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Note

Spherical two-distance sets

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A R T I C L E   I N F O

A B S T R A C T

A set S of unit vectors in n-dimensional Euclidean space is called spherical two-distance set, if there are two numbers a and b so that the inner products of distinct vectors of S are either a or b. It is known that the largest cardinality $g(n)$ of spherical two-distance sets does not exceed $n(n+3)/2$. This upper bound is known to be tight for $n = 2, 6, 22$. The set of mid-points of the edges of a regular simplex gives the lower bound $L(n) = n(n+1)/2$ for $g(n)$.

In this paper using the so-called polynomial method it is proved that for nonnegative $a + b$ the largest cardinality of S is not greater than $L(n)$. For the case $a + b < 0$ we propose upper bounds on $|S|$ which are based on Delsarte's method. Using this we show that $g(n) = L(n)$ for $6 < n < 22$, $23 < n < 40$, and $g(23) = 276$ or 277.

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1. Introduction

A set S in Euclidean space $\mathbb{R}^n$ is called a *two-distance set*, if there are two distances c and d, and the distances between pairs of points of S are either c or d. If a two-distance set S lies in the unit sphere $\mathbb{S}^{n-1}$, then S is called *spherical two-distance set.* In other words, S is a set of unit vectors, there are two real numbers a and b, $-1 \leqslant a, b < 1$, and inner products of distinct vectors of S are either a or b.

The ratios of distances of two-distance sets are quite restrictive. Namely, Larman, Rogers, and Seidel [8] have proved the following fact: if the cardinality of a two-distance set S in $\mathbb{R}^n$, with distances c and d, $c < d$, is greater than $2n + 3$, then the ratio c^2/d^2 equals $(k-1)/k$ for an integer k with

$$2 \leqslant k \leqslant \frac{1 + \sqrt{2n}}{2}.$$

Einhorn and Schoenberg [6] proved that there are finitely many two-distance sets S in $\mathbb{R}^n$ with cardinality $|S| \geqslant n+2$. Delsarte, Goethals, and Seidel [5] proved that the largest cardinality of spherical two-distance sets in $\mathbb{R}^n$ (we denote it by $g(n)$) is bounded by $n(n+3)/2$, i.e.,

$$g(n) \leqslant \frac{n(n+3)}{2}.$$

Moreover, they give examples of spherical two-distance sets with $n(n+3)/2$ points for $n = 2, 6, 22$. (Therefore, in these dimensions we have equality $g(n) = n(n+3)/2$.) Blockhuis [2] showed that the cardinality of (Euclidean) two-distance sets in $\mathbb{R}^n$ does not exceed $(n+1)(n+2)/2$.

The standard unit vectors $e_1, \ldots, e_{n+1}$ form an orthogonal basis of $\mathbb{R}^{n+1}$. Denote by Δ_n the regular simplex with vertices $2e_1, \ldots, 2e_{n+1}$. Let Λ_n be the set of points $e_i + e_j$, $1 \leqslant i < j \leqslant n+1$. Since Λ_n lies in the hyperplane $\sum x_k = 2$, we see that Λ_n represents a spherical two-distance set in $\mathbb{R}^n$. On the other hand, Λ_n is the set of mid-points of the edges of Δ_n. Thus,

$$g(n) \geqslant |\Lambda_n| = \frac{n(n+1)}{2}.$$

For $n < 7$ the largest cardinality of Euclidean two-distance sets is $g(n)$, where $g(2) = 5$, $g(3) = 6$, $g(4) = 10$, $g(5) = 16$, and $g(6) = 27$ (see [10]). However, for $n = 7, 8$ Lisoněk [10] discovered non-spherical maximal two-distance sets of the cardinality 29 and 45, respectively.

In this paper we prove that

$$g(n) = \frac{n(n+1)}{2}, \quad \text{where } 6 < n < 40, \ n \neq 22, 23,$$

and $g(23) = 276$ or 277. This proof (Section 4) is based on the new sharp upper bound $\binom{n+1}{2}$ for spherical two-distance sets with $a + b \geqslant 0$ (Section 2), and on the Delsarte bounds for spherical two-distance sets in the case $a + b < 0$.

