Original article

Method of corrections by higher order differences for Poisson equation with nonlocal boundary conditions

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Abstract

We consider the Bitsadze–Samarskii type nonlocal boundary value problem for Poisson equation in a unit square, which is solved by a difference scheme of second-order accuracy. Using this approximate solution, we correct the right-hand side of the difference scheme. It is shown that the solution of the corrected scheme converges at the rate $O(|h|^s)$ in the discrete $L_2$-norm provided that the solution of the original problem belongs to the Sobolev space with exponent $s \in [2, 4]$.

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1. Introduction

Finite difference method is a significant tool in the numerical solution of problems posed for differential equations. In order to minimize the amount of calculations it is desirable for the difference scheme to be sufficiently good on coarse meshes, i.e. to have high order accuracy. In the present work, for improving the accuracy of the approximate solution, we study two-stage finite difference method. We consider Bitsadze–Samarskii type nonlocal boundary value problem for Poisson’s equation.

At the first stage we solve the difference scheme $\Delta_h \tilde{U} = \varphi$, which has the second order of approximation. Using the solution $\tilde{U}$ the right-hand side of the difference scheme is corrected, $\Delta_h U = \varphi + R\tilde{U}$, and solved again on the same mesh.

This approach for some boundary value problems posed for Poisson and Laplace equations has been studied in Volkov’s papers (see, e.g. [1–3]), where the input data were chosen so as to ensure that the exact solution belongs to the Hölder class $C_{6,\lambda}(\bar{\Omega})$. 
For establishing the convergence we use the methodology of obtaining the compatible estimates of convergence rate of difference schemes. This methodology develops from the works of Samarskii, Lazarov and Makarov (see, e.g., [4–6]), and later in the works of other authors (see, e.g., [7,8]). For the elliptic problems such estimates have the form

\[ \| U - u \|^2_{W_2^s(\Omega)} \leq c|h|^{s-k} \| u \|^2_{W_2^s(\Omega)}, \quad s > k \geq 0, \]

where \( u \) is the solution of original problem, \( U \) is the approximate solution, \( k \) and \( s \) are integer and real numbers, respectively, \( W_2^k(\Omega) \) and \( W_2^s(\Omega) \) are the Sobolev norms on the set of functions with discrete and continuous arguments. Here and below \( c \) denotes a positive generic constant, independent of \( h \) and \( u \).

It is proved that the solution \( U \) of the corrected difference scheme converges at rate \( O(h^s) \) in the discrete \( L_2 \)-norm, when the exact solution belongs to the Sobolev space \( W_2^s, \ s \in [2, 4] \).

The generalization of the Bitsadze–Samarskii problem [9] was investigated by many authors (see, e.g., [10–13]). In [11] for a Poisson equation it is considered a difference scheme, which converges by the rate \( O(h^2) \) in the discrete \( W_2^2 \)-norm to the exact solution from the class \( C^2(\bar{\Omega}) \).

In [13] difference scheme is considered for a second order elliptic equation with variable coefficients and the compatible estimate of convergence rate in discrete \( W_2^1 \)-norm is obtained.

Results, analogous to those given in the present work, are obtained in [14] for the Dirichlet problem posed for an elliptic equation, and also in [15] for the mixed problem with third kind conditions.

One of the methods for obtaining compact high order approximations is the Mehrstellen method ("Mehrstellenverfahren"), defined by Collatz (see [16]). Instead of approximating only the left hand side of the differential equation, he proposes to take several points of the right hand side as well. In the case of two-dimensional problem, the differential operator is approximated on a 9-point stencil with the fourth order accuracy.

The advantage of the Mehrstellen schemes over ordinary (second order) accuracy schemes on a coarse grid is obvious.

The advantage of our method is:

(a) It needs to approximate the differential operator on minimally acceptable stencil (5-point stencil for a two-dimensional problem). Therefore, the condition number of this operator is better as compared with the Mehrstellen schemes, which is notable on a fine grid.

(b) It is a two-stage method, nevertheless it requires matrix inversion only once (on the second stage we change only the right-hand side of the equation, while the operator is kept unchanged).

(c) The method of correction is handy even in the case when construction of high precision schemes is impossible.

