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Higher-order infinite horizon variational problems in discrete quantum calculus

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ABSTRACT

We obtain necessary optimality conditions for higher-order infinite horizon problems of the calculus of variations via discrete quantum operators.

1. Introduction

Quantum difference operators are receiving an increase of interest due to their applications in physics, economics, and the calculus of variations; see [1–5] and the references therein. Here, we develop quantum variational calculus in the infinite horizon case. Let $q > 1$, and denote by $\mathcal{Q}$ the set $\mathcal{Q} := q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$. In what follows, σ denotes the function defined by $\sigma(t) := qt$ for all $t \in \mathcal{Q}$. For any $k \in \mathbb{N}$, $\sigma^k := \sigma \circ \sigma^{k-1}$, where $\sigma^0 = id$. It is clear that $\sigma^k(t) = q^k t$. For $f : \mathcal{Q} \to \mathbb{R}$, we define $f^{\sigma^k} := f \circ \sigma^k$. Fix $a \in \mathcal{Q}$ and $r \in \mathbb{N}$. We are concerned with the following higher-order q-variational problem:

$$\begin{aligned} &\mathcal{J}(x(\cdot)) = \int_a^{+\infty} L\left(t, (x \circ \sigma^r)(t), D_q[x \circ \sigma^{r-1}](t), \ldots, D_q^{r-1}[x \circ \sigma](t), D_q^r[x](t)\right) d_q t \longrightarrow \max \\ &x(a) = \alpha_0, \quad D_q[x](a) = \alpha_1, \ \ldots, \ D_q^{r-1}[x](a) = \alpha_{r-1}, \end{aligned} \tag{1}$$

where $(u_1, \ldots, u_r, u_{r+1}) \to L(t, u_1, \ldots, u_{r+1})$ is a $C^1(\mathbb{R}^{r+1}, \mathbb{R})$ function for any $t \in \mathcal{Q}$, and $\alpha_0, \ldots, \alpha_{r-1}$ are given real numbers. The results of the paper are trivially generalized for the case of functions $x : \mathcal{Q} \to \mathbb{R}^n$, $n \in \mathbb{N}$, but for simplicity of presentation we restrict ourselves to the scalar case, i.e., $n = 1$. In Section 2, we present some preliminary results and basic definitions. The main results appear in Section 3: in Section 3.1, we prove some fundamental lemmas of the calculus of variations for infinite horizon q-variational problems; an Euler–Lagrange type equation and transversality conditions for (1) are obtained in Section 3.2.

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2. Preliminaries

Let f be a function defined on $\mathcal{Q}$. By D_q we denote the Jackson q-difference operator:

$$D_q[f](t) := \frac{f(qt) - f(t)}{(q-1)t} \quad \forall t \in \mathcal{Q}. \tag{2}$$

The higher-order q-derivatives are defined in the usual way: the rth q-derivative, $r \in \mathbb{N}$, of $f: \mathcal{Q} \to \mathbb{R}$ is the function $D_q^r[f]: \mathcal{Q} \to \mathbb{R}$ given by $D_q^r[f] := D_q[D_q^{r-1}[f]]$, where $D_q^0[f] := f$.

The Jackson q-difference operator (2) satisfies the following properties.

Theorem 1 (*cf.* [6]). *Let f and g be functions defined on $\mathcal{Q}$ and $t \in \mathcal{Q}$. One has the following:*

1. $D_q[f] \equiv 0$ *on I if and only if f is constant;*
2. $D_q[f+g](t) = D_q[f](t) + D_q[g](t);$
3. $D_q[fg](t) = D_q[f](t)\, g(t) + f(qt)\, D_q[g](t);$
4. $D_q\left[\frac{f}{g}\right](t) = \frac{D_q[f](t)g(t) - f(t)D_q[g](t)}{g(t)g(qt)}$ *if* $g(t)\, g(qt) \neq 0$.

Let $a \in \mathcal{Q}$ and $b := aq^n \in \mathcal{Q}$ for some $n \in \mathbb{N}$. The q-integral of f from a to b is defined by

$$\int_a^b f(t) d_q t := a(q-1) \sum_{k=0}^{n-1} q^k f(aq^k).$$

Theorem 2 (*cf.* [6]). *If $a, b, c \in \mathcal{Q}$, $a \le c \le b$, $\alpha, \beta \in \mathbb{R}$, and $f, g: \mathcal{Q} \to \mathbb{R}$, then*

1. $\int_a^b (\alpha f(t) + \beta g(t))\, d_q t = \alpha \int_a^b f(t) d_q t + \beta \int_a^b g(t) d_q t;$
2. $\int_a^b f(t) d_q t = -\int_b^a f(t) d_q t;$
3. $\int_a^a f(t) d_q t = 0;$
4. $\int_a^b f(t) d_q t = \int_a^c f(t) d_q t + \int_c^b f(t) d_q t;$
5. *If* $f(t) > 0$ *for all* $a \le t < b$, *then* $\int_a^b f(t) d_q t > 0;$
6. $\int_a^b f(t) D_q[g](t) d_q t = [f(t)g(t)]_{t=a}^{t=b} - \int_a^b D_q[f](t) g(qt) d_q t$ *(q-integration by parts formula);*
7. $\int_a^b D_q[f](t) d_q t = f(b) - f(a)$ *(fundamental theorem of q-calculus);*
8. $D_q\left[s \to \int_a^s f(\tau) d_q \tau\right](t) = f(t).$

As usual, we define

$$\int_a^{+\infty} f(t) d_q t := \lim_{b \to +\infty} \int_a^b f(t) d_q t$$

provided this limit exists (in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$). We say that the improper q-integral converges if this limit is finite; otherwise, we say that the improper q-integral diverges.

