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Computing desirable partitions in additively separable hedonic games

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A B S T R A C T

An important aspect in systems of multiple autonomous agents is the exploitation of synergies via coalition formation. *Additively separable hedonic games* are a fundamental class of coalition formation games in which each player has a value for any other player and the value of a coalition to a particular player is simply the sum of the values he assigns to the members of his coalition. In this paper, we consider a number of solution concepts from cooperative game theory, welfare theory, and social choice theory as criteria for desirable partitions in hedonic games. We then conduct a detailed computational analysis of computing, checking the existence of, and verifying stable, fair, optimal, and popular partitions for additively separable hedonic games.

1. Introduction

Topics concerning coalitions and coalition formation have come under increasing scrutiny of computer scientists. The reason for this may be obvious. For the proper operation of distributed and multiagent systems, cooperation may be required. At the same time, collaboration in very large groups may also lead to unnecessary overhead, which may even exceed the positive effects of cooperation. To model such situations formally, concepts from the social and economic sciences have proved to be very helpful and thus provide the mathematical basis for a better understanding of the issues involved.

Coalition formation games, as introduced by Drèze and Greenberg [18], provide a simple but versatile formal model that allows one to focus on coalition formation. In many situations it is natural to assume that a player's appreciation of a coalition structure only depends on the coalition he is a member of and not on how the remaining players are grouped. Initiated by Banerjee et al. [5] and Bogomolnaia and Jackson [9], much of the work on coalition formation now concentrates on these so-called *hedonic games*. Hedonic games are relevant in modeling many settings such as the formation of groups, clubs and societies [9] and online social networking [19]. The main focus in hedonic games has been on notions of stability for coalition structures such as *Nash stability*, *individual stability*, *contractual individual stability*, or *core stability*. *Additively separable hedonic games* (*ASHGs*) constitute a particularly natural and succinctly representable class of hedonic games. Each player in an ASHG has a value for any other player and the value of a coalition to a particular player is simply the sum of the values he assigns to the members of his coalition.

In this paper, we present a systematic investigation of stability, fairness, optimality, and popularity concepts in hedonic games. After presenting a cohesive bigger picture of the relationships between these concepts, we focus on ASHGs and characterize the complexity of computing and verifying stable, fair, optimal, and popular partitions. Apart from examining standard stability notions, we also analyze concepts from fair division and social choice theory in the context of coalition formation games and examine various standard criteria from the social sciences: *Pareto optimality*, *utilitarian social welfare*, *egalitarian social welfare*, *envy-freeness*, and *popularity*.

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In Section 4, we present a polynomial-time algorithm to compute a contractually individually stable partition. This is the first positive algorithmic result (with respect to one of the standard stability concepts put forward by Bogomolnaia and Jackson [9]) for general ASHGs with no restrictions on the preferences.

We strengthen the recent results of Sung and Dimitrov [37] by proving in Section 5 that checking whether the core or the strict core exists is NP-hard, even if the preferences of the players are symmetric.

In Section 6, we consider the complexity of computing welfare maximizing partitions. We show that computing a partition with maximum egalitarian social welfare is NP-hard. Similarly, computing a partition with maximum utilitarian social welfare is NP-hard, even when preferences are symmetric and strict. In contrast, we show that it can be checked efficiently whether there exists a partition in which each player is in one of his most favored coalitions.

In Section 7, Pareto optimality and the related stability concept of the contractual strict core (CSC) are studied. It is shown that verifying whether a partition is in the CSC is coNP-complete, even if the partition under question consists of the grand coalition. This is the first computational hardness result concerning CSC stability in hedonic games of any representation. The proof can be used to show that verifying whether the partition consisting of the grand coalition is Pareto optimal is coNP-complete. Furthermore, checking whether a given partition is Pareto optimal is coNP-complete, even when preferences are strict and symmetric. By contrast, we present a polynomial-time algorithm for computing a Pareto optimal partition when preferences are strict. Thus, we identify a natural problem in coalitional game theory where verifying a possible solution is presumably harder than actually finding one. Interestingly, computing an individually rational *and* Pareto optimal partition is NP-hard in general.

In Section 8, we consider complexity questions regarding envy-free partitions. It is observed that envy-freeness and individual rationality together can be easily achieved. Therefore, we turn to the combination of envy-freeness and other desirable criteria. Checking whether there exists a partition which is both Pareto optimal and envy-free is shown to be Σ_2^p-complete. We construct an example which exhibits the tension between stability and envy-freeness and use the example to prove that checking whether there exists a partition which is both envy-free and Nash stable is NP-complete, even when preferences are symmetric.

We finally consider the notion of popularity in Section 9. Popularity has previously been examined in resource allocation and captures the idea that any change in the outcome requires the approval of a majority of the players. We show that in contrast to results in matching theory and resource allocation [27,8], the problems of computing and verifying a popular partition are intractable for ASHGs.

ASHGs are a simple and fundamental class of coalition formation games and many of our computational results carry over to other classes of coalition formation games. For example, all of our computational hardness results imply computational hardness of the equivalent questions for *hedonic coalition nets*—a general representation scheme for hedonic games [19].

2. Related work

There has been considerable work in hedonic games on identifying restrictions on preferences that guarantee the existence of partitions that satisfy various notions of stability (see, e.g., [9,12]). Sung and Dimitrov [36] presented a taxonomy of stability concepts which includes the *contractual strict core*, the most general stability concept that is guaranteed to exist. Hedonic games encapsulate well-studied settings in matching theory such as stable marriage and stable roommates problems in which only coalitions of size two are admissible [32,25]. We refer to Hajduková [26] for a critical overview of hedonic games.

More recently, hedonic games have been examined from an algorithmic perspective. The focus has been on the computational complexity of computing stable or optimal partitions for different classes of hedonic games (see, e.g., [4,17,13,3]). Cechlárová [13] surveyed the algorithmic problems related to stable partitions in hedonic games in various representations. Ballester [4] showed that for hedonic games represented by *individually rational list of coalitions*, the complexity of checking whether core stable, Nash stable, or individual stable partitions exist is NP-complete. He also proved that every hedonic game admits a contractually individually stable partition. Coalition formation games have also received attention in the artificial intelligence community where the focus has generally been on computing optimal partitions for general *transferable utility* coalitional games without any combinatorial structure (see, e.g., [31,33]). In contrast, hedonic games are a simple class of *non-transferable utility* coalitional games. Elkind and Wooldridge [19] proposed a fully-expressive model to represent hedonic games which encapsulates well-known representations such as *individually rational list of coalitions* and *additive separability*.

Additive separability satisfies a number of desirable axiomatic properties [7,20]. Moreover, ASHGs are the non-transferable utility generalization of *graph games* as studied by Deng and Papadimitriou [15]. Due to their succinct and natural representation, ASHGs have recently attracted increased interest by computer scientists. Olsen [29] showed that checking whether a nontrivial Nash stable partition exists in an ASHG is NP-complete if preferences are non-negative and symmetric. This result was improved by Sung and Dimitrov [37] who showed that checking whether a core stable, strict core stable, Nash stable, or individually stable partition exists in a general ASHG is NP-hard. Dimitrov et al. [17] obtained positive algorithmic results for subclasses of ASHGs in which each player merely divides other players into friends and enemies. In another paper, Branzei and Larson [11] examined the tradeoff between stability and social welfare in ‘coalitional affinity games' which are equivalent to ASHGs. Finally, Gairing and Savani [21,22] showed that for ASHGs with symmetric preferences, computing partitions that satisfy some variants of individual-based stability is PLS-complete.

3. Preliminaries

In this section, we provide the terminology and notation required for our results.

A *hedonic coalition formation game* is a pair $(N, \succsim)$ where N is a set of players and $\succsim$ is a *preference profile* which specifies for each player $i \in N$ the preference relation $\succsim_i$, a reflexive, complete, and transitive binary relation on the set $\mathcal{N}_i = \{S \subseteq N \mid i \in S\}$. The statement $S \succ_i T$ denotes that i strictly prefers S over T whereas $S \sim_i T$ means that i is indifferent between coalitions S and T. A *partition* π is a partition of players N into disjoint coalitions. By $\pi(i)$, we denote the coalition of π that includes player i.

A game $(N, \succsim)$ is *separable* if for any player $i \in N$ and any coalition $S \in \mathcal{N}_i$ and for any player j not in S we have the following: $S \cup \{j\} \succ_i S$ if and only if $\{i, j\} \succ_i \{i\}$; $S \cup \{j\} \prec_i S$ if and only if $\{i, j\} \prec_i \{i\}$; and $S \cup \{j\} \sim_i S$ if and only if $\{i, j\} \sim_i \{i\}$.

In an *additively separable hedonic game* (*ASHG*) $(N, \succsim)$, each player $i \in N$ has value $v_i(j)$ for player j being in the same coalition as i and if i is in coalition $S \in \mathcal{N}_i$, then i gets utility $\sum_{j\in S\setminus\{i\}} v_i(j)$. For coalitions $S, T \in \mathcal{N}_i$, $S \succsim_i T$ if and only if $\sum_{j\in S\setminus\{i\}} v_i(j) \geqslant \sum_{j\in T\setminus\{i\}} v_i(j)$. Therefore an ASHG can be represented as (N, v).

A preference profile is *symmetric* if $v_i(j) = v_j(i)$ for any two players $i, j \in N$ and is *strict* if $v_i(j) \neq 0$ for all $i, j \in N$. We note that even when $v_i(j) \neq 0$ for all $i, j \in N$, a player can be indifferent between two coalitions. For any player i, let $F(i, A) = \{j \in A \mid v_i(j) > 0\}$ be the set of friends of player i within $A \subseteq N$.

Unless mentioned otherwise, all our results are for ASHGs. We now define important stability concepts used in the context of coalition formation games.

- We say that a partition π is *individually rational* (*IR*) if each player does as well as by being alone, i.e., for all $i \in N$, $\pi(i) \succsim_i \{i\}$. Individual rationality is a minimal requirement of stability.
- A partition π is *Nash stable* (*NS*) if no player can benefit by moving from his coalition to another (possibly empty) coalition, i.e., for all $i \in N$, $\pi(i) \succsim_i S \cup \{i\}$ for all $S \in \pi \cup \{\emptyset\}$.
- A partition π is *individually stable* (*IS*) if no player can benefit by moving from his coalition to another existing (possibly empty) coalition while not making the members of that coalition worse off, i.e., for all $i \in N$ if there exists a (possibly empty) coalition $S \neq \pi(i)$ s.t. $S \cup \{i\} \succ_i \pi(i)$ then there exists a $j \in S$ with $S \succ_j S \cup \{i\}$.
- A partition π is *contractually individually stable* (*CIS*) if no player can benefit by moving from his coalition to another existing (possibly empty) coalition while making no member of either coalition worse off. Formally, for every $i \in N$ if there exists a (possible empty) coalition $S \neq \pi(i)$ s.t. $S \cup \{i\} \succ_i \pi(i)$ then there exists a $j \in S$ with $S \succ_j S \cup \{i\}$ or there exists a $j' \in \pi(i)$ with $\pi(i) \succ_{j'} \pi(i) \setminus \{i\}$.
- We say that a coalition $S \subseteq N$ *strongly blocks* a partition π, if each player $i \in S$ strictly prefers S to his current coalition $\pi(i)$ in the partition π. A partition which admits no blocking coalition is said to be in the *core* (*C*).
- We say that a coalition $S \subseteq N$ *weakly blocks* a partition π, if each player $i \in S$ weakly prefers S to $\pi(i)$ and there exists at least one player $j \in S$ who strictly prefers S to his current coalition $\pi(j)$. A partition which admits no weakly blocking coalition is in the *strict core* (*SC*).
- A partition π is in the *contractual strict core* (*CSC*) if any weakly blocking coalition S makes at least one player $j \in N \setminus S$ worse off when breaking off.

