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Inequalities for the Perimeter of an Ellipse

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1. INTRODUCTION

Let a, b denote the lengths of the semimajor and -minor axes of an ellipse with eccentricity $e = (1/a)\sqrt{a^2 - b^2}$ and whose perimeter is $L(a, b)$. The problem of approximating $L(a, b)$ is an ancient one. An excellent account of this problem is found in an article by Almkvist and Berndt [1]. In the article [1], several approximations that have been proposed over a period of nearly 4 centuries are presented. The approximation for $L(a, b)/(\pi(a + b))$,

$$A(a, b) = \frac{2}{a+b}\left(\frac{a^{3/2} + b^{3/2}}{2}\right)^{2/3},$$

was given by Muir in 1883. In 1996, Vuorinen in the paper "Hypergeometric Functions in Geometric Function Theory" [6] raised the question

(Q1) Question. *Is it true that $A(a, b)$ is an approximation to $L(a, b)/(\pi(a + b))$ from* below *throughout the entire range of eccentricity e?*

As is often the case for a mathematical conjecture, the insights gained in the course of its resolution and extended ramifications of the result supersede the original question. In this paper we develop a technique to answer Question (Q1) that readily extends to a broader class of inequalities, all of which are motivated by the problem of approximating the elliptical perimeter. The verification of each inequality is accomplished by showing the positivity of an infinite series. The proof of the positivity of the infinite series is achieved by utilizing a computer algebra system to execute a Sturm sequence argument.

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In verifying (Q1) it will be shown that $A(a,b)$ exhibits a monotonic dependence on eccentricity. As a consequence of this property, bounds on the error, $E = L(a,b)/(\pi(a+b)) - A(a,b)$, are readily established.

2. APPROXIMATIONS OF ELLIPTIC PERIMETER

Recall that if an ellipse is described by the parametric equations $x = a\cos\phi$ and $y = b\sin\phi$, $0 \le \phi \le 2\pi$, then the perimeter L of the ellipse is given by

$$L = L(a,b) = \int_0^{2\pi} \sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)\,d\phi}.$$

The perimeter of the ellipse can be expressed exactly in terms of Gauss's ordinary hypergeometric series

$$F(a,b;c:x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \qquad |x| < 1,$$

where a, b, and c denote arbitrary complex numbers and $(\alpha)_k$ is defined by

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1).$$

The following expressions for the elliptical perimeter are due, respectively, to Maclaurin (1742), Euler (1773), and Ivory (1796) (see [1] and the references therein) and illustrate the connection between the elliptical perimeter and the hypergeometric functions.

PROPOSITION 1. *Let $x = a\cos\phi$ and $y = b\sin\phi$, $0 \le \phi \le 2\pi$, and let $e = (1/a)\sqrt{a^2 - b^2}$ be the eccentricity of the ellipse. Then*

$$\begin{aligned} L(a,b) &= 2\pi a F\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1 : e^2\right) \\ &= \pi\sqrt{2(a^2+b^2)}\, F\left(-\tfrac{1}{4}, \tfrac{1}{4}; 1 : \left(\tfrac{a^2-b^2}{a^2+b^2}\right)^2\right) \\ &= \pi(a+b) F\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1 : \lambda^2\right), \end{aligned}$$

where

$$\lambda = \frac{a-b}{a+b}.$$

The approximations presented in [1] are reproduced in Table I. In the second column of the table the approximation $A(\lambda)$ to the "normalized" length $L(a, b)/[\pi(a+b)]$ is given. The third column provides the first nonzero term in the power series for the difference $L(a, b)/[\pi(a+b)] - A(\lambda)$.

Let us now reconsider the question (Q1). If we utilize Maclaurin's exact expression for the elliptical perimeter given in the proposition, then Muir's approximation amounts to the statement that

$$2\pi a F\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1 : e^2\right) \approx 2\pi \left(\frac{a^{3/2} + b^{3/2}}{2}\right)^{2/3}.$$

Without loss of generality we may assume that the semimajor axis $a = 1$ and then set $x = 1 - b^2$, so that the entire range of eccentricity is represented by $0 < x < 1$. The explicit statement of (Q1) as posed by Vuorinen is given by the problem

Problem. Is it true for $0 < x < 1$ that we have

$$F\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1 : x\right) - \left(\frac{1 + (1-x)^{3/4}}{2}\right)^{2/3} > 0?$$

Let us denote the error between the normalized elliptical perimeter and the algebraic approximation by

$$\mathscr{E}(x) = F\left(\tfrac{1}{2}, -\tfrac{1}{2}; 1 : x\right) - A(x), \tag{1}$$

where for Muir's approximation

$$A(x) = \left(\frac{1 + (1-x)^{\frac{3}{4}}}{2}\right)^{2/3}.$$

In the course of answering (Q1) we will in fact show the stronger result that

$$.00006 < \frac{\mathscr{E}(x)}{x^4} < \mathscr{E}(1) < .00666, \qquad 0 < x < 1. \tag{2}$$

This bound on the error illustrates the surprising accuracy of Muir's approximation.