In what follows, all intervals are q-intervals; that is, for $a, b \in \mathcal{Q}$, $[a, b] := \{t \in \mathcal{Q}: a \le t \le b\}$ and $[a, +\infty[:= \{t \in \mathcal{Q}: a \le t < +\infty\}$.

Definition 1. We say that $x: [a, +\infty[\to \mathbb{R}$ is an admissible path for problem (1) if $x(a) = \alpha_0, D_q[x](a) = \alpha_1, \ldots, D_q^{r-1}[x](a) = \alpha_{r-1}$.

There are several definitions of optimality for problems with unbounded domain (see, e.g., [7–10]). Here, we follow Brock's notion of optimality.

Definition 2. Suppose that $a, T, T' \in \mathcal{Q}$ are such that $T' \ge T > a$. We say that x_* is weakly maximal to problem (1) if and only if x_* is an admissible path and

$$\lim_{T \to +\infty} \inf_{T' \ge T} \int_a^{T'} \Big[L(t, (x \circ \sigma^r)(t), D_q[x \circ \sigma^{r-1}](t), \ldots, D_q^{r-1}[x \circ \sigma](t), D_q^r[x](t)) - L(t, (x_* \circ \sigma^r)(t), D_q[x_* \circ \sigma^{r-1}](t), \ldots, D_q^{r-1}[x_* \circ \sigma](t), D_q^r[x_*](t))\Big] d_q t \le 0$$

for all admissible x.

Note that, in the case where the functional $\mathcal{J}$ of problem (1) converges for all admissible paths, the weak maximal path is optimal in the sense of the usual definition of optimality. However, if every admissible function x yields an infinite value to the functional, using the usual definition of optimality, each admissible path is an optimal path, showing that the standard definition is not appropriate for problems with an unbounded domain.

Lemmas 1 and 2 are an immediate consequence of the definition of the Jackson q-difference operator.

Lemma 1. *For any $f: \mathcal{Q} \to \mathbb{R}$ and $t \in \mathcal{Q}$, $D_q[f](\sigma(t)) = \frac{1}{q} D_q[f \circ \sigma](t)$.*

Lemma 2. *Assume that $\eta: [a, +\infty[\to \mathbb{R}$ is such that $D_q^i[\eta](a) = 0$ for all $i = 0, 1, \ldots, r$. Then, $D_q^{i-1}[\eta \circ \sigma](a) = 0$ for each $i = 1, \ldots, r$.*

The following basic result will be useful in the proof of our main result (Theorem 4).

Theorem 3 (*cf.* [11])**.** *Let S and T be subsets of a normed vector space. Let f be a map defined on $T \times S$, having values in some complete normed vector space. Let v be adherent to S and w adherent to T. Assume that*

1. $\lim_{x\to v} f(t, x)$ *exists for each* $t \in T$;
2. $\lim_{t\to w} f(t, x)$ *exists uniformly for* $x \in S$.

Then the limits $\lim_{t\to w} \lim_{x\to v} f(t, x)$, $\lim_{x\to v} \lim_{t\to w} f(t, x)$, and $\lim_{(t,x)\to(w,v)} f(t, x)$ all exist and are equal.

3. Main results

Before proving our main result (Theorem 4), we need several preliminary results. Namely, we prove in Section 3.1 a higher-order q-integration by parts formula and three higher-order fundamental lemmas for the q-calculus of variations.

3.1. Fundamental lemmas

In our results, we use the standard convention that $\sum_{k=1}^{j} \gamma(k) = 0$ whenever $j = 0$.

Lemma 3 (*Higher-Order q-Integration by Parts Formula*)**.** *Let $r \in \mathbb{N}$, $a, b \in \mathcal{Q}$, $a < b$, and $f, g: [a, \sigma^r(b)] \to \mathbb{R}$. For each $i = 1, 2, \ldots, r$, we have*

$$\int_a^b f(t) D_q^i[g \circ \sigma^{r-i}](t) d_q t = (-1)^i \int_a^b \left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f](t) g^{\sigma^r}(t) d_q t + \left[f(t) D_q^{i-1}[g \circ \sigma^{r-i}](t) + \sum_{k=1}^{i-1} (-1)^k D_q^k[f](t) D_q^{i-1-k}[g \circ \sigma^{r-i+k}](t) \cdot \prod_{j=1}^{k} \left(\frac{1}{q}\right)^{i-j} \right]_a^b.$$