We now formulate concepts from the social sciences, especially the literature on fair division, for the context of hedonic games. For a utility-based hedonic game $(N, \succsim)$ and partition π, we will denote the utility of player $i \in N$ by $u_\pi(i)$.

The different notions of fair, optimal, or popular partitions are defined as follows.[1]

- We say that a partition π is *perfect* if $\pi(i)$ is a most preferred coalition for all players $i \in N$ [3].
- The *utilitarian social welfare* of a partition is defined as the sum of individual utilities of the players: $u_{\text{util}}(\pi) = \sum_{i\in N} u_\pi(i)$. A *maximum utilitarian partition* maximizes the utilitarian social welfare.
- The *elitist social welfare* is given by the utility of the player that is best off: $u_{\text{elite}}(\pi) = \max\{u_\pi(i) \mid i \in N\}$. A *maximum elitist partition* maximizes the elitist social welfare.
- The *egalitarian social welfare* is given by the utility of the agent that is worst off: $u_{\text{egal}}(\pi) = \min\{u_\pi(i) \mid i \in N\}$. A *maximum egalitarian partition* maximizes the egalitarian social welfare.
- A partition π of N is *Pareto optimal* if there exists no partition π' of N which *Pareto dominates* π, that is for all $i \in N$, $\pi'(i) \succsim_i \pi(i)$ and there exists at least one player $j \in N$ such that $j \in N$, $\pi'(j) \succ_j \pi(j)$.
- *Envy-freeness* is a notion of fairness. In an *envy-free* (*EF*) partition, no player has incentive to replace another player. More formally, a partition π is envy-free if for all $i, j \in N$ such that $\pi(i) \neq \pi(j)$, it is the case that $\pi(i) \succsim_i (\pi(j) \setminus \{j\}) \cup \{i\}$.

[1] All welfare notions considered in this paper (utilitarian, elitist, and egalitarian) are based on the interpersonal comparison of utilities.

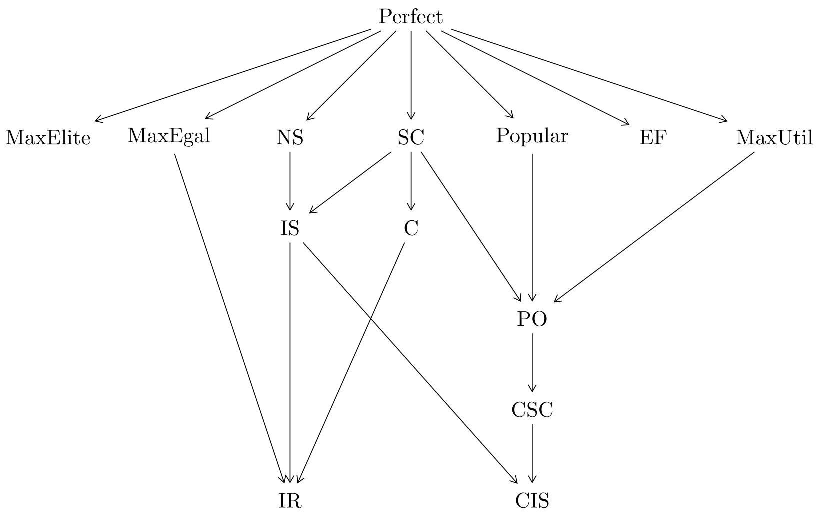

Fig. 1. Inclusion relationships between stability, fairness, optimality, and popularity concepts for hedonic coalition formation games. For example, every Nash stable partition is also individually stable.

- Let $\mathcal{D}(\pi, \pi') = |P(\pi, \pi')| - |P(\pi', \pi)|$ where $P(\pi, \pi')$ is the set of players who strictly prefer partition π to π'. Then, partition π is *popular* if $\mathcal{D}(\pi, \pi') \geqslant 0$ for all other partitions π'.

The inclusion relationships between stability concepts depicted in Fig. 1 follow from their definitions.

For a given stability, optimality, or fairness property α, the following natural computational problems can be formulated for hedonic games.

VERIFICATION: Given $(N, \succsim)$ and a partition π of N, does π satisfy α?
EXISTENCE: Does a partition satisfying α for a given $(N, \succsim)$ exist?
COMPUTATION: If a partition satisfying α for a given $(N, \succsim)$ exists, find one.

We consider ASHGs (additively separable hedonic games) in this paper. Unless mentioned otherwise, all our results are for ASHGs. Throughout the paper, we assume familiarity with basic concepts of computational complexity (see, e.g., [30]).

Observation 1. *It follows from the definitions that there exist partitions which yield maximum utilitarian social welfare, elitist social welfare, and egalitarian social welfare, respectively. Therefore,* EXISTENCE *trivially holds for any notion of maximum welfare.*

Similarly, the following observation indicates that VERIFICATION is easy for a number of solution concepts.

Observation 2. *It can be checked in polynomial time whether a partition is one of the following: individually rational, Nash stable, individually stable, contractual individually stable, and envy-free. For individual rationality, simply check whether each player has a non-negative payoff. For Nash stability, individual stability, and contractual individual stability, check for each player whether he has an incentive to move to another coalition in the partition. Finally, for envy-freeness, we need to check for each player whether he wants to replace another player in another coalition.*

4. Contractual individual stability

It is known that computing or even checking the existence of Nash stable or individually stable partitions in an ASHG is NP-hard [37]. On the other hand, a potential function argument can be used to show that at least one CIS partition exists for every hedonic game [4]. The potential function argument does not imply that a CIS partition can be computed in polynomial time. There are many cases in hedonic games, where a solution is guaranteed to exist but *computing* it is not feasible. For example, Bogomolnaia and Jackson [9] presented a potential function argument for the existence of a Nash stable partition for ASHGs with symmetric preferences. However, there are no known polynomial-time algorithms to *compute* such partitions and there is evidence that there may not be a polynomial-time algorithm [21]. In this section, we show that a CIS partition can be computed in polynomial time for ASHGs. The algorithm is formally described as Algorithm 1.

Theorem 1. *A CIS partition can be computed in polynomial time.*

Proof. Our algorithm to compute a CIS partition can be viewed as successively giving a priority token to players to form the best possible coalition among the remaining players or join the best possible coalition which tolerates the player. More

Algorithm 1 CIS partition of an ASHG.

```
Input: ASHG (N, v).
Output: CIS partition.
  i ← 0
  R ← N
  while R ≠ ∅ do
    a ∈ R
    h ← ∑_{b∈F(a,R)} v_a(b)
    z ← i + 1
    for k ← 1 to i do
      h' ← ∑_{b∈S_k} v_a(b)
      if (h < h') ∧ (∀b ∈ S_k, v_b(a) = 0) then
        h ← h'
        z ← k
      end if
    end for
    if z ≠ i + 1 then {a is latecomer}
      S_z ← {a} ∪ S_z
      R ← R \ {a}
    else {a is leader}
      i ← z
      S_i ← {a}
      S_i ← S_i ∪ F(a, R) {add leader's helpers}
      R ← R \ S_i
    end if
    while ∃j ∈ R such that ∀i ∈ S_z, v_i(j) ⩾ 0 and ∃i ∈ S_z, v_i(j) > 0 do
      R ← R \ {j}
      S_z ← S_z ∪ {j} {add needed players}
    end while
  end while
  return {S_1, ..., S_i}
```

precisely, the algorithm works as follows. Set variable R to N, S_0 to $\emptyset$, and consider an arbitrary player $a \in R$. Call a the *leader* of the first coalition S_i with $i = 1$. Move any player j such that $v_a(j) > 0$ from R to S_i. Such players are called the *leader's helpers*. Then keep moving any player from R to S_i which is tolerated by all players in S_i and strictly liked by at least one player in S_i. Call such players *needed players*. Now increment i and take another player a from among the remaining players R and check the maximum utility he can get from among R. If this utility is less than the utility which can be obtained by joining a previously formed coalition in $\{S_1, \ldots, S_{i-1}\}$, then send the player to such a coalition where he can get the maximum utility (as long as all players in the coalition tolerate the incoming player). Such players are called *latecomers*. Otherwise, form a new coalition S_i around a which is the best possible coalition for player a taking only players from the remaining players R. Repeat the process until all players have been dealt with and $R = \emptyset$. We prove by induction on the number of coalitions formed that no CIS deviation can occur in the resulting partition. The hypothesis is the following:

Consider the first $k+1$ formed coalitions $S_0, S_1, \ldots, S_k$. Then, the following two statements hold.

(i) *There is no CIS deviation for any player in $\bigcup_{i\in\{0,\ldots,k\}} S_i$.*
(ii) *There is no CIS deviation for any player in $N \setminus \bigcup_{i\in\{0,\ldots,k\}} S_i$ to a non-empty coalition in $\{S_0, S_1, \ldots, S_k\}$.*

Base case. Clearly, the statement is trivially satisfied if $k = 0$.

Induction step. Assume that the hypothesis is true. Then we prove that the same holds for the formed coalitions $S_0, \ldots, S_k, S_{k+1}$. By the hypothesis, we know that players cannot leave coalitions $S_0, \ldots, S_k$. Now consider S_{k+1}. The leader a of S_{k+1} is either not allowed to join one of the coalitions in $\{S_1, \ldots, S_k\}$ or if he is, he has no incentive to join it. Player a would already have been member of S_i for some $i \in \{1, \ldots, k\}$ if one of the following was true:

- There is some $i \in \{1, \ldots, k\}$ such that the leader of S_i likes a.
- There is some $i \in \{1, \ldots, k\}$ such that for all $b \in S_i$, $v_b(a) \geqslant 0$ and there exists $b \in S_i$ such that $v_b(a) > 0$.
- There is some $i \in \{1, \ldots, k\}$, such that for all $b \in S_i$, $v_b(a) = 0$ and $\sum_{b\in S_i} v_a(b) > \sum_{b\in F(i, N\setminus\bigcup_{j=1}^{k} S_j)} v_a(b)$ and $\sum_{b\in S_i} v_a(b) \geqslant \sum_{b\in S_j} v_a(b)$ for all $j \in \{1, \ldots, k\}$.

Therefore a has no incentive or is not allowed to move to another S_j for $j \in \{1, \ldots, k\}$. Also a will have no incentive to move to any coalition formed after $S_1, \ldots, S_{k+1}$ because he can do strictly better in S_{k+1}. Similarly, a's helpers are not allowed to leave S_{k+1} even if they have an incentive to. Their movement out of S_{k+1} will cause a to become less happy. Also each needed player in S_{k+1} is not allowed to leave because at least one player in S_k likes him. Now consider a latecomer l in S_{k+1}. Latecomer l gets strictly less utility in any coalition $C \subseteq N \setminus \bigcup_{i=0}^{k+1} S_i$. Therefore l has no incentive to leave S_{k+1}.

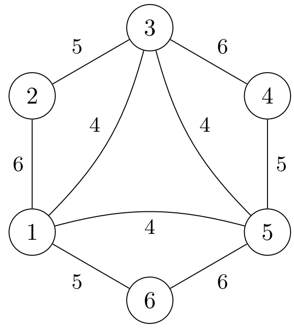

Fig. 2. Graphical representation of Example 1. All edges not shown in the figure have weight −33.

Finally, we prove that there exists no player $x \in N \setminus \bigcup_{i=0}^{k+1} S_i$ such that x has an incentive to and is allowed to join S_i for $i \in \{1, \ldots, k+1\}$. By the hypothesis, we already know that x does not have an incentive or is allowed to a join a coalition S_i for $i \in \{1, \ldots, k\}$. Since x is not a latecomer for S_{k+1}, x either does not have an incentive to join S_{k+1} or is disliked by at least one player in S_{k+1}. □

Algorithm 1 may also prove useful as a preprocessing or intermediate step in algorithms for computing other types of stable partitions in hedonic games.