It is natural to ask whether an estimate of this type can be obtained for each of the approximations in Table I. The positivity of the first term in the Taylor expansion of $L(a, b)/(\pi(a+b)) - A(\lambda)$ is consistent with the conjecture that Muir's approximation to $L(a, b)$ could indeed be an approximation from below. Furthermore, the power of the first nonzero term in the Taylor expansion suggests that one consider $\mathscr{E}(x)/x^4$ in proving the result in (2). For those approximations in Table I for which the first term in the Taylor expansion is positive, i.e., for the approximations of Kepler, Lindner, Ramanujan(a,b), Bronshtein, Selmer(a,b), and Jacobsen,

TABLE I

Sources (see [1])	$A(\lambda)$	$\frac{L(a,b)}{\pi(a+b)} - A(\lambda)$
Kepler (1609)	$\frac{2\sqrt{ab}}{a+b} = \sqrt{1-\lambda^2}$	$\frac{3}{4}\lambda^2$
Euler (1773)	$\frac{\sqrt{2(a^2+b^2)}}{a+b} = \sqrt{1+\lambda^2}$	$-\frac{1}{4}\lambda^2$
Sipos (1792)	$\frac{2(a+b)}{(\sqrt{a}+\sqrt{b})^2} = \frac{2}{1+\sqrt{1-\lambda^2}}$	$-\frac{7}{64}\lambda^4$
Peano (1889)	$\frac{3}{2} - \frac{\sqrt{ab}}{a+b} = \frac{3}{2} - \frac{1}{2}\sqrt{1-\lambda^2}$	$-\frac{3}{64}\lambda^4$
Muir (1883)	$\frac{2}{a+b}\left(\frac{a^{3/2}+b^{3/2}}{2}\right)^{2/3} = \frac{1}{2^{2/3}}((1+\lambda)^{\frac{3}{2}} + (1-\lambda)^{\frac{3}{2}})^{\frac{2}{3}}$	$\frac{1}{64}\lambda^4$
Lindner (1904)	$\left\{1 + \frac{1}{8}\left(\frac{a-b}{a+b}\right)^2\right\}^2 = \left(1 + \frac{1}{8}\lambda^2\right)^2$	$\frac{1}{2^8}\lambda^6$
Ramanujan (1914a)	$\frac{3-\sqrt{(a+3b)(3a+b)}}{a+b} = 3 - \sqrt{4-\lambda^2}$	$\frac{1}{2^9}\lambda^6$
Ramanujan (1914b)	$1 + \frac{3((a-b)(a+b))^2}{10+\sqrt{4-3((a-b)/(a+b))^2}}$ $= 1 + \frac{3\lambda^2}{10+\sqrt{4-3\lambda^2}}$ $= 2\frac{(1+\sqrt{1-\lambda^2})^2+\lambda^2\sqrt{1-\lambda^2}}{(1+\sqrt{1-\lambda^2})(1+\sqrt[4]{1-\lambda^2})^2}$	$\frac{3}{2^{17}}\lambda^{10}$
Bronshtein (1964)	$\frac{1}{16}\frac{64(a+b)^4-3(a-b)^4}{(a+b)^2(3a+b)(a+3b)} = \frac{64-3\lambda^4}{64-16\lambda^2}$	$\frac{9}{2^{14}}\lambda^8$
Selmer (1975,a)	$1 + \frac{4(a-b)^2}{(5a+3b)(3a+5b)} = 1 + \frac{\lambda^2}{4}\frac{16}{16-\lambda^2}$	$\frac{3}{2^{10}}\lambda^6$
Selmer (1975,b)	$\frac{1}{8}\left(12 + \left(\frac{a-b}{a+b}\right)^2 - \frac{2\sqrt{2(a^2+6ab+b^2)}}{a+b}\right)$ $= \frac{3}{2} + \frac{1}{8}\lambda^2 - \frac{1}{2}\sqrt{1-\frac{1}{2}\lambda^2}$	$\frac{5}{2^{14}}\lambda^8$
Almkvist (1978)	$2\frac{2(a+b)^2-(\sqrt{a}-\sqrt{b})^4}{(a+b)[(\sqrt{a}+\sqrt{b})^2+2\sqrt{2}\sqrt{a+b}\sqrt[4]{ab}]}$	$-\frac{15}{2^{14}}\lambda^8$
Jacobsen (1985)	$\frac{256-48\lambda^2-21\lambda^4}{256-112\lambda^2+3\lambda^4}$	$\frac{33}{2^{18}}\lambda^{10}$