Proof. We prove the lemma by mathematical induction. If $r = 1$, the result is obviously true from the q-integration by parts formula. Assuming that the result holds for degree $r > 1$, we will prove it for $r+1$. Fix some $i = 1, 2, \ldots, r$. By the induction hypothesis, we get

$$\begin{aligned}
\int_a^b f(t) D_q^i[g \circ \sigma^{r+1-i}](t) d_q t &= \int_a^b f(t) D_q^i[g^\sigma \circ \sigma^{r-i}](t) d_q t \\
&= \left[f(t) D_q^{i-1}[g^\sigma \circ \sigma^{r-i}](t) + \sum_{k=1}^{i-1} (-1)^k D_q^k[f](t) D_q^{i-1-k}[g^\sigma \circ \sigma^{r-i+k}](t) \cdot \prod_{j=1}^{k} \left(\frac{1}{q}\right)^{i-j} \right]_a^b \\
&\quad + (-1)^i \int_a^b \left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f](t) (g^\sigma)^{\sigma^r}(t) d_q t \\
&= \left[f(t) D_q^{i-1}[g \circ \sigma^{r+1-i}](t) + \sum_{k=1}^{i-1} (-1)^k D_q^k[f](t) D_q^{i-1-k}[g \circ \sigma^{r+1-i+k}](t) \cdot \prod_{j=1}^{k} \left(\frac{1}{q}\right)^{i-j} \right]_a^b \\
&\quad + (-1)^i \int_a^b \left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f](t) g^{\sigma^{r+1}}(t) d_q t.
\end{aligned}$$

It remains to prove that the result is true for $i = r + 1$. Note that

$$\int_a^b f(t) D_q^{r+1}[g](t) d_q t = \int_a^b f(t) D_q^r[D_q[g]](t) d_q t$$

and, by the induction hypothesis for degree r and $i = r$,

$$\int_a^b f(t)D_q^{r+1}[g](t)d_qt = (-1)^r \int_a^b \left(\frac{1}{q}\right)^{\frac{r(r-1)}{2}} D_q^r[f](t)D_q[g](\sigma^r(t))d_qt + \left[f(t)D_q^{r-1}[D_q[g]](t) + \sum_{k=1}^{r-1}(-1)^k D_q^k[f](t)D_q^{r-1-k}[D_q[g]\circ\sigma^k](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{r-j}\right]_a^b.$$

From Lemma 1, we can write that

$$\int_a^b f(t)D_q^{r+1}[g](t)d_qt = \left[f(t)D_q^r[g](t) + \sum_{k=1}^{r-1}(-1)^k D_q^k[f](t)D_q^{r-k}[g\circ\sigma^k](t)\cdot\left(\frac{1}{q}\right)^k\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{r-j}\right]_a^b + (-1)^r\int_a^b\left(\frac{1}{q}\right)^{\frac{r(r-1)}{2}}\left(\frac{1}{q}\right)^r D_q^r[f](t)D_q[g\circ\sigma^r](t)d_qt$$

and, by the q-integration by parts formula,

$$\int_a^b f(t)D_q^{r+1}[g](t)d_qt = \left[f(t)D_q^r[g](t) + \sum_{k=1}^{r-1}(-1)^k D_q^k[f](t)D_q^{r-k}[g\circ\sigma^k](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{r+1-j}\right]_a^b + \left[(-1)^r D_q^r[f](t)g^{\sigma^r}(t)\left(\frac{1}{q}\right)^{\frac{r(r+1)}{2}}\right]_a^b - (-1)^r\int_a^b\left(\frac{1}{q}\right)^{\frac{r(r+1)}{2}} D_q^{r+1}[f](t)g^{\sigma^{r+1}}(t)d_qt.$$

We conclude that

$$\int_a^b f(t)D_q^{r+1}[g](t)d_qt = \left[f(t)D_q^r[g](t) + \sum_{k=1}^{r}(-1)^k D_q^k[f](t)D_q^{r-k}[g\circ\sigma^k](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{r+1-j}\right]_a^b + (-1)^{r+1}\int_a^b\left(\frac{1}{q}\right)^{\frac{r(r+1)}{2}} D_q^{r+1}[f](t)g^{\sigma^{r+1}}(t)d_qt,$$

proving that the result is true for $i = r + 1$. □

The following lemma follows easily (by contradiction and the properties of the q-integral).

Lemma 4. *Suppose that $a \in \mathcal{Q}$ and that $f\colon [a, +\infty[\to \mathbb{R}$ is a function such that $f \geq 0$. If*

$$\lim_{T\to+\infty}\inf_{T'\geq T}\int_a^{T'} f(t)d_qt = 0,$$

then $f = 0$ on $[a, +\infty[$.

We now present two first-order fundamental lemmas of the q-calculus of variations for infinite horizon variational problems.

Lemma 5. *Let $a \in \mathcal{Q}$ and let $f\colon [a, +\infty[\to \mathbb{R}$. If*

$$\lim_{T\to+\infty}\inf_{T'\geq T}\int_a^{T'} f(t)D_q[\eta](t)d_qt = 0 \quad \text{for all } \eta\colon [a, +\infty[\to \mathbb{R} \text{ such that } \eta(a) = 0,$$

then $f(t) = c$ for all $t \in [a, +\infty[$, where $c \in \mathbb{R}$.