2. Statement of the problem and some auxiliary estimate

As usual, by symbol \( W_2^s(\Omega) \), \( s \geq 0 \) we denote the Sobolev space. For integer \( s \) the norm in \( W_2^s(\Omega) \) is given by formula

\[ \| u \|^2_{W_2^s(\Omega)} = \sum_{j=0}^{s} |u|^2_{W_2^j(\Omega)}, \quad |u|^2_{W_2^j(\Omega)} = \sum_{|\nu|=j} \| D^\nu u \|^2_{L_2(\Omega)}, \]

where \( D^\nu = \partial^{\nu_1}/(\partial x_1^{\nu_1}) \partial^{\nu_2}/(\partial x_2^{\nu_2}) \), \( \nu = (\nu_1, \nu_2) \) is multi-index with non-negative integer components, \( |\nu| = \nu_1 + \nu_2 \).

If \( s = \bar{s} + \bar{\varepsilon} \), where \( \bar{s} \) is an integer part of \( s \) and \( 0 < \varepsilon < 1 \), then

\[ \| u \|^2_{W_2^s(\Omega)} = \| u \|^2_{W_2^\bar{s}(\Omega)} + |u|^2_{W_2^\bar{\varepsilon}(\Omega)}, \]

where

\[ |u|_{W_2^\bar{\varepsilon}(\Omega)} = \sum_{|\nu|=\bar{s}} \int_{\Omega} \int_{\Omega} \frac{|D^\nu u(x) - D^\nu u(y)|^2}{|x-y|^{2+2\bar{\varepsilon}}} \, dx \, dy. \]

Particularly, for \( s = 0 \) we have \( W_2^0 = L_2 \).
Let \( \hat{\Omega} = \{(x_1, x_2) : 0 \leq x_\alpha \leq 1, \ \alpha = 1, 2\} \) be a unit square with a boundary \( \Gamma; \) \( \Gamma_0 = \Gamma \setminus \{(1, x_2) : 0 < x_2 < 1\}; \) \( \xi_k \) be fixed points from interval \((0, 1), 0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1. \) Denote \( \xi_0 = 0, \xi_{m+1} = 1. \)

Consider the problem

\[
\Delta u = f(x), \quad x \in \Omega, \quad u \big|_{\Gamma_0} = 0, \quad u(1, x_2) = \sum_{k=1}^{m} \alpha_k u(\xi_k, x_2), \quad 0 < x_2 < 1
\]

(1)

where the coefficients \( \alpha_k \) are real numbers satisfying conditions

\[
\kappa := \sum_{k=1}^{m} |\alpha_k|^{1/\xi_k} < 1.
\]

It was shown in [12] that, for \( f(x) \in L^2(\Omega, \rho), \) there exists a unique strong solution of problem (1) in the weighted Sobolev space \( W^s_2(\Omega, \rho). \) Throughout the following, we assume that the function \( f(x) \) provides the unique solvability of problem (1) in the \( W^s_2(\Omega), 2 \leq s \leq 4. \)

Consider the following grid domains in \( \hat{\Omega}: \)

\[
\tilde{\omega}_k = \{x_k = i_k h : i_k = 0, 1, \ldots, n, \ h = 1/n\}, \quad \omega_k = \tilde{\omega}_k \cap (0, 1), \omega_k^+ = \tilde{\omega}_k \cap (0, 1), \quad k = 1, 2, \quad \omega = \omega_1 \times \omega_2, \quad \tilde{\omega} = \tilde{\omega}_1 \times \tilde{\omega}_2, \quad \gamma_0 = \Gamma_0 \cap \tilde{\omega}.
\]

We assume that the points \( \xi_k \) coincide with grid nodes

\[
\xi_k = n_k h, \quad k = 1, 2, \ldots, m,
\]

where \( n_k \) are nonnegative integers \( 0 < n_1 < n_2 < \cdots < n_m < n. \) We suppose also that

\[
h/2 \leq 1 - \xi_m - v, \quad v = \text{const} > 0.
\]

For grid functions we define difference quotients in \( x_k \) directions as follows

\[
V_{x_k} = \frac{(V^{(+1)}) - V}{h}, \quad V_{\xi_k} = \frac{(V - V^{(-1)})}{h}
\]

where

\[
V = V(x), \quad V^{(+1)} = V(x_1 \pm h, x_2), \quad V^{(-1)} = V(x_1, x_2 \pm h).
\]