it is natural to ask whether they are approximations from below and if a result analogous to (2) can be obtained. What we will show in this article is that this is indeed the case.

An obvious analogy would suggest that the results due to Euler, Sipos, Peano, and Almkvist are approximations from above. Somewhat surprisingly, the techniques developed in this article are not directly applicable to these approximations. However, using deeper methods, it has been shown by Barnard *et al.* [4], that even stronger results can be obtained which imply that inequalities analogous to (2) hold, and hence the approximations of Euler, Sipos, Peano, and Almkvist are indeed from above.

The verification of (2) is obtained by showing the positivity of an infinite series in the form

$$\mathcal{E}(x) = \sum_{n=0}^{\infty} b_n(x).$$

We first verify that exists an N for which $b_n(x) > 0,\ n \geq N$. Consequently, the proof of (2), and hence the resolution of (Q1), is reduced to examining the positivity of the polynomial

$$p(x) = \sum_{n=0}^{N} b_n(x).$$

Because the coefficients of the polynomial p are integers, the positivity of p may be verified by a Sturm sequence argument that involves only exact arithmetic. In a subsequent section, we will elaborate on the details of this approach but it should be emphasized at this point that these calculations, though computationally formidable, are readily carried out using a computer algebra system. Nevertheless, care must be exercised to ensure that only exact arithmetic and algebra are performed.

3. MUIR'S APPROXIMATION AND VERIFICATION OF VUORINEN'S CONJECTURE

In this section we will verify (2) and hence answer (Q1). The arguments needed for Muir's approximation turn out to be exceptional among the list of approximations in Table I. Nevertheless, the arguments used in this case suggest a more general approach to the verification of inequalities of the type (2) that are relevant to the other approximations.

In order to avoid the complications associated with a choice of a branch cut, we first make the substitution $x \to 1 - x^4$ and define

$$G(x) = \frac{\mathcal{E}(1 - x^4)}{(1 - x^4)^4}. \tag{3}$$

The result is established by showing that $G'(x)$ does not change sign and so G is monotone. More precisely, we will show that $G'(x) < 0,\ 0 < x < 1$. It is convenient to calculate G' using a computer algebra system (CAS) such as Mathematica or Maple. Using Maple, we obtain

$$G'(x) = \frac{N1 + N2 + P}{D},$$

where

$$N1 = -4x\sqrt[3]{\frac{1}{2} + \frac{1}{2}x^3}\ K\left(\sqrt{1 - x^4}\right)$$
$$N2 = -28x\sqrt[3]{\frac{1}{2} + \frac{1}{2}x^3}\ E\left(\sqrt{1 - x^4}\right)$$
$$P = \pi(1 + 7x^4 + 8x)$$
$$D = \frac{\pi(-1 + x^4)^5\ \sqrt[3]{\frac{1}{2} + \frac{1}{2}x^3}}{x^2},$$

and K, E denote elliptical integrals of the first and second kind respectively. It should be noted that the above expression for $\mathscr{G}'(x)$ can be checked by recalling the definitions of $F\left(\frac{1}{2}, -\frac{1}{2}; 1 : x\right)$, K, and E.