Proof. Fix $T, T' \in \mathcal{Q}$ such that $T' \geq T > a$. Let c be a constant defined by the condition

$$\int_a^{T'} (f(\tau) - c)\, d_q\tau = 0,$$

and let

$$\eta(t) = \int_a^t (f(\tau) - c)\, d_q\tau.$$

Clearly, $D_q[\eta](t) = f(t) - c$, and

$$\eta(a) = \int_a^a (f(\tau) - c)\, d_q\tau = 0 \quad \text{and} \quad \eta(T') = \int_a^{T'} (f(\tau) - c)\, d_q\tau = 0.$$

Observe that

$$\int_a^{T'} (f(t) - c)\, D_q[\eta](t) d_q t = \int_a^{T'} (f(t) - c)^2\, d_q t$$

and

$$\int_a^{T'} (f(t) - c)\, D_q[\eta](t) d_q t = \int_a^{T'} f(t) D_q[\eta](t) d_q t - c \int_a^{T'} D_q[\eta](t) d_q t = \int_a^{T'} f(t) D_q[\eta](t) d_q t.$$

Hence,

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} f(t) D_q[\eta](t) d_q t = \lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} (f(t) - c)^2\, d_q t = 0,$$

which shows, by Lemma 4, that $f(t) - c = 0$ for all $t \in [a, +\infty[$. □

Lemma 6. *Let* $f, g\colon [a, +\infty[\to \mathbb{R}$. *If*

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \big(f(t)\eta(qt) + g(t) D_q[\eta](t)\big)\, d_q t = 0$$

for all $\eta\colon [a, +\infty[\to \mathbb{R}$ *such that* $\eta(a) = 0$, *then* $D_q[g](t) = f(t)$ *for all* $t \in [a, +\infty[$.

Proof. Fix $T, T' \in \mathcal{Q}$ such that $T' \geq T > a$, and define $A(t) = \int_a^t f(\tau) d_q\tau$. Then $D_q[A](t) = f(t)$ for all $t \in [a, +\infty[$, and

$$\int_a^{T'} A(t) D_q[\eta](t) d_q t = [A(t)\eta(t)]_a^{T'} - \int_a^{T'} D_q[A](t)\eta(qt) d_q t = A(T')\eta(T') - \int_a^{T'} f(t)\eta(qt) d_q t.$$

Restricting η to those such that $\eta(T') = 0$, we obtain

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \big(f(t)\eta(qt) + g(t) D_q[\eta](t)\big)\, d_q t = \lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} (-A(t) + g(t))\, D_q[\eta](t) d_q t = 0.$$

By Lemma 5, we may conclude that there exists $c \in \mathbb{R}$ such that $-A(t) + g(t) = c$ for all $t \in [a, +\infty[$. Therefore, $D_q[A](t) = D_q[g](t)$ for all $t \in [a, +\infty[$, proving the desired result. □

Lemma 7 (*Higher-Order Fundamental Lemma of the q-Calculus of Variations I*)**.** *Let* $f_0, f_1, \ldots, f_r\colon [a, +\infty[\to \mathbb{R}$. *If*

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(\sum_{i=0}^{r} f_i(t) D_q^i[\eta \circ \sigma^{r-i}](t)\right) d_q t = 0$$

for all $\eta\colon [a, +\infty[\to \mathbb{R}$ *such that* $\eta(a) = 0, D_q[\eta](a) = 0, \ldots, D_q^{r-1}[\eta](a) = 0$, *then*

$$\sum_{i=0}^{r} (-1)^i \left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f_i](t) = 0 \quad \forall t \in [a, +\infty[.$$

Proof. We proceed by mathematical induction. If $r = 1$, the result is true by Lemma 6. Assume that the result is true for some $r > 1$. We prove that the result is also true for $r + 1$. Suppose that

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(\sum_{i=0}^{r+1} f_i(t) D_q^i[\eta \circ \sigma^{r+1-i}](t)\right) d_q t = 0$$

for all $\eta\colon [a, +\infty[\to \mathbb{R}$ such that $\eta(a) = 0, D_q[\eta](a) = 0, \ldots, D_q^r[\eta](a) = 0$. We need to prove that

$$\sum_{i=0}^{r+1} (-1)^i \left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f_i](t) = 0 \quad \forall t \in [a, +\infty[.$$

Note that

$$\int_a^{T'} \left(\sum_{i=0}^{r+1} f_i(t) D_q^i[\eta \circ \sigma^{r+1-i}](t)\right) d_q t = \int_a^{T'} \left(\sum_{i=0}^{r} f_i(t) D_q^i[\eta \circ \sigma^{r+1-i}](t)\right) d_q t + \int_a^{T'} f_{r+1}(t) D_q[D_q^r[\eta]](t) d_q t.$$

Using the q-integration by parts formula in the last integral, we obtain that

$$\int_a^{T'} f_{r+1}(t) D_q[D_q^r[\eta]](t) d_q t = \left[f_{r+1}(t) D_q^r[\eta](t)\right]_a^{T'} - \int_a^{T'} D_q[f_{r+1}](t) D_q^r[\eta](qt) d_q t.$$

Since $D_q^r[\eta](a) = 0$ and we can restrict ourselves to those η such that $D_q^r[\eta](T') = 0$,

$$\int_a^{T'} f_{r+1}(t) D_q[D_q^r[\eta]](t) d_q t = -\int_a^{T'} D_q[f_{r+1}](t) D_q^r[\eta](\sigma(t)) d_q t.$$

By Lemma 1,

$$\int_a^{T'} f_{r+1}(t) D_q[D_q^r[\eta]](t) d_q t = -\int_a^{T'} D_q[f_{r+1}](t) \left(\frac{1}{q}\right)^r D_q^r[\eta \circ \sigma](t) d_q t.$$