For functions, defined on \( \Omega, \) we need the following averaging operators:

\[
T_1 u(x) := \frac{1}{h^2} \int_{x_1-h_1}^{x_1+h_1} (h_1 - |x_1 - t_1|) u(t_1, x_2) \, dt_1.
\]

Analogously is defined operator \( T_2. \) Note that these operators commute and

\[
T_k \frac{\partial^2 u}{\partial x_k^2} = u_{\xi_k x_k}, \quad k = 1, 2.
\]

Define the following weight functions

\[
r(x_1) = 1 - x_1, \quad \rho(x_1) = 1 - x_1 - \sum_{k=1}^{m} \kappa \sigma_k \chi(\xi_k - x_1),
\]

where

\[
\sigma_k = \frac{|\alpha_k|}{\sqrt{\xi_k}}, \quad \chi(t) = \begin{cases} t, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}
\]

Let

\[
\tilde{r} = (r + r^{(-1)})/2, \quad \tilde{\rho} = (\rho + \rho^{(-1)})/2.
\]
Notice that the following inequality
\[(1 - \chi^2)r(x_1) \leq \rho(x_1) \leq r(x_1)\] holds.
Indeed, the right-hand side inequality is obvious. The left-hand side inequality can be verified as follows:
\[
\rho(x_1) = 1 - x_1 - \chi \sum_{k=j+1}^{m} \sigma_k(\xi_k - x_1) \geq \left(1 - \chi \sum_{k=j+1}^{m} \sigma_k\xi_k\right)(1 - x_1)
\]
\[
\geq (1 - \chi^2)(1 - x_1), \quad x_1 \in (\xi_j, \xi_{j+1}).
\]

**Remark.** Introduction of auxiliary (equivalent to \(r\)) weight function \(\rho\) gives possibility to state the positive definiteness of the difference scheme operator.

Let \(H = H(\omega)\) be the set of grid functions defined on \(\omega\) with the inner product and norm
\[
(U, V)_r = \sum_{x \in \omega} h^2r(x_1)U(x)V(x), \quad \|V\|_r = \|V\|_{L^2(\omega, r)} = (V, V)_r^{1/2}.
\]
Moreover, let
\[
(U, V) = \sum_{x \in \omega} h^2U(x)V(x), \quad \|V\| = (V, V)^{1/2}.
\]
Inner product and norm, involving \(\rho\) in index will make similar to the expression with index \(r\) sense.

Denote by \(\tilde{H} = \tilde{H}(\tilde{\omega})\) the set of grid functions \(V(x)\), given on \(\tilde{\omega}\) and satisfying conditions
\[
V(x) = 0, \quad x \in \gamma_0, \quad V(1, x_2) = \sum_{k=1}^{m} \alpha_k V(\xi_k, x_2), \quad x_2 \in \omega_2.
\]

**Lemma 1.** For each function, defined on mesh \(\tilde{\omega}\), which equals zero on \(x_1 = 0\) and satisfies the nonlocal condition from (3), the following inequalities
\[
-\sum_{\omega_1} h\rho Y_{\xi_1x_1} Y \geq \sum_{\omega_1} h \tilde{\rho} Y_{\xi_1}^2,
\]
\[
\sum_{\omega_1} hr Y^2 \leq 4 \sum_{\omega_1} h \tilde{r}(Y_{\xi_1})^2
\]
hold.