Since $D < 0$, to verify that $\mathscr{G}'(x) < 0$ it suffices to show that

$$\mathscr{H}(x) = \frac{-N1 - N2}{P} \leq 1.$$

Since $\mathscr{H}(1) = 1$, we need only verify that $\mathscr{H}'(x) \geq 0$. After converting the elliptical integrals to hypergeometric functions and simplifying, the problem is reduced to showing that

$$p_1(x)F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1 : 1 - x^4\right) + p_2(x)F\left(-\tfrac{1}{2}, \tfrac{1}{2}; 1 : 1 - x^4\right) \geq 0, \tag{4}$$

where

$$p_1(x) = 252x^9 - 504x^8 + 756x^7 - 502x^6 + 248x^5 + 6x^4 - 4x^3 + 6x^2 - 4x + 2,$$
$$p_2(x) = -238x^6 + 196x^5 - 154x^4 - 112x^3 + 94x^2 - 52x + 10.$$

A crucial manipulation involves the conversion of the above expression to series form. One may then express the left-hand side of (4) as

$$p_1(x)\sum_{k=0}^{\infty}\left(\frac{\left(\frac{1}{2}\right)_k}{k!}\right)^2(1 - x^4)^k + p_2(x)\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{k!k!}(1 - x^4)^k.$$

These series may be combined as

$$p_1(x) + p_2(x) + \sum_{k=1}^{\infty} \left(\frac{\left(\frac{1}{2}\right)_k}{k!} \right)^2 [p_1(x) - p_2(x)/(2k-1)]\,(1-x^4)^k. \qquad (5)$$

It is at this point that a Sturm sequence argument is applied in order to verify the needed positivity. More precisely, we shall use this terminology to refer to a generalized form of Descartes' rule of signs which gives the precise number of zeros of a polynomial in an interval. This is achieved by utilizing the Euclidean algorithm and a notion of *Sturm sequences* [5, 7]. In particular, given a polynomial $P(x)$ and a real number a, define the function $V(a)$ as the number of sign variations in the numbers $\{p_0(a), p_1(a), \ldots, p_n(a)\}$. The *Sturm* polynomials $\{p_0, p_1, \ldots, p_n\}$ are defined by

$$p_0(x) = p(x),$$
$$p_1(x) = p'(x)$$

and, for each $k \geq 2$, $p_k(x)$ is the unique polynomial of degree less than that of $p_{k-1}(x)$ such that

$$p_{k-2}(x) = q_k(x)p_{k-1}(x) - p_k(x),$$

where $q_k(x)$ is a polynomial. The Sturm Theorem states that if $P(a) \neq 0$ and $P(b) \neq 0$, then the number of distinct roots of $P(x) = 0$ in the interval $[\mathrm{a}, \mathrm{b}]$ is exactly $V(a) - V(b)$. In applying Strum's theorem to an arbitrary polynomial, a rounding error may affect the signs of $p_i(a)$ and $p_i(b)$ and hence may limit the usefulness of the procedure. However, for our purposes, the polynomials to which the method is always applied have integer coefficients and with the exact arithmetic provided by a computer algebra system accuracy is never compromised. The method is implemented by calling on the procedure *sturm* within Maple. This procedure uses Strum's theorem to return the number of real roots in the interval $(\mathrm{a}, \mathrm{b}]$ of a polynomial $P(x)$.

For the series given in (5), a Strum sequence argument applied to $p_1(x) - p_2(x)/(2k-1)$ shows that when $k = 4$ this polynomial has no zeros. It then follows that

$$p_1(x) - \frac{p_2(x)}{2k-1} > 0, \qquad k > 4. \qquad (6)$$

The positivity of the infinite series will be confirmed if the 61st degree polynomial with rational coefficients

$$P_1(x) + p_2(x) + \sum_{k=1}^{13} \left(\frac{\left(\frac{1}{2}\right)_k}{k!} \right)^2 \left[p_1(x) - \frac{p_2(x)}{2k-1} \right] (1 - x^4)^k \tag{7}$$

is positive. This is readily verified by another Sturm sequence argument.

Because of the topic in Vuorinen's article [6], the question he posed regarding Muir's approximation was expressed in terms of the eccentricity and hence inequality (2) was given as a function of e. If the notation of [1] is adopted, it is more natural to express the error in terms of $\lambda = (a-b)/(a+b)$. What we shall show in the next section is that the methods that have been utilized in verifying (2) can be generalized to obtain analogous results. As argued above, the proof will ultimately be reduced to verifying the positivity of an infinite series as in (5) and, subsequently, the positivity of a polynomial analogous to (7).