Hence,

$$\begin{aligned}
&\int_a^{T'} \left(\sum_{i=0}^{r+1} f_i(t) D_q^i[\eta \circ \sigma^{r+1-i}](t)\right) d_q t \\
&= \int_a^{T'} \left(\sum_{i=0}^{r} f_i(t) D_q^i[\eta \circ \sigma^{r+1-i}](t)\right) d_q t - \int_a^{T'} D_q[f_{r+1}](t) \left(\frac{1}{q}\right)^r D_q^r[\eta \circ \sigma](t) d_q t \\
&= \int_a^{T'} \left(\sum_{i=0}^{r-1} f_i(t) D_q^i[\eta^\sigma \circ \sigma^{r-i}](t) + \left(f_r(t) - D_q[f_{r+1}](t)\left(\frac{1}{q}\right)^r\right) D_q^r[\eta \circ \sigma](t)\right) d_q t
\end{aligned}$$

and, therefore,

$$\begin{aligned}
&\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(\sum_{i=0}^{r+1} f_i(t) D_q^i[\eta \circ \sigma^{r+1-i}](t)\right) d_q t \\
&= \lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(\sum_{i=0}^{r-1} f_i(t) D_q^i[\eta^\sigma \circ \sigma^{r-i}](t) + \left(f_r(t) - D_q[f_{r+1}](t)\left(\frac{1}{q}\right)^r\right) D_q^r[\eta \circ \sigma](t)\right) d_q t = 0.
\end{aligned}$$

By Lemma 2, $\eta^\sigma(a) = 0, D_q[\eta \circ \sigma](a) = 0, \ldots, D_q^{r-1}[\eta \circ \sigma](a) = 0$. Then, by the induction hypothesis, we conclude that

$$\sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f_i](t) \;+\; (-1)^r \left(\frac{1}{q}\right)^{\frac{r(r-1)}{2}} D_q^r\left[f_r - \left(\frac{1}{q}\right)^r D_q[f_{r+1}]\right](t) = 0 \quad \forall t \in [a, +\infty[,$$

which is equivalent to $\sum_{i=0}^{r+1} (-1)^i \left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f_i](t) = 0$ for all $t \in [a, +\infty[$. □

Lemma 8 (*Higher-Order Fundamental Lemma of the q-Calculus of Variations II*). *Let* $f_0, f_1, \ldots, f_r \colon [a, +\infty[\to \mathbb{R}$. *If*

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(\sum_{i=0}^{r} f_i(t) D_q^i[\eta \circ \sigma^{r-i}](t)\right) d_q t = 0$$

for all $\eta \colon [a, +\infty[\to \mathbb{R}$ *such that* $\eta(a) = 0, D_q[\eta](a) = 0, \ldots, D_q^{r-1}[\eta](a) = 0$, *then*

$$\lim_{T\to+\infty} \inf_{T'\geq T} \left\{f_r(T') \cdot D_q^{r-1}[\eta](T')\right\} = 0.$$

Proof. Note that

$$\int_a^{T'} \left(\sum_{i=0}^{r} f_i(t) D_q^i[\eta \circ \sigma^{r-i}](t) d_q t\right) = \int_a^{T'} f_0(t) \eta^{\sigma^r}(t) d_q t + \sum_{i=1}^{r} \left(\int_a^{T'} f_i(t) D_q^i[\eta \circ \sigma^{r-i}](t) d_q t\right)$$

$$= \int_a^{T'} f_0(t)\eta^{\sigma^r}(t)d_qt + \sum_{i=1}^{r}\left((-1)^i \int_a^{T'} \left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f_i](t)\eta^{\sigma^r}(t)d_qt\right)$$
$$+ \sum_{i=1}^{r}\left[f_i(t)D_q^{i-1}[\eta\circ\sigma^{r-i}](t) + \sum_{k=1}^{i-1}(-1)^k D_q^k[f_i](t)D_q^{i-1-k}[\eta\circ\sigma^{r-i+k}](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{i-j}\right]_a^{T'}$$
$$= \int_a^{T'}\left(f_0(t) + \sum_{i=1}^{r}(-1)^i\left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f_i](t)\right)\cdot\eta^{\sigma^r}(t)d_qt$$
$$+ \sum_{i=1}^{r}\left[\left(f_i(t)D_q^{i-1}[\eta\circ\sigma^{r-i}](t) + \sum_{k=1}^{i-1}(-1)^k D_q^k[f_i](t)D_q^{i-1-k}[\eta\circ\sigma^{r-i+k}](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{i-j}\right)\right]_a^{T'}$$
$$= \int_a^{T'}\left(\sum_{i=0}^{r}(-1)^i\left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i[f_i](t)\right)\cdot\eta^{\sigma^r}(t)d_qt$$
$$+ \sum_{i=1}^{r-1}\left[\left(f_i(t)D_q^{i-1}[\eta\circ\sigma^{r-i}](t) + \sum_{k=1}^{i-1}(-1)^k D_q^k[f_i](t)D_q^{i-1-k}[\eta\circ\sigma^{r-i+k}](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{i-j}\right)\right]_a^{T'}$$
$$+ \left[f_r(t)D_q^{r-1}[\eta](t) + \sum_{k=1}^{r-1}(-1)^k D_q^k[f_r](t)D_q^{r-1-k}[\eta\circ\sigma^k](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{r-j}\right]_a^{T'},$$