**Proof.** After simple computations, we obtain
\[
-\sum_{\omega_1} h\rho Y_{\xi_1x_1} Y = \sum_{\omega_1} h \tilde{\rho} Y_{\xi_1}^2 - \frac{1}{2} Y^2(1, x_2) - \frac{1}{2} \sum_{\omega_1} h Y^2 \rho_{\xi_1x_1}.
\]
Taking into account
\[
\sum_{\omega_1} h Y^2 \rho_{\xi_1x_1} = - \sum_{\omega_1} h Y^2 \sum_{k=1}^{m} \chi \sigma_k \frac{1}{h} \delta(x_1, \xi_k) = - \sum_{k=1}^{m} Y^2(\xi_k, x_2) \sigma_k \chi,
\]
where \(\delta(, , \cdot)\) is the Kronecker delta, and
\[
Y^2(1, x_2) \leq \left(\sum_{k=1}^{m} \sqrt{\alpha_k^2 \xi_k} \sqrt{\alpha_k^2 / \xi_k} |Y(\xi_k, x_2)|\right)^2 \leq \chi \sum_{k=1}^{m} \frac{\alpha_k}{\sqrt{\xi_k}} Y^2(\xi_k, x_2),
\]
we obtain (4).
One can show that
\[
\sum_{\omega_1} h\tilde{r}_1^2(Y^2)_{\tilde{x}_1} = \sum_{\omega_1} hr Y^2 + \frac{h^2}{8} Y^2(1, x_2).
\]
(7)

On the other hand,
\[
\sum_{\omega_1} h\tilde{r}_1^2(Y^2)_{\tilde{x}_1} = \sum_{\omega_1} h\tilde{r}_1^2 Y_{\tilde{x}_1}(Y + Y^{(-1)})
\]
\[
\leq \left( \sum_{\omega_1} h\tilde{r}(Y_{\tilde{x}_1})^2 \right)^{1/2} \left( \sum_{\omega_1} h\tilde{r}(Y + Y^{(-1)})^2 \right)^{1/2}
\]
\[
\leq \varepsilon \sum_{\omega_1} h\tilde{r}(Y_{\tilde{x}_1})^2 + \frac{1}{2\varepsilon} \sum_{\omega_1} h\tilde{r}(Y + Y^{(-1)})^2.
\]
Whence, choosing \(\varepsilon = 4\) we obtain
\[
\sum_{\omega_1} h\tilde{r}_1^2(Y^2)_{\tilde{x}_1} \leq 2 \sum_{\omega_1} h\tilde{r}(Y_{\tilde{x}_1})^2 + \frac{1}{8} \sum_{\omega_1} h\tilde{r}(Y + Y^{(-1)})^2
\]
\[
= 2 \sum_{\omega_1} h\tilde{r}(Y_{\tilde{x}_1})^2 + \frac{h^2}{8} Y^2(1, x_2) + \frac{1}{2} \sum_{\omega_1} hr Y^2.
\]
(8)

(7), (8) prove the inequality (5). Lemma 1 is proved. \(\square\)

3. Difference scheme, correction procedure, and main result

At the first stage, we approximate problem (1) by the difference scheme

\[
\tilde{U}_{x_1} + \tilde{U}_{x_2} = \varphi(x), \quad x \in \omega, \quad \tilde{U} \in \overset{\circ}{H},
\]
(9)

where \(\varphi = T_1 T_2 f\) is the average of function \(f\).

Define the operators

\[
A := A_1 + A_2, \quad A_k Y := -\overset{\circ}{Y}_{x_k}, \quad k = 1, 2, \ x \in \omega,
\]

where

\[
Y \in H, \quad Y \in \overset{\circ}{H} \quad \text{and} \quad Y(x) = \overset{\circ}{Y}(x) \quad \text{for} \ x \in \omega.
\]

The difference scheme (9) can be rewritten in the form of operator equation

\[
-A\tilde{U} = \varphi(x), \quad x \in \omega, \quad \tilde{U} \in H.
\]
(10)

Operator \(A\) maps \(H\) onto \(H\). Indeed, it suffices to show that operator \(A_1\) on near-boundary point \((1 - h, x_2)\) has the form

\[
A_1 Y(1 - h, x_2) = -\overset{\circ}{Y}_{x_1}(1 - h, x_2)
\]
\[
= -\left( \overset{\circ}{Y}(1, x_2) - 2\overset{\circ}{Y}(1 - h, x_2) + \overset{\circ}{Y}(1 - 2h, x_2) \right) / h^2
\]
\[
= - \left( \sum_{\xi_k=1}^{m} a_k Y(\xi_k, x_2) - 2Y(1 - h, x_2) + Y(1 - 2h, x_2) \right) / h^2.
\]

According to the estimates (2), (4) and (5) we obtain the inequality

\[
(A_1 Y, Y)_\rho \geq c \|Y\|^2_\rho, \quad Y \in H.
\]
In addition, it is well known that $A_2$ is a self-adjoint and positive definite operator, $A_2 = A_2^*$, $(A_2 Y, Y) \geq c\|Y\|_\rho^2$. Therefore, the operator $A$ is positive definite on the space $H$,

$$(AY, Y) \geq \|Y\|_\rho^2,$$

and hence the scheme (10) (i.e. (9)) is uniquely solvable.