5. Core and strict core

The core and the strict core are two of the most fundamental stability concepts in cooperative game theory. For transferable utility cooperative games, both concepts coincide. For hedonic games, this is not necessarily the case. Recently, Sung and Dimitrov [37] showed that for ASHGs checking whether a core stable or strict core stable partition exists is NP-hard in the strong sense. Their reduction relied on the asymmetry of the players' preferences. We prove that even with symmetric preferences, checking whether a core stable or a strict core stable partition exists is NP-hard in the strong sense. Symmetry is a natural, but rather strong condition, that yields more positive existence results and can often be exploited algorithmically. For example, it is known that for ASHGs, computing a Nash stable partition is NP-hard whereas the same problem is PLS-complete if the preferences are symmetric [9,21].

We first present an example of a six-player ASHG with symmetric preferences for which the core (and thereby the strict core) is empty.

Example 1. Consider a six player symmetric ASHG adapted from an example by Banerjee et al. [5] where

- $v_1(2) = v_3(4) = v_5(6) = 6$;
- $v_1(6) = v_2(3) = v_4(5) = 5$;
- $v_1(3) = v_3(5) = v_1(5) = 4$;
- $v_1(4) = v_2(5) = v_3(6) = -33$; and
- $v_2(4) = v_2(6) = v_4(6) = -33$;

as depicted in Fig. 2.

It can be checked that no partition is core stable for the game. Note that if $v_i(j) = -33$, then i and j cannot be in the same coalition of a core stable partition. Also, players can do better than in a partition of singleton players. We note that the following are the individually rational coalitions: $\{1,2\}$, $\{1,3\}$, $\{1,5\}$, $\{1,6\}$, $\{1,2,3\}$, $\{1,3,5\}$, $\{1,5,6\}$, $\{2,3\}$, $\{3,4\}$, $\{3,4,5\}$, $\{3,5\}$, $\{4,5\}$ and $\{5,6\}$.

Consider the partition

$$\pi = \big\{\{1,2\}, \{3,4,5\}, \{6\}\big\}.$$

Then, $u_\pi(1) = u_\pi(2) = 6$, $u_\pi(3) = 10$, $u_\pi(4) = 11$, $u_\pi(5) = 9$, and $u_\pi(6) = 0$.

Out of the individually rational coalitions listed above, the only weakly (and also strongly) blocking coalition is $\{1,5,6\}$ in which player 1 gets utility 9, player 5 gets utility 10, and player 6 gets utility 11. We note that the coalition $\{1,2,3\}$ is not a weakly or strongly blocking coalition because player 3 gets utility 9. Similarly $\{1,3,5\}$ is not a weakly or strongly blocking coalition because both player 3 and player 5 are worse off. One way to prevent that $\{1,5,6\}$ is weakly blocking is to provide some incentive for player 6 not to deviate with 1 and 5. This idea will be used in the proof of Theorem 2.

We now define a problem that is NP-complete is the strong sense.

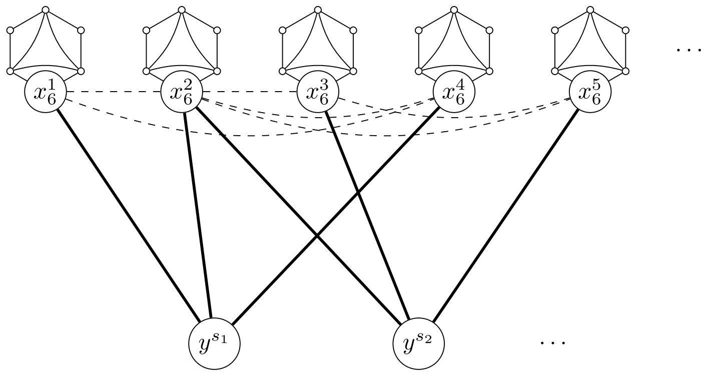

Fig. 3. Graphical representation of an ASHG derived from an instance of E3C in the proof of Theorem 2. Symmetric utilities other than −33 are given as edges. Thick edges indicate utility $10\frac{1}{4}$ and dashed edges indicate utility 1/2. Each hexagon at the top looks like the one in Fig. 2.

Name: ExactCoverBy3Sets (E3C)
Instance: A pair (R, S), where R is a set and S is a collection of subsets of R such that $|R| = 3m$ for some positive integer m and $|s| = 3$ for each $s \in S$.
Question: Is there a sub-collection $S' \subseteq S$ that is a partition of R?

It is known that E3C remains NP-complete even if each $r \in R$ occurs in at most three members of S [24]. We will use this assumption in the proof of Theorem 2, which will be shown by a reduction from E3C.

Theorem 2. *Checking whether a core stable or a strict core stable partition exists is NP-hard in the strong sense, even when preferences are symmetric.*

Proof. Let (R, S) be an instance of E3C where $r \in R$ occurs in at most three members of S. We reduce (R, S) to an ASHG with symmetric preferences (N, v) in which there is a player y^s corresponding to each $s \in S$ and there are six players $x_1^r, \dots, x_6^r$ corresponding to each $r \in R$. These players have preferences over each other in exactly the way players $1, \dots, 6$ have preferences over each other as in Example 1.

So, $N = \{x_1^r, \dots, x_6^r \mid r \in R\} \cup \{y^s \mid s \in S\}$. We assume that all preferences are symmetric. The player preferences are as follows (see also Fig. 3):

- For $i \in R$,
  $v_{x_1^i}(x_2^i) = v_{x_3^i}(x_4^i) = v_{x_5^i}(x_6^i) = 6$;
  $v_{x_1^i}(x_6^i) = v_{x_2^i}(x_3^i) = v_{x_4^i}(x_5^i) = 5$; and
  $v_{x_1^i}(x_3^i) = v_{x_3^i}(x_5^i) = v_{x_1^i}(x_5^i) = 4$;
- For any $s = \{k, l, m\} \in S$,
  $v_{x_6^k}(x_6^l) = v_{x_6^l}(x_6^k) = v_{x_6^k}(x_6^m) = v_{x_6^m}(x_6^k) = v_{x_6^l}(x_6^m) = v_{x_6^m}(x_6^l) = 1/2$; and
  $v_{x_6^k}(y^s) = v_{x_6^l}(y^s) = v_{x_6^m}(y^s) = 10\frac{1}{4}$;
- $v_i(j) = -33$ for any $i, j \in N$ for valuations not defined above.

We prove that (N, P) has a non-empty strict core (and thereby also a non-empty core) if and only if there exists an $S' \subseteq S$ such that S' is a partition of R.

Assume that there exists an $S' \subseteq S$ such that S' is a partition of R. Then we prove that there exists a strict core stable (and thereby core stable) partition π where π is defined as follows:

$$\bigl\{\{x_1^i, x_2^i\}, \{x_3^i, x_4^i, x_5^i\} \bigm| i \in R\bigr\} \cup \bigl\{\{y^s\} \bigm| s \in S \setminus S'\bigr\} \cup \bigl\{\{y^s \cup \{x_6^i \bigm| i \in s\}\} \bigm| s \in S'\bigr\}.$$

For all $i \in R$, $u_\pi(x_1^i) = u_\pi(x_2^i) = 6$, $u_\pi(x_3^i) = 10$, $u_\pi(x_4^i) = 11$, $u_\pi(x_5^i) = 9$, and $u_\pi(x_6^i) = 1/2 + 1/2 + 10\frac{1}{4} = 11\frac{1}{4} > 11$.

Also $u_\pi(y^s) = 3 \cdot (10\frac{1}{4}) = 30\frac{3}{4}$ for all $s \in S'$ and $u_\pi(y^s) = 0$ for all $s \in S \setminus S'$. We see that each player's utility is non-negative. Therefore there is no incentive for any player to deviate and form a singleton coalition. From Example 1 we also know that the only possible strongly blocking (and weakly blocking) coalition is $\{x_1^i\, x_5^i, x_6^i\}$ for any $i \in R$. However, x_6^i has no incentive to be part $\{x_1^i, x_5^i, x_6^i\}$ because $u_\pi(x_6^i) = 11\frac{1}{4}$ and $v_{x_6^i}(x_5^i) + v_{x_6^i}(x_1^i) = 6 + 5 = 11$. Also x_1^i and x_5^i have no incentive to join $\pi(x_6^i)$ because their new utility will become negative because of the presence of the y^s player. Assume for the sake of contradiction that π is not core stable and x_6^i can deviate with some x_6^js. x_6^i, however, can only deviate with a maximum of six other players of type x_6^j because $i \in R$ is present in a maximum of three elements in S. In this case x_6^i gets a maximum utility of only 1. Therefore π is in the strict core (and thereby in the core).

We now assume that there exists a partition which is core stable. Then we prove that there exists an $S' \subseteq S$ such that S' is a partition of R. For any $s = \{k, l, m\} \in S$, the new utilities created due to the reduction gadget are only beneficial to y^s, x_6^k, x_6^l, and x_6^m. We already know that the only way the partition is core stable is if x_6^i can be provided disincentive to deviate with x_5^i and x_1^i. The claim is that each x_6^i needs to be in a coalition with exactly one y^s such that $i \in s \in S$ and exactly two other players x_6^j and x_6^k such that $\{i, j, k\} = s \in S$. We first show that x_6^i needs to be with exactly one y^s such that $i \in s \in S$. Player x_6^i needs to be with at least one such y^s. If x_6^i is only with other x_6^js, then we know that x_6^i gets a maximum utility of only $6 \cdot 1/2 = 3$. Also, player x_6^i cannot be in a coalition with y^s and $y^{s'}$ such that $i \in s$ and $i \in s'$ because both y^s and $y^{s'}$ then get negative utility. Each x_6^i also needs to be with at least 2 other players x_6^j and x_6^k where j and k are also members of s. If x_6^i is with at least three players x_6^j, x_6^k and x_6^k, then there is one element among $a \in \{j, k, l\}$ such that $a \notin s$. Therefore y^s and x_6^a hate each other and the coalition $\{y^s, x_6^i, x_6^j, x_6^k, x_6^k\}$ is not even individually rational. Therefore for the partition to be core stable each x_6^i has to be with exactly one y^s such that $i \in s$ and least 2 other players x_6^j and x_6^k where j and k are also members of s. This implies that there exists an $S' \subseteq S$ such that S' is a partition of R. □

We now turn to the problem of verifying core or strict core stable partitions. For ASHGs, the problem of testing the core membership of a partition is coNP-complete [35]. The same reduction as in [35] can also be used to prove that testing strict core membership of a partition is coNP-complete, even when preferences are symmetric.

Theorem 3. *Verifying whether a partition is strict-core stable is coNP-complete, even if preferences are symmetric and $v_i(j) \in \{1, -n\}$ for all $i \neq j$.*

6. Maximizing social welfare

In this section, we examine the complexity of maximizing social welfare in ASHGs. We first examine utilitarian welfare. It is seen that the problems of computing and verifying a maximum utilitarian partition are computationally intractable.

Theorem 4. *For maximum utilitarian partitions, the following statements hold.*

(i) *Computing a maximum utilitarian partition is NP-hard in the strong sense, even when preferences are symmetric and strict and $v_i(j) \in \{-1, +1\}$ for all $i, j \in N$.*
(ii) *Verifying a maximum utilitarian partition is coNP-complete in the strong sense.*

Proof. We prove Theorem 4 by a reduction from the MaxCut problem. Before defining this problem, recall that a *cut* is a partition of the vertices of a graph into two disjoint subsets. The *cut-set* of the cut is the set of edges whose end points are in different subsets of the partition. In a weighted graph, the *weight of the cut* is the sum of the weights of the edges in the cut-set. Then, MaxCut is the following problem:

Name: MaxCut
Instance: An undirected weighted graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}^+$ and an integer k.
Question: Does there exist a cut of weight at least k in G?