where in the second equality we use Lemma 3. Applying now Lemma 7, we get

$$\int_a^{T'}\left(\sum_{i=0}^{r} f_i(t)D_q^i[\eta\circ\sigma^{r-i}](t)d_qt\right)$$
$$= \sum_{i=1}^{r-1}\left[\left(f_i(t)D_q^{i-1}[\eta\circ\sigma^{r-i}](t) + \sum_{k=1}^{i-1}(-1)^k D_q^k[f_i](t)D_q^{i-1-k}[\eta\circ\sigma^{r-i+k}](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{i-j}\right)\right]_a^{T'}$$
$$+ \left[f_r(t)D_q^{r-1}[\eta](t) + \sum_{k=1}^{r-1}(-1)^k D_q^k[f_r](t)D_q^{r-1-k}[\eta\circ\sigma^k](t)\cdot\prod_{j=1}^{k}\left(\frac{1}{q}\right)^{r-j}\right]_a^{T'}.$$

Therefore, restricting the variations η to those such that

$$D_q^{k-1}[\eta\circ\sigma^{r-k}](T') = D_q^{k-1}[\eta\circ\sigma^{r-k}](a) = 0, \quad \forall k = 1, 2, \ldots, r-1,$$
$$D_q^{r-1-k}[\eta\circ\sigma^{k}](T') = D_q^{r-1-k}[\eta\circ\sigma^{k}](a) = 0, \quad \forall k = 1, 2, \ldots, r-1,$$

we get

$$\lim_{T\to+\infty}\inf_{T'\geq T}\int_a^{T'}\left(\sum_{i=0}^{r} f_i(t)D_q^i[\eta\circ\sigma^{r-i}](t)\right)d_qt = 0 \Rightarrow \lim_{T\to+\infty}\inf_{T'\geq T}\left\{f_r(T')D_q^{r-1}[\eta](T')\right\} = 0,$$

proving the desired result. □

Lemma 9 (*Higher-Order Fundamental Lemma of the q-Calculus of Variations III*). *Let* $f_0, f_1, \ldots, f_r\colon [a, +\infty[\to\mathbb{R}$. *If*

$$\lim_{T\to+\infty}\inf_{T'\geq T}\int_a^{T'}\left(\sum_{i=0}^{r} f_i(t)D_q^i[\eta\circ\sigma^{r-i}](t)\right)d_qt = 0$$

for all $\eta\colon [a, +\infty[\to\mathbb{R}$ *such that* $\eta(a) = 0$, $D_q[\eta](a) = 0, \ldots, D_q^{r-1}[\eta](a) = 0$, *then*

$$\lim_{T\to+\infty}\inf_{T'\geq T}\left\{\left(f_{r-(k-1)}(T') + \sum_{i=1}^{k-1}(-1)^i D_q^i[f_{r-(k-1)+i}](T')\cdot\prod_{j=1}^{i}\left(\frac{1}{q}\right)^{r-(k-1)+(j-1)}\right)\cdot D_q^{r-k}[\eta\circ\sigma^{k-1}](T')\right\} = 0$$

for $k = 1, 2, \ldots, r$.

Proof. We prove the lemma by mathematical induction. For $r = 1$, using the q-integration by parts formula and Lemma 7, we obtain $\lim_{T\to+\infty} \inf_{T'\geq T} f_1(T')\eta(T') = 0$, showing that the result is true for $r = 1$. Assuming that the result holds for degree $r > 1$, we will prove it for $r + 1$. Suppose that

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(\sum_{i=0}^{r+1} f_i(t) D_q^i[\eta \circ \sigma^{r+1-i}](t)\right) d_q t = 0$$

for all $\eta: [a, +\infty[\to \mathbb{R}$ such that $\eta(a) = 0, D_q[\eta](a) = 0, \ldots, D_q^r[\eta](a) = 0$. We need to prove that

$$\lim_{T\to+\infty} \inf_{T'\geq T} \left\{\left(f_{r+1-(k-1)}(T') + \sum_{i=1}^{k-1}(-1)^i D_q^i[f_{r+1-(k-1)+i}](T') \cdot \prod_{j=1}^{i}\left(\frac{1}{q}\right)^{r+1-(k-1)+(j-1)}\right) \cdot D_q^{r+1-k}[\eta \circ \sigma^{k-1}](T')\right\} = 0 \tag{3}$$

for $k = 1, 2, \ldots, r, r + 1$. Fix some $k = 2, \ldots, r, r + 1$. The main idea of the proof is that the k-transversality condition for the variational problem of order $r + 1$ is obtained from the $k - 1$ transversality condition for the variational problem of order r. Using the same techniques as in Lemma 7, we prove that

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(\sum_{i=0}^{r+1} f_i(t) D_q^i[\eta \circ \sigma^{r+1-i}](t)\right) d_q t = 0$$
$$\Rightarrow \lim_{T\to+\infty} \inf_{T'\geq T} \left\{\int_a^{T'} \left(\sum_{i=0}^{r-1} f_i(t) D_q^i[\eta^\sigma \circ \sigma^{r-i}](t) + \left(f_r(t) - D_q[f_{r+1}](t)\left(\frac{1}{q}\right)^r\right) D_q^r[\eta \circ \sigma](t)\right) d_q t\right\} = 0.$$