At the second stage, we use the earlier-found solution of the difference scheme (10), define the correction term

$$\mathcal{R}\hat{U} := \frac{h^2}{6} \hat{U}_{\tilde{x}_1\tilde{x}_2}$$

and solve the difference scheme

$$-AU = \varphi - \mathcal{R}\hat{U}, \quad x \in \omega, \quad U \in H$$

(11)

on the same grid.

The following assertion is the main result of the present paper.

**Theorem 1.** Let the solution of problem (1) belong to the space $W^s_2(\Omega)$, $s \geq 2$. Then the convergence rate of the corrected difference scheme (11) in the discrete $L_2$-norm is defined by the estimate

$$\|U - u\|_{L_2(\omega, r)} \leq c h^s \|u\|_{W^s_2(\Omega)}, \quad 2 \leq s \leq 4.$$

4. A priori error estimates. Proof of Theorem 1

Let

$$\zeta_{3-k} = T_k u - u, \quad \eta_{3-k} = T_k u - u - \frac{h^2}{12} u_{\tilde{x}_k}, \quad k = 1, 2.$$

By $\tilde{Z} = \tilde{U} - u$ and $Z = U - u$ we denote the errors in the solution of the schemes (10) and (11) respectively. First, notice that these functions represent solutions of the following problems:

$$-A\tilde{Z} = (\zeta_1)_{\tilde{x}_1\tilde{x}_1} + (\zeta_2)_{\tilde{x}_2\tilde{x}_2}, \quad x \in \omega, \quad \tilde{Z} \in H$$

(12)

and

$$-AZ = (\eta_1)_{\tilde{x}_1\tilde{x}_1} + (\eta_2)_{\tilde{x}_2\tilde{x}_2} - (h^2/6)\tilde{Z}_{\tilde{x}_1\tilde{x}_1\tilde{x}_2\tilde{x}_2}, \quad x \in \omega, \quad Z \in H.$$

(13)

Indeed, we have

$$-AZ = -AU + Au = \varphi - \mathcal{R}\hat{U} + Au = -\mathcal{R}\tilde{Z} + T_1 T_2 f - \mathcal{R}u + Au,$$

whence using the relation

$$T_1 T_2 \Delta u = (T_2 u)_{\tilde{x}_1\tilde{x}_1} + (T_1 u)_{\tilde{x}_2\tilde{x}_2}$$

and the expressions for the operators $Au$ and $\mathcal{R}u$, we obtain (13). Eq. (12) is obtained analogously.

**Lemma 2.** For the solutions of problems (12), (13) there hold the following a priori estimates

$$\|\tilde{Z}_{\tilde{x}_1\tilde{x}_1}\|_\rho \leq c (\|\zeta_1\|_{\tilde{x}_1\tilde{x}_1} + \|\zeta_2\|_{\tilde{x}_2\tilde{x}_2}),$$

$$\|Z\|_\rho \leq c (\|\eta_1\| + \|\eta_2\| + h^2 \|\tilde{Z}_{\tilde{x}_1\tilde{x}_1}\|_\rho).$$

(14)

(15)

**Proof.** From (12) it follows

$$\left(\tilde{Z}_{\tilde{x}_1\tilde{x}_1}, \tilde{Z}_{\tilde{x}_1\tilde{x}_1}\right)_\rho + (\tilde{Z}_{\tilde{x}_2\tilde{x}_2}, \tilde{Z}_{\tilde{x}_1\tilde{x}_1})_\rho = -((\zeta_1)_{\tilde{x}_1\tilde{x}_1} + (\zeta_2)_{\tilde{x}_2\tilde{x}_2}, \tilde{Z}_{\tilde{x}_1\tilde{x}_1})_\rho.$$

