Exact Controllability of the Euler–Bernoulli Equation with Boundary Controls for Displacement and Moment*

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We consider the Euler–Bernoulli problem (1.1) in the solution $w(\cdot, \cdot)$ with boundary controls $g_1$ and $g_2$ acting in the Dirichlet traces for $w$ and $\partial w$, respectively. We show two main exact controllability results both in the spaces of maximal regularity of the solution at $t = T$ (see I. Lasiecka and R. Triggiani, Regularity theory for a class of Euler Bernoulli equations: a cosine operator approach, *Boll. Unione Mat. Ital.* (7), 3–B (1989)) and both without geometrical conditions on the open bounded domain $\Omega$ (except for smoothness of its boundary $\Gamma$): one with $g_1 \in L^2(0, T; L^2(\Gamma))$ and $g_2 \in [H^1(0, T; L^2(\Gamma))]'$ for any $T > 0$ arbitrarily short; and one with $g_1 \in H_0^1(0, T; L^2(\Gamma))$ and $g_2 \in L^2(0, T; L^2(\Gamma))$ for all $T > 0$ sufficiently large. An interpolation result between these two cases is also presented. A direct approach is given based on two main steps. First, by means of an operator model (I. Lasiecka and R. Triggiani, Regularity theory for a class of Euler Bernoulli equations: a cosine operator approach, *Boll. Unione Mat. Ital.* (7), 3–B (1989)) for problem (1.1) and a functional analytic approach, the question of exact controllability is shown to be equivalent to an a-priori inequality for the corresponding homogeneous problem. Next, this a-priori inequality is proved to hold true by means of multiplier techniques. These are inspired by recent progress on the maximal regularity, exact controllability and uniform stabilization questions for second order hyperbolic equations.

1. Introduction, Statement of Main Results, Literature

Throughout this paper $\Omega$ is an open bounded domain in $\mathbb{R}^n$, typically $n > 1$, with sufficiently smooth boundary $\partial \Omega = \Gamma$, see Remark 1.2. In $\Omega$ we consider the following non homogeneous problem for the Euler–Bernoulli equation in the solution $w(t, x)$:

\begin{align*}
    w_{tt} + A^2 w &= 0 \quad \text{in } (0, T] \times \Omega = Q, & (1.1a) \\
    w(0, \cdot) &= w^0; \quad w_t(0, \cdot) = w^1 \quad \text{in } \Omega & (1.1b) \\
    w|\Sigma &= g_1 \quad \text{in } (0, T] \times \Gamma \subseteq \Sigma. & (1.1c) \\
    A w|\Sigma &= g_2 \quad \text{in } \Sigma & (1.1d)
\end{align*}

with control functions $g_1, g_2$. The regularity question for problem (1.1) was recently studied in [L-T.8], whose major results (relevant to the present paper) will be recalled below as Theorem 1.0. The aim of this article is to study the exact controllability question for (1.1). Qualitatively this means: given $\Omega$, we ask whether there exists some $T_0 > 0$, such that if $T > T_0$, the following steering property of (1.1) holds true: for all initial data $w^0, w^1$ in some preassigned space $Z = Z_1 \times Z_2$ based on $\Omega$, there exist suitable control functions $g_1$ and $g_2$ on some preassigned space $V_x = V_{1x} \times V_{2x}$ based on $\Gamma \times (0, T]$, whose corresponding solution of (1.1) satisfies $w(T, \cdot) = 0$, $w_t(T, \cdot) = 0$. We then say that the dynamics (1.1) is exactly controllable on the space $Z$ over the interval $[0, T]$ by means of control functions in $V_x$. We shall consider a few natural choices of pairs $[Z, V_x]$ of spaces. In fact, these will be selected according to the regularity results established in [L-T.8]. In order to report these results, we need to introduce some preliminary background. Throughout the paper we let $A : L^2(\Omega) \rightarrow D(A) \rightarrow L^2(\Omega)$ be the positive, self-adjoint operator defined by

$$A h = A^2 h, \quad D(A) = \{ h \in L^2(\Omega) : A^2 h \in L^2(\Omega), h|_\Gamma = A h|_\Gamma = 0 \} = \{ h \in H^4(\Omega) : h|_\Gamma = A h|_\Gamma = 0 \}. \quad (1.2)$$

It is expedient to record the known result that

$$A^{1/2} = A_D, \quad (1.3)$$

where $A_D$ is the (positive, self-adjoint) operator defined by

$$A_D h = -A h, \quad D(A_D) = H_\Delta(\Omega) \cap H_0^1(\Omega) \quad (1.4)$$

the subscript "D" reminding us that $A_D$ has homogeneous boundary conditions of Dirichlet type. The following space identifications are known (with equivalent norms) [G.1] [L–M.1]
EULER-BERNOULLI EQUATION

\[ D(A^\theta) = \{ h \in H^4(\Omega) : h |_F = 0 \} = H_0^4(\Omega), \quad \frac{1}{8} < \theta < \frac{5}{8} \]  
(1.5)

\[ D(A^\theta) = \{ h \in H^4(\Omega) : h |_F = \Delta h |_F = 0 \}, \quad \frac{5}{8} < \theta < 1. \]  
(1.6)

The following specialization thereof will be needed below

\[ \theta = \frac{1}{4} : D(A^{1/4}) = H_0^1(\Omega) \text{ (with equivalent norms); for } f \in H_0^1(\Omega) \]  
(1.7a)

\[ \text{norm } \| f \|_{D(A^{1/4})} = \| A^{1/4} f \|_{H^2(\Omega)}, \text{equivalent to } \| f \|_{H_0^1(\Omega)}, \]  
in turn equivalent to the gradient-norm  
(1.7b)

\[ \left\{ \int_\Omega |\nabla f|^2 \, d\Omega \right\}^{1/2}, \text{ by } \text{Poincare inequality}. \]

\[ \theta = \frac{3}{4} : D(A^{3/4}) = V \text{ (with equivalent norms) } \]

\[ V = \{ h \in H^3(\Omega) : h |_F = \Delta h |_F = 0 \} \]  
(1.8a)

\[ \text{norm } \| f \|_{D(A^{3/4})} = \| A^{3/4} f \|_{L^2(\Omega)} = \| A^{1/4} A D f \|_{L^2(\Omega)} = \| A^{1/4} (\Delta f) \|, \]  
equivalent to \[ \left\{ \int_\Omega |\nabla (\Delta f)|^2 \, d\Omega \right\}^{1/2}, \text{ by } (1.7b). \]

In general, natural norms are

\[ \| x \|_{D(A^\beta)} = \| A^\beta x \|_{L^2(\Omega)}; \quad \| x \|_{[D(A^\theta)]'} = U A^{-\beta} x \|_{L^2(\Omega)} \]  
(1.9)

for \( \beta \geq 0 \), where \([D(A^\theta)]'\) denotes the dual space of \( D(A^\theta) \) with respect to the \( L^2(\Omega) \)-topology.

**Theorem 1.0** (Regularity [L–T.8]). (i) Consider problem \( (1.1) \) subject to

\[ \{ w_0, w_1 \} \in [D(A^{1/4})]' \times [D(A^{3/4})]' = H^{-1}(\Omega) \times V' \quad (1.10) \]

\[ g_1 \in L^2(\Sigma), \quad g_2 \in [H^1(0, T; L^2(\Gamma))]'. \]  
(1.11)

Then the map

\[ \{ w_0, w_1, g_1, g_2 \} \rightarrow \{ w(T), w_1(T) \} \in H^{-1}(\Omega) \times V' \]  
(1.12)

is continuous. Moreover, the map

\[ \{ w_0, w_1, g_1, g_2 \} \rightarrow \{ w(t), w_1(t) \} \]  
(1.13a)

\[ [H^1(0, T; L^2(\Gamma))]' \text{ is the space dual to } H^1(0, T; L^2(\Gamma)) \text{ with respect to the } H^0(0, T; L^2(\Gamma))\text{-topology. Since } H^1_0(0, T; L^2(\Gamma)) \subsetneq H^1(0, T; L^2(\Gamma)), \text{ we then have } [H^1(0, T; L^2(\Gamma))]' \subset H^{-1}(0, T; L^2(\Gamma)). \]
is likewise continuous into
\[ C([0, T]; H^{-1}(\Omega)) \times L^2(0, T; V'), \tag{1.13b} \]

where \(L^2\) in (1.13b) cannot be replaced by \(C\).

(ii) Consider problem (1.1) subject to
\[ \{w^0, w^1\} \in D(\{A^{1/4}\}) \times [D(\{A^{3/4}\})]' = H^1_0(\Omega) \times H^{-1}(\Omega) \tag{1.14} \]
\[ g_1 \in H^1_0(0, T; L^2(\Gamma)); \quad g_2 \in L^2(\Sigma). \tag{1.15} \]

Then the map
\[ \{w^0, w^1, g_1, g_2\} \rightarrow \{w(T), w_1(T)\} \in H^1_0(\Omega) \times H^{-1}(\Omega) \tag{1.16} \]
is continuous. Moreover, the map
\[ \{w^0, w^1, g_1, g_2\} \rightarrow \{w(t), w_1(t)\} \tag{1.17a} \]
is continuous into
\[ C([0, T]; H^{1/2}(\Omega)) \times C([0, T]; H^{-1}(\Omega)), \tag{1.17b} \]

where \(H^{1/2}(\Omega)\) cannot be replaced by \(H^1_0(\Omega)\).

We can now state our main exact controllability results. They do not require geometrical conditions on \(\Omega\).

**THEOREM 1.1** (Exact controllability on \([D(\{A^{1/4}\})]' \times [D(\{A^{3/4}\})]' = H^{-1}(\Omega) \times V'\)). For any \(T > 0\), given any pair of initial data
\[ \{w^0, w^1\} \in [D(\{A^{1/4}\})]' \times [D(\{A^{3/4}\})]' = H^{-1}(\Omega) \times V' \tag{1.18} \]
there exist boundary controls
\[ g_1 \in L^2(\Sigma); \quad g_2 \in [H^1(0, T; L^2(\Gamma))]' \tag{1.19} \]
such that the corresponding solution to problem (1.1) satisfies
\[ w(T) = w_1(T) = 0 \tag{1.20} \]
as well as (1.13).

**THEOREM 1.2** (Exact controllability on \(D(\{A^{1/4}\}) \times [D(\{A^{1/4}\})]' \equiv H^1_0(\Omega) \times H^{-1}(\Omega)\)). For any \(T > 0\) given any initial data
\[ \{w^0, w^1\} \in D(\{A^{1/4}\}) \times [D(\{A^{1/4}\})]' = H^1_0(\Omega) \times H^{-1}(\Omega) \tag{1.21} \]
there exist boundary controls

\[ g_1 \in H_0^1(0, T; L^2(\Gamma)); \quad g_2 \in L^2(\Sigma) \]  

(1.22)

such that the corresponding solution to problem (1.1) satisfies

\[ w(T) = w_t(T) = 0 \]  

(1.23)
as well as (1.17).

