Applications of the Theory of Semi-embeddings to Banach Space Theory

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Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ an injective bounded linear operator. $T$ is called a semi-embedding if $T$ maps the closed unit ball of $X$ to a closed subset of $Y$. (This concept was introduced by Lotz, Peck, and Porta, Proc. Edinburgh Math. Soc. 22 (1979), 233-240.) It is proved that if $X$ semi-embeds in $Y$, and $X$ is separable, then $X$ has the Radon-Nikodym property provided $Y$ does. It is shown that if $L^1$ semi-embeds in $Y$, then $Y$ fails the Schur property and contains a subspace isomorphic to $l^1$. As a consequence of the proof, it is shown that if $X$ is a subspace of $L^1$, either $L^1$ embeds in $X$ or $l^1$ embeds in $L^1;X$. The simpler result that $L^1$ does not semi-embed in $c_0$ is treated separately. This result is used to deduce the classic result of Menchoff that there exists a singular probability measure on the circle with Fourier coefficients vanishing at infinity. Some generalizations of the notion of semi-embedding are given, and several complements and open questions are discussed.

INTRODUCTION

Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ an operator (meaning a bounded linear map). $T$ is called a semi-embedding if $T$ is one-one and $T\overline{Ba(X)}$ is closed, where $Ba(X)$, the unit ball of $X$, equals the set of $x \in X$ with $\|x\| \leq 1$. We say that $X$ semi-embeds in $Y$ if there exists a semi-embedding mapping $X$ into $Y$. (The concept of semi-embedding was introduced by Lotz et al. [16].) $T$ is called an embedding or isomorphism if there is a $\delta > 0$ so that $\|Tx\| \geq \delta \|x\|$ for all $x \in X$. Usually the notion of a semi-embedding is much weaker than that of an embedding.

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In the first section we investigate the relationship between semi-embeddings and the Radon-Nikodym Property (the RNP). Our main structural result (Theorem 1.1) asserts that a separable Banach space has the RNP provided it semi-embeds in a space with the RNP. The proof is given rather quickly, and most of the first section is devoted to applications and complements of this result. For example, it is a simple exercise that the dual of any separable Banach space semi-embeds in Hilbert space (see Proposition 1.2). We thus obtain a simple proof that separable duals have the RNP; in fact, any separable space which semi-embeds in a separable dual has the RNP (Corollary 1.3).

There are known examples of separable Banach spaces which semi-embed in separable duals, yet fail to embed in any separable dual. We discuss these briefly in Proposition 1.4, and also prove in Proposition 1.5 that no separable $L^\infty$-space of infinite dimension semi-embeds in a separable dual. Bourgain and Delbaen have constructed such a space with the RNP [3].) We also introduce the class $\mathcal{R}$ consisting of the smallest family of separable Banach spaces closed under the operation of semi-embeddings and containing Hilbert space. Evidently every space in $\mathcal{R}$ has the RNP by Theorem 1.1. We briefly review Delbaen's result that also no separable $L^\infty$-space belongs to $\mathcal{R}$ [6]. Thus $\mathcal{R}$ does not exhaust the family of separable spaces with the RNP. In spite of this, it still seems worthwhile to obtain some "independent" characterizations of the spaces in $\mathcal{R}$.

In Proposition 1.6 we present an elegant result of Saint-Raymond characterizing semi-embeddings under an equivalent norm on the domain space. Semi-embeddings suffer from the following defect: the restriction of a semi-embedding to a closed linear subspace need not be a semi-embedding, even under an equivalent norm.

In Proposition 1.8 we show that if $T: X \to Y$ is a semi-embedding, then $T$ has the following property provided $X$ is separable:

$$TK \text{ is a } G_\delta \text{ for all closed bounded } K. \quad (*)$$

We define an injective operator $T: X \to Y$ to be a $G_\delta$-embedding provided it satisfies $(*)$. Evidently the restriction of a $G_\delta$-embedding is also a $G_\delta$-embedding. We present some equivalences to $(*)$ in Proposition 1.9. In particular, we obtain a result important for our work in Section 2: If $T: X \to Y$ is a $G_\delta$-embedding and $K$ is a closed bounded nonempty subset of $X$, there is a $k \in K$ so that $Tk$ is a point of continuity for $T^{-1} | TK$. We also list a number of open questions concerning $G_\delta$-embeddings. For example, we do not know the answer to the following question: Does a separable Banach space have the RNP provided it $G_\delta$-embeds into an RNP-space?

In the second section, we mainly treat semi-embeddings of $L^1$ (after a "warm-up" result which shows that $G_\delta$-embeddings of $c_0$ are automatically embeddings while $G_\delta$-embeddings of $C([0,1])$ have a restriction which
The theory of semi-embeddings embeds an isomorph of $C([0,1])$, Proposition 2.2). Because of its applications to harmonic analysis, we first treat the case of operators from $L^1$ to $c_0$. In fact, $L^1$ does not semi-embed in $c_0$ (Theorem 2.3). This fact alone allows us to deduce the theorem of Menchoff [18] that there exists a singular probability measure on the circle with Fourier coefficients tending to zero (Corollary 2.4). In fact, 2.3 shows that the measure may be chosen to be singular with respect to any pre-assigned probability measure with nonvanishing Fourier coefficients (Corollary 2.5).

We complete the proof of Theorem 2.3 after some preliminary work which yields some general principles concerning semi-embeddings on $L^1$ (Lemma 2.6 and Corollary 2.7). In fact, however, the main problems concerning semi-embeddings of $L^1$ remain unsolved. They are as follows: Let $T: L^1 \to X$ be a semi-embedding.

(A) Does $L^1$ embed in $X$?

(B) Is there a subspace $Y$ of $L^1$ isomorphic to $L^1$ so that $T|Y$ is an embedding?

We may summarize our knowledge of these problems at this point as follows:

**Theorem.** Let $T: L^1 \to X$ be a $G_δ$-embedding, $X$ a given Banach space.

(a) $L^1$ embeds in $X$ if $X$ is isomorphic to a dual Banach space or if $X$ itself embeds in $L^1$. Moreover in the latter case, question (B) has an affirmative answer.

(b) $X$ fails the RNP and the Schur property.

(c) There is a subspace $Y$ of $L^1$ with $Y$ isomorphic to $l^1$ so that $T|Y$ is an embedding. (So $l^1$ embeds in $X$.)

We note that Bourgain has recently proved that $L^1$ embeds in $L^1/H^1$ [2] thus eliminating one possible counterexample to question (A). Also Rosenthal has proved that if $L^1$ $G_δ$-embeds in $X$, then $X$ fails to have an unconditional basis [22].

The proof of (c) of the theorem lies considerably deeper than the other results of Section 2, and we devote all of Section 3 to its proof. The proof yields in addition that (c) holds provided $T: L^1 \to X$ is a one-one operator with $T: \mathcal{P}$ closed, where $\mathcal{P}$ denotes the set of (equivalence classes of) probability densities in $L^1$. We deduce the special case that $T: \mathcal{P}$ is not closed if $T: L^1 \to c_0$ is a one-one operator in Theorem 2.9, inemploying a martingale result (Lemma 2.10) which also provides an alternate proof of Theorem 1.1'.

The proof of (c) yields the following result (Corollary 2.15): If $X$ is a closed linear subspace of $L^1$, then either $L^1$ embeds in $X$ or $l^1$ embeds in $L^1/X$. 
The entire third section is devoted to the proof of (c) of the theorem. The argument yields a considerably stronger result (by virtue of Lemma 2.6) which implies the following: Let $T : L^1 \to X$ be an operator with the following property: there is a $\delta > 0$ so that $\|Tf\| \geq \delta$ whenever $f$ is a function with $|f| = 1$ and $f$ is a sum of a sequence of disjointly supported $L^\infty$-normalized mean zero Haar functions. Then $T$ fixes a copy of $l^1$. Rosenthal [22] has shown that there exist $X$ and $T$ satisfying this hypothesis so that $X$ has an unconditional basis. Thus the hypothesis is strictly weaker than semi-embeddability of $L^1$ in $X$, and produces a new class of Banach space containing $l^1$. *

1. SEMI-EMBEDDINGS AND THE RNP

Our main structural result is as follows:

**Theorem 1.1.** Let $X$ be a separable Banach space and suppose $X$ semi-embeds in a Banach space with the RNP. Then $X$ has the RNP.

We first prove this result, then pass to several applications. We require a convenient form of the RNP (cf. Diestel and Uhl [7]). Let $L^1$ denote the usual space of equivalence classes of Lebesgue integrable functions on the unit interval. An operator (meaning bounded linear operator) $T$ from $L^1$ to a Banach space $X$ is representable provided there is a bounded strongly measurable function $\varphi : [0, 1] \to X$ with $Tf = \int f \varphi \, dt$ for all $f \in L^1$. Our formulation of the RNP: $X$ has the RNP if and only if every operator from $L^1$ to $X$ is representable. It is of interest also to consider the RNP for closed bounded convex sets of a Banach space. Let $\mathcal{P}$ denote the set of all $f \in L^1$ with $f \geq 0$ a.e. and $\int f \, dt = 1$. (Thus, $\mathcal{P}$ is the set of (equivalence classes of) probability densities on $[0, 1]$.) Let $K$ be a closed bounded convex subset of a Banach space $X$. Then $K$ has the RNP if and only if every operator $T : L^1 \to X$ with $T: \mathcal{P} \subset K$ is representable by a $K$-valued strongly measurable function $\varphi$. It follows easily from these formulations that if $X$ has the RNP, then $K$ does for every closed bounded convex subset $K$ of $X$. Indeed, suppose that $T : L^1 \to X$ is represented by $\varphi$. Then $\varphi$ is valued in $T(\mathcal{P})$ almost everywhere. One way of seeing this is to let

$$\varphi_n = \sum_{j=1}^{2^n} T(2^n \chi_{1((j-1)/2^n, j/2^n)}) \chi_{1((j-1)/2^n, j/2^n)}$$

$$= \sum_{j=1}^{2^n} \left(2^n \int_{(j-1)/2^n}^{j/2^n} \varphi \, dt \right) \chi_{1((j-1)/2^n, j/2^n)}.$$

* Note added in proof. The ideas of this paper have been developed somewhat further in the articles of H. Rosenthal entitled "Sign-Embeddings of $L^1$" (Proc. Univ. Conn. Year in Analysis 1982, in press) and "Some Results concerning Sign-Embeddings" (Functional Analysis Seminar, Paris VII, 1981–1982, in press).
Then $\phi_n \to \phi$ in $L^1(X)$; so $\phi_{n_j} \to \phi$ a.e. for some subsequence $n_1 < n_2 < \cdots$, whence $\phi$ is valued in $T^\circ \mathcal{F}$ a.e. (Of course $\phi_n \to \phi$ a.e. by the martingale convergence theorem, but this is not really needed.)

Let $Ba(X)$ equal the closed unit ball of $X$. (Of course $X$ has the RNP if and only if $Ba(X)$ does.) In view of the above comments, Theorem 1.1 is an immediate consequence of the following slightly stronger result:

**Theorem 1.1'.** Let $X$ be a separable Banach space. $Y$ a Banach space. and $T: X \to Y$ a semi-embedding such that $TBa(X)$ has the RNP. Then $X$ has the RNP.

**Proof.** We first observe that $TU$ is a Borel set for all open sets $U$. Indeed, if $W$ is a closed ball in $X$, then $TW$ is closed. Since $U$ is a countable union of open balls, $TU$ is in fact an $F_\sigma$ set. Now let $S: L^1 \to X$ be a given operator: let us assume without loss of generality that $\|S\| \leq 1$. The operator $TS$ is representable by a function $\psi$ since $TS$ is valued in the RNP-set $TBa(X)$; moreover $\psi$ can be chosen to be valued in $TBa(X)$. Now set $\tilde{\phi} = T^{-1} \psi$. (Since $T$ is one-one, $T^{-1}$ is defined on $TX$.) If $U$ is an open set, then $\phi^{-1}(U) = \psi^{-1}(TU)$ is a measurable subset of $[0, 1]$ since $TU$ is a Borel set and $\psi$ is strongly measurable. Thus $\phi$ is also strongly measurable. It remains to show that $\phi$ does indeed represent $S$. Let $f \in L^1$. We must show that

$$Sf = \int f \phi \, dt.$$  

(1)

Now since $T$ is one-one. it suffices to show that

$$TSf = T \int f \phi \, dt.$$  

(2)

But $T \int f \phi \, dt = \int fT \phi \, dt = \int f \psi \, dt$. Thus (2) holds since $TS$ is represented by $\psi$. and the proof is complete.

**Remark.** Rather than using the semi-embedding property to show that $TU$ is Borel if $U$ is open, we may instead appeal to the classical theorem of Lusin: if $K$ is a complete separable metric space, $Y$ is a metric space, and $T: K \to Y$ is a one-one continuous map, then $TU$ is a Borel subset of $Y$ if $U$ is a Borel subset of $K$ (cf. [11, p. 238]). Our proof of Theorem 1.1' thus yields the following generalization: Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ a bounded linear map. Suppose $K$ is a closed bounded convex separable subset of $X$ so that $T$ is one-one on $K$ and $TK$ is closed. Then $K$ has the RNP provided $TK$ does. (We also present an alternate proof of this in the second section, Lemma 2.10.)

We pass now to immediate consequences of the main result. Starting from
the fact that Hilbert space has the RNP, we easily deduce the standard result that separable dual Banach spaces have the RNP. We require only the following elementary result:

**Proposition 1.2.** Let $X$ be a separable Banach space. Then $X^*$ semi-embeds in $l^2$.

**Proof.** Let $x_1, x_2, \ldots$ be a countable dense subset of the unit ball of $X$. Define $S: l^2 \to X$ by $Sf = \sum_{j=1}^{\infty} (f(j)/\|x_j\|)x_j$ for all $f \in l^2$. Then $S$ is a compact operator with dense range, hence $S^*$ is a one-one (compact) operator. Since $S^*Ba(X^*)$ is closed, $S^*$ is the desired semi-embedding.

**Corollary 1.3.** Let $X$ be a separable Banach space. If $X$ semi-embeds in a separable dual space, then $X$ has the RNP.

The proof follows immediately from Theorem 1.1 and Proposition 1.2 and the fact that $l^2$ has the RNP.

There are known examples of Banach spaces which semi-embed in separable dual spaces, yet do not embed in separable duals. We wish to indicate briefly the form of these spaces (constructed by McCartney and O'Brian [17] and later by Johnson and Lindenstrauss [13]). Given $Y_1, Y_2, \ldots$, Banach spaces, $(\oplus Y_j)_{\text{**}}$ denotes the Banach space of all sequences $(y_j)$ with $y_j \in Y_j$ for all $j$ and $\|\sum y_j\| < \infty$. We require a simple result.

**Proposition 1.4.** Let $B_1, B_2, \ldots$, and $X_1, X_2, \ldots$, be sequences of Banach spaces so that $X_i$ is isomorphic to $B_i^*$ for all $i$. Let $X = (\oplus X_i)_{\text{**}}$ and $Y = (\oplus B_i^*)_{\text{**}}$. Then $X$ semi-embeds in $Y$.

**Proof.** We identify $B_i^*$ with the corresponding canonical subspace of $Y$. For each $i$, let $T_i: X_i \to B_i^*$ be a surjective isomorphism with $\|T_i\| = 1$ for all $i$. (We assume none of the $B_i$'s is the 0 space.) Define $T: X \to Y$ by $T((x_i)) = (T_i x_i)$ for all $(x_i) \in X$. We simply check that $T$ is a semi-embedding. (It is evident that $T$ is a norm-one linear operator and $T$ is one-one.) Suppose $(x^