(16)
Summing up by parts, we get
\[
(\tilde{Z}_{\bar{x}_{2}x_2}, \tilde{Z}_{\bar{x}_{1}x_1})_\rho = \sum_{\omega^+} h^2 \tilde{\rho}(\tilde{Z}_{\bar{x}_{1}\bar{x}_{2}})^2 - \sum_{\omega^+} \frac{h}{2} (\tilde{Z}_{\bar{x}_{2}}(1, x_2))^2 - \frac{1}{2} \sum_{\omega_1 \times \omega_2^+} h^2 \rho \tilde{x}_{1,x_1}(\tilde{Z}_{\bar{x}_{2}})^2
\]
\[
= \sum_{\omega^+} h^2 \tilde{\rho}(\tilde{Z}_{\bar{x}_{1}\bar{x}_{2}})^2 - \sum_{\omega^+} \frac{h}{2} (\tilde{Z}_{\bar{x}_{2}}(1, x_2))^2 - \frac{1}{2} \sum_{k=1}^{m} \frac{x^k}{\sqrt{k}} (\tilde{Z}_{\bar{x}_{2}}(\xi_k, x_2))^2.
\]

Using analogous to the estimate (6), written for \(\tilde{Z}_{\bar{x}_{2}}\), we obtain
\[
(\tilde{Z}_{\bar{x}_{2}x_2}, \tilde{Z}_{\bar{x}_{1}x_1})_\rho \geq \sum_{\omega^+} h^2 \tilde{\rho}(\tilde{Z}_{\bar{x}_{1}\bar{x}_{2}})^2 \geq 0.
\]

Therefore, from (16) we obtain the validity of (14).

Now, represent the solution of the problem (13) in the form of sum
\[
Z = Z^{(1)} + Z^{(2)},
\]
where \(Z^{(k)}, k = 1, 2\), are the solutions of the following problems
\[
-AZ^{(1)} = (\eta_1)_{\bar{x}_{1}x_1}, \quad x \in \omega, \quad Z^{(1)} \in H,
\]
\[
-AZ^{(2)} = (\eta_2)_{\bar{x}_{2}x_2} - \frac{h^2}{6} \tilde{Z}_{\bar{x}_{1}x_1\bar{x}_{2}x_2}, \quad x \in \omega, \quad Z^{(2)} \in H.
\]

From (17) we have
\[
Z^{(1)} + A^{-1}_1 A_2 Z^{(1)} = -\eta_1,
\]
\[
\|Z^{(1)}\|^2_\rho + (A^{-1}_1 A_2 Z^{(1)}, Z^{(1)})_\rho = -\langle \eta_1, Z^{(1)} \rangle_\rho.
\]

The operator \(A_2\) is self-adjoint and positive definite, therefore, there exists quadratic root \(A^{1/2}_2\), which is self-adjoint and commutable with \(A^{-1}_1\). Thus
\[
(A^{-1}_1 A_2 Z^{(1)}, Z^{(1)})_\rho = (A^{-1}_1 (A^{1/2}_2 Z^{(1)}), (A^{1/2}_2 Z^{(1)})_\rho \geq 0
\]
and
\[
\|Z^{(1)}\|_\rho \leq \|\eta_1\|.
\]

From (18) it follows
\[
A^{-1}_2 A_1 Z^{(2)} + Z^{(2)} = -\eta_2 + (h^2/6) \tilde{Z}_{\bar{x}_{1}x_1},
\]
and since
\[
(A^{-1}_2 A_1 Z^{(2)}, Z^{(2)})_\rho = (A_1 (A^{-1/2}_2 Z^{(2)}), (A^{-1/2}_2 Z^{(2)})_\rho \geq 0,
\]
we obtain
\[
\|Z^{(2)}\|_\rho \leq \|\eta_2\| + (h^2/6) \|\tilde{Z}_{\bar{x}_{1}x_1}\|_\rho.
\]

(19) and (20) prove (15). □

To determine the rate of convergence of the two-stage finite difference method with the help of Lemma 2, it is sufficient to estimate the terms on the right-hand sides of (18), (19). For that purpose we use the following lemma.