4. APPROXIMATIONS OF ELLIPTICAL PERIMETER FROM BELOW

With the exception of Muir's, the algebraic approximations from below given in Table I may be denoted as $A(\lambda^2)$. There is a polynomial $\phi(x)$ so that with the change of variable $\lambda^2 = \phi(x)$ we obtain a rational function $f(x) = A(\phi(x))$. In particular, for the approximation of Kepler, $\phi(x) = 1 - x^2$; for those of Lindner, Selmer(a), Bronshstein, and Jacobsen, $\phi(x) = x$; for Ramanujan(a, b), $\phi(x) = (4 - x^2)$ and $\phi(x) = (4 - x^2)/3$ respectively; while for Selmer(b), $\phi(x) = 2(1 - x^2)$. Let N denote the power of the first nonzero term in the Taylor series for the difference $L(a, b)/(\pi(a+b)) - A(\lambda^2)$.

The result, analogous to (3), that will be verified is

$$\frac{d}{dx} \left(\frac{F\left(-\frac{1}{2}, -\frac{1}{2}; 1 : \phi(x)\right) - f(x)}{[\phi(x)]^M} \right) \neq 0. \tag{8}$$

Here $M = N/2$, and the interval on which (8) will hold is determined by the change of variable $\lambda^2 = \phi(x)$.

We begin by computing

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{F\left(-\frac{1}{2},-\frac{1}{2};1:\phi(x)\right)-f(x)}{[\phi(x)]^M}\right)\\
&\quad=\frac{1}{[\phi(x)]^{M+1}}[Mf(x)\phi'(x)-\phi(x)f'(x)]\\
&\qquad+\frac{1}{[\phi(x)]^{M+1}}\left[\frac{\phi(x)\phi'(x)}{4}\sum_{k=0}^{\infty}\frac{\left(\left(\frac{1}{2}\right)_k\right)^2}{(2)_k}\frac{\phi^k(x)}{k!}\right.\\
&\qquad\left.-M\phi'(x)\sum_{k=0}^{\infty}\left(\frac{(-\frac{1}{2})_k}{k!}\right)^2\phi^k(x)\right]\\
&\quad=D1+D2.
\end{aligned}$$

For D2 we have

$$\begin{aligned}
D2&=\frac{1}{[\phi(x)]^{M+1}}\left[\frac{\phi(x)\phi'(x)}{4}-M\phi'(x)\right]\\
&\quad+\frac{1}{[\phi(x)]^{M+1}}\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\left[\frac{\phi(x)\phi'(x)}{4(k+1)}-\frac{M\phi'(x)}{(2k-1)^2}\right]\phi(x)^k\\
&=D21+D22.
\end{aligned}$$

A further computation shows that

$$\begin{aligned}
D22&=\frac{\phi'(x)}{[\phi(x)]^{M+1}}\left[\frac{\phi(x)}{4}\left(-1+\sum_{k=0}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\frac{\phi^k(x)}{(k+1)}\right)\right.\\
&\qquad\left.-\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\frac{M\phi^k(x)}{(2k-1)^2}\right]\\
&=\frac{\phi'(x)}{[\phi(x)]^{M+1}}\left[\frac{\phi(x)}{4}\left(-1+\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_{k-1}}{(k-1)!}\right)^2\frac{\phi^{(k-1)}(x)}{k}\right)\right.\\
&\qquad\left.-\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\frac{M\phi^k(x)}{(2k-1)^2}\right]\\
&=\frac{\phi'(x)}{[\phi(x)]^{M+1}}\left[\frac{-\phi(x)}{4}+\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\frac{k[\phi(x)]^k}{(2k-1)^2}\right.\\
&\qquad\left.-\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\frac{M\phi^k(x)}{(2k-1)^2}\right].
\end{aligned}$$

Combining these results gives us

$$\frac{d}{dx}\left(\frac{F\left(-\frac{1}{2},-\frac{1}{2};1:\phi(x)\right)-f(x)}{[\phi(x)]^M}\right)$$
$$=\frac{1}{[\phi(x)]^{M+1}}\left[M\phi'(x)(f(x)-1)-f'(x)\phi(x)\right.$$
$$\left.+\phi'(x)\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\frac{(k-M)\phi^k(x)}{(2k-1)^2}\right].$$

The necessary inequality that is analogous to (6) is trivially met if $k > M$. To verify (8), one need only determine $K \geq M$ and apply a sturm sequence argument to show that the polynomial

$$Mf(x)\phi'(x)-f'(x)\phi(x)-M\phi'(x)+\phi'(x)\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\frac{(k-M)\phi^k(x)}{(2k-1)^2}$$

has the same sign as

$$\phi'(x)\sum_{k=1}^{\infty}\left(\frac{(\frac{1}{2})_k}{k!}\right)^2\frac{(k-M)\phi^k(x)}{(2k-1)^2}$$