It is well known that MaxCut is an NP-complete problem. It follows that computing a maxcut is NP-hard. It can also be shown that the verification problem VerifyMaxCut is coNP-complete: given a feasible edge cut, decide whether the edge cut is a maxcut. This follows from a general argument by Schulz [34, p. 20] concerning the optimization of a linear function over a 0/1-polytope.

We present a polynomial-time reduction from MaxCut to the problem of checking whether there exists a partition with utilitarian social welfare of at least k and thereby to the problem of computing a maximum utilitarian partition. It also serves as a polynomial-time reduction from VerifyMaxCut to the problem of verifying a maximum utilitarian partition.

Consider a connected undirected graph $G = (V, E)$ and positive weights $w(i, j)$ for each edge (i, j). Let $W = \sum_{(i,j)\in E} w(i, j)$. Consider the following method which in polynomial time reduces $G = (V, E)$ to an ASHG (N, v) with $|V| + 2$ players $N = \{m_1, \ldots, m_{|V|}, s_1, s_2\}$. For any two players m_i and m_j, $v_{m_i}(m_j) = v_{m_j}(m_i) = -w(i, j)$. For any player m_i and player s_j, $v_{m_i}(s_j) = v_{s_j}(m_i) = W$. Also $v_{s_1}(s_2) = v_{s_2}(s_1) = -W(|V| + 1)$. Then, it is easy to show that (A, B) is a cut of weight k if and only if partition $\pi' = \{\{s_1 \cup \{m_i \mid i \in A\}\}, \{s_2 \cup \{m_i \mid i \in B\}\}\}$ is a partition with utilitarian social welfare $W(3|V| + 1) + k$, for game (N, v).

Thus, we have established that computing and verifying a maximum utilitarian partition is NP-hard and coNP-complete, respectively. For the restricted case when preferences are symmetric and strict and $v_i(j) \in \{-1, +1\}$ for all $i, j \in N$, *computing* a maximum utilitarian partition is still NP-hard. This follows from the observation that computing a maximum utilitarian partition for strict and symmetric preferences is NP-hard because it is equivalent to the NP-hard problem of maximizing agreements in the context of correlation clustering [6]. □

In contrast, computing and verifying a maximum elitist partition is much easier.

Theorem 5. *A maximum elitist partition can be computed or verified in polynomial time.*

Proof. Recall that for any player i, $F(i,N) = \{j \in N \mid v_i(j) > 0\}$. Let $f(i) = \sum_{j \in F(i,N)} v_i(j)$. Both $F(i,N)$ and $f(i)$ can be computed in linear time. Let $k \in N$ be the player such that $f(k) \geqslant f(i)$ for all $i \in N$. Then $\pi = \{\{\{k\} \cup F(k,N)\}, N \setminus \{\{k\} \cup F(k,N)\}\}$ is a partition which maximizes the elitist social welfare. As a corollary, we can verify whether a partition π has maximum elitist social welfare by computing a partition π^* with maximum elitist social welfare and comparing $u_{\text{elite}}(\pi)$ with $u_{\text{elite}}(\pi^*)$. □

We now turn our attention to maximum egalitarian partition. Just like maximizing the utilitarian social welfare, maximizing the egalitarian social welfare is computationally hard.

Theorem 6. *For maximum egalitarian partitions, the following statements hold.*

(i) *Computing a maximum egalitarian partition is NP-hard in the strong sense.*
(ii) *Furthermore, verifying a maximum egalitarian partition is coNP-complete.*

Proof.

(i) We provide a polynomial-time reduction from the NP-hard problem MaxMinMachineCompletionTime [16,38]:

Name: MaxMinMachineCompletionTime
Instance: A set of m identical machines $M = \{M_1, \ldots, M_m\}$, a set of n independent jobs $J = \{J_1, \ldots, J_n\}$ where job J_i has processing time p_i.
Output: Allot jobs to the machines such that the minimum processing time (without machine idle times) of all machines is maximized.

Let I be an instance of MaxMinMachineCompletionTime and let $P = \sum_{i=1}^{n} p_i$. We assume that $n \geqslant m$, because otherwise any allocation of jobs to machines results in a minimum processing time of zero. From I we construct an ASHG (N, v) with $N = \{i \mid M_i \in M\} \cup \{s_i \mid J_i \in J\}$ and the preferences of the players are as follows: for all $i = 1, \ldots, m$ and all $j = 1, \ldots, n$ let $v_i(s_j) = p_j$ and $v_{s_j}(i) = P$. Also, for $1 \leqslant i, i' \leqslant m, i \neq i'$ let $v_i(i') = -(P+1)$ and for $1 \leqslant j, j' \leqslant n$, $j \neq j'$ let $v_{s_j}(v_{s_{j'}}) = 0$. Each player i corresponds to machine M_i and each player s_j corresponds to job J_j.
Let π be the partition which maximizes $u_{\text{egal}}(\pi)$. We show that players $1, \ldots, m$ are in separate coalitions and each player s_j is in $\pi(i)$ for some $1 \leqslant i \leqslant m$. We can do so by proving two claims. The first claim is that for $i, j \in \{1, \ldots, m\}$ such that $i \neq j$, we have that $i \notin \pi(j)$. Assume there exist exactly two players i and j for which this is not the case. Then we know that $u_\pi(i) = -(P+1) + \sum_{s_j \in \pi(i)} p_j$. Since $\sum_{s_j \in \pi(i)} p_j \leqslant P$, we know that $u_\pi(i) = u_\pi(j) < 0$, $u_\pi(a) \geqslant 0$ for all $a \in N \setminus \{i, j\}$ and thus $u_{\text{egal}}(\pi) < 0$. However, if i deviates and forms a singleton coalition in new partition π', then $u_{\pi'}(i) = 0$ and $u_{\pi'}(j) \geqslant 0$ and the utility of other players has not decreased. Therefore, $u_{\text{egal}}(\pi') \geqslant 0$, a contradiction.
The second claim is that each player s_j is in a coalition with a player i. Assume this was not the case so that there exists a non-empty set S' of such players. Since we already know that all is are in separate coalitions, then $u_\pi(a) > 0$ for all $a \in N \setminus S'$ and $u_{\text{egal}}(\pi) = u_\pi(s_j) = 0$ for all $s_j \in S'$. But then each such s_j can deviate and join $\pi(i)$ for any $1 \leqslant i \leqslant m$ to form a new partition π'. In doing so, the utility of no player decreases and $u_{\pi'}(s_j) > 0$. If this is done for all players from S', we have $u_{\text{egal}}(\pi') > 0$ for the new partition π' which is a contradiction.
A job allocation Alloc(π) corresponds to a partition π where s_j is in $\pi(i)$ if job J_j is assigned to M_i for all j and i. Note that the utility $u_\pi(i) = \sum_{s_j \in \pi(i)} v_i(s_j) = \sum_{s_j \in \pi(i)} p_j$ of a player corresponds to the total completion time of all jobs assigned to M_i according to Alloc(π). Let π^* be a maximum egalitarian partition. Assume that there is another partition π' and Alloc(π') induces a strictly greater minimum completion time. We know that $u_{\pi^*}(s_j) = u_{\pi''}(s_j) = P$ for all $1 \leqslant j \leqslant n$ and $u_{\pi^*}(i) \leqslant P$ for all $1 \leqslant i \leqslant m$. But then from the assumption we have $u_{\text{egal}}(\pi') > u_{\text{egal}}(\pi^*)$ which is a contradiction.

(ii) We now prove that verifying a maximum egalitarian partition is coNP-complete. Both MaxMinMachineCompletionTime and computing a maximum egalitarian partition do not appear to be problems which involve the optimization of a linear function over a 0/1-*polytope*. Nonetheless Deuermeyer et al. [16] observed that MaxMinMachineCompletionTime is NP-hard by a reduction from integer partition—a well-known NP-complete problem.

Name: Partition
Instance: A set of k positive integer weights $A = \{a_1, \ldots, a_k\}$ such that $\sum_{a_i \in A} a_i = W$.
Question: Is it possible to partition A, into two subsets $A_1 \subseteq A$, $A_2 \subseteq A$ so that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$ and $\sum_{a_i \in A_1} a_i = \sum_{a_i \in A_2} a_i = W/2$?

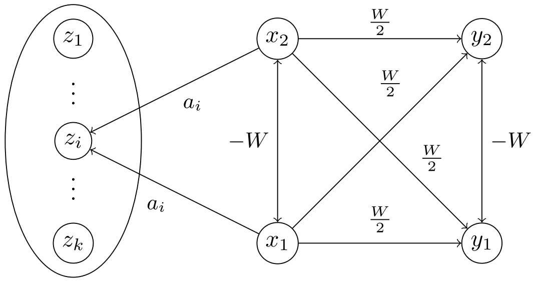

Fig. 4. Graphical representation of the ASHG in the proof of Theorem 8. For all $i \in \{1, \dots, k\}$, an edge from x_1 to z_i and from x_2 to z_i has weight a_i. All other edges not shown in the figure have weight zero.

The problem Partition is equivalent to the following optimization problem over a 0/1-polytope: compute a subset $S \subseteq A$ such that $\sum_{a \in S} a \geqslant W/2$ and which minimizes $\sum_{a \in S} a$. Therefore, we can again utilize the general argument by Schulz [34, p. 20] concerning the optimization of a linear function over a 0/1-polytope that if computing an optimal solution is NP-hard, then verifying the optimality of the feasible solution is as hard. It follows that verifying a maximum egalitarian partition is coNP-complete. □

Recall that a partition π is *perfect* if $\pi(i)$ is a most preferred coalition for all players i.

Theorem 7. *For separable hedonic games, the existence of a perfect partition can be checked in polynomial time. Moreover, a perfect partition can be verified in polynomial time.*

Proof. The idea behind the algorithm is to build up coalitions and ensure that player i and $F(i, N)$, all the player i likes, are in the same coalition. While ensuring this, if there is a player j and a player $j' \in E(j)$ (disliked by j) in the same coalition, then return 'no'. In each step, either a player gets all the players he likes or it is found that some player is in the same coalition as a player he strictly dislikes.

For a given partition π, it can easily be checked whether for all $i \in N$, $\pi(i) \succsim_i S$ for all $S \in \mathcal{N}_i$. This is only possible if $F(i, N) \subseteq \pi(i)$ and $\pi(i) \cap E(i) = \emptyset$. □

7. Contractual strict core and Pareto optimality

In this section, we present a number of results concerning CSC stability and Pareto optimality. The complexity of Pareto optimality has already been considered in several settings such as house allocation (see, e.g., [2]). Bouveret and Lang [10] examined the complexity of Pareto optimal allocations in resource allocation problems. Although the resource allocation model with additive utilities has some similarities with ASHGs, there are some distinct differences. The problem of computing Pareto optimal allocations is already trivial in resource allocation: give each object to the agent who values it the most. In the context of coalition formation, the question is more interesting. Furthermore, our hardness results for the grand coalition (Theorem 9) or symmetric preferences (Theorem 10) have no equivalent in the context of resource allocation.

Firstly, we prove that verifying whether a partition is CSC stable is coNP-complete. Interestingly, coNP-completeness holds even if the partition in question consists of the grand coalition. The proof of Theorem 8 is by a reduction from the weakly NP-complete problem Partition.