2. Linearly independent polynomials

The upper bound $n(n+3)/2$ for spherical two-distance sets [5], the bound $\binom{n+2}{2}$ for Euclidean two-distance sets [2], as well as the bound $\binom{n+s}{s}$ for s-distance sets [1,3] were obtained by the polynomial method. The main idea of this method is the following: to associate sets to polynomials and show that these polynomials are linearly independent as members of the corresponding vector space.

Now we apply this idea to improve upper bounds for spherical two-distance sets with $a + b \geqslant 0$.

Theorem 1. *Let S be a spherical two-distance set in $\mathbb{R}^n$ with inner products a and b. If $a + b \geqslant 0$, then*

$$|S| \leqslant \frac{n(n+1)}{2}.$$

Proof. Let

$$F(t) := \frac{(t-a)(t-b)}{(1-a)(1-b)}.$$

For a unit vector $y \in \mathbb{R}^n$ we define the function $F_y : \mathbb{S}^{n-1} \to \mathbb{R}$ by

$$F_y(x) := F(\langle x, y \rangle), \quad x \in \mathbb{R}^n, \ \|x\| = 1.$$

Let $S = \{x_1, \ldots, x_m\}$ be an m-element set. Denote $f_i(x) := F_{x_i}(x)$. Since

$$f_i(x_j) = \delta_{i,j}, \tag{1}$$

the quadratic polynomials f_i, $i = 1, \ldots, m$, are linearly independent.

Let $e_1, \ldots, e_n$ be a basis of $\mathbb{R}^n$. Let $L_i(x) := \langle x, e_i \rangle$, $x \in \mathbb{S}^{n-1}$. Then the linear polynomials $L_1, \ldots, L_n$ are also linearly independent.

Now we show that if $a + b \geqslant 0$, then $f_1, \ldots, f_m, L_1, \ldots, L_n$ form a linearly independent system of polynomials. Assume the converse. Then

$$\sum_{k=1}^{n} d_k L_k(x) = \sum_{i=1}^{m} c_i f_i(x),$$

where there are nonzero d_k and c_i.

Let

$$v = d_1 e_1 + \ldots + d_n e_n.$$

Then

$$\langle x, v \rangle = \sum_i c_i f_i(x). \tag{2}$$

For $x = x_i$ in (2), using (1), we get

$$c_i = \langle x_i, v \rangle.$$

Take $x = v$ and $x = -v$ in (2). Then we have

$$\|v\|^2 = \sum_i c_i f_i(v) = \sum_i c_i F(c_i), \tag{3}$$

$$-\|v\|^2 = \sum_i c_i f_i(-v) = \sum_i c_i F(-c_i). \tag{4}$$

Subtracting (3) from (4), we obtain

$$-\|v\|^2 = \frac{a+b}{(1-a)(1-b)} \sum_i c_i^2.$$

This yields $v = 0$, a contradiction.

Note that the dimension of the vector space of quadratic polynomials on the sphere $\mathbb{S}^{n-1}$ is $n(n+3)/2$. Therefore,

$$\dim\{f_1, \ldots, f_m, L_1, \ldots, L_n\} = m + n \leqslant \frac{n(n+3)}{2}.$$

Thus $|S| = m \leqslant n(n+1)/2$. $\quad\square$

Denote by $\rho(n)$ the largest possible cardinality of spherical two-distance sets in $\mathbb{R}^n$ with $a + b \geqslant 0$.

Theorem 2. *If $n \geqslant 7$, then*

$$\rho(n) = \frac{n(n+1)}{2}.$$

Proof. Theorem 1 implies that $\rho(n) \leqslant n(n+1)/2$. On the other hand, the set of mid-points of the edges of a regular simplex has $n(n+1)/2$ points and $a + b \geqslant 0$ for $n \geqslant 7$. Indeed, for this spherical two-distance set we have

$$a = \frac{n-3}{2(n-1)}, \qquad b = \frac{-2}{n-1}.$$

Thus

$$a + b = \frac{n-7}{2(n-1)} \geqslant 0. \quad\square$$

3. Delsarte's method for two-distance sets

Delsarte's method is widely used in coding theory and discrete geometry for finding bounds for error-correcting codes, spherical codes, and sphere packings (see [4,5,7]). In this method upper bounds for spherical codes are given by the following theorem:

Theorem 3. *(See [5,7].) Let T be a subset of the interval $[-1, 1]$. Let S be a set of unit vectors in $\mathbb{R}^n$ such that the set of inner products of distinct vectors of S lies in T. Suppose a polynomial f is a nonnegative linear combination of Gegenbauer polynomials $G_k^{(n)}(t)$, i.e.,*

$$f(t) = \sum_k f_k G_k^{(n)}(t), \quad \text{where } f_k \geqslant 0.$$

If $f(t) \leqslant 0$ for all $t \in T$ and $f_0 > 0$, then

$$|S| \leqslant \left\lfloor \frac{f(1)}{f_0} \right\rfloor.$$

There are many ways to define Gegenbauer (or ultraspherical) polynomials $G_k^{(n)}(t)$. $G_k^{(n)}$ are a special case of Jacobi polynomials $P_k^{(\alpha,\beta)}$ with $\alpha = \beta = (n-3)/2$ and with normalization $G_k^{(n)}(1) = 1$. Another way to define $G_k^{(n)}$ is the recurrence formula:

$$G_0^{(n)} = 1, \quad G_1^{(n)} = t, \quad \dots, \quad G_k^{(n)} = \frac{(2k+n-4)t G_{k-1}^{(n)} - (k-1)G_{k-2}^{(n)}}{k+n-3}.$$

For instance,

$$G_2^{(n)}(t) = \frac{nt^2 - 1}{n-1},$$

$$G_3^{(n)}(t) = \frac{(n+2)t^3 - 3t}{n-1},$$

$$G_4^{(n)}(t) = \frac{(n+2)(n+4)t^4 - 6(n+2)t^2 + 3}{n^2 - 1}.$$

Now for given n, a, b we introduce polynomials $P_i(t)$, $i = 1, \dots, 5$.

$i = 1$: $P_1(t) = (t-a)(t-b) = f_0^{(1)} + f_1^{(1)}t + f_2^{(1)}G_2^{(n)}(t)$.

$i = 2$: $P_2(t) = (t-a)(t-b)(t+c) = f_0^{(2)} + f_1^{(2)}t + f_2^{(2)}G_2^{(n)}(t) + f_3^{(2)}G_3^{(n)}(t)$, where c is defined by the equation $f_1^{(2)} = 0$.

$i = 3$: $P_3(t) = (t-a)(t-b)(t+a+b) = f_0^{(3)} + f_1^{(3)}t + f_2^{(3)}G_2^{(n)}(t) + f_3^{(3)}G_3^{(n)}(t)$. Note that $f_2^{(3)} = 0$.

$i = 4$: $P_4(t) = (t-a)(t-b)(t^2+ct+d) = \sum f_k^{(4)}G_k^{(n)}(t)$, where c and d are defined by the equations $f_1^{(4)} = f_2^{(4)} = 0$.

$i = 5$: $P_5(t) = (t-a)(t-b)(t^2+ct+d) = \sum f_k^{(5)}G_k^{(n)}(t)$, where c and d are defined by the equations $f_2^{(5)} = f_3^{(5)} = 0$.

Denote by $D_i^{(n)}$ the set of all pairs (a, b) such that the polynomial $P_i(t)$ is well defined, all $f_k^{(i)} \geqslant 0$, and $f_0^{(i)} > 0$. For instance,

$$D_1^{(n)} = \left\{ (a, b) \in \mathbf{I}^2 : \ f_1^{(1)} = -a - b \geqslant 0, \ f_0^{(1)} = ab + \frac{1}{n} > 0 \right\},$$

$$D_2^{(n)} = \left\{ (a, b) \in \mathbf{I}^2 : \ a + b \neq 0, \ c \geqslant a + b, \ f_0^{(2)} = abc + \frac{c - a - b}{n} > 0 \right\},$$

where

$$\mathbf{I} = [-1, 1), \qquad c = \frac{ab(n+2)+3}{(n+2)(a+b)}.$$

Let

$$U_i^{(n)}(a, b) := \frac{P_i(1)}{f_0^{(i)}}.$$

Note that we have $P_i(a) = P_i(b) = 0$. Then Theorem 3 yields

Theorem 4. *Let S be a spherical two-distance set in $\mathbb{R}^n$ with inner products a and b. Suppose $(a, b) \in D_i^{(n)}$ for some i, $1 \leqslant i \leqslant 5$. Then*