Since, by Lemma 2, $\eta^\sigma(a) = 0, D_q[\eta \circ \sigma](a) = 0, \ldots, D_q^{r-1}[\eta \circ \sigma](a) = 0$, then, by the induction hypothesis for $k - 1$, we conclude that

$$\lim_{T\to+\infty} \inf_{T'\geq T} \left\{\left(f_{r-(k-2)}(T') + \sum_{i=1}^{k-3}(-1)^i D_q^i[f_{r-(k-2)+i}](T') \cdot \prod_{j=1}^{i}\left(\frac{1}{q}\right)^{r-(k-2)+(j-1)}\right.\right.$$
$$+ (-1)^{k-2} D_q^{k-2}[f_r](T') \cdot \prod_{j=1}^{k-2}\left(\frac{1}{q}\right)^{r-(k-2)+(j-1)}$$
$$\left.\left. + \ (-1)^{k-1} D_q^{k-1}[f_{r+1}](T') \cdot \prod_{j=1}^{k-2}\left(\frac{1}{q}\right)^{r-(k-2)+(j-1)}\left(\frac{1}{q}\right)^r\right) \cdot D_q^{r-(k-1)}[\eta^\sigma \circ \sigma^{k-2}](T')\right\} = 0,$$

which is equivalent to

$$\lim_{T\to+\infty} \inf_{T'\geq T} \left\{\left(f_{r-(k-2)}(T') + \sum_{i=1}^{k-1}(-1)^i D_q^i[f_{r-(k-2)+i}](T') \cdot \prod_{j=1}^{i}\left(\frac{1}{q}\right)^{r-(k-2)+(j-1)}\right) \cdot D_q^{r-(k-1)}[\eta^{\sigma^{k-1}}](T')\right\} = 0,$$

and proves Eq. (3) for $k = 2, 3, \ldots, r, r + 1$. It remains to prove (3) for $k = 1$. This condition follows from Lemma 8. □

3.2. Euler–Lagrange equation and transversality conditions

We are now in a position to prove a first-order necessary optimality condition for the higher-order infinite horizon q-variational problem. In what follows, $\partial_i L$ denotes the partial derivative of L with respect to its ith argument. For simplicity of expressions, we introduce the operator $\langle\cdot\rangle$, defined by

$$\langle x\rangle(t) := \left(t, (x \circ \sigma^r)(t), D_q[x \circ \sigma^{r-1}](t), \ldots, D_q^{r-1}[x \circ \sigma](t), D_q^r[x](t)\right).$$

Theorem 4. *Suppose that the optimal path to problem (1) exists and is given by x_*. Let $\eta: [a, +\infty[\to \mathbb{R}$ be such that $\eta(a) = 0, D_q[\eta](a) = 0, \ldots, D_q^{r-1}[\eta](a) = 0$. Define*

$$A(\varepsilon, T') := \int_a^{T'} \frac{L\langle x_* + \epsilon\eta\rangle(t) - L\langle x_*\rangle(t)}{\varepsilon} d_q t,$$
$$V(\varepsilon, T) := \inf_{T'\geq T} \int_a^{T'} (L\langle x_* + \epsilon\eta\rangle(t) - L\langle x_*\rangle(t))\, d_q t,$$
$$V(\varepsilon) := \lim_{T\to+\infty} V(\varepsilon, T).$$

Suppose that

1. $\lim_{\varepsilon\to 0} \frac{V(\varepsilon,T)}{\varepsilon}$ *exists for all* T*;*
2. $\lim_{T\to+\infty} \frac{V(\varepsilon,T)}{\varepsilon}$ *exists uniformly for* ε*;*
3. *for every* $T' > a$, $T > a$, *and* $\varepsilon \in \mathbb{R}\setminus\{0\}$, *there is a sequence* $\left(A(\varepsilon, T'_n)\right)_{n\in\mathbb{N}}$ *such that* $\lim_{n\to+\infty} A(\varepsilon, T'_n) = \inf_{T'\geq T} A(\varepsilon, T')$ *uniformly for* ε.

Then x_* *satisfies the Euler–Lagrange equation*

$$\sum_{i=0}^{r}(-1)^i\left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i\left[\partial_{i+2}L\langle x\rangle\right](t) = 0 \tag{4}$$

for all $t \in [a, +\infty[$, *and the* r *transversality conditions*

$$\lim_{T\to+\infty}\inf_{T'\geq T}\left\{\left(\partial_{r+2-(k-1)}L\langle x\rangle(T') + \sum_{i=1}^{k-1}(-1)^i D_q^i\left[\partial_{r+2-(k-1)+i}L\langle x\rangle\right](T')\cdot \Psi_i\right)\cdot D_q^{r-k}[x\circ\sigma^{k-1}](T')\right\} = 0, \tag{5}$$

$k = 1, 2, \ldots, r$, *where* $\Psi_i = \prod_{j=1}^{i}\left(\frac{1}{q}\right)^{r-(k-1)+(j-1)}$.