**Remark 1.1.** Consider the following homogeneous problem

\[ \phi_{tt} + A^2 \phi = 0 \quad \text{in } Q \]  

(1.24a)

\[ \phi |_{t=0} = \phi^0; \quad \phi_t |_{t=0} = \phi^1 \quad \text{in } \Omega \]  

(1.24b)

\[ \phi |_{\Sigma} = 0 \quad \text{in } \Sigma \]  

(1.24c)

\[ \Delta \phi |_{\Sigma} = 0 \quad \text{in } \Sigma. \]  

(1.24d)

The proof of Theorem 1.1 shows that exact controllability as stated there is equivalent to the following inequality: there is \( C_T > 0 \) such that for all \( \{\phi^0, \phi^1\} \in D(A^{3/4}) \times D(A^{1/4}) \) we have

\[ \int_\Sigma \left( \frac{\partial (\Delta \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi_t}{\partial v} \right)^2 + \left( \frac{\partial \phi}{\partial v} \right)^2 \, d\Sigma \geq C_T \| \{\phi^0, \phi^1\} \|_{D(A^{3/4}) \times D(A^{1/4})}^2 \]  

(1.25)

see the (backward in time) problem (2.18) in Lemma 2.1. Next, Lemma 2.4 and Remark 2.1 at the end of Section 2 yield that inequality (1.25) is in fact equivalent to the following inequality: there is \( C_T^* > 0 \) such that for all \( \{\phi^0, \phi^1\} \in D(A^{3/4}) \times D(A^{1/4}) \) we have

\[ \int_\Sigma \left( \frac{\partial (\Delta \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi_t}{\partial v} \right)^2 \, d\Sigma \geq C_T^* \| \{\phi^0, \phi^1\} \|_{D(A^{3/4}) \times D(A^{1/4})}^2 \]  

(1.26)
i.e., the lower order term on \( \partial \phi / \partial v \) can be absorbed by the other boundary terms. Finally, Proposition 2.2 shows that inequality (1.26) always holds true for any \( T > 0 \) and with no geometrical conditions on \( \Omega \) (except for smoothness of \( \Omega \) as in Remark 1.2 below).

On the other hand, the opposite inequality

\[ \int_\Sigma \left( \frac{\partial (\Delta \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi_t}{\partial v} \right)^2 \, d\Sigma \leq CT \| \{\phi^0, \phi^1\} \|_{D(A^{3/4}) \times D(A^{1/4})}^2 \]  

(1.27)
is likewise always true for all \( T > 0 \) \([L-T.8].\) Thus, for any \( T > 0 \)

\[ \left\{ \int_\Sigma \left( \frac{\partial (\Delta \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi_t}{\partial v} \right)^2 \, d\Sigma \right\}^{1/2} \]  

(1.28)
defines a norm on the space \( D(A^{3/4}) \times D(A^{1/4}) = V \times H^1_0(\Omega) \), which is equivalent to the norm
\[
\| \{ \phi^0, \phi^1 \} \|_{D(A^{3/4}) \times D(A^{1/4})}
\]
so that J. L. Lions' space \( F = D(A^{3/4}) \times D(A^{1/4}) = V \times H^1_0(\Omega) \).

By interpolation between Theorem 1.1 and Theorem 1.2 we obtain part (ii) of the next corollary.

**Corollary 1.3.** (i) For any \( T > 0 \), inequality (2.21) [which characterizes exact controllability of problem (1.1) with \( g_1 \in L^2(\Sigma), g_2 \in [H^1(0, T; L^2(\Gamma))]' \subset H^{-1}(0, T; L^2(\Gamma)) \) on the space \([D(A^{1/4})]' \times [D(A^{3/4})]'\) over \([0, T]\)] and inequality (3.10) [which characterizes exact controllability of problem (1.1) with \( g_1 \in H^1_0(0, T; L^2(\Gamma)), g_2 \in L^2(\Sigma) \) on the space \( D(A^{1/4}) \times [D(A^{3/4})]' \) over \([0, T]\)] are equivalent conditions.

(ii) When part (i) holds, we deduce by interpolation [L–M.1, p. 29, 64–66], that problem (1.1) with controls
\[
\begin{align*}
g_1 & \in H^1_0(0, T; L^2(\Gamma)), & 0 < \theta < 1, \theta \neq \frac{1}{2} \\
g_1 & \in H^{1/2}_{00}(0, T; L^2(\Gamma)), & \theta = \frac{1}{2} \\
g_2 & \in [H^{\theta}(0, T; L^2(\Gamma))]', & 0 < \theta < 1,
\end{align*}
\]
is exactly controllable on the space \( D(A^{1/4} - \theta/2) \times D(A^{-1/4} + \theta/2) \),
\[
(1.32)
\]
where we are using the convention that
\[
D(A^{-\beta}), \quad \beta \geq 0 \text{ means } [D(A^{\beta})]'.
\]

**Remark 1.2 (On the smoothness of \( \Gamma \)).** The proofs given below require the existence of a dense set of initial data for which the solutions of the corresponding homogeneous problem (1.24) possess the regularity required to carry out the actual computations in the multiplier methods of Sections 2–3. This condition is satisfied if \( \Gamma \) is sufficiently smooth.

**Remark 1.3.** Once exact controllability is established, then an elementary argument provides the minimal norm, steering control (see the abstract operator argument in [T.2, Appendix B], [L–T.3, Appendix B] and its specialization there to the wave equation, as well as [L–T.7, Appendix D; L.6] and its specialization to plate problems).

**Literature.** This paper is part of the present effort on exact controllability problems for "plate equations," which has followed the recent progress and understanding on questions of maximal regularity [L.1; L–T.2; L–T.4; L–T–T.1], uniform stabilization [L.5; L–T.1] and exact controllability for second order hyperbolic equations [L.2; L.3; H.1;...
K.1; T.2]. (See also [F–L–T.1]; [L–T.5], for Riccati equations corresponding to these problems.) Here, for the Euler–Bernoulli problem (1.1), we pursue the same general direct "ontoness" approach to exact controllability which we have already carried out for (i) second order hyperbolic equations with Dirichlet [T.2] or Neumann [L–T.3] boundary control, and also for (ii) Euler–Bernoulli problems with different types of boundary conditions [L–T.6; L–T.7]. Remark 1.3 then provides an explicit steering control, in fact the minimal-norm control. This is the one used by J. L. Lions in his approach. Our Theorem 1.1 is slightly stronger than a similar result sketched by J. L. Lions in [L.2] in the sense that in our present paper the boundary control $g_2$ is taken in the strictly smaller space $[H^1(0, T; L^2(\Gamma))]'$ than the space $H^{-1}(0, T; L^2(\Gamma))$ considered in [L.2]. Moreover, the approach in [L.2] and our present ontoness approach are different. Theorem 1.2 and the interpolation result Corollary 1.3 are new. Theorem 1.2 passes through a new feature (the term $K_{1\Gamma}$ in the boundary integral of the characterizing inequality (3.10) which is not encountered in second order hyperbolic equations or in the plate problems considered in [L.2]). It is not surprising that our exact controllability results are achieved for any $T$ arbitrarily small [L–T.7], [Z.1]. For other work on plate equations we refer to [L–L.1] for different boundary conditions and to [L.4] for smoother spaces.

2. PROOF OF THEOREM 1.1: EXACT CONTROLLABILITY ON $[D(A^{1/4})'] \times [D(A^{3/4})']$ (COROLLARY 1.3)

Throughout this section we set for convenience

$Z = [D(A^{1/4})'] \times [D(A^{3/4})']$, \quad $U = L^2(\Sigma) \times [H^1(0, T; L^2(\Gamma))]'$,

where for $y = [y_1, y_2]$, $z = [z_1, z_2]$ in $Z$, the inner product in $Z$ is defined by

$$(y, z)_Z = (A^{-1/4}y_1, A^{-1/4}z_1)_{L^2(\Gamma)} + (A^{-3/4}y_2, A^{-3/4}z_2)_{L^2(\Gamma)}.$$ (2.0b)

The inner product on $[H^1(0, T; L^2(\Gamma))]'$ is defined as follows. Let $A$ be an isomorphism $H^1(0, T; L^2(\Gamma)) \rightarrow H^0(0, T; L^2(\Gamma))$, self-adjoint on $H^0(0, T; L^2(\Gamma))$. Then

$$(f, g)_{H^1(0, T; L^2(\Gamma))} = (Af, Ag)_{L^2(0, T; L^2(\Gamma))}$$ (2.0c)

$$(f, g)_{[H^1(0, T; L^2(\Gamma))]} = (A^{-1}f, A^{-1}g)_{L^2(0, T; L^2(\Gamma))}.$$ (2.0d)

Moreover from (2.0c)

$$\|f\|^2_{H^1(0, T; L^2(\Gamma))} = \|f\|^2_{L^2(\Sigma)} + \left\| \frac{df}{dt} \right\|^2_{L^2(\Sigma)} = \|Af\|^2_{L^2(\Sigma)}.$$ (2.0e)
Step 0. In line with the authors' approach to time invariant problems for second order and fourth order differential operators in the space variables [L–T.1–L–T.8], we recall that the solution at time $T$ to problem (1.1) with $w^0 = w^1 = 0$ may be written explicitly as (see [L–T.8]):

$$
\left[ \begin{array}{c} w(T; t = 0; w^0 = 0, w^1 = 0) \\ w_t(T; t = 0; w^0 = 0, w^1 = 0) \end{array} \right] = L_{1T} g_1 + L_{2T} g_2 = L_T \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \quad (2.1)
$$

$$
L_{1T} g_1 = \begin{vmatrix} A \int_0^T S(T-t) G_1 g_1(t) dt \\ A \int_0^T C(T-t) G_1 g_1(t) dt \end{vmatrix} \quad (2.2)
$$

$$
L_{2T} g_2 = \begin{vmatrix} A \int_0^T S(T-t) G_2 g_2(t) dt \\ A \int_0^T C(T-t) G_2 g_2(t) dt \end{vmatrix} \quad (2.3)
$$

Here, $-A$ is the (negative self-adjoint) operator defined in (1.2) which generates a s.c. cosine operator $C(t)$ on $L^2(\Omega)$, with $S(t) = \int_0^t C(\tau) d\tau, t \in R$. Moreover, $G_1$ and $G_2$ are Green maps defined as follows:

$$
\begin{cases}
A^2 y = 0 & \text{in } \Omega \\
y|_\Gamma = g_1 & \text{on } \Gamma \\
A y|_\Gamma = 0 & \text{on } \Gamma
\end{cases} \quad (2.4)
$$

$$
\begin{cases}
A^2 y = 0 & \text{in } \Omega \\
y|_\Gamma = 0 & \text{on } \Gamma \\
A y|_\Gamma = g_2 & \text{on } \Gamma
\end{cases} \quad (2.5)
$$

By elliptic theory we have for any $s \in R$ [L–M.1, Vol. I, p. 188–189]

$$
G_1: \text{continuous } H^s(\Gamma) \to H^{s + 1/2}(\Omega) \quad (2.6)
$$

$$
G_2: \text{continuous } H^s(\Gamma) \to H^{s + 5/2}(\Omega) \quad (2.7)
$$

Let now $G_i^*$ be the adjoint of $G_i$ in the sense

$$
(G_i g, v)_{L^2(\Omega)} = (g, G_i^* v)_{L^2(\Gamma)}, \ g \in L^2(\Gamma), \ v \in L^2(\Omega). \quad (2.8)
$$

The following Lemma can be established by use of Green's second theorem for its parts (ii), (iii).
**Lemma 2.0** [L–T.8]. We have

(i)

\[ G_1 = D; \quad G_2 = -A_D^{-1}D, \]  

where \( A_D = A^{1/2} \), see (1.3), and where

\[ Dg = \zeta \Leftrightarrow A\zeta = 0 \text{ in } \Omega; \quad \zeta = g \text{ on } \Gamma; \]  

