{\nu\alpha} - g^{\nu\alpha} J^{\mu\beta} + g^{\nu\beta} J^{\mu\alpha} \right), \tag{A.28}$$

$$\left\{ x^\mu, J^{\alpha\beta} \right\} = \frac{1}{p^2} \left(J^{\mu\alpha} p^\beta - J^{\mu\beta} p^\alpha \right). \tag{A.29}$$

BMT-sector: take $s^\mu = \frac{1}{4\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} p_\nu J_{\alpha\beta}$, then

$$\{s^{\mu}, s^{\nu}\} = -\frac{1}{\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} p_{\alpha} s_{\beta} = \frac{1}{2} J^{\mu\nu}, \tag{A.30}$$

$$\{x^{\mu}, s^{\nu}\} = -\frac{s^{\mu} p^{\nu}}{p^2} = -\frac{1}{4\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} J_{\alpha\beta} - \frac{p^{\mu} s^{\nu}}{p^2}. \tag{A.31}$$

Other Dirac brackets vanish. In Eqs. (A.25) and (A.28) it has been denoted

$$g^{\mu}{}_{\nu} \equiv \delta^{\mu}{}_{\nu} - \frac{p^{\mu} p_{\nu}}{p^2}. \tag{A.32}$$

Together with $\tilde{g}^{\mu}{}_{\nu} \equiv \frac{p^{\mu} p_{\nu}}{p^2}$, this forms a pair of projectors

$$g + \tilde{g} = 1, \quad g^2 = g, \quad \tilde{g}^2 = \tilde{g}, \quad g\tilde{g} = 0. \tag{A.33}$$

The free Dirac brackets which take into account the canonical gauge are as follows. Basic variables of spin:

$$\{\omega^{\mu}, \omega^{\nu}\}_{\tau} = 0, \qquad \{\omega^{\mu}, \pi^{\nu}\}_{\tau} = g^{\mu\nu} - \frac{\omega^{\mu} \omega^{\nu}}{\omega^2}, \qquad \{\pi^{\mu}, \pi^{\nu}\}_{\tau} = -\frac{1}{2\omega^2} J^{\mu\nu}, \tag{A.34}$$

$$\{x^{\mu}, \omega^{\nu}\}_{\tau} = \frac{-\omega^{\mu} p^{\nu}}{p^2} + \frac{\omega^0 p^{\mu} p^{\nu}}{p^0 p^2}, \tag{A.35}$$

$$\{x^{\mu}, \pi^{\nu}\}_{\tau} = \frac{-\pi^{\mu} p^{\nu}}{p^2} + \frac{\pi^0 p^{\mu} p^{\nu}}{p^0 p^2}. \tag{A.36}$$

Spacial sector:

$$\{x^{\mu}, x^{\nu}\}_{\tau} = \frac{-1}{2p^0 p^2} \left(p^0 J^{\mu\nu} - p^{\mu} J^{0\nu} + p^{\nu} J^{0\mu} \right), \tag{A.37}$$

$$\{x^{\mu}, p^{\nu}\}_{\tau} = \eta^{\mu\nu} - \frac{p^{\mu}}{p^0} \eta^{0\nu}, \quad \{p^{\mu}, p^{\nu}\} = 0. \tag{A.38}$$

Frenkel sector:

$$\{J^{\mu\nu}, J^{\alpha\beta}\}_{\tau} = 2 \left(g^{\mu\alpha} J^{\nu\beta} - g^{\mu\beta} J^{\nu\alpha} - g^{\nu\alpha} J^{\mu\beta} + g^{\nu\beta} J^{\mu\alpha} \right), \tag{A.39}$$

$$\{x^{\mu}, J^{\alpha\beta}\}_{\tau} = \frac{-1}{p^0 p^2} \left(p^{\mu} J^{0[\alpha} p^{\beta]} - p^0 J^{\mu[\alpha} p^{\beta]} \right). \tag{A.40}$$

BMT-sector:

$$\{s^{\mu}, s^{\nu}\}_{\tau} = -\frac{1}{\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} p_{\alpha} s_{\beta} = \frac{1}{2} J^{\mu\nu}, \tag{A.41}$$

$$\{x^{\mu}, s^{\nu}\}_{\tau} = -\frac{1}{4\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} J_{\alpha\beta} + \frac{p^{\mu}}{4p^0\sqrt{-p^2}} \epsilon^{0\nu\alpha\beta} J_{\alpha\beta} = \frac{(s^0 p^{\mu} p^{\nu} - s^{\mu} p^{\nu} p^0)}{p^0 p^2}. \tag{A.42}$$

Other Dirac brackets vanish.

Here we define Dirac brackets for all phase space variables. After transition to the Dirac brackets the second-class constraints can be used as strong equalities, therefore it is enough to consider Dirac brackets at the constraint surface only. Then, explicit form of the Dirac brackets depends on the choice of independent variables. For instance, in the free theory considered in the gauge of physical time we can present s^0_{BMT} and p^0 in terms of independent variables $\mathbf{s}_{BMT}$, $\mathbf{p}$, $\mathbf{x}$

$$s^0 = \frac{(\mathbf{s}\,\mathbf{p})}{\sqrt{\mathbf{p}^2 + (mc)^2}}, \qquad p^0 = \sqrt{\mathbf{p}^2 + (mc)^2}. \tag{A.43}$$

The non-vanishing Dirac brackets are

$$\{x^i, x^j\}_\tau = \frac{\epsilon^{ijk} s_k}{mcp^0}, \qquad \{x^i, p^j\}_\tau = \delta^{ij}, \qquad \{p^i, p^j\}_\tau = 0, \tag{A.44}$$

$$\{s^i, s^j\}_\tau = \frac{p^0}{mc} \epsilon^{ijk} \left(s_k - \frac{(\mathbf{s}\,\mathbf{p}) p_k}{p_0^2} \right), \tag{A.45}$$

$$\{x^i, s^j\}_D = \left(s^i - \frac{(\mathbf{s}\,\mathbf{p}) p^i}{p_0^2} \right) \frac{p^j}{(mc)^2}. \tag{A.46}$$

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