Proof. Using the notion of weak maximality, if x_* is optimal, then $V(\varepsilon) \leq 0$ for every $\varepsilon \in \mathbb{R}$. Since $V(0) = 0$, then 0 is an extremal of V. We prove that V is differentiable at $t = 0$; hence $V'(0) = 0$. Note that

$$\begin{aligned}
V'(0) &= \lim_{\varepsilon\to 0}\frac{V(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0}\lim_{T\to+\infty}\frac{V(\varepsilon,T)}{\varepsilon}\\
&= \lim_{T\to+\infty}\lim_{\varepsilon\to 0}\frac{V(\varepsilon,T)}{\varepsilon} \quad \text{(by hypotheses 1 and 2 and Theorem 3)}\\
&= \lim_{T\to+\infty}\lim_{\varepsilon\to 0}\inf_{T'\geq T} A(\varepsilon,T')\\
&= \lim_{T\to+\infty}\lim_{\varepsilon\to 0}\lim_{n\to+\infty} A(\varepsilon,T'_n) \quad \text{(by hypothesis 3)}\\
&= \lim_{T\to+\infty}\lim_{n\to+\infty}\lim_{\varepsilon\to 0} A(\varepsilon,T'_n) \quad \text{(by hypothesis 3 and Theorem 3)}\\
&= \lim_{T\to+\infty}\inf_{T'\geq T}\lim_{\varepsilon\to 0} A(\varepsilon,T') \quad \text{(by hypothesis 3)}\\
&= \lim_{T\to+\infty}\inf_{T'\geq T}\lim_{\varepsilon\to 0}\int_a^{T'}\frac{L\langle x_*+\epsilon\eta\rangle(t) - L\langle x_*\rangle(t)}{\varepsilon}\,d_qt\\
&= \lim_{T\to+\infty}\inf_{T'\geq T}\int_a^{T'}\lim_{\varepsilon\to 0}\frac{L\langle x_*+\epsilon\eta\rangle(t) - L\langle x_*\rangle(t)}{\varepsilon}\,d_qt\\
&= \lim_{T\to+\infty}\inf_{T'\geq T}\int_a^{T'}\left(\sum_{i=0}^{r}\partial_{i+2}L\langle x_*\rangle(t)\cdot D_q^i[\eta\circ\sigma^{r-i}](t)\right)d_qt,
\end{aligned}$$

and hence

$$\lim_{T\to+\infty}\inf_{T'\geq T}\int_a^{T'}\left(\sum_{i=0}^{r}\partial_{i+2}L\langle x_*\rangle(t)\cdot D_q^i[\eta\circ\sigma^{r-i}](t)\right)d_qt = 0.$$

Using Lemma 7, we conclude that

$$\sum_{i=0}^{r}(-1)^i\left(\frac{1}{q}\right)^{\frac{i(i-1)}{2}} D_q^i\left[\partial_{i+2}L\langle x_*\rangle\right](t) = 0$$

for all $t \in [a, +\infty[$, proving that x_* satisfies the Euler–Lagrange equation (4). By Lemma 9, for $k = 1, 2, \ldots, r$,

$$\lim_{T\to+\infty}\inf_{T'\geq T}\left\{\left(\partial_{r+2-(k-1)}L\langle x_*\rangle(T') + \sum_{i=1}^{k-1}(-1)^i D_q^i\left[\partial_{r+2-(k-1)+i}L\langle x_*\rangle\right](T')\cdot \Psi_i\right)\cdot D_q^{r-k}[\eta\circ\sigma^{k-1}](T')\right\} = 0, \tag{6}$$

where $\Psi_i = \prod_{j=1}^{i}\left(\frac{1}{q}\right)^{r-(k-1)+(j-1)}$. Consider η defined by $\eta(t) = \alpha(t)x_*(t)$, $t \in [a, +\infty[$, where $\alpha\colon [a, +\infty[\to \mathbb{R}$ satisfies $\alpha(a) = 0$, $D_q[\alpha](a) = 0, \ldots, D_q^{r-1}[\alpha](a) = 0$, and there exists $T_0 \in \mathcal{Q}$ such that $\alpha(t) = \beta \in \mathbb{R}\setminus\{0\}$ for all $t > T_0$. Note that

$\eta(a) = 0, D_q[\eta](a) = 0, \ldots, D_q^{r-1}[\eta](a) = 0$. Substituting η in Eq. (6), we conclude that

$$\lim_{T\to+\infty} \inf_{T'\geq T} \left\{ \left(\partial_{r+2-(k-1)} L\langle x_* \rangle (T') + \sum_{i=1}^{k-1} (-1)^i D_q^i \left[\partial_{r+2-(k-1)+i} L\langle x_* \rangle \right] (T') \cdot \Psi_i \right) \cdot D_q^{r-k} [x_* \circ \sigma^{k-1}](T') \right\} = 0,$$

proving that x_* satisfies the transversality condition (5) for all $k = 1, 2, \ldots, r$. □

Remark 1. For the simplest case $r = 1$, we obtain from Theorem 4 the Euler–Lagrange equation

$$D_q \left[s \to \partial_3 L \left(s, x(qs), D_q[x](s) \right) \right] (t) = \partial_2 L \left(t, x(qt), D_q[x](t) \right)$$

and the transversality condition $\lim_{T\to+\infty} \inf_{T'\geq T} \left\{ \partial_3 L \left(T', x(qT'), D_q[x](T') \right) \cdot x(T') \right\} = 0$. However, when $r > 1$, Theorem 4 gives more than one transversality condition. Indeed, for an infinite horizon variational problem of order r, one has r transversality conditions and, for each $k = 1, 2, \ldots, r$, the kth transversality condition has exactly k terms. This improves the results of [12].

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