$$|S| \leqslant U_i^{(n)}(a, b).$$

Let S be a spherical two-distance set in $\mathbb{R}^n$ with inner products a and b, where $a > b$. Let $c = \sqrt{2 - 2a}$, $d = \sqrt{2 - 2b}$. Then c and d are the Euclidean distances of S.
     Let

$$b_k(a) = \frac{ka - 1}{k - 1}.$$

If k is defined by the equation $b_k(a) = b$, then $(k-1)/k = c^2/d^2$. Therefore, if $|S| > 2n + 3$, then k is an integer number and $k \in \{2, \ldots, K(n)\}$ [8]. Here, $K(n) = \lfloor \frac{1+\sqrt{2n}}{2} \rfloor$.

Denote by $D_{i,k}^{(n)}$ the set of all real numbers a such that $(a, b_k(a)) \in D_i^{(n)}$. Let

$$R_{i,k}^{(n)}(a) := \begin{cases} U_i^{(n)}(a, b_k(a)) & \text{for } a \in D_{i,k}^{(n)}, \\ \infty & \text{for } a \notin D_{i,k}^{(n)}, \end{cases}$$

$$Q_k^{(n)}(a) := \min_i \{ R_{i,k}^{(n)}(a) \}$$

Then Theorem 4 yields the following bound for $|S|$.

Theorem 5. *Let S be a spherical two-distance set in $\mathbb{R}^n$ with inner products a and $b_k(a)$. Then*

$$|S| \leqslant Q_k^{(n)}(a).$$

Consider the case $a + b_k(a) < 0$. Since $b_k(a) \geqslant -1$, we have

$$a \in I_k := \left[\frac{2-k}{k}, \frac{1}{2k-1} \right).$$

Remark 1. Actually, the polynomials P_i are chosen such that the maximum of $Q_k^{(n)}(a)$ on I_k minimize the Delsarte bound (Theorem 3). Clearly, $Q_k^{(n)}(a)$ is a piecewise rational function on I_k. It is not hard to find explicit expressions for $Q_k^{(n)}(a)$ and to compute its maximum on I_k numerically. For instance, $\max \{ Q_3^{(25)}(a) : a \in I_3 = [-1/3, 1/5] \} \approx 284.14$ (see Fig. 1).

4. Maximal spherical two-distance sets

In this section we use Theorem 5 to bound the cardinality of a spherical two-distance set with $a + b < 0$.

Let S, $|S| > 2n + 3$, be a spherical two-distance set in $\mathbb{R}^n$ with inner products a and $b_k(a)$. Then $k \in \{2, \ldots, K(n)\}$, and $-1 \leqslant b_k(a) < a < 1$.
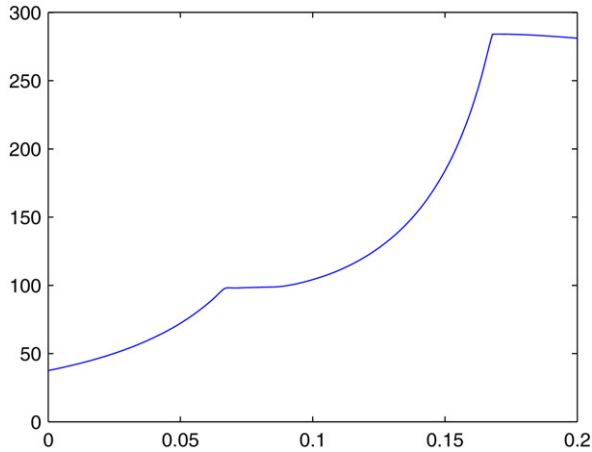
Fig. 1. The graph of the function $Q_3^{(25)}(a)$.

Let $\widetilde{K}(n) := \max\{K(n), 2\}$. For given n and $k = 2, \ldots, \widetilde{K}(n)$, we denote by $\Omega(n, k)$ the set of all spherical two-distance sets S in $\mathbb{R}^n$ with $a + b_k(a) < 0$. Denote by $\omega(n, k)$ the largest cardinality of $S \in \Omega(n, k)$.