(ii)

\[ G^*_1Af = D^*A^2Df = \frac{\partial (Af)}{\partial v}, \quad f \in \mathcal{D}(A) \]  

\[ G^*_1A^{1/2}f = D^*A_Df = -\frac{\partial f}{\partial v}, \quad f \in \mathcal{D}(A^{1/2}); \]  

(iii)

\[ G^*_2Af = -D^*A_Df = \frac{\partial f}{\partial v}, \quad f \in \mathcal{D}(A^{1/2}). \]

**Step 1.** The (regularity) Theorem 1.0 gives that (recall (2.0a))

\[ L_T \equiv [L_{1T}, L_{2T}]: \text{continuous } U \rightarrow Z \]  

By time reversibility of problem (1.1), exact controllability of problem (1.1) on the space \( Z \) over \([0, T]\) by means of controls in \( U \) is equivalent to the condition that \( L^*_T \) has a continuous inverse, i.e., \([T–L.1]\) there is \( C_T > 0 \) such that

\[ \| L^*_T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \|_U \geq C_T \| \{z_1, z_2\} \|_Z \]  

for all \( z = [z_1, z_2] \in Z \), where for \([g_1, g_2] \in U\)

\[ \left( L_T \begin{pmatrix} g_1 \\ g_2 \\ z_1 \\ z_2 \end{pmatrix} \right)_Z = \begin{pmatrix} g_1 \\ g_2 \\ L^*_1 z_1 \\ L^*_2 z_2 \end{pmatrix}_U \]

\[ = (L_{1T}g_1 + L_{2T}g_2, z)_Z \]

\[ = (g_1, L^*_1 z_1)_{L^2(\Sigma)} + (g_2, L^*_2 z_2)_{[H^1(0, T; L^2(\Gamma))]}. \]

\[ = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} L^*_1 z_1 \\ L^*_2 z_2 \end{pmatrix} \]  

\[ L^*_T z = \begin{pmatrix} L^*_1 z_1 \\ L^*_2 z_2 \end{pmatrix}; \quad \| L^*_T z \|_U^2 = \| L^*_1 z_1 \|_{L^2(\Sigma)}^2 + \| L^*_2 z_2 \|_{[H^1(0, T; L^2(\Gamma))]}. \]

\[ \| L^*_T z \|_U^2 = \| L^*_1 z_1 \|_{L^2(\Sigma)}^2 + \| L^*_2 z_2 \|_{[H^1(0, T; L^2(\Gamma))]}. \]

recalling (2.0d).
Step 2. An equivalent partial differential equation characterization of inequality (2.15) is given by the following Lemma.

**Lemma 2.1.** For \( z \in Z = \left[ D(A^{1/4}) \right]' \times \left[ D(A^{3/4}) \right]' \) we have:

(i) Let

\[
(L_T^* z)(t) = \frac{\partial (A \phi(t))}{\partial v} \quad \text{on } \Sigma, \tag{2.17}
\]

where \( \phi(t) = \phi(t, \phi_0, \phi_1) \) is the solution of the following homogeneous problem, backward in time

\[
\phi_{tt} + A^2 \phi = 0 \quad \text{in } Q \tag{2.18a}
\]

\[
\phi|_{t=T} = \phi_0; \quad \phi_t|_{t=T} = \phi_1 \quad \text{in } \Omega \tag{2.18b}
\]

\[
\phi|_{\Sigma} = 0 \quad \text{in } \Sigma \tag{2.18c}
\]

\[
\Delta \phi|_{\Sigma} = 0 \quad \text{in } \Sigma \tag{2.18d}
\]

with

\[
\phi_0 = A^{-3/2} z_2 \in D(A^{3/4}); \quad \phi_1 = -A^{-1/2} z_1 \in D(A^{1/4}) \tag{2.18e}
\]

explicitly given by

\[
\phi(t) = C(t - T) \phi_0 + S(t - T) \phi_1 \in C([0, T]; D(A^{3/4})) \tag{2.18f}
\]

\[
\phi, (t) = -AS(t - T) \phi_0 + C(t - T) \phi_1 \in C([0, T]; D(A^{1/4})). \tag{2.18g}
\]

(ii) With \( A \) the self-adjoint isomorphism introduced in (2.0c)

\[
(A^{-2} L_{T}^* z)(t) = \frac{\partial}{\partial v} \phi(t), \tag{2.19}
\]

where \( \phi(t) \) solves (2.18a)-(2.18e); moreover

\[
\|L_{T}^* z\|^2_{L^2(\Omega; L^2(\Sigma))} = \left\| \frac{\partial \phi(t)}{\partial v} \right\|_{L^2(\Sigma)}^2 + \left\| \frac{\partial \phi_t(t)}{\partial v} \right\|_{L^2(\Sigma)}^2. \tag{2.20}
\]

(iii) For any \( 0 < T < \infty \), inequality (2.15) is equivalent to: there is \( C_T > 0 \) such that

\[
\int_{\Sigma} \left( \frac{\partial (A \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi}{\partial v} \right)^2 + \left( \frac{\partial \phi_t}{\partial v} \right)^2 \; d\Sigma \geq C_T \| \{ \phi_0, \phi_1 \} \|^2_{D(A^{1/4}) \times D(A^{1/4})}. \tag{2.21}
\]
for all \( \{ \phi^0, \phi^1 \} \in D(A^{3/4}) \times D(A^{1/4}) \). Moreover, the map \( T \to C_T \) is monotone increasing.

**Proof of Lemma 2.1.** (i) We let \( g_1 \in L^2(\Sigma) \) and \( z \in Z \) and proceed as in [T.2; L–T.7, L–T.3] by use of (2.2), (2.0b)

\[
(L_{1T}^* g_1, z)_{L^2(\Gamma)} = \left( \int_0^T S(T-t) G_1 g_1(t) dt, A^{1/2} z_1 \right)_{L^2(\Omega)} + \left( \int_0^T C(T-t) G_1 g_1(t) dt, A^{-1/2} z_2 \right)_{L^2(\Gamma)}
\]

\[
= \int_0^T (g_1(t), G_1^* [S(T-t) A^{1/2} z_1 + C(T-t) A^{-1/2} z_2])_{L^2(\Gamma)} dt.
\]

(2.22)

Hence, by (2.16a) and (2.22) and using that \( C(\cdot) \) is even and \( S(\cdot) \) is odd:

\[
(L_{1T}^* z)(t) = G_1^* [C(t-T) A^{-1/2} z_2 + S(t-T)(-A^{1/2} z_1)] = G_1^* A [C(t-T) A^{-3/2} z_2 + S(t-T)(-A^{-1/2} z_1)]
\]

(by (2.11))

\[
= \frac{\partial (\Delta \phi(t, \phi^0, \phi^1))}{\partial v},
\]

where \( \phi(t) = \phi(t, \phi^0, \phi^1) \) solves problem (2.18a–e) and part (i) is proved

(ii) Similarly, by virtue of (2.3) we compute with \( g_2 \in [H^1(0, T; L^2(\Gamma))]' \) and \( z = [z_1, z_2] \in Z \)

\[
(L_{2T}^* g_2, z)_{L^2(\Gamma)} = \left( \int_0^T S(T-t) G_2 g_2(t) dt, A^{1/2} z_1 \right)_{L^2(\Omega)} + \left( \int_0^T C(T-t) G_2 g_2(t) dt, A^{-1/2} z_2 \right)_{L^2(\Omega)}
\]

\[
= \int_0^T (g_2(t), G_2^* [S(T-t) A^{1/2} z_1 + C(T-t) A^{-1/2} z_2])_{L^2(\Gamma)} dt
\]

(2.24a)

[counterpart of (2.22), and by (2.16a), (2.0d)]

\[
= (g_2, L_{2T}^* z)_{[H^1(0, T; L^2(\Gamma))]'} = (A^{-1} g_2, A^{-1} L_{2T}^* z)_{L^2(\Omega)} = (g_2, A^{-2} L_{2T}^* z)_{L^2(\Omega)}.
\]

(2.24b)
Hence, by comparing (2.24a) and (2.24b) we obtain

\[(A^{-1}L^*_{2T}z)(t) = G_z^* [S(T-t)A^{1/2}z_1 + C(T-t)A^{-1/2}z_2]\]

\[= G_z^* A \left[ C(t-T)A^{-3/2}z_2 + S(t-T)(-A^{-1/2}z_1) \right]\]

(by (2.13))

\[= \frac{\partial}{\partial V} \phi(t, \phi^0, \phi^1), \quad (2.25)\]

where \(\phi(t, \phi^0, \phi^1)\) solves problem (2.18a–e). Hence by (2.5)

\[(A^{-1}L^*_{2T}z)(t) = A \frac{\partial}{\partial V} \phi(t, \phi^0, \phi^1) \quad (2.26)\]

and by (2.0d), (2.26), and (2.0e)

\[\|L^*_{2T}z\|_{L^2(0, T; L^2(\Gamma)))}^2 = \|A^{-1}L^*_{2T}z\|_{L^2(\Sigma)}^2 = \left\| \frac{\partial \phi(t, \phi^0, \phi^1)}{\partial V} \right\|_{L^2(\Sigma)}^2 \]

\[= \left\| \frac{\partial \phi(t)}{\partial V} \right\|_{L^2(\Sigma)}^2 + \left\| \frac{\partial \phi(t)}{\partial V} \right\|_{L^2(\Sigma)}^2 \quad (2.27)\]

and part (ii) is proved. Part (iii) then is an immediate consequence of (2.15), (2.16b), (2.17), (2.20) = (2.27) and of the following identity (recall (2.18e) and (2.0b):

Since, Eqs. (2.18e) and (2.0b):

\[\|z_1 = A^{-1/2}\phi^1, z_2 = A^{3/2}\phi^0\|_{L^2(\Omega)}^2 = \| -A^{1/4}\phi^1, A^{3/4}\phi^0\|_{L^2(\Omega)}^2 \]

\[= \|\phi^0, \phi^1\|_{D(A^{3/4}) \times D(A^{1/4})}^2 \quad (2.28)\]

To prove that the map \(T \rightarrow C_T^r\) is monotone increasing, we first note (from (2.15), (2.16b), (2.17), (2.20), (2.28)) that we may take

\[C_T^r = \|L^*_{T}^{-1}\|\]

in the uniform norm from \(F \equiv D(A^{3/4}) \times D(A^{1/4}) \) into \(U \) (see (2.0a)). Now, as \(y\) runs over all of \(F\), the functions \(u(t) = (L^*_{T}^{-1}y)(t) 0 \leq t \leq T\), once extended by zero over \(T < t < T_1\), are competitors in the computation of \(C_{T_1}^r = \|L^*_{T_1}^{-1}\|\). Thus it follows easily that \(T < T_1\) implies \(C_T^r < C_{T_1}^r\). The proof of Lemma 2.1 is complete. \(\blacksquare\)

Step 3. It remains to show if, or when, inequality (2.21) holds true. The following Proposition is the key technical issue of the exact controllability problem of the present section for the dynamics (1.1).
PROPOSITION 2.2. For any \( T > 0 \) there is \( C_T > 0 \) such that for all \( \{ \phi^0, \phi^1 \} \in D(A^{3/4}) \times D(A^{1/4}) \) we have

\[
\int_\Sigma \left( \frac{\partial (A\phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi_t}{\partial v} \right)^2 \, d\Sigma \geq C_T \| \{ \phi^0, \phi^1 \} \|_{D(A^{3/4}) \times D(A^{1/4})}^2,
\]

(2.28a)

where, by time reversal in (2.18), we may take \( \phi \) in (2.28) to be the solution of problem (2.18a, c, d, e) and

\[
\phi \big|_{t=0} = \phi^0 \in D(A^{3/4}); \quad \phi_t \big|_{t=0} = \phi^1 \in D(A^{1/4}).
\]

(2.28b)

Thus, (2.28a) a fortiori proves inequality (2.21).