**Lemma 3.** Assume that the linear functional \(l(u)\) is bounded in \(W^s(E)\), where \(s = \bar{s} + \epsilon, \bar{s} \) is an integer, \(0 < \epsilon \leq 1\), and \(l(P) = 0\) for every polynomial \(P\) of degree \(\leq \bar{s}\) in two variables. Then, there exists a constant \(c\), independent of \(u\), such that \(|l(u)| \leq c\|u\|_{W^s(E)}\).
This lemma is a particular case of the Dupont–Scott approximation theorem [17] and represents a generalization of the Bramble–Hilbert lemma [18].

Quantities \((\zeta_k)_{\bar{x}_1x_1}\), as a linear functionals with respect to \(u\), vanish on the third order polynomials and are bounded in \(W_2^2(\Omega)\), \(s \geq 2\). Using the well known methodology (see, e.g., [6, Ch. 4, §1]), based on Lemma 3, for them we obtain the estimates

\[
\|\xi_k\| \leq ch\|u\|_{W_2^2(\Omega)}, \quad k = 1, 2,
\]

\[
\|\eta_k\| \leq ch^2\|u\|_{W_2^2(\Omega)}, \quad k = 1, 2.
\]

Due to Lemma 2

\[
\|Z\|_\rho \leq c \left( \|\eta_1\| + \|\eta_2\| + h^2\|\zeta_{\bar{x}_1x_1}\| + h^2\|\zeta_{\bar{x}_2x_2}\| \right),
\]

which together with the estimates (21), (22) accomplishes the proof of Theorem 1.

5. Numerical experiments

Now, we present some numerical results to demonstrate the convergence order of the proposed method. The experimental order of convergence in the discrete \(L_2(\omega, r)\) and \(L_2(\omega)\) norms is computed by formulas

\[
Ord(Y) = \log_2 \frac{\|Y_h - u\|_r}{\|Y_{h/2} - u\|_r}, \quad Ord(Y) = \log_2 \frac{\|Y_h - u\|}{\|Y_{h/2} - u\|},
\]

where \(u\) is the exact solution of original problem, while \(Y_h\) denotes the solution of the difference scheme on the grid with step \(h\).

Below, in the examples the symbols \(\bar{U}, U\) denote solutions of the difference schemes (10), (11), respectively.

The results of calculations are given by Tables 1, 2.

Consider the following problem

\[
\Delta u = f, \quad x \in (0, 1)^2, \quad u|_{\Gamma_0} = 0, \quad u(1, x_2) = u(0.5, x_2), \quad 0 < x_2 < 1,
\]

where

\[
f(x) = -\frac{13\pi^2}{9} \sin \left( \frac{2\pi x_1}{3} \right) \sin(\pi x_2).
\]

The exact solution \(u(x) = \sin(\frac{2\pi x_1}{3}) \sin(\pi x_2)\) of the problem (23) belongs to the space \(W_2^4\), therefore, theoretical convergence rate of the difference scheme equals 4.

The right-hand side of the scheme is calculated by the formula

\[
\varphi(x) = T_1 T_2 f = -\frac{13\pi^2}{9}\lambda_1^2 \lambda_2^2 \sin \left( \frac{2\pi i h}{3} \right) \sin(\pi j h),
\]

\[
\lambda_1 = \frac{3}{\pi h} \sin \left( \frac{\pi h}{3} \right), \quad \lambda_2 = \frac{2}{\pi h} \sin \left( \frac{\pi h}{2} \right).
\]
Table 2
Experimental order of convergence in $L_2(\omega)$-norm.

<table>
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<th>$h$</th>
<th>$| U_h - u |$</th>
<th>$| U_h - u |$</th>
<th>$\text{Ord}(U)$</th>
<th>$\text{Ord}(\tilde{U})$</th>
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</thead>
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<td>4.15297e-03</td>
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<td>8.91061e-10</td>
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</tbody>
</table>

6. Conclusion

For solution of the Bitsadze–Samarskii type nonlocal problem posed in unit square for Poisson equation it is used a finite-difference scheme. Using the solution, obtained by the method with second order accuracy, we correct the right-hand side of the scheme and solve it again on the same grid. It is proved that if the solution of original problem belongs to the Sobolev space with fractional exponent $s \in [2; 4]$, then the corrected scheme converges with the rate $O(|h|^s)$. The theoretical results are supported by numerical experiments. The obtained results can be extended to the nonlocal problem posed for general elliptic equations, and also to three-dimensional case.

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