Let

$$\varphi(n, k) := \sup_{a \in I_k}\{Q_k^{(n)}(a)\},$$

$$\widehat{\omega}(n, k) := \max\{\lfloor \varphi(n, k) \rfloor, 2n + 3\}.$$

Let us denote by $\widehat{\omega}(n)$ the maximum of numbers $\widehat{\omega}(n, 2), \ldots, \widehat{\omega}(n, \widetilde{K}(n))$, and by $\omega(n)$ we denote the largest cardinality of a two-distance set S in $\mathbb{S}^{n-1}$ with $a + b < 0$. Then $g(n) = \max\{\omega(n), \rho(n)\}$.

Since Theorem 5 implies $\omega(n, k) \leqslant \widehat{\omega}(n, k)$, we have

Theorem 6. $g(n) \leqslant \max\{\widehat{\omega}(n), \rho(n)\}$.

Finally, for $g(n)$ we have the following bounds: $\rho(n) \leqslant g(n) \leqslant \max\{\widehat{\omega}(n), \rho(n)\}$. Recall that $\rho(n) = n(n + 1)/2$ for $n \geqslant 7$. For $\widehat{\omega}(n)$, $7 \leqslant n \leqslant 40$, we obtain the computational results gathered in Table 1.

Since $\widehat{\omega}(n) \leqslant \rho(n)$ for $6 < n < 40$, $n \neq 22, 23$, for these cases we have $g(n) = \rho(n)$. For $n = 23$ we obtain $g(23) \leqslant 277$. But $g(23) \geqslant \rho(23) = 276$. This proves the following theorem.

Theorem 7. If $6 < n < 22$ or $23 < n < 40$, then

$$g(n) = \frac{n(n + 1)}{2}.$$

For $n = 23$ we have

$$g(23) = 276 \quad or \quad 277.$$

Remark 2. The case $n = 23$ is very interesting. In this dimension the maximal number of equiangular lines (or equivalently, the maximal cardinality of a two-distance set with $a + b = 0$) is 276 [9]. On the other hand, $|\Lambda_{23}| = 276$. So in 23 dimensions we have two very different two-distance sets with 276 points.

Note that $\max\{Q_3^{(23)}(a): a \in I_3\} \approx 277.095$. So this numerical bound is not far from 277. Perhaps stronger tools, such as semidefinite programming bounds, are needed here to prove that $g(23) = 276$.

Table 1
$\widehat{\omega}(n)$ and $\rho(n)$. The last column gives the k with $\widehat{\omega}(n) = \widehat{\omega}(n,k)$.

n	$\widehat{\omega}$	ρ	k
7	28	28	2
8	31	36	2
9	34	45	2
10	37	55	2
11	40	66	2
12	44	78	2
13	47	91	2
14	52	105	2
15	56	120	2
16	61	136	2
17	66	153	2
18	76	171	3
19	96	190	3
20	126	210	3
21	176	231	3
22	275	253	3
23	277	276	3
24	280	300	3
25	284	325	3
26	288	351	3
27	294	378	3
28	299	406	3
29	305	435	3
30	312	465	3
31	319	496	3
32	327	528	3
33	334	561	3
34	342	595	3
35	360	630	2
36	416	666	2
37	488	703	2
38	584	741	2
39	721	780	2
40	928	820	2

Remark 3. Our numerical calculations show that the barrier $n = 40$ is in fact fundamental: LP bounds are incapable of resolving the $n \geqslant 40$, $k = 2$ case. That means a new idea is required to deal with $n \geqslant 40$.

Remark 4. It is known that for $n = 3, 7, 23$ maximal spherical two-distance sets are not unique, and for $n = 2, 6, 22$, when $g(n) = n(n+3)/2$, these sets are unique up to isometry. Lisoněk [10] confirmed the maximality and uniqueness of previously known sets for $n = 4, 5, 6$. For all other n the problem of uniqueness of maximal two-distance sets is open. We think that for $7 < n < 46$, $n \neq 22, 23$ maximal spherical two-distance sets in $\mathbb{R}^n$ are unique and congruent to Λ_n.

Acknowledgments

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