Indeed \( C_T \) may be taken as

\[
C_T = \frac{T - 2\hat{c}c_{h,c}}{\text{const}_{h,c}}
\]

(2.28c)

in the notation of (2.70a) in the proof below.

Proof of Proposition 2.2. Step (i). Let \( h(x) \) be, for the time being, a \( C^2(\Omega) \)-vector field. With reference to Remark 1.2 we multiply Eq. (2.18a) by \( h \cdot \nabla(A\phi) \) and integrate by parts over \( Q \). We obtain the following identity (see Appendix A for details):

\[
\int_\Sigma \frac{\partial (A\phi)}{\partial v} h \cdot \nabla(A\phi) \, d\Sigma + \int_\Sigma \frac{\partial \phi_t}{\partial v} h \cdot \nabla \phi_t \, d\Sigma + \int_\Sigma \frac{\partial \phi_t}{\partial \phi_t} h \cdot \nabla h \, d\Sigma
\]

\[
- \frac{1}{2} \int_\Sigma |\nabla(A\phi)|^2 h \cdot v \, d\Sigma - \frac{1}{2} \int_\Sigma |\nabla \phi_t|^2 h \cdot v \, d\Sigma - \int_\Sigma \phi_t \nabla \phi_t \cdot h \cdot v \, d\Sigma
\]

\[
= \int_Q H\nabla(A\phi) \cdot \nabla(A\phi) \, dQ + \int_Q H\nabla \phi_t \cdot \nabla \phi_t \, dQ
\]

\[
+ \frac{1}{2} \int_Q \{ |\nabla \phi_t|^2 - |\nabla(A\phi)|^2 \} \, \text{div} \, dQ
\]

\[
+ \int_Q \phi \nabla \cdot \nabla h \cdot \nabla \phi_t \, dQ - [(\phi_t, h \cdot \nabla(A\phi))_0]_0^T.
\]

(2.29)

where \( H = H(x) \) is the matrix

\[
H(x) = \begin{vmatrix}
\frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_n} \\
\frac{\partial h_2}{\partial x_1} & \ldots & \frac{\partial h_n}{\partial x_n} \\
\vdots & & \vdots \\
\frac{\partial h_n}{\partial x_1} & \ldots & \frac{\partial h_n}{\partial x_n}
\end{vmatrix}
\]

(2.30)
We next use the boundary conditions (2.18c)--(2.18d) on $\Sigma$:

$$\phi \equiv 0; \quad \phi_r \equiv 0; \quad |\nabla \phi_r| = \left| \frac{\partial \phi_r}{\partial v} \right|; \quad \nabla \phi_r \text{ parallel to } v$$ \hfill (2.31a)

$$A\phi \equiv 0; \quad A\phi_r \equiv 0; \quad |\nabla (A\phi_r)| = \left| \frac{\partial (A\phi)}{\partial v} \right|; \quad \nabla (A\phi) \text{ parallel to } v.$$ \hfill (2.31b)

Hence the terms on $\Sigma$ marked by an arrow cancel in (2.29). Moreover, writing

$$h = (h \cdot v)v + (h \cdot \tau)\tau$$

on $\Gamma$, $\tau =$ unit tangent vector we obtain by (2.31a) and (2.31b) respectively

$$h \cdot \nabla (A\phi) = \nabla (A\phi) \cdot v (h \cdot v) = \frac{\partial (A\phi)}{\partial v} h \cdot v$$ \hfill (2.31c)

$$h \cdot \nabla \phi_r = \nabla \phi_r (h \cdot v) = \frac{\partial \phi_r}{\partial v} h \cdot v.$$ \hfill (2.31d)

Thus using (2.31a, b, c, d) in the left hand side (LHS) of (2.29) we obtain

$$\text{LHS of (2.29)} = \frac{1}{2} \int_{\Sigma} \left( \frac{\partial (A\phi)}{\partial v} \right)^2 h \cdot v \, d\Sigma + \frac{1}{2} \int_{\Sigma} \left( \frac{\partial \phi_r}{\partial v} \right)^2 h \cdot v \, d\Sigma$$

$$\frac{3C_h}{2} \int_{\Sigma} \left( \frac{\partial (A\phi)}{\partial v} \right)^2 \left( \frac{\partial \phi_r}{\partial v} \right)^2 \, d\Sigma \geq \text{LHS of (2.29)}. \hfill (2.32)$$

**Step (ii).** We specialize to the radial vector field $h(x) = x - x_0$, $x_0 \in \mathbb{R}^n$, so that by (2.30)

$$H(x) \equiv \text{identity}; \quad \text{div } h \equiv n = \dim \Omega; \quad \nabla (\text{div } h) = 0. \hfill (2.33)$$

Moreover, we multiply Eq. (2.18a) by $A\phi$ and obtain (see Appendix B for details)

$$\int_{Q} \left\{ |\nabla \phi_r|^2 - |\nabla (A\phi)|^2 \right\} \, dQ$$

$$= \left[ \int_{\Omega} \nabla \phi \cdot \nabla \phi_r \, d\Omega - \int_{r} \frac{\partial \phi_r}{\partial r} d\Gamma \right]_{0}$$

$$+ \int_{\Sigma} \frac{\partial \phi_r}{\partial v} \phi_r \, d\Sigma - \int_{\Sigma} \frac{\partial (A\phi)}{\partial v} A\phi \, d\Sigma \hfill (2.34)$$
after using the boundary conditions (2.18c–d), hence (2.31). Thus, using (2.33), (2.34) in the right hand side (RHS) of (2.29) results in

\[
\text{RHS of (2.29)} = \int_{\Omega} |\nabla(\Delta \phi)|^2 + |\nabla \phi|^2 \, d\Omega + \beta_{0,T},
\]

(2.35)

where \(\beta_{0,T}\) (boundary terms at \(t=0\) and \(t=T\)) is given by

\[
\beta_{0,T} = \frac{n}{2} \left[ \int_{\Omega} \nabla \phi \cdot \nabla \phi, \, d\Omega \right]_0^T - \left[ (\phi, \, h \cdot \nabla(\Delta \phi))_\Omega \right]_0^T.
\]

(2.36)

**Step (iii)** (Conservation of “energy”). Multiplying Eq. (2.18a) by \(\Delta \phi\), and integrating by parts via Green’s theorems we find

\[
\frac{1}{2} \frac{\partial}{\partial t} \left\{ \int_{\Omega} |\nabla \phi_1|^2 + |\nabla(\Delta \phi)|^2 \, d\Omega \right\} = \int_{\Gamma} \frac{\partial(\Delta \phi)}{\partial n} \Delta \phi, \, d\Gamma = 0 \quad (2.37a)
\]

using the boundary condition (2.31b); hence

\[
E(t) \equiv \int_{\Omega} |\nabla \phi_1(t)|^2 + |\nabla(\Delta \phi(t))|^2 \, d\Omega
\]

\[
= \int_{\Omega} |\nabla \phi_1|^2 + |\nabla(\Delta \phi^0)|^2 \, d\Omega \equiv E(0)
\]

(2.37b)

for all \(t \in R\).

**Step (iv).**

**Lemma 2.3.** With reference to (2.36) we have for any \(\varepsilon > 0\) and for \(C_{h,n} = \max[\frac{n}{2}, M_h]\):

\[
|\beta_{0,T}| \leq \frac{n}{\sqrt{\varepsilon}} \|\nabla \phi\|_{C([0,T]; L^2(\Omega))} + \frac{2M_h}{\varepsilon} \|\phi_1\|_{C([0,T]; L^2(\Omega))}^2 + 2\varepsilon C_{h,n} E(0). \quad (2.38)
\]

**Proof of Lemma 2.3.** We estimate each term of (2.36) separately by Schwarz inequality (here below all norms are \(L^2(\Omega)\)-norms):

\[
\left[ \int_{\Omega} \nabla \phi \cdot \nabla \phi, \, d\Omega \right]_0^T \leq \| \nabla \phi(T) \| + \| \nabla \phi_1(T) \| + \| \nabla \phi_0 \| \| \nabla \phi^1 \|
\]

\[
\leq \frac{1}{2} \left\{ \frac{1}{\varepsilon} \left[ \| \nabla \phi(T) \|^2 + \| \nabla \phi_1(T) \|^2 \right] + \varepsilon \| \nabla \phi_1(T) \| \| \nabla \phi^1 \| \right\}
\]

\[
\leq \frac{1}{\varepsilon} \| \nabla \phi \|_{C([0,T]; L^2(\Omega))}^2 + \frac{\varepsilon}{2} \left[ \| \nabla \phi_1(T) \|^2 + \| \nabla \phi^1 \|^2 \right]. \quad (2.39)
\]
Similarly, with \( 2M_h \equiv \max_{\Omega} |h| \):

\[
\| [(\phi^r, h \cdot \nabla(\Delta \phi))_\Omega]_0 \| \\
\leq 2M_h \left\{ \| \phi^r(T) \| \| \nabla(\Delta \phi(T)) \| + \| \phi^l \| \| \nabla(\Delta \phi^0) \| \right\} \\
\leq M_h \left\{ \frac{1}{\varepsilon} \left[ \| \phi^r(T) \|^2 + \| \phi^l \|^2 \right] \\
+ \varepsilon \left[ \| \nabla(\Delta \phi(T)) \|^2 + \| \nabla(\Delta \phi^0) \|^2 \right] \right\}
\]

\[
\leq \frac{2M_h}{\varepsilon} \| \phi^r \|^2 _{C([0,T];L^2(\Omega))} \\
+ \varepsilon M_h \left[ \| \nabla(\Delta \phi(T)) \|^2 + \| \nabla(\Delta \phi^0) \|^2 \right]. \tag{2.40}
\]

Hence, using (2.36), (2.39), (2.40) we obtain with \( C_{n,h} = \max \left[ n/2, M_n \right] \):

\[
|\beta_0| \leq \frac{n}{2\varepsilon} \| \nabla \phi \|^2 _{C([0,T];L^2(\Omega))} + \frac{2M_h}{\varepsilon} \| \phi^l \|^2 _{C([0,T];L^2(\Omega))} \\
+ C_{n,h} \varepsilon \left[ \| \nabla \phi^r(T) \|^2 + \| \nabla(\Delta \phi(T)) \|^2 \right] \\
+ \| \nabla \phi^l \|^2 + \| \nabla(\Delta \phi^0) \|^2 \right] \tag{2.41}
\]

and (2.38) follows from (2.41) by use of the conservation of energy (2.37b). Lemma 2.3 is proved. \( \blacksquare \)

**Step (v).** Using (2.38) and (2.37b) in (2.38) we obtain for the right hand side of (2.29):

\[
\text{RHS of (2.29)} \geq \int_0^T E(t) \, dt - 2\varepsilon C_{n,h} E(0)
\]

\[
- 2 \frac{C_{n,h}}{\varepsilon} \left[ \| \nabla \phi \|^2 _{C([0,T];L^2(\Omega))} + \| \phi^l \|^2 _{C([0,T];L^2(\Omega))} \right] \\
- \left[ T - 2\varepsilon C_{n,h} \right] E(0) \\
- 2 \frac{C_{n,h}}{\varepsilon} \left[ \| \nabla \phi \|^2 _{C([0,T];L^2(\Omega))} + \| \phi^l \|^2 _{C([0,T];L^2(\Omega))} \right]. \tag{2.42}
\]

Combining (2.32) with (2.42) we finally arrive at

\[
\text{Constr}_{c,n} \left[ \| \nabla \phi \|^2 _{C([0,T];L^2(\Omega))} + \| \phi^l \|^2 _{C([0,T];L^2(\Omega))} \right] \\
+ \frac{3C_n}{2} \int _\Sigma \left( \left( \frac{\partial(\nabla \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi^l}{\partial v} \right)^2 \right) \, d\Sigma \geq \left[ T - 2\varepsilon C_{n,h} \right] E(0). \tag{2.43}
\]
Step (vi). To complete the proof of Proposition 2.2, we need the following Lemma, of the type already used in [L.3], in [L.2], in [L-T.3], [L-T.7], etc.