Theorem 8. *Verifying whether the partition consisting of the grand coalition is CSC stable is weakly coNP-complete.*

Proof. The problem is clearly in coNP because a partition π' resulting from a CSC deviation from $\{N\}$ is a succinct certificate that $\{N\}$ is not CSC stable. We prove NP-hardness of deciding whether the grand coalition is *not* CSC stable by a reduction from Partition. We can reduce an instance I of Partition to an instance $I' = ((N, v), \pi)$ where $\pi = \{N\}$ and (N, v) is an ASHG defined in the following way (see also Fig. 4):

- $N = \{x_1, x_2, y_1, y_2, z_1, \dots, z_k\}$;
- $v_{x_1}(y_1) = v_{x_1}(y_2) = v_{x_2}(y_1) = v_{x_2}(y_2) = W/2$;
- $v_{x_1}(z_i) = v_{x_2}(z_i) = a_i$ for all $i \in \{1, \dots, k\}$;
- $v_{x_1}(x_2) = v_{x_2}(x_1) = -W$;
- $v_{y_1}(y_2) = v_{y_2}(y_1) = -W$; and
- $v_a(b) = 0$ for any $a, b \in N$ for which $v_a(b)$ is not already defined.

We see that $u_\pi(x_1) = u_\pi(x_2) = W$, $u_\pi(y_1) = u_\pi(y_2) = -W$, $u_\pi(z_i) = 0$ for all $i \in \{1, \dots, k\}$. We show that π is not CSC stable if and only if I is a 'yes' instance of Partition. Assume I is a 'yes' instance of Partition and there exists an $A_1 \subseteq A$ such that $\sum_{a_i \in A_1} a_i = W/2$. Then, form the partition

$$\pi' = \big\{\{x_1, y_1\} \cup \{z_i \mid a_i \in A_1\}, \{x_2, y_2\} \cup \{z_i \mid a_i \in N \setminus A_1\}\big\}.$$

Hence,

- $u_{\pi'}(x_1) = u_{\pi'}(x_2) = W$;
- $u_{\pi'}(y_1) = u_{\pi'}(y_2) = 0$; and
- $u_{\pi}(z_i) = 0$ for all $i \in \{1, \dots, k\}$.

The coalition $C_1 = \{x_1, y_1\} \cup \{z_i \mid a_i \in A_1\}$ can be considered as a coalition that leaves the grand coalition so that all players in N do as well as before and at least one player in C_1, namely y_1, gets strictly more utility. Also, the departure of C_1 does not make any player in $N \setminus C_1$ worse off.

Assume that I is a 'no' instance of PARTITION and there exists no $A_1 \subseteq A$ such that $\sum_{a_i \in A_1} a_i = W/2$. We show that no CSC deviation is possible from π by considering all different possibilities for a CSC blocking coalition C:

(i) $x_1, x_2, y_1, y_2 \notin C$;
(ii) $x_1, x_2 \notin C$ and there exists $y \in \{y_1, y_2\}$ such that $y \in C$;
(iii) $x_1, x_2, y_1, y_2 \in C$;
(iv) $x_1, x_2 \in C$ and $|C \cap \{y_1, y_2\}| \leqslant 1$; and
(v) there exists $x \in \{x_1, x_2\}$ and $y \in \{y_1, y_2\}$ such that $x, y \in C$, $\{x_1, x_2\} \setminus x \nsubseteq C$, and $\{y_1, y_2\} \setminus y \nsubseteq C$.

We show that in each of the cases, C is a not a valid CSC blocking coalition.

(i) If C is empty, then there exists no CSC blocking coalition. If C is not empty, then x_1 and x_2 get strictly less utility when a subset of $\{z_1, \dots, z_k\}$ deviates.
(ii) In this case, both x_1 and x_2 get strictly less utility when $y \in \{y_1, y_2\}$ leaves N.
(iii) If $\{z_1, \dots, z_k\} \subset C$, then there is no deviation as $C = N$. If there exists a $z_i \in \{z_1, \dots, z_k\}$ such that $z_i \notin C$, then x_1 and x_2 get strictly less utility than in N.
(iv) If $|C \cap \{y_1, y_2\}| = 0$, then the utility of no player increases. If $|C \cap \{y_1, y_2\}| = 1$, then the utility of y_1 and y_2 increases but the utility of x_1 and x_2 decreases.
(v) Consider $C = \{x, y\} \cup S$ where $S \subseteq \{z_1, \dots, z_k\}$. Without loss of generality, we can assume that $x = x_1$ and $y = y_1$. We know that y_1 and y_2 get strictly more utility because they are now in different coalitions. Since I is a 'no' instance of PARTITION, we know that there exists no S such that $\sum_{a \in S} v_{x_1}(a) = W/2$. If $\sum_{a \in S} v_{x_1}(a) > W/2$, then $u_{\pi}(x_2) < W$. If $\sum_{a \in S} v_{x_1}(a) < W/2$, then $u_{\pi}(x_1) < W$.

Thus, if I is a 'no' instance of PARTITION, then there exists no CSC deviation. □

From the proof of Theorem 8, it can be seen that π is not Pareto optimal if and only if I is a 'yes' instance of PARTITION.

Theorem 9. *Verifying whether the partition consisting of the grand coalition is Pareto optimal is coNP-complete.*

We show that checking whether a partition is Pareto optimal is hard even under severely restricted settings.

Theorem 10. *The problem of checking whether a partition is Pareto optimal is coNP-complete in the strong sense, even when preferences are symmetric and strict.*

Proof. The problem is clearly in coNP as another partition which Pareto dominates the given partition π is a witness that π is not Pareto optimal. The reduction is from the NP-complete problem E3C (EXACT-3-COVER) to deciding whether a given partition is Pareto dominated by another partition or not. Recall that in E3C, an instance is a pair (R, S), where $R = \{1, \dots, |R|\}$ is a set and S is a collection of subsets of R such that $|R| = 3m$ for some positive integer m and $|s| = 3$ for each $s \in S$. The question is whether there is a sub-collection $S' \subseteq S$ which is a partition of R.

It is known that E3C remains NP-complete even if each $r \in R$ occurs in at most three members of S [24]. Let (R, S) be an instance of E3C. (R, S) can be reduced to an instance $((N, v), \pi)$, where (N, v) is an ASHG defined in the following way. Let $N = \{w^s, x^s, y^s \mid s \in S\} \cup \{z^r \mid r \in R\}$. The players preferences are symmetric and strict and are defined as follows (as also depicted in Fig. 5):

- $v_{w^s}(x^s) = v_{x^s}(y^s) = 3$ for all $s \in S$;
- $v_{y^s}(w^s) = v_{y^s}(w^{s'}) = -1$ for all $s, s' \in S$;
- $v_{y^s}(z^r) = 1$ if $r \in s$ and $v_{y^s}(z^r) = -7$ if $r \notin s$;
- $v_{z^r}(z^{r'}) = 1/(|R| - 1)$ for any $r, r' \in R$ such that $r \neq r'$; and
- $v_a(b) = -7$ for any $a, b \in N$ and $a \neq b$ for which $v_a(b)$ is not already defined.

The partition π in the instance $((N, v), \pi)$ is $\{\{x^s, y^s\}, \{w^s\} \mid s \in S\} \cup \{\{z^r \mid r \in R\}\}$. We see that the utilities of the players are as follows: $u_\pi(w^s) = 0$ for all $s \in S$; $u_\pi(x^s) = u_\pi(y^s) = 3$ for all $s \in S$; and $u_\pi(z^r) = 1$ for all $r \in R$.

Assume that there exists $S' \subseteq S$ such that S' is a partition of R. Then we prove that π is not Pareto optimal and there exists another partition π' of N which Pareto dominates π. We form another partition $\pi' = \{\{x^s, w^s\} \mid s \in S'\} \cup \{\{y^s, z^i, z^j, z^k\} \mid s \in S' \wedge i, j, k \in s\} \cup \{\{x^s, y^s\}, \{w^s\} \mid s \in (S \setminus S')\}$. In that case, $u_{\pi'}(w^s) = 3$ for all $s \in S'$; $u_{\pi'}(w^s) = 0$ for all

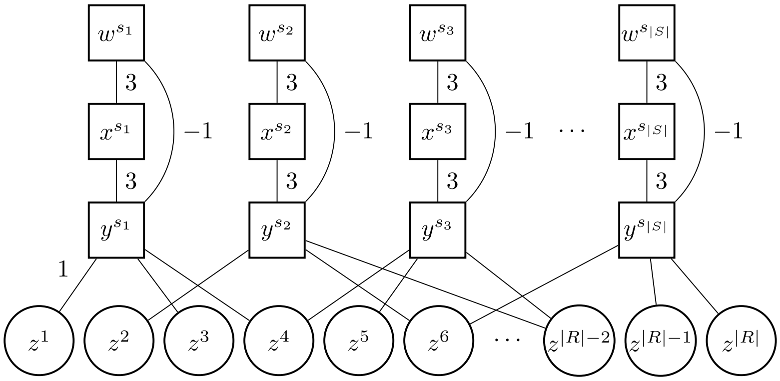

Fig. 5. A graph representation of an ASHG derived from an instance of E3C. The (symmetric) utilities are given as edge weights. Some edges and labels are omitted: All edges between any y^s and z^r have weight 1 if $r \in s$. All $z^{r'}$, $z^{r''}$ with $r' \neq r''$ are connected with weight $\frac{1}{|R|-1}$. All missing edges have weight −7.

$s \in S \setminus S'$; $u_\pi(x^s) = u_\pi(y^s) = 3$ for all $s \in S$; and $u_\pi(z^r) = 1 + 2/(|R| - 1)$ for all $r \in R$. Whereas the utilities of no player in π' decreases, the utility of some players in π' is more than in π. Since π' Pareto dominates π, π is not Pareto optimal.

We now show that if there exists no $S' \subseteq S$ such that S' is a partition of R, then π is Pareto optimal. We note that −7 is a sufficiently large negative valuation to ensure that if $v_a(b) = v_b(a) = -7$, then $a, b \in N$ cannot be in the same coalition in a Pareto optimal partition. Assume for the sake of contradiction that π is not Pareto optimal and there exists a partition π' which Pareto dominates π. We will see that if there exists a player $i \in N$ such that $u_{\pi'}(i) > u_\pi(i)$, then there exists at least one $j \in N$ such that $u_{\pi'}(j) < u_\pi(j)$. The only players whose utility can increase (without causing some other player to be less happy) are $\{x^s \mid s \in S\}$, $\{w^s \mid s \in S\}$ or $\{z^r \mid r \in R\}$. We consider these player classes separately. If the utility of player x^s increases, it can only increase from 3 to 6 so that x^s is in the same coalition as y^s and w^s. However, this means that y^s's utility is decreased. The utility of y^s can increase or stay the same only if it forms a coalition with some z^rs. However in that case, to satisfy all z^rs, there needs to exist an $S' \subseteq S$ such that S' is a partition of R.

Assume the utility of a player w^s for $s \in S$ increases. This is only possible if w^s is in the same coalition as x^s. Clearly, the coalition formed is $\{w^s, x^s\}$ because coalition $\{w^s, x^s, y^s\}$ brings a utility of 2 to y^s. In that case y^s needs to form a coalition $\{y^s, z^i, z^j, z^k\}$ where $s = \{i, j, k\}$. If y^s forms a coalition $\{y^s, z^i, z^j, z^k\}$, then all players $y^{s'}$ for $s' \in (S \setminus \{s\})$ need to form coalitions of the form $\{y^{s'}, z^{i'}, z^{j'}, z^{k'}\}$ such that $s' = \{i', j', k'\}$. Otherwise, their utility of 3 decreases. This is only possible if there exists a set $S' \subseteq S$ of R such that S' is a partition of R.

Assume that there exists a partition π' that Pareto dominates π and the utility of a player $u_{\pi'}(z^r) > u_\pi(z^r)$ for some $r \in R$. This is only possible if each z^r forms the coalition of the form $\{z^r, z^{r'}, z^{r''}, y^s\}$ where $s = \{r, r', r''\}$. This can only happen if there exists a set $S' \subseteq S$ of R such that S' is a partition of R. Thus we have proved that π is not Pareto optimal if and only if (R, S) is a 'yes' instance. □

The fact that checking whether a partition is Pareto optimal is coNP-complete has no obvious implications on the complexity of *computing* a Pareto optimal partition. In fact there is a simple polynomial-time algorithm to compute a partition which is Pareto optimal for strict preferences.