From the monotonicity of $[F\left(-\frac{1}{2},-\frac{1}{2};1:\phi(x)\right)-A(\phi(x))]/[\phi(x)]^M$, one readily obtains, analogous to inequality (2), the bounds (α, β) on

$$\frac{\mathscr{E}(\lambda)}{\lambda^N}=\frac{F\left(-\frac{1}{2},-\frac{1}{2};1:\lambda^2\right)-A(\lambda^2)}{\lambda^N}.$$

In particular, α is obtained as the limit of $\mathscr{E}(\lambda)/\lambda^N$ as $\lambda \to 0$, while β is determined by evaluation at $\lambda = 1$. Table II illustrates the accuracy of the approximations from below by specifying α, β, where

$$\alpha < \frac{\mathscr{E}(\lambda)}{\lambda^N} < \mathscr{E}(1) < \beta, \qquad 0 < \lambda < 1. \tag{9}$$

To illustrate the use of these bounds in obtaining error approximations first note that

$$\frac{a-b}{a+b}=\frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}=\lambda(e).$$

It follows that the error in elliptical perimeter approximation

$$\widetilde{E}(\lambda) := L(a,b)-\pi(a+b)F\left(\frac{-1}{2},\frac{-1}{2};1:\lambda^2\right)$$

satisfies

$$a\pi\alpha\left(1+\sqrt{1-e^2}\right)[\lambda(e)]^N < \widetilde{E}(\lambda) < a\pi\beta\left(1+\sqrt{1-e^2}\right)[\lambda(e)]^N. \tag{10}$$

TABLE II

Sources (see [1])	(α, β)
Kepler (1609)	(.75, 1.2732)
Lindner (1904)	(.003906, .007614)
Ramanujan (1914a)	(.001953, .005290)
Ramanujan (1914b)	(.000022, .000512)
Bronshtein (1964)	(.002929, .006572)
Selmer (1975a)	(.000549, .002406)
Selmer (1975b)	(.000305, .001792)
Jacobsen (1978)	(.000125, .001130)

5. IMPLICATIONS AND FURTHER DIRECTIONS

It can be shown that the remaining approximations in Table I are from above. To verify this requires considerably more sophisticated methods. In fact, what has been shown by Barnard *et al.* [4] is that, with the exception of Euler's, the coefficients of the Taylor expansions of $\mathscr{E}(x)$ are all of the same sign. As a consequence of their results one can deduce not only that (8) holds, but more generally, for every integer $n \geq 0$, that

$$\frac{d^n \mathscr{E}(x)}{dx^n} \neq 0.$$

Of course, the results of this paper are trivial consequences of this more general result, but the methods that are presented here are of interest in their own right. For instance, the results presented here, and in particular Eq. (10), provide a relatively simple derivation of estimates that historically appeared in the context of astronomical considerations and the application of Landen's transformation [1].

In addition, there is an obvious intriguing issue associated with the study of these inequalities. Is there a geometrical significance to these algebraic approximations that accounts for the monotone properties that these approximations possess? It has been suggested by Berndt [2] that Ramanujan's formulas have their origins in the representation of analytic functions as continued fractions. As for the other approximations, little is known as to their motivation.

REFERENCES

1. G. Almkvist and B. Berndt, Gauss, Landen, Ramanujan, the arithmetic–geometric mean, ellipses, π, and the *Ladies Diary, Amer. Math. Monthly* **95** (1988), 585–608.

2. B. Berndt, "Ramanujan's Notebooks, Part III," Springer-Verlag, New York/Berlin, 1985.
3. R. W. Barnard, K. Pearce, and K. Richards, An inequality involving the generalized hypergeometric function and the arc length of an ellipse, *SIAM J. Math. Anal.* **32** (2000), 403–419.
4. K. Devlin, The logical structure of computer-aided mathematical reasoning, *Amer. Math. Monthly* **104** (1997), 632–646.
5. P. Henrici, "Applied and Computational Complex Analysis," Wiley, New York, 1974.
6. M. Vuorinen, Hypergeometric functions in geometric function theory, *in* "Proceedings of Special Functions and Differential Equations," (K. Srinivasa Roa, P. Jagannthan, and G. Van der Jeugy, Eds.), Allied Publishers, 1998.
7. D. Young and R. Gregory, "A Survey of Numerical Mathematics," Dover, New York, 1973.