**Lemma 2.4.** (i) Inequality (2.43) implies: for any $T > 0$, there is $C_T > 0$ such that for all $\{\phi^0, \phi^1\} \in D(A^{3/4}) \times D(A^{1/4})$ we have

$$\|\nabla \phi\|_{C([0, T]; L^2(\Omega))}^2 + \|\phi_t\|_{C([0, T]; L^2(\Omega))}^2 \leq C_T \int_{\Sigma} \left( \left( \frac{\partial (\Delta \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi_t}{\partial v} \right)^2 \right) d\Sigma;$$

(2.44a)

(ii) for any sequence $T_n \uparrow \infty$ we have

$$\lim \inf C_{T_n} = 0 \text{ so that } \sup_T C_T \equiv C < \infty.$$  

(2.44b)

**Proof of Lemma 2.4. Part (i).** The proof is by contradiction. Let there exist a sequence $\{\phi_n(t)\}$ of solutions to problem (2.18a, c, d, e) and (2.28) over $[0, T]$:

$$\phi''_n + \Delta^2 \phi_n = 0 \quad \text{in } \Omega \quad (2.45a)$$

$$\phi_n(0, \cdot) = \phi^0 \in D(A^{3/4}), \quad \phi'_n(0, \cdot) - \phi^1 \in D(A^{1/4}) \quad \text{in } \Omega \quad (2.45b)$$

$$\phi_n |_{\Sigma} \equiv 0 \quad \text{in } \Sigma \quad (2.45c)$$

$$\Delta \phi_n |_{\Sigma} \equiv 0 \quad \text{in } \Sigma \quad (2.45d)$$

($d/dt = \cdot$), given explicitly by

$$\phi_n(t) = C(t) \phi^0 + S(t) \phi^1 \in C([0, T]; D(A^{3/4})) \quad (2.46a)$$

$$\phi'_n(t) = -\Delta S(t) \phi^0 + C(t) \phi^1 \in C([0, T]; D(A^{1/4})), \quad (2.46b)$$

such that

$$\|\nabla \phi_n\|_{C([0, T]; L^2(\Omega))}^2 + \|\phi'_n\|_{C([0, T]; L^2(\Omega))}^2 = 1$$

(2.47a)

$$\int_{\Sigma} \left( \frac{\partial (\Delta \phi_n)}{\partial v} \right)^2 + \left( \frac{\partial \phi'_n}{\partial v} \right)^2 d\Sigma \to 0, \quad \text{as } n \to \infty. \quad (2.47b)$$

By the preceding steps (i)-(vi), each solution $\phi_n(t)$ satisfies inequality (2.43) and thus we have

$$E_n(0) = \int_{\Omega} |\nabla \phi_n|^2 + |\nabla (\Delta \phi^0)|^2 d\Omega \leq \text{Const}, \quad \text{uniformly in } n. \quad (2.48)$$
Recalling (1.7)–(1.8) $H_0^1 = D(A^{1/4})$, we see that there is a subsequence, still subindexed by $n$, such that
\[ \phi_n^0 \to \text{some function } \phi^0 \quad \text{in } H_0^1(\Omega) = D(A^{1/4}) \text{ weakly} \] (2.49a)
\[ \phi_n^1 \to \text{some function } \phi^1 \quad \text{in } V = D(A^{3/4}) \text{ weakly}. \] (2.49b)

We then consider the solution to problem (2.18a, c, d, e), (2.28) with initial data found in (2.49)
\[ \bar{\phi}(t) = C(t) \phi^0 + S(t) \phi^1 \in C([0, T]; D(A^{3/4})) \] (2.50a)
\[ \bar{\phi}'(t) = -AS(t) \phi^0 + C(t) \phi^1 \in C([0, T]; D(A^{1/4})). \] (2.50b)

Then (see details, e.g., in [L–T.3, Sect. 2] in a similar situation corresponding to the wave equation), it follows that
\[ \phi_n(t) \to \bar{\phi}(t) \quad \text{in } L^\infty(0, T; V = D(A^{3/4})) \text{ weak star} \] (2.51a)
\[ \phi'_n(t) \to \bar{\phi}'(t) \quad \text{in } L^\infty(0, T; H_0^1(\Omega) = D(A^{1/4})) \text{ weak star}. \] (2.51b)

Then (2.51) implies in turn that $\phi_n(t)$ and $\phi'_n(t)$ are uniformly bounded in $L^\infty(0, T; V)$ and $L^\infty(0, T; D(A^{1/4}))$, respectively. This fact, along with the compactness of $V \to D(A^{1/4}) = H_0^1(\Omega)$ and of $D(A^{1/4}) \to L^2(\Omega)$, implies [S.1, Corollary 4] that there is a subsequence, still subindexed by $n$, such that
\[ \phi_n(t) \to \bar{\phi}_n(t) \quad \text{strongly in } L^\infty(0, T; H_0^1(\Omega)) \] (2.52)
\[ \phi'_n(t) \to \bar{\phi}'(t) \quad \text{strongly in } L^\infty(0, T; L^2(\Omega)). \] (2.53)

A fortiori, from (2.47a) and (2.52)–(2.53) we obtain
\[ 1 \leq \|\nabla \phi_n\|_{C([0, T]; L^2(\Omega))}^2 + \|\phi_n'\|_{C([0, T]; L^2(\Omega))}^2 \]
\[ \leq \|\nabla \bar{\phi}\|_{C([0, T]; L^2(\Omega))}^2 + \|\bar{\phi}'\|_{C([0, T]; L^2(\Omega))}^2 = 1. \] (2.54)

Moreover, by (2.47c)
\[ \frac{\partial(\Delta \bar{\phi})}{\partial v} \bigg|_\Sigma = 0 \quad \text{and} \quad \frac{\partial \bar{\phi}'}{\partial v} \bigg|_\Sigma = 0. \] (2.55)

Thus $\bar{\phi}(t)$ satisfies
\[ \bar{\phi}_n + \Delta^2 \bar{\phi} = 0 \quad \text{in } \Omega \]
\[ \bar{\phi} |_{\Sigma} \equiv 0, \quad \Delta \bar{\phi} |_{\Sigma} \equiv 0 \] \{ from (2.50a) \}
\[ \frac{\partial(\Delta \bar{\phi})}{\partial v} \bigg|_\Sigma = 0, \quad \frac{\partial \bar{\phi}'}{\partial v} \bigg|_\Sigma = 0 \] \{ from (2.55) \} (2.56)
and, by differentiation in $t$, also
\[ F_{tt} + \Delta^2 F = 0 \]
\[ F |_{\Sigma} \equiv 0, \quad \Delta F |_{\Sigma} \equiv 0 \quad (2.57) \]
on $[0, T]$, with $0 < T < \infty$ arbitrary but fixed. Holmgren's uniqueness theorem applied to (2.57) yields then $F \equiv 0$ on $Q$, hence $F \equiv \text{const}$ in $Q$. By the boundary condition $\tilde{F} |_{\Sigma} \equiv 0$ in (2.56), we then conclude that $F \equiv 0$ in $\bar{Q}$, a contradiction with (2.54). The proof of Part (i) of Lemma 2.4 is complete.

**Proof of part (ii).** For each $\{\phi^0, \phi^1\} \in D(A^{3/4}) \times D(A^{1/4}) \equiv F$ and each $T > 0$ we set for convenience
\[ N_T(\phi^0, \phi^1) \equiv \| \nabla \phi \|_{C([0, T]; L^2(\Omega))}^2 + \| \phi \|_{C([0, T]; L^2(\Omega))}^2 \quad (2.58) \]
\[ D_T(\phi^0, \phi^1) \equiv \int_0^T \int_\Omega \left( \frac{\partial (\Delta \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi}{\partial v} \right)^2 \, d\Gamma \, dt. \quad (2.59) \]
Then we can take $C_T$ in (2.44) as defined by
\[ \infty > C_T \equiv \sup_{0 \neq \{\phi^0, \phi^1\} \in F} \frac{N_T(\phi^0, \phi^1)}{D_T(\phi^0, \phi^1)} \quad (2.60) \]
which is finite by Part (i) for each $0 < T < \infty$.

Now, by contradiction, let there be a sequence $T_m \uparrow \infty$ such that $C_{T_m} \geq \rho > 0$ for all $m$. Since $C_{T_m}$ is finite for each $m$, definition (2.60) implies that: for each $m$ large enough, there is a pair $\{\phi_m^0, \phi_m^1\} \in F$ of initial data such that
\[ \frac{N_{T_m}(\phi_m^0, \phi_m^1)}{D_{T_m}(\phi_m^0, \phi_m^1)} \geq C_{T_m} - \frac{1}{m} \geq \rho - \frac{1}{m}. \quad (2.61) \]
By normalization if necessary, we may achieve that such $\{\phi_m^0, \phi_m^1\}$ gives rise to
\[ N_{T_m}(\phi_m^0, \phi_m^1) \equiv \| \nabla \phi \|_{C([0, T_m]; L^2(\Omega))}^2 + \| \phi \|_{C([0, T_m]; L^2(\Omega))}^2 \equiv 1 \quad (2.62) \]
and hence, by (2.61), also to
\[ D_{T_m}(\phi_m^0, \phi_m^1) \equiv \int_0^{T_m} \int_\Omega \left( \frac{\partial (\Delta \phi(t, \phi_m^0, \phi_m^1))}{\partial v} \right)^2 + \left( \frac{\partial \phi(t, \phi_m^0, \phi_m^1)}{\partial v} \right)^2 \, d\Gamma \, dt \leq \text{const}, \quad \text{for all } m. \quad (2.63) \]
But the solution $\phi(t, \phi_m^0, \phi_m^1)$ was shown earlier to satisfy inequality (2.43).
Thus by (2.62)–(2.63) used in (2.43) we obtain (with \( \varepsilon' = 2\varepsilon C_{h,n} \)):

\[
[T_m - \varepsilon'] E_m(0) \leq \text{const, uniformly in } m
\]

(2.64)

\[
E_m(0) = \int_{\Omega} |\nabla \phi_m^1|^2 + |\nabla (\Delta \phi_m^1)|^2 \, d\Omega,
\]

(by (2.37b)).