Theorem 11. *For strict preferences, a Pareto optimal partition can be computed in polynomial time.*

Proof. The statement follows from an application of *serial dictatorship* (see, e.g., [1]). *Serial dictatorship* is a well-known mechanism in resource allocation in which an arbitrary player is chosen as the 'dictator' who is then given his most favored allocation and the process is repeated until all players or resources have been dealt with. In the context of coalition formation, serial dictatorship is well defined if preferences of players over coalitions are strict. Serial dictatorship is also well defined for ASHGs with strict preferences as the dictator forms a coalition with all the players he strictly likes who have not been considered as dictators or are not already in some dictator's coalition. The resulting partition π is such that for any other partition π', at least one dictator will strictly prefer π to π'. Therefore π is Pareto optimal. □

A standard criticism of Pareto optimality is that it admits inherently unfair allocations. To address this criticism, the algorithm can be modified to obtain less lopsided partitions. Whenever an arbitrary player is selected to become the dictator among the remaining players, choose a player that does not get extremely high elitist social welfare among the remaining players. Nevertheless, even this modified algorithm may output a partition that fails to be individually rational.

We know that the set of partitions which are both Pareto optimal and individually rational is non-empty. Repeated Pareto improvements on an individually rational partition consisting of singletons leads to a Pareto optimal and individually rational partition. We show that computing a Pareto optimal and individually rational partition for ASHGs is weakly NP-hard. To prove the statement, we first prove the following theorem.

Theorem 12. *Computing a CSC stable and individually rational partition is weakly NP-hard.*

Proof. Consider the decision problem SubsetSum in which an instance consists of a set of k integer weights $A = \{a_1, \dots, a_k\}$ and the question is whether there exists a non-empty $S \subseteq A$ such that $\sum_{s\in S} s = 0$?

Name: SubsetSum
Instance: A set of k integer weights $A = \{a_1, \dots, a_k\}$.
Question: Does there exist a non-empty $S \subseteq A$ such that $\sum_{s\in S} s = 0$?

Since SubsetSum for positive integers is NP-complete, it follows that MaximalSubsetSum, the problem of finding a maximal cardinality subset $S \subseteq A$ such that $\sum_{s\in S} s = 0$ is NP-hard.

We prove the theorem by a reduction from MaximalSubsetSum. Reduce an instance I of MaximalSubsetSum to an ASHG (N, v) defined in the following way:

- $N = \{x, y_1, y_2\} \cup Z$ where $Z = \{z_i \mid i \in \{1, \dots, k\}\}$;
- $v_x(y_1) = v_x(y_2) = k + 1$; $v_x(z_i) = 1$ for all $i \in \{1, \dots, k\}$;
- $v_{y_1}(z_i) = -v_{z_i}(y_1) = -v_{y_2}(z_i) = v_{z_i}(y_2) = a_i$ for all $i \in \{1, \dots, k\}$; and
- $v_a(b) = 0$ for any $a, b \in N$ for which $v_a(b)$ is not already defined.

First, we show that in an individually rational partition π, no player except x gets positive utility, i.e., $u_\pi(b) = 0$ for all $b \in N \setminus \{x\}$. Assume that without loss of generality y_1 gets positive utility in π. This implies that there exists a subset $Z' = Z \cap \pi(y_1)$ such that $\sum_{z\in Z'} v_{y_1}(z) > 0$. Then there exists $z \in Z'$ such that $v_{y_1}(z) > 0$ which means that $v_z(y_1) < 0$. Due to individual rationality, $y_2 \in \pi(z) = \pi(y_1)$. But if $y_1 \in \pi(y_2)$, then $u_\pi(y_2) = \sum_{z\in Z'} -v_{y_1}(z) < 0$ and π is not individually rational.

Assume that there exists a $z_i \in Z$ such that $u_\pi(z_i) > 0$. Then without loss of generality $v_{z_i}(y_1) > 0$ and due to individual rationality $y_1 \in \pi(z_i)$. Again due to individual rationality, y_1 needs to be with another z_j such that $v_{y_1}(z_j) > 0$. And again due to individual rationality, z_j needs to be with y_2. This means, that for each $z_l \in \pi(z_i) \cap Z$, $u_\pi(z_l) = a_l - a_l = 0$.

We show that in every CSC stable and individually rational partition π, we have $y_1, y_2 \in \pi(x)$. For any other partition π', in which this does not hold, $u_{\pi'}(x) \leqslant 2k + 1 < 2k + 2 = u_\pi(x)$.

Consider an $S \subseteq A$ and let π_Z^S be any partition of $\{z_i \mid a_i \in A \setminus S\}$. The claim is that π is a CSC stable and individually rational partition if and only if π is of the form $\{\{x, y_1, y_2\} \cup \{z_i \mid a_i \in S\}\} \cup \pi_Z^S$ where $S \subseteq A$ is the maximal subset such that $\sum_{s\in S} s = 0$.

Assume that $S \subseteq A$ is not a maximal subset such that $\sum_{s\in S} s = 0$. If $\sum_{s\in S} s \neq 0$, there exists a $y \in \{y_1, y_2\}$ such that $u_\pi(y) < 0$. If S is not maximal then there is a larger set S' and a corresponding partition $\pi' = \{\{x, y_1, y_2\} \cup \{z_i \mid a_i \in S'\}\} \cup \pi_Z^{S'}$ with $u_\pi(x) = |S| < |S'| = u_{\pi'}(x)$ and $u_\pi(b) = u_{\pi'}(b)$ for all $b \in N \setminus \{x\}$. For any other $S' \subseteq A$ such that $|S'| > |S|$, we know that $\sum_{s'\in S'} s' \neq 0$ which implies that there is a $y \in \{y_1, y_2\}$ which gets negative utility. □

As a corollary, we get the following.

Corollary 1. *Computing a Pareto optimal and individually rational partition is weakly NP-hard.*

Proof. Observe that a partition which is Pareto optimal and individually rational is also CSC stable and individually rational. □

8. Envy-freeness

Envy-freeness is a desirable property in resource allocation, especially in *cake cutting* settings. Lipton et al. [28] proposed different variants of envy-minimization and examined the complexity of minimizing envy in resource allocation. Bogomolnaia and Jackson [9] mentioned envy-freeness in the context of hedonic games but focused on stability.

Envy-freeness resembles Nash stability in that no player has an incentive to move to another coalition. However, one can produce simple examples to show that envy-freeness does not imply Nash stability and Nash stability does not imply envy-freeness.

Example 2. A partition that satisfies envy-freeness may not be Nash stable. Take the game (N, v) where $N = \{1, 2\}$ and where v is specified by $v_1(2) = v_2(1) = 1$. Then the partition $\pi = \{\{1\}, \{2\}\}$ satisfies envy-freeness but it is not Nash stable. Similarly, a Nash stable partition may not satisfy envy-freeness. Take the game (N, v) where $N = \{1, 2, 3\}$ where v is specified by: $v_1(2) = 1$, $v_1(3) = -1$, $v_2(3) = v_3(2) = 2$, and $v_2(1) = v_3(1) = 0$. Consider the partition $\pi = \{\{1\}, \{2, 3\}\}$ which is Nash stable. However, π does not satisfy envy-freeness because player 1 is envious of player 3 and would prefer to replace him to be with player 2.

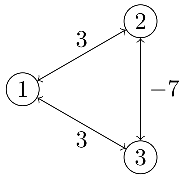

Fig. 6. Graphical representation of the ASHG in Example 3. No partition is both envy-free and Nash stable.

Unlike Nash stability, we already know that envy-freeness can be easily achieved.

Observation 3. *The partition of singletons is envy-free and individually rational.*

Therefore, in conjunction with envy-freeness, we seek to satisfy other properties such as stability or Pareto optimality. For symmetric ASHGs, it is known that Nash stable partitions always exist and they correspond to partitions for which the utilitarian social welfare is a local optimum (see, e.g., [9]). We now show that for symmetric ASHGs, there may not exist any partition that is both envy-free and Nash stable.

Example 3. Consider an ASHG (N, v) where $N = \{1, 2, 3\}$ and v is defined as follows (see also Fig. 6): $v_1(2) = v_2(1) = 3$, $v_1(3) = v_3(1) = 3$, and $v_2(3) = v_3(2) = -7$. Then there exists no partition which is both envy-free and Nash stable.

We use the game in Example 3 as a gadget to prove the following.[2]

Theorem 13. *Checking whether there exists a partition which is both envy-free and Nash stable is NP-complete in the strong sense, even when preferences are symmetric.*

Proof. The problem is clearly in NP since envy-freeness and Nash stability can be verified in polynomial time. We reduce the problem from E3C. Let (R, S) be an instance of E3C where R is a set and S is a collection of subsets of R such that $|R| = 3m$ for some positive integer m and $|s| = 3$ for each $s \in S$. We will use the fact that E3C remains NP-complete even if each $r \in R$ occurs in at most three members of S. (R, S) can be reduced to an instance (N, v) where (N, v) is an ASHG defined in the following way. Let $N = \{y^s \mid s \in S\} \cup \{z_1^r, z_2^r, z_3^r \mid r \in R\}$. We set all preferences as symmetric. The players preferences are as follows:

- $v_{z_1^r}(z_2^r) = v_{z_2^r}(z_1^r) = 3$, $v_{z_1^r}(z_3^r) = 3$ and $v_{z_2^r}(z_3^r) = v_{z_3^r}(z_2^r) = -7$ for all $r \in R$;
- $v_{z_1^i}(z_1^j) = v_{z_1^i}(z_1^k) = v_{z_1^j}(z_1^k) = 1/10$ and $v_{y^s}(z_1^i) = v_{y^s}(z_1^j) = v_{y^s}(z_1^k) = 28/10$ for all $s = \{i, j, k\} \in S$; and
- $v_a(b) = v_b(a) = -7$ for all $a, b \in N$ for which valuations have not been defined.

We note that -7 is a sufficiently large negative valuation to ensure that if $v_a(b) = v_b(a) = -7$, then a and b will get negative utility if they are in the same coalition. We show that there exists an envy-free and Nash stable partition for (N, v) if and only if (R, S) is a 'yes' instance of E3C.

Assume that there exists $S' \subseteq S$ such that S' is a partition of R. Then there exists a partition $\pi = \{\{y^s, z_1^i, z_1^j, z_1^k\} \mid s = \{i, j, k\} \in S'\} \cup \{\{z_2^r\}, \{z_3^r\} \mid r \in R\} \cup \{\{s\} \mid s \in S \setminus S'\}$. It is easy to see that partition π is Nash stable and envy-free. Players z_1^r and z_3^r both had an incentive to be with each other when they are singletons. However, each z_1^r now gets utility 3 by being in a coalition with $z_1^{r'}$, $z_1^{r''}$ and y^s where $s = \{r, r', r''\} \in S$. Therefore z_1^r has no incentive to be with z_3^r and z_3^r has no incentive to join $\{z_1^r, z_1^{r'}, z_1^{r''}, y^s\}$ because $v_{z_3^r}(z_1^{r'}) = v_{z_3^r}(z_1^{r''}) = v_{z_3^r}(y^s) = -7$. Similarly, no player is envious of another player.

Assume that there exists no partition $S' \subseteq S$ of R such that S' is a partition of R. Then, there exists at least one $r \in R$ such that z_1^i is not in the coalition of the form $\{z_1^r, z_1^{r'}, z_1^{r''}, y^s\}$ where $s = \{r, r', r''\} \in S$. Then the only individually rational coalitions which z_1^r can form and get utility at least 3 are the following $\{z_1^r, z_3^r\}$, $\{z_1^r, z_2^r\}$. In the first case, z_1^r wants to deviate to $\{z_3^r\}$. In the second case, z_2^r is envious and wants to replace z_3^r. Therefore, there exists no partition which is both Nash stable and envy-free. □

While the existence of a Pareto optimal partition and an envy-free partition is guaranteed, we show that checking whether there exists a partition which is both envy-free and Pareto optimal is hard.

Theorem 14. *Checking whether there exists a partition which is both Pareto optimal and envy-free is Σ_2^p-complete.*

[2] Example 3 and the proof of Theorem 13 also apply to the combination of envy-freeness and IS (individual stability).

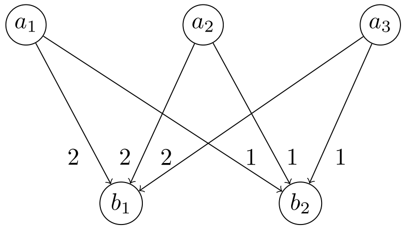

Fig. 7. A graphical representation of the ASHG in Example 4 with the utilities of a_i for b_j. All upward edges (b_j, a_i) have weight 1 while all still unspecified edges have utility −4. For this ASHG, no popular partition exists.

Proof. An instance is a 'yes' instance if it admits an envy-free partition that Pareto dominates every other partition. Therefore the problem is in the complexity class $\mathrm{NP}^{\mathrm{coNP}} = \Sigma_2^p$.

We prove hardness by a reduction from a problem concerning resource allocation (with additive utilities) [14]. A resource allocation problem is a tuple (I, X, w) where I is a set of agents, X is a set of indivisible objects and $w : I \times X \to \mathbb{R}$ is a weight function. An $a : I \to 2^X$ is an allocation if for all $i, j \in I$ such that $i \neq j$, we have $a(i) \cap a(j) = \emptyset$. The resultant utility of each agent $i \in I$ is then $\sum_{x \in a(i)} w(i, x)$. It was shown by de Keijzer et al. [14] that the problem ∃-EEF-ADD of checking the existence of an envy-free and Pareto optimal allocation is Σ_2^p-complete.

Now, consider an instance (I, X, w) of ∃-EEF-ADD and reduce it to an instance (N, v) of an ASHG where $N = I \cup X$ and v is specified by the following values:

- $v_i(x_j) = w(i, x_j)$ and $v_{x_j}(i) = 0$ for all $i \in I$, $x_j \in X$;
- $v_{x_k}(x_j) = v_{x_j}(x_k) = 0$ for all x_j, x_k; and
- $v_i(j) = v_j(i) = -W \cdot |I \cup X|$ for all $i, j \in I$ where $W = \sum_{i \in I, x_j \in X} |w(i, x_j)|$.

It can then be shown that there exists a Pareto optimal and envy-free partition in (N, v) if and only if (I, X, w) is a 'yes' instance of ∃-EEF-ADD. It is clear that for any Pareto optimal partition π, there exist no $i, j \in I \subset N$ such that $i \neq j$ and $j \in \pi(i)$. Assume that this were not the case and there exist $i, j \in I \subset N$ such that $i \neq j$ and $j \in \pi(i)$. Then i and j both get negative value because $\sum_{k \in \pi(i)} v_i(k) = \sum_{k \in (\pi(i) \setminus \{j\})} v_i(k) - W < 0$ and $\sum_{k \in \pi(i)} v_j(k) = \sum_{k \in (\pi(i) \setminus \{i\})} v_j(k) - W < 0$. Then i and j can be separated to form singletons to get another partition π', where the value of every other player $k \in (N \setminus \{i, j\})$ gets the same value while i and j get at least zero value. Therefore there is a one-to-one correspondence between any such partition π and allocation a where $a(i) = \pi(i) \setminus \{i\}$. It now easy to see that π is Pareto optimal and envy-free in G if and only if a is a Pareto optimal and envy-free allocation. □

The results of this section show that, even though envy-freeness can be trivially satisfied on its own, it becomes much more delicate when considered in conjunction with other desirable properties.

9. Popularity

In this section, we consider the complexity of verifying and checking the existence of popular partitions for additively separable hedonic games (ASHGs). If changing an outcome requires the approval of a majority of players, then popularity can also be considered a notion of stability. The idea of using popularity in matching theory was initiated by Gärdenfors [23]. Popular matchings were then studied in the context of *assignment problems* (in which objects, posts or houses) are allocated among agents such that each agent receives at most one object (see, e.g., [27]). Biró et al. [8] considered popular outcomes in the context of marriage games and roommate games.

The following is an example of an ASHG which does not admit a popular partition.

Example 4. Consider the following ASHG (see also Fig. 7): $N = \{a_1, a_2, a_3, b_1, b_2\}$ such that $v_{a_i}(b_1) = 2$ and $v_{a_i}(b_2) = 1$ for all $i = 1, 2, 3$; $v_{b_i}(a_j) = 1$ for all $i = 1, 2$ and $j = 1, 2, 3$; and $v_x(y) = -4$ for all other x and y. Then, there exists no popular partition. For example in $\{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3\}\}$, a_2 and a_3 can both strictly improve their utility.

In fact, not only may an ASHG not admit a popular partition, but checking whether there exists a popular partition is NP-hard. Whereas verifying a popular allocation is already known to be polynomial-time solvable for roommate games [8], we show that the same problem is coNP-complete for ASHGs.

Theorem 15. *For popular partitions, the following statements hold.*

(i) *Checking whether there exists a popular partition is NP-hard.*
(ii) *Verifying whether a partition is popular is coNP-complete.*

Proof.

(i) The reduction is from E3C to deciding whether there exists a popular partition. In our reduction, we use as a gadget the ASHG discussed in Example 4. Let (R, S) be an instance of E3C where R is a set and S is a collection of subsets of R such that $|R| = 3m$ for some positive integer m, $|s| = 3$ for each $s \in S$, and each $r \in R$ occurs in at most three members of S. (R, S) can be reduced to an instance (N, v), where (N, v) is an ASHG defined in the following way.
Let $N = \{a_1^r, a_2^r, a_3^r, b_1^r, b_2^r \mid r \in R\} \cup \{y^s, z_1^s, z_2^s \mid s \in S\}$ and v be as follows:
- $v_{a_i^r}(b_1^r) = 2$ and $v_{a_i^r}(b_2^r) = 1$ for all $i = 1, 2, 3$ and $r \in R$;
- $v_{b_i^r}(a_j^r) = 1$ for all $i = 1, 2,\ j = 1, 2, 3$ and $r \in R$;
- $v_{a_3^r}(a_3^{r'}) = 1/10$ for all $r, r' \in s \in S$ such that $r \neq r'$;
- $v_{a_3^r}(y^s) = 4/5$ and $v_{y^s}(a_3^r) = 0$ for all $s \in S$ and $r \in R$ such that $r \in s$;
- $v_{y^s}(z_1^s) = v_{y^s}(z_2^s) = v_{z_1^s}(y^s) = v_{z_2^s}(y^s) = 1/2$ and $v_{z_1^s}(z_2^s) = v_{z_2^s}(z_1^s) = 0$ for all $s \in S$; and
- $v_x(y) = -4$ for all other x and y.

The main idea of the reduction is that if (R, S) is a 'yes' instance of E3C, then players of the form a_3^r will be sufficiently happy to not disrupt the coalitions $\{a_1^r, b_1^r\}$ and $\{a_2^r, b_2^r\}$. If (R, S) is a 'no' instance of E3C, then players of the form a_3^r will be able to disrupt the coalitions $\{a_1^r, b_1^r\}$ and $\{a_2^r, b_2^r\}$. We show that there exists a popular partition of (N, v) if and only if (R, S) is a 'yes' instance of E3C.
Assume (R, S) is a 'yes' instance of E3C. Then, there exists $S' \subseteq S$ such that S' is a partition of R.
Consider the following partition π.
- $\{\{a_1^r, b_1^r\} \mid r \in R\} \cup \{\{a_2^r, b_2^r\} \mid r \in R\}$;
- $\{\{y^s, a_3^{r_i}, a_3^{r_j}, a_3^{r_k}\} \mid s = \{i, j, k\} \in S'\}$;
- $\{\{z_1^s, z_2^s\} \mid s \in S'\}$;
- $\{\{y^s, z_1^s, z_2^s\} \mid s \in N \setminus S'\}$.

It can be shown that the partition π is popular. If a player a_3^r is in a coalition with a player b_k^r for $r \in \{1, 2\}$, then that partition is not popular. No player a_1^r can become happier. Any player a_2^r can become happier but only at the expense of a_1^r. Each player a_3^r gets utility one and is sufficiently happy to not disrupt the coalitions $\{a_1^r, b_1^r\}$ and $\{a_2^r, b_2^r\}$. A player a_3^r can become happier by being with at least y^s and $y^{s'}$ such that $r \in s$ and $r \in s'$ but in that coalition, the number of players who are strictly happier is not more than the number of players who are strictly less happy. The only new coalition which can form in which the number of players who are strictly happier is more than the number of players who are strictly less happy is of the form $\{z_1^s, z_2^s, y^s\}$. A y^s which is in coalition $\{y^s, a_3^{r_i}, a_3^{r_j}, a_3^{r_k}\}$ can move to a coalition $\{z_1^s, z_2^s\}$ but he leaves behind $a_3^{r_i}, a_3^{r_j}, a_3^{r_k}$ who are less happy.
Assume that there exists a popular partition π. Then, we show that (R, S) is a 'yes' instance of E3C. The only type of player from among a_1^r, a_2^r, a_3^r to benefit from the new players created in the reduction is a_3^r. Each a_3^r needs to be with at least two $a_3^{r'}$s and at least one likable y^s to get a payoff of at least $4/5 + 2(1/10) = 1$. If a_3^r is only with likable $a_3^{r'}$s, it can get a maximum utility of $6 \cdot 1/10 = 6/10$ due to the initial assumption about the number of occurrences of r in S. This is not enough as a_3^r got utility 1 in π. If a_3^r is with at least two likable y^ss and between zero to six likable $a_3^{r'}s$ in a coalition S, then in each case, at least $|S|$ players (including some z_1^s and z_2^ss) improve their utilities and less than $|S| - 1$ players get less utility if the unhappy y^ss join their corresponding z_1^s and z_2^ss. If a_3^r is with one likable y^s and at least three likable $a_3^{r'}$s in a coalition T, then y^s and at least one $a_3^{r'}$ in T get negative utility. A partition π' in which all coalitions are exactly the same but y^s moves away from T and joins z_1^s and z_2^s is more popular than π. Therefore each a_3^r needs to be with exactly two likable $a_3^{r'}$s and exactly one likable y^s in partition π. But this means that there exists an $S' \subseteq S$ such that S' is a partition of R. This completes the proof.

(ii) We introduce the following notation for the proof: $\mathcal{D}_T(\pi, \pi') = |P_T(\pi, \pi')| - |P_T(\pi', \pi)|$ where $T \subseteq N$ and $P_T(\pi, \pi')$ is the set of players in T who strictly prefer allocation π to π'.
Let the partition in question be π. The problem is clearly in coNP as a partition which is more popular than π is a polynomial-time certificate that π is not popular.
The reduction is from E3C to the problem $((N, v), \pi)$ of deciding whether partition π is popular for game (N, v). For an instance (R, S), let $Z = \{1, \ldots, |R| + |S|\}$ and the sets in S are assumed to have a natural ordering so that $S = \{s_1, \ldots, s_{|S|}\}$. Then, N is defined as follows: $N = \{a_i \mid i \in Z\} \cup \{b_i \mid i \in Z\} \cup \{c_r \mid r \in R\} \cup \{d_{i,j} \mid i \in S, j \in Z\}$.
We will refer to $\{a_i \mid i \in Z\}$ by A, $\{b_i \mid i \in Z\}$ by B, $\{c_r \mid r \in R\}$ by C, and $\{d_{i,j} \mid s_i \in S, j \in Z\}$ by D. Let $\gamma(i) = \{j \neq i \mid i, j \in s \in S\}$ and $\beta(i) = \{j \mid i \in s_j \in S\}$. We will refer to $\{d_{k,j} \mid s_k \in S, j \in Z\}$ by D_k. The preferences v are defined as follows:
- $v_{a_i}(b_i) = 0$ for all $i \in Z$;
- $v_{b_1}(a_1) = |Z| \cdot |R| + 1$;
- $v_{b_i}(a_i) = |Z| \cdot |R|$ for all $i \in Z \setminus \{1\}$;
- $v_b(c) = 1$ for all $b \in B$ and $c \in C$;
- $v_b(b') = |R|$ for all $b, b' \in B$;
- $v_{c_i}(c_j) = 1$ if $i, j \in s \in S$ and $v_{c_i}(c_j) = 0$ otherwise;
- $v_{c_k}(b) = \frac{|Z|^3 - |\gamma(k)|}{|Z|}$ for all $b \in B$;

- $v_{c_k}(d_{i,j}) = (|Z|^3 - 2)/|Z|$ if $k \in s_i \in S$;
- $v_{d_{i,j}}(c_k) = 0$ if $k \in s_i \in S$;
- $v_{d_{i,j}}(d_{i,j'}) = 1$ $j \neq j'$ and $i \in \{1, \ldots, |S|\}$; and
- $v_x(y) = -|Z|^5$ for all other x and y.

The value of $-|Z|^5$ is a sufficiently large negative value so that if $v_x(y) = -|Z|^5$, then x and y get negative utility whenever they are together in the same coalition.

The partition π in question is defined as $\pi = \{\{a\} \mid a \in A\} \cup \{B \cup C\} \cup \{D_m \mid m \in \{1, \ldots, |S|\}\}$ so that

- $u_\pi(a) = 0$ for all $a \in A$;
- $u_\pi(b) = |Z| \cdot |R|$ for all $b \in B$;
- $u_\pi(c) = |Z|^3$ for all $c \in C$; and
- $u_\pi(d) = |Z| - 1$ for all $d \in D$.

We show that π is not popular if and only if (R, S) is a 'yes' instance of E3C. If (R, S) is a 'yes' instance of E3C, and there exists an $S' \subseteq S$, a proper partitioning of N, then the following partition π' is more popular than and a Pareto improvement over π:

$$\pi' = \bigl\{\{a_i, b_i\} \bigm| i \in Z\bigr\} \cup \bigl\{\bigl\{D_m \cup \{c_i \mid i \in s_m\}\bigr\} \bigm| s_m \in S'\bigr\} \cup \bigl\{\{D_m\} \bigm| s_m \in S \setminus S'\bigr\}.$$

In partition π', the players get utility as follows:

- $u_{\pi'}(a) = 0$ for all $a \in A$;
- $u_{\pi'}(b) = |Z| \cdot |R|$ for all $b \in B \setminus \{b_1\}$;
- $u_{\pi'}(b_1) = |Z| \cdot |R| + 1$;
- $u_{\pi'}(c) = |Z|^3$ for all $c \in C$; and
- $u_{\pi'}(d) = |Z| - 1$ for all $d \in D$.

We now show that if π is not popular, then (R, S) is a 'yes' instance of E3C. If π is not popular, then there exists another partition $\pi'' \neq \pi'$ which is more popular than π. Then, it follows that there exists at least one coalition $T \in \pi''$ such that $\mathcal{D}_T(\pi, \pi'') > 0$. If $\{a_1, b_1\} \in \pi''$, then a_1 is indifferent between π and π'' but b_1 strictly prefers π'' to π'. We prove that $\{a_1, b_1\}$ *is the only possible coalition in which a majority of players prefers* π'' *to* π. To prove the claim, we show that for any coalition $T \in \pi''$, if there is a player $i \in T$ such that i prefers π'' to π, then $\mathcal{D}_T(\pi, \pi'') < 0$.

(a) Consider the case that $a \in A$ strictly prefers π'' to π. This is not possible since no player in A can improve because they do not like anyone.

(b) Let $b_i \in B \setminus \{b_1\}$ be a player who prefers π'' to π. Then $\pi''(b_i) = \{b_i\} \cup X$ where $X \subseteq B \cup C \cup \{a_i\}$. If there exists an $x \in N \setminus ((B \setminus \{b_i\}) \cup C \cup \{a_i\})$, such that $x \in \pi''(b_i)$, then $u_{\pi''}(b_i) < 0$. If $X \subset B \cup C$, then b_i does not prefer π'' to π. Therefore X contains a_i and at least one element from $((B \setminus \{i\}) \cup C$. But then, every player in $\pi''(b_i) \setminus \{b_i\}$ gets negative utility and prefers π to π''.

(c) Let $c_i \in C$ be a player who is happier in π'' than in π. Then, $\pi''(c_i) \setminus \{c_i\} \subseteq C \cup B \cup \{d_{k,j} \mid k \in \beta(i)\})$. If this were not that case, then c_i gets negative utility. Note that c_i likes the following different types of sets of players: $\{c_j \mid j \in \gamma(i)\}$ (there are between two to six players of this type), D_k where $i \in s_k \in S$ (there are between one to three sets of this type), and B.

Player c_i may prefer π'' to π if $\pi''(c_i)$ contains a sufficient number of players from the sets of players outlined above. We will show that if c_i prefers π'' to π, then $\mathcal{D}_{\pi''(c_i)}(\pi, \pi'') < 0$.

We know that players from different sets D_k and $D_{k'}$ will get negative utility if they are in the same coalition.

Similarly, if players from B and D_k are together in $\pi''(c_i)$, they will all get negative utility.

Also, let us say that $j \notin s_k$ but $j \in \gamma(i)$. Then if c_j is in $\pi''(c_i)$ along with players from D_k, then c_j and all the players in D_k get negative utility. Therefore, the best c_i can do is to be with *(i)* B and $\{c_j \mid j \in \gamma(i)\}$ together or *(ii)* one of the sets D_k along with those c_js such that $j \in s_k \in S$. In either case, c_i is indifferent between π and π'' as he gets utility $|Z|^3$ in both partitions.

(d) No player in $d \in D$ can prefer another partition π'' to π as d is already in a coalition with all the player he likes.

If π'' is more popular than π, we already know that $\{a_1, b_1\} \in \pi''$. In order to ensure that π'' is more popular than π, each player $N \setminus \{b_1\}$ should be indifferent between π and π''. This is because we already know that no player $x \in N \setminus \{b_1\}$ can be strictly happier in π'' without making the majority of the players in $\pi''(x)$ strictly less happy.

We know that players in B are not together in the same coalition in π''. Since players in B cannot be together it must be that each b_i is with its corresponding a_i. Similarly, since players in C cannot be together with players in B anymore, they must utilize their positive valuation for players in some set D_m. We already saw that each c_i must be in a coalition with players in D_m and two other players c_j and c_k where $\{i, j, k\} = s_m \in S$. Therefore, we have shown that there exists a subset $D' \subseteq \{D_1, \ldots, D_{|S|}\}$ such that each set D_a in D' hosts three players $\{c_i, c_j, c_k\}$ from C such that $\{i, j, k\} = s_a \in S$. Therefore, there exists a partition $S' \subseteq S$ of N and (R, S) is a 'yes' instance of E3C. This completes the proof. □

An interesting open problem is whether natural restricted classes of ASHGs always admit a popular partition.

Table 1
Complexity of ASHGs. We use the following abbreviations: *symm* (for symmetric preferences), *strict* (for strict preferences), *grand* (when the partition in question consists of the grand coalition), *IR* (when the partition is also required to be individually rational), and *trivial* (when the problem is trivial because existence is already guaranteed).

Concept	VERIFICATION	EXISTENCE	COMPUTATION
NS	in P (Obs. 2)	trivial ([9], symm) NP-complete [37]	PLS-complete ([9,21], symm) NP-hard [37]
IS	in P (Obs. 2)	trivial ([9], symm) NP-complete [37]	PLS-complete ([22], symm) NP-hard [37]
CIS	in P (Obs. 2)	trivial [4]	in P (Th. 1)
C	coNP-complete ([35], symm)	NP-hard (Th. 2, symm)	NP-hard (Th. 2, symm)
SC	coNP-complete (Th. 3, symm)	NP-hard (Th. 2, symm)	NP-hard (Th. 2, symm)
PO	coNP-complete (Th. 10, strict & symm) coNP-complete (Th. 9, GC)	trivial	NP-hard (Cor. 1, IR) in P (Th. 11, strict)
CSC	coNP-complete (Th. 8, GC)	trivial [36]	NP-hard (Th. 12, IR)
Perfect	in P (Th. 7)	in P (Th. 7)	in P (Th. 7)
MaxUtil	coNP-complete (Th. 4)	trivial (Obs. 1)	NP-hard (Th. 4)
MaxEgal	coNP-complete (Th. 6)	trivial (Obs. 1)	NP-hard (Th. 6)
MaxElite	trivial (Th. 5)	in P (Obs. 1)	in P (Th. 5)
EF	in P (Obs. 2)	trivial (Obs. 3)	in P (Obs. 3)
EF & NS	in P (Obs. 2)	NP-complete (Th. 13, symm)	NP-hard (Th. 13, symm)
EF & PO	coNP-complete (Th. 10)	Σ_2^p-complete (Th. 14)	Σ_2^p-hard (Th. 14)
Popular	coNP-complete (Th. 15)	NP-hard (Th. 15)	NP-hard (Th. 15)

10. Conclusion and discussion

We presented a number of new computational results concerning stable, fair, optimal, and popular partitions of ASHGs. Both new and existing results are summarized in Table 1. We saw that considering CSC deviations facilitates arguments about more complex Pareto optimal improvements. As a result, we present similar computational results for CSC stability and Pareto optimality. It was shown that under various restrictions of preferences, verifying, checking the existence, and computing stable, fair, optimal and popular partitions is computationally intractable. On a more positive note, we proposed a polynomial-time algorithm for computing a contractually individually stable (CIS) partition. It is also seen that the existence of a perfect partition can be checked efficiently, and that for strict preferences, a Pareto optimal partition can be computed efficiently.

There are some interesting contrasts in the results. For example, whereas a Pareto optimal partition can be computed in polynomial time when preferences are strict, checking whether a given partition is Pareto optimal is coNP-complete even in the restricted setting of strict and symmetric preferences. Even though the existence of an envy-free partition and the existence of a Nash stable partition are guaranteed under symmetric preferences, checking the existence of a partition which satisfies both properties simultaneously is computationally hard. Finally, verifying popular outcomes is coNP-complete whereas the same problem is computationally easy for house allocation and even the stable roommates setting.

A number of new questions arise as a result of our study. The complexity of computing a Pareto optimal partition for ASHGs with unrestricted preferences is still open. We note that Algorithm 1 for computing a CIS partition may very well return a partition that fails to satisfy individual rationality, i.e., players may get negative utility. It is an open question how to efficiently compute a CIS partition that is guaranteed to satisfy individual rationality. Furthermore, the complexity of computing a CSC stable partition which is not necessarily IR is still open.

We highlighted the logical relationships between different stability, fairness, optimality, and popularity concepts from cooperative game theory, social choice and welfare theory. It will be interesting to examine these relationships in other domains, in particular with respect to strategic issues. Other directions for future research include approximation algorithms to compute maximum utilitarian or egalitarian social welfare for different representations of hedonic games. Finally, there is scope for further work on the *dynamics* of deviations in various classes of hedonic games.

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