(2.65)

Then (2.64) implies \( E_m(0) \downarrow 0 \), i.e., by (2.65) and (1.7b)–(1.8b)

\[
\{\phi_m^0, \phi_m^1\} \to (0, 0) \quad \text{in } D(A^{3/4}) \times D(A^{1/4}).
\]

(2.66)

It then plainly follows from (2.66) that the solution

\[
\phi(t, \phi_m^0, \phi_m^1) \equiv C(t) \phi_m^0 + S(t) \phi_m^1
\]

(2.67)

satisfies

\[
\phi(t, \phi_m^0, \phi_m^1) \to \text{zero function, in } C([0, \infty]; D(A^{1/4}))
\]

(2.68)

\[
\phi(t, \phi_m^0, \phi_m^1) \to \text{zero function, in } C([0, \infty]; L^2(\Omega))
\]

(2.69)

since \( \|C(t)\|, \|A^{1/2}S(t)\| \leq \text{const, for all } t \in \mathbb{R} \) in the uniform norm of \( L^2(\Omega) \). But then (2.68) (2.69) imply \( \lim_m N_{T_m}(\phi_m^0, \phi_m^1) = 0 \), and this contradicts \( \lim_m N_{T_m}(\phi_m^0, \phi_m^1) \equiv 1 \) which follows from (2.62). Lemma 2.4 is fully proved. 

Step (vii). We use Lemma 2.4 in (2.43) and obtain

**Corollary 2.5.** For any \( T > 0 \) and \( \varepsilon \) sufficiently small we have the inequality

\[
\text{Const}_{h,\varepsilon} \int_{\Sigma} \left( \frac{\partial (\Delta \phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi}{\partial v} \right)^2 \, d\Sigma \geq \left[ T - 2\varepsilon C_{h,n} \right] E(0),
\]

(2.70a)

with (see 2.37b):

\[
E(0) = \int_{\Omega} |\nabla \phi|^2 + |\nabla (\Delta \phi^0)|^2 \, d\Omega \text{ equivalent to } \{\phi^0, \phi^1\}^2_{D(A^{3/4}) \times D(A^{1/4})}
\]

(2.70b)

by (1.7) (1.8). Then, inequality (2.70a) a fortiori implies inequality (2.21) for \( T > 0 \) arbitrarily small. The proof of Proposition 2.2 is now complete.

**Remark 2.1.** At no extra effort over the proof of Lemma 2.4, we may
strengthen its statement to read: inequality (2.43) implies that for any $T > 0$ there is $C_T > 0$ such that all $\{\phi^0, \phi^1\} \in D(A^{3/4}) \times D(A^{1/4})$ we have
\[
\|\nabla \phi\|_{C([0,T]; L^2(\Omega))}^2 + \|\phi_T\|_{C([0,T]; L^2(\Omega))}^2 + \int_{\Sigma} \left( \frac{\partial \phi}{\partial v} \right)^2 d\Sigma \leq C_T \int_{\Sigma} \left( \frac{\partial (A\phi)}{\partial v} \right)^2 + \left( \frac{\partial \phi_T}{\partial v} \right)^2 d\Sigma.
\]
(2.71)
Thus, a fortiori (2.71) implies that the characterization (2.21) in Lemma 2.1(iii) for exact controllability in the present section is equivalent to inequality (2.28) in Proposition 2.2.

3. PROOF OF THEOREM 1.2: EXACT CONTROLLABILITY ON 
$H^1_0(\Omega) \times H^{-1}(\Omega) \equiv D(A^{1/4}) \times \left[ D(A^{1/4}) \right]'$

We parallel and complement the proof of Theorem 1.1, by working this time on different spaces.

Step 1. We return to the input-solution operator $L_T = [L_{1T}, L_{2T}]$ in (2.1) and compute its adjoint $L_T^*$ for $z = [z_1, z_2] \in D(A^{1/4}) \times \left[ D(A^{1/4}) \right]'$
\[
\begin{pmatrix}
L_T & \left| g_1 \right|_{D(A^{1/4}) \times \left[ D(A^{1/4}) \right]}' \\
g_2 \\
z_1 & \left| z_1 \right| \\
z_2
\end{pmatrix},
\]
(3.1a)
\[
= \begin{pmatrix}
g_1 & \left| z_1 \right|_{H_0^0(0,T; L^2(\Gamma)) \times L^2(\Sigma)} \\
g_2 & \left| L_{1T}^* z_1 \right|_{H_0^0(0,T; L^2(\Gamma)) \times L^2(\Sigma)} \\
z_1 & \left| L_{2T}^* z_2 \right|_{H_0^0(0,T; L^2(\Gamma)) \times L^2(\Sigma)}
\end{pmatrix}
\]
(3.1b)
as in (2.16a), so that
\[
\| L_T^* z \|_{H_0^0(0,T; L^2(\Gamma)) \times L^2(\Sigma)}^2 = \| L_{1T}^* z \|_{H_0^0(0,T; L^2(\Gamma))}^2 + \| L_{2T}^* z \|_{L^2(\Sigma)}^2.
\]
(3.2)
as in (2.16c). The (regularity) Theorem 1.0 gives
\[
L_T = [L_{1T}, L_{2T}]: \text{continuous } H_0^0(0,T; L^2(\Gamma)) \\
\times L^2(\Sigma) \to D(A^{1/4}) \times \left[ D(A^{1/4}) \right]'
\]
(3.3)
and exact controllability of problem (1.1) in the present section means that the above map in (3.3) is onto, or equivalently that: there is $C_T > 0$ such that
\[
\left\| L_T^* \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} \right\|_{H_0^0(0,T; L^2(\Gamma)) \times L^2(\Sigma)}^2 \geq C_T \left\{ z_1, z_2 \right\}_{D(A^{1/4}) \times \left[ D(A^{1/4}) \right]}'^2
\]
(3.4a)
or recalling (3.2)

\[
\begin{align*}
\left\| \frac{d}{dt} L_{1T} z \right\|_{L^2(\Sigma)}^2 + \left\| L_{2T} z \right\|_{L^2(\Sigma)}^2 & \geq C_7 \left\| \{ z_1, z_2 \} \right\|_{\mathcal{D}(A^{1/4}) \times \mathcal{D}(A^{1/4})}. \\
\end{align*}
\]  

(3.4b)

where we have used that the $H^1_0(0, T)$-norm is equivalent to the "gradient norm."

**Step 2.** An equivalent partial differential equation characterization of inequality (3.4b) is given by the following lemma.

**Lemma 3.1.** For $z \in D(A^{1/4}) \times [D(A^{1/4})]'$ we have:

(i)

\[
\begin{align*}
(L_{1T} z)(t) &= G_1^* \left[ C(t - T) A^{-1/2} z_2 + S(t - T)(-A^{1/2} z_1) \right] \\
K_1 + K_2 &\in H^1_0(0, T; L^2(\Gamma)) \\
K_1 &= K_{1T} = \frac{G_1^*}{T} \{ [C(T) - I] A^{-1/2} z_2 + S(T) A^{1/2} z_1 \} \\
K_2 &= K_{2T} = -G_1^* [C(T) A^{-1/2} z_2 + S(T) A^{1/2} z_1] \\
\frac{d(L_{1T} z)}{dt} &= -\frac{\partial(\Delta \phi(t))}{\partial \nu} + K_{1T},
\end{align*}
\]

(3.5a) \hspace{1cm} (3.5b) \hspace{1cm} (3.5c) \hspace{1cm} (3.6)

where $\phi(t) = \phi(t, \phi^0, \phi^1)$ is the solution of the following homogeneous problem, backward in time

\[
\begin{align*}
\phi_{tt} + A^2 \phi &= 0 \hspace{1cm} (3.7a) \\
\phi |_{t = T} &= \phi^0; \ \phi |_{t = T} = \phi^1 \hspace{1cm} (3.7b) \\
\phi |_{\Sigma} &\equiv 0 \hspace{1cm} (3.7c) \\
A \phi |_{\Sigma} &\equiv 0 \hspace{1cm} (3.7d)
\end{align*}
\]

with

\[\phi^0 = A^{-1/2} z_1; \ \ \phi^1 = A^{-1/2} z_2\] (3.7e)

explicitly given by

\[
\begin{align*}
\phi(t) &= C(t - T) \phi^0 + S(t - T) \phi^1 \\
\phi_t(t) &= A S(t - T) \phi^0 + C(t - T) \phi^1.
\end{align*}
\]

(3.8a) \hspace{1cm} (3.8b)
(ii) 
\[(L_{2T}^*z)(t) = \frac{\partial}{\partial v} \phi(t), \quad (3.9)\]

where \(\phi(t)\) solves (3.7).

(iii) For any \(0 < T < \infty\), inequality (3.4b) [which characterizes exact controllability of problem (1.1) on the space \(D(A^{1/4}) \times \overline{D(A^{1/4})}'\) by means of controls \([g_1, g_2] \in H^1_0(0, T; L^2(\Gamma)) \times L^2(\Sigma)\) over \([0, T]\)] is equivalent to: there is \(C_T > 0\) such that

\[\int_{\Sigma} \left[-\frac{\partial (A\phi)}{\partial v} + K_{1T}\right]^2 d\Sigma + \int_{\Sigma} (\frac{\partial \phi}{\partial v})^2 d\Sigma \geq C_T \|\{\phi^0, \phi^1\}\|^2_{D(A^{1/4}) \times D(A^{1/4})}\]

\[(3.10a)\]

where

\[K_{1T} = \frac{G^*_1}{T} \{[C(T) - I]\phi^1 + AS(T)\phi^0\}\]

\[= \frac{1}{T} \frac{\partial}{\partial v} A\{[C(T) - I] A^{-1}\phi^1 + S(T)\phi^0\}. \quad (3.10b)\]

Moreover, the map \(T \to C_T\) is monotone increasing.

Proof of Lemma 3.1. (i) With \(g_1 \in H^1_0(0, T; L^2(\Gamma))\), on the one hand we have

\[(L_{1T} g_1, z)_{D(A^{1/4}) \times \overline{D(A^{1/4})}'} = (g_1, L_{1T}^*z)_{H^1_0(0, T; L^2(\Gamma))}\]

\[= \left(\frac{dg_1}{dt}, \frac{dL_{1T}^*z}{dt}\right)_{L^2(\Sigma)} = \left(g_1, -\frac{d^2}{dt^2} (L_{1T}^*z)\right)_{L^2(\Sigma)} \quad (3.11)\]

since \(g_1\) vanishes at \(t = 0\) and \(t = T\); on the other hand, using (2.2) we readily compute as usual (Lemma 2.1)

\[(L_{1T} g_1, z)_{D(A^{1/4}) \times \overline{D(A^{1/4})}'} = \left(\int_0^T S(T-t) G_1 g_1(t) dt, A^{3/2}z_1\right)_{L^2(\Omega)} + \left(\int_0^T C(T-t) G_1 g_1(t) dt, A^{1/2}z_2\right)_{L^2(\Omega)} + \left(g_1(t), G^* [S(T-t) A^{3/2}z_1 + C(T-t) A^{1/2}z_2]\right)_{L^2(\Omega)} dt \quad (3.12)\]
so that comparing (3.11) with (3.12) and using that $C(\cdot)$ is even while $S(\cdot)$ is odd
\[ -\frac{d^2}{dt^2} (L_{1T}^* z) = G_1^* \left[ C(t - T) A^{1/2} z_2 + S(t - T) (- A^{3/2} z_1) \right]. \tag{3.13} \]

Integrating in $t$ we readily find
\[ \frac{d}{dt} (L_{1T}^* z)(t) = -G_1^* \left[ S(t - T) A^{1/2} z_2 - A^{-1} C(t - T) (- A^{3/2} z_1) \right] + K_1 \tag{3.14} \]
\[ (L_{1T}^* z)(t) = G_1^* \left[ A^{-1} S(t - T) (- A^{3/2} z_1) \right. \]
\[ + A^{-1} C(t - T) A^{1/2} z_2 \left. \right] + K_1 t + K_2. \tag{3.15} \]

By imposing that $(L_{1T}^* z)(t)$ vanishes at $t = 0$ and $t = T$, so that $L_{1T}^* z \in H^1_0(0, T; L^2(\Omega))$, we readily identify the operators $K_1$ and $K_2$ as in (3.5a–b), and (3.15) then becomes (3.5a). For the purposes of (3.4b) we now re-write (3.14) as
\[ \frac{d(L_{1T}^* z)(t)}{dt} = -G_1^* A \left[ C(t - T) A^{-1/2} z_1 + S(t - T) A^{-1/2} z_2 \right] + K_1 T \]
\[ = \frac{\partial (A(\phi(t))}{\partial v} + K_{1T}, \tag{3.16} \]

where in the last step we have used (2.11) and (3.8a).

(ii) With $g_2 \in L^2(\Sigma)$ we compute from (2.3) as before
\[ (L_{2T}^* g_2, z)_{D(A^{1/2}) \times [D(A^{1/2})]} = \left( \int_0^T S(T - t) G_2 g_2(t) \ dt, A^{3/2} z_1 \right)_{L^2(\Omega)} \]
\[ + \left( \int_0^T C(T - t) G_2 g_2(t) \ dt, A^{1/2} z_2 \right)_{L^2(\Omega)} \]
\[ = \int_0^T \left( g_2(t), G_2^* \left[ S(T - t) A^{3/2} z_1 + C(T - t) A^{1/2} z_2 \right] \right)_{L^2(\Omega)} dt \]
\[ = (g_2, L_{2T}^* z)_{L^2(\Sigma)}. \tag{3.17} \]

Thus
\[ (L_{2T}^* z)(t) = G_2^* \left[ C(t - T) A^{1/2} z_2 + S(t - T) (- A^{3/2} z_1) \right] \]
\[ = G_2^* A \left[ C(t - T) A^{-1/2} z_2 - AS(t \ T) A^{-1/2} z_1 \right] = \frac{\partial \phi_2(t)}{\partial v}, \tag{3.18} \]
where in the last step we have used (2.13) and (3.8b), (3.7c). Part (ii) is proved. As to Part (iii), we first note that by (3.7e)

\[ \| z_1 = A^{1/2} \phi^0, z_2 = A^{1/2} \phi^1 \|_{D(A^{1/4}) \times D(A^{1/4})} = \| A^{3/4} \phi^0, A^{1/4} \phi^1 \|_{L^2(\Omega) \times L^2(\Omega)} - \| \phi^0, \phi^1 \|_{D(A^{1/4}) \times D(A^{1/4})}. \]  

(3.19)

Moreover, from (3.5b) and (3.7e)

\[ K_{1T} = \frac{G^*_1}{T} \{ [C(T) - I] A^{-1/2} z_2 + AS(T) A^{-1/2} z_1 \} \]

\[ = \frac{G^*_1}{T} \{ [C(T) - I] \phi^1 + AS(T) \phi^0 \}. \]  

(3.20)

Then, (3.4b) becomes (3.10a–b) as desired, by use of (3.6), (3.20), (3.9), and (3.19). The proof that \( T \to C'_T \) is monotone increasing is conceptually the same as the one given for inequality (2.21), just below (2.28). The proof of Lemma 3.1 is complete.

Step 3. We “absorb” the lower order term \( K_{1K} \), the boundary vector given by (3.10b), in the lefthand side of inequality (3.10a), through an argument of the same type as the one in Lemma 2.4.

Lemma 3.2. Inequality (3.10a)—which characterizes exact controllability over \([0, T]\) on the space \( \mathcal{D}(A^{1/4}) \times [\mathcal{D}(A^{1/4})]' \) by means of controls \([g_1, g_2] \in H^1_0(0, T; L^2(\Gamma)) \times L^2(\Sigma)\)—is equivalent to inequality (1.26) = (2.28a)—which characterizes exact controllability over \([0, T]\) on the space \( [\mathcal{D}(A^{1/4})]' \times [\mathcal{D}(A^{3/4})]' \) by means of controls \([g_1, g_2] \in L^2(\Sigma) \times [H^1(0, T; L^2(\Gamma))]\).

Proof. We assume inequality (3.10a) and we wish to show that, in fact, there exists a constant \( C_T > 0 \) such that

\[ \int_{\Sigma} \left( -\frac{\partial (\Delta \phi)}{\partial v} + K_{1T} \right)^2 d\Sigma \leq C_T \int_{\Sigma} \left( \frac{\partial (\Delta \phi)}{\partial v} \right)^2 d\Sigma \]  

(3.21)

so that inequality (1.26) = (2.28a) will likewise hold true. Suppose by contradiction that there exists a sequence \( \{ \phi_n(t) \} \) of solutions to problem (1.24)

\[ \phi'' + \Delta^2 \phi_n = 0 \quad \text{in } \Omega \]

\[ \phi_n|_{t=0} = \phi_{n0} \in \mathcal{D}(A^{3/4}), \quad \phi'_n|_{t=0} = \phi_{n1} \in \mathcal{D}(A^{1/4}) \quad \text{in } \Omega \]  

(3.22)

\[ \phi_n|_{\Sigma} = \Delta \phi_n|_{\Sigma} \equiv 0 \quad \text{in } \Sigma \]
such that
\[
\left\| -\frac{\partial (A\phi_n)}{\partial v} + K_{1n}\phi_n \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial \phi_n}{\partial v} \right\|_{L^2(\Omega)}^2 \equiv 1 \quad (3.23)
\]
\[
\left\| \frac{\partial (A\phi_n)}{\partial v} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial \phi_n}{\partial v} \right\|_{L^2(\Omega)}^2 \to 0 \quad \text{as } n \to \infty \quad (3.24)
\]

By assumption, all the \(\{\phi_n(t)\}\) satisfy inequality (3.10a) and so by (3.23), there is a subsequence \(\phi_{n_0}, \phi_{n_1}\) converging to some \(\bar{\phi}_0, \bar{\phi}_1\) weakly in \(\mathcal{D}(A^{3/4}) \times \mathcal{D}(A^{1/4})\) and hence, by compactness of \(A^{-1}\), strongly in \(\mathcal{W}(A^{3/4} - \delta, A^{1/4} - \delta)\) for \(\delta > 0\). As a consequence, we return to (3.10b) and see that
\[
\text{converges strongly in } L^2(\Gamma) \text{ to}
\]
\[
K_{1Tn} = \frac{G^*_1}{T} A_{-1}^{(3/4) - \delta} \left\{ C(T) - I \right\} A^{1/4} - \delta \phi_{1n} + A^{1/2} S(T) A^{3/4} - \delta \phi_{0n} \quad (3.25)
\]
converges strongly in \(L^2(\Gamma)\) to
\[
\bar{K}_{1T} = \frac{G^*_1}{T} \left\{ C(T) - I \right\} \bar{\phi}_1 + A S(T) \bar{\phi}_1 \]

since \(G^*_1\) is a bounded operator \(L^2(\Omega) \to L^2(\Gamma)\). Using (3.23), (3.24), (3.25) we conclude that
\[
\|\bar{K}_{1T}\|_{L^2(\Omega)} = 1 \quad (3.26)
\]

On the other hand, \(\phi(t) = C(t)\bar{\phi}_0 + S(t)\bar{\phi}_1\) satisfies
\[
\phi_{tt} + A^2 \phi = 0 \quad \text{in } Q
\]
\[
\phi|_{\Sigma} = A \phi|_{\Sigma} \equiv 0 \quad \text{in } \Sigma \quad (3.27)
\]
\[
\frac{\partial \phi}{\partial v}|_{\Sigma} = \frac{\partial (A\phi)}{\partial v}\bigg|_{\Sigma} \equiv 0 \quad \text{from (3.24)} \quad \text{in } \Sigma
\]

for \(0 \leq t \leq T\). After differentiating (3.27) in \(t\), we apply Holmgren uniqueness Theorem [H.2, p. 129] and conclude that \(\phi' \equiv 0\) in \(Q\), hence \(\phi \equiv 0\) in \(Q\) by (3.27b), i.e., \(\phi_0 = \phi_1 = 0\). Finally, by (3.25) we obtain \(\bar{K}_{1T} = 0\). But this contradicts (3.26).

The proof that inequality (1.26) = (2.28a) implies inequality (3.10a) is identical. \(\square\)

**Proof of Corollary 1.3.** (i) This part is proved in Lemma 3.2. By Remark 2.1, inequalities (2.21) and (2.28) are equivalent. By Lemma 3.2, inequalities (2.28) and (3.10) are equivalent.
(ii) We apply the interpolation Theorem [L–M.1, Theorem 5.1, p. 27] to the operator $L_f^{\ast -1}$ which, by part (i) is bounded between the space in (1.32) and the space in (1.30) at the end points $\theta = 0$ and $\theta = 1$. Hence, $L_f^{\ast -1}$ is continuous between the interpolation spaces

$$[D(A^{1/4}) \times [D(A^{1/4})], [D(A^{1/4})]' \times [D(A^{3/4})]'_\theta$$

and the interpolation spaces

$$[H_0^1(0, T; L^2(\Gamma)) \times L^2(\Sigma), L^2(\Sigma) \times [H^1(0, T; L^2(\Gamma))]'_{\theta}$$

$0 < \theta < 1$, which means that $L_f$ is onto in the opposite direction. Then [L–M.1, p. 64–66] gives (1.30) while [L–M.1, p. 29] gives (1.31).

**APPENDIX A: PROOF OF IDENTITY (2.1)**

Let $h(x) \in C^2(\Omega)$. With reference to Remark 1.2, we multiply Eq. (2.20a) by $h \cdot \nabla(\Delta \phi)$ and integrate over $Q$. We shall use the identity

$$\int_{\Omega} h \cdot \nabla q \, d\Omega = \int_{\Gamma} q h \cdot \nu \, d\Gamma - \int_{\Omega} q \, \text{div} \, h \, d\Omega$$

obtained from $\text{div}(qh) = h \cdot \nabla q + q \, \text{div} \, h$, $q$ scalar function, and the divergence theorem. In addition, we shall use the identity

$$\int_{Q} A \psi (h \cdot \nabla \psi) \, dQ = \int_{\Sigma} \frac{\partial \psi}{\partial \nu} (h \cdot \nabla \psi) \, d\Sigma - \frac{1}{2} \int_{\Sigma} |\nabla \psi|^2 \, h \cdot \nu \, d\Sigma$$

$$- \int_{Q} H \, \nabla \psi \cdot \nabla \psi \, dQ + \frac{1}{2} \int_{Q} |\nabla \psi|^2 \, \text{div} \, h \, dQ$$

(A.2)

already proved in, say, [T.3, Eq. (A.3) of Appendix A] (with similar multiplier techniques) where $H = H(x)$ is the transpose of the Jacobian matrix of $h(x)$:

$$H(x) = \begin{vmatrix} \frac{\partial h_1}{\partial x_1}, \ldots, \frac{\partial h_1}{\partial x_n} \\ \vdots \\ \frac{\partial h_n}{\partial x_1}, \ldots, \frac{\partial h_n}{\partial x_n} \end{vmatrix}$$

(A.3)

**Term $\phi^\prime, h \cdot \nabla(\Delta \phi)$.** Integrating at first by parts in $t$

$$\int_{Q} \phi^\prime, h \cdot \nabla(\Delta \phi) \, dQ - \left[ \int_{\Omega} \phi^\prime, h \cdot \nabla(\Delta \phi) \, d\Omega \right]_0^T = - \int_{Q} \phi, h \cdot \nabla(\Delta \phi^\prime) \, dQ$$
[using (A.1) with \( h \) there replaced by \( \phi \), \( h \) now, with \( q = \Delta \phi \), and with
\[
\text{div}(\phi, h) = \nabla \phi \cdot h + \phi \cdot \text{div} h
\]
\[
= \left[ \int_{\Omega} \phi, h \cdot \nabla(\Delta \phi) \, d\Omega \right]_0^T - \int_{\Sigma} \phi, \Delta \phi, h \cdot v \, d\Sigma
\]
\[
+ \int_{Q} \Delta \phi, h \cdot \nabla \phi, dQ + \int_{Q} \Delta \phi, \phi, \text{div} h \, dQ. \quad (A.4)
\]

[Using identity (A.2) with \( \psi = \phi \), for the third integral on the right of
(A.4),
\[
\int_{\Omega} \int_{0}^{T} \phi, h \cdot \nabla(\Delta \phi) \, dt \, d\Omega = \left[ \int_{\Omega} \phi, h \cdot \nabla(\Delta \phi) \, d\Omega \right]_0^T - \int_{\Sigma} \phi, \Delta \phi, h \cdot v \, d\Sigma
\]
\[
- \frac{1}{2} \int_{\Sigma} |\nabla \phi,|^2 h \cdot v \, d\Sigma + \int_{\Sigma} \frac{\partial \phi}{\partial v} h \cdot \nabla \phi, \, d\Sigma
\]
\[
- \int_{Q} H \nabla \phi, \cdot \nabla \phi, \, dQ + \frac{1}{2} \int_{Q} |\nabla \phi,|^2 \text{div} h \, dQ
\]
\[
+ \int_{Q} \Delta \phi, \phi, \text{div} h \, dQ. \quad (A.5)
\]

Using Green's first theorem on the last integral at the right of (A.5) along
with the identity
\[
\nabla \phi, \cdot \nabla(\phi) \cdot \nabla h) = \phi, \nabla(\text{div} h), \nabla \phi, + |\nabla \phi,|^2 \text{div} h
\]
we finally obtain from (A.5)
\[
\int_{Q} \phi, h \cdot \nabla(\Delta \phi) \, dQ
\]
\[
= \left[ (\phi, h \cdot \nabla(\Delta \phi))_Q \right]_0^T - \int_{\Sigma} \phi, \Delta \phi, h \cdot v \, d\Sigma
\]
\[
- \frac{1}{2} \int_{\Sigma} |\nabla \phi,|^2 h \cdot v \, d\Sigma + \int_{\Sigma} \frac{\partial \phi}{\partial v} h \cdot \nabla \phi, \, d\Sigma + \int_{\Sigma} \frac{\partial \phi}{\partial \phi} \phi, \text{div} h \, d\Sigma
\]
\[
- \int_{Q} H \nabla \phi, \cdot \nabla \phi, \, dQ - \frac{1}{2} \int_{Q} |\nabla \phi,|^2 \text{div} h \, dQ
\]
\[
- \int_{Q} \phi, \nabla(\text{div} h), \nabla \phi, \, dQ \quad (A.6)
\]
Term $\Delta^2 \phi h \cdot \nabla(\Delta \phi)$. Using identity (A.2) this time with $\psi = \Delta \phi$ we obtain

$$
\int_Q \Delta(\Delta \phi) h \cdot \nabla(\Delta \phi) \, dQ \\
= \int_S \frac{\partial (\Delta \phi)}{\partial \nu} h \cdot \nabla(\Delta \phi) \, dS - \frac{1}{2} \int_S |\nabla(\Delta \phi)|^2 h \cdot \nu \, dS \\
- \int_Q H \nabla(\Delta \phi) \cdot \nabla(\Delta \phi) \, dQ + \frac{1}{2} \int_Q |\nabla(\Delta \phi)|^2 \text{div} \, h \, dQ. \quad (A.7)
$$

Summing up (A.6) and (A.7) and recalling (1.2a) we finally obtain

$$
\int_S \frac{\partial (\Delta \phi)}{\partial \nu} h \cdot \nabla(\Delta \phi) \, dS + \int_S \frac{\partial \phi_i}{\partial \nu} h \cdot \nabla \phi_i \, dS + \int_S \frac{\partial \phi_i}{\partial \nu} \phi_i \, \text{div} \, h \, dS \\
- \frac{1}{2} \int_S |\nabla \phi_i|^2 h \cdot \nu \, dS - \frac{1}{2} \int_S |\nabla(\Delta \phi)|^2 h \cdot \nu \, dS - \int_S \phi_i \Delta \phi_i h \cdot \nu \, dS \\
= \int_Q H \nabla(\Delta \phi) \cdot \nabla(\Delta \phi) \, dQ + \int_Q H \nabla \phi_i \cdot \nabla \phi_i \, dQ \\
+ \frac{1}{2} \int_Q \{ |\nabla \phi_i|^2 - |\nabla(\Delta \phi)|^2 \} \, \text{div} \, h \, dQ \\
+ \int_Q \phi_i \nabla(\text{div} \, h) \cdot \nabla \phi_i \, dQ \\
- [(\phi_i, h \cdot \nabla(\Delta \phi))_\alpha]_\sigma^T + \int_Q f h \cdot \nabla(\Delta \phi) \, dQ \quad (A.8)
$$

which is the sought after identity for $\phi$ satisfying (1.2a).

**Specialization of Left Hand Side of (A.8) to $\phi$ which Satisfies also the Boundary Conditions (2.20c–d).** Recalling (2.20c–d) we have

$$
\phi_i |_{\Sigma} = 0; \nabla \phi \perp \Gamma \text{ and } |\nabla \phi| = \left| \frac{\partial \phi}{\partial \nu} \right| \equiv 0 \quad \text{on } \Sigma \text{ by (2.20d)} \quad (a)
$$

$$
\frac{\partial \phi_i}{\partial \nu} |_{\Sigma} = 0; \nabla \phi_i \perp \Gamma \text{ and } |\nabla \phi_i| = \left| \frac{\partial \phi_i}{\partial \nu} \right| \equiv 0 \quad \text{in } \Sigma. \quad (b)
$$
Thus, using (2.20c–d) and (A.9a–b) in the left hand side (LHS) of (A.8) we find that this simplifies to

\[
\text{LHS of (A.8)} = \int_{\Sigma} \frac{\partial(A\phi)}{\partial v} h \cdot \nabla(A\phi) d\Sigma - \frac{1}{2} \int_{\Sigma} \nabla(A\phi)^2 h \cdot v d\Sigma. \tag{A.10}
\]

**Specialization of the Right Hand side of (A.8) to Radial Vector Fields**

For \( h(x) = x - x_0 \). In this case, recalling (A.3) we obtain

\[
H(x) \equiv \text{identity matrix}; \quad \text{div} \, h \equiv n = \dim \Omega \tag{A.11}
\]

which used in the right hand side (RHS) of (A.8) yield

\[
\text{RHS of (A.8)} = \int_{\Omega} \{ |\nabla \phi|^2 + |\nabla(A\phi)|^2 \} dQ
\]

\[+ \frac{n}{2} \int_{\Omega} \{ |\nabla \phi|^2 - |\nabla(A\phi)|^2 \} dQ \]

\[\times \int_{\Omega} \phi h \cdot \nabla(A\phi) dQ - \left[ (\phi, h \cdot \nabla(A\phi))_0 \right]. \tag{A.12}
\]

Combining (A.10) and (A.12) proves (2.1), as desired.

**APPENDIX B: PROOF OF IDENTITY (2.34)**

Again, we shall first obtain an identity, (B.3) below, for \( \phi \) which solves only (2.18a) and for arbitrary smooth vector field \( h \in C^2(\bar{\Omega}) \). Next we shall specialize this identity (B.3) to the case where \( \phi \) satisfies in addition also the boundary conditions (2.18c–d) and, moreover, the vector field is radial.

We multiply Eq. (2.18a) by \( \Delta \phi \) \text{div} \( h \) and integrate over \( \Omega \) by parts in \( t \) and by Green’s first theorem:

\[
\int_{\Omega} \int_{0}^{T} \phi_{tt} \Delta \phi \text{div} \, h \, dt \, d\Omega
\]

\[= \left[ \int_{\Omega} \Delta \phi \, \phi_{t} \text{div} \, h \, d\Omega \right]_{0}^{T} - \int_{0}^{T} \int_{\Omega} \Delta \phi \phi_{t} \text{div} \, h \, d\Omega \, dt
\]

\[= \left[ \int_{\Omega} \phi_{t} \frac{\partial \phi}{\partial v} \phi_{t} \text{div} \, h \, d\Gamma - \int_{\Omega} \nabla \phi \cdot \nabla(\phi \text{div} \, h) \, d\Omega \right]_{0}^{T}
\]

\[- \int_{\Sigma} \frac{\partial \phi_{t}}{\partial v} \phi_{t} \text{div} \, h \, d\Sigma + \int_{\Omega} |\nabla \phi_{t}|^2 \text{div} \, h \, dQ + \int_{\Omega} \phi_{t} \nabla(\text{div} \, h) \cdot \nabla \phi_{t} \, dQ \tag{B.1}
\]
also
\[
\int_{\Omega} \frac{\partial (\Delta \phi)}{\partial v} A \Delta \phi \, dv \, d\Omega \, dt
\]
\[
= \int_{\Sigma} \frac{\partial (\Delta \phi)}{\partial v} A \Delta \phi \, dv \, d\Sigma
\]
\[
- \int_{Q} |\nabla (\Delta \phi)|^2 \, dv \, dQ - \int_{Q} \Delta \phi \nabla (\Delta \phi) \cdot \nabla (\Delta \phi) \, dQ. \tag{B.2}
\]

Summing up (B.1) and (B.2) we find the identity
\[
\int_{Q} \left\{ |\nabla \phi|^2 - |\nabla (\Delta \phi)|^2 \right\} \, dv \, dQ
\]
\[
= \int_{\Sigma} \frac{\partial \phi}{\partial v} \phi, \, dv \, d\Sigma - \int_{\Sigma} \frac{\partial (\Delta \phi)}{\partial v} \Delta \phi \, dv \, d\Sigma
\]
\[
+ \int_{Q} \Delta \phi \nabla (\Delta \phi) \cdot \nabla (\Delta \phi) \, dQ - \int_{Q} \phi, \nabla (\Delta \phi) \cdot \nabla (\Delta \phi) \, dQ
\]
\[
+ \left[ \int_{\Omega} \Delta \phi \cdot (\nabla h) \, dQ - \int_{\gamma} \frac{\partial \phi}{\partial v} \phi, \, dv \, d\Gamma \right] \tag{B.3}
\]

for \( \phi \) satisfying (2.18a).

If now \( h(x) \) is a radial vector field, then (see (2.33)), (B.3) specializes to (2.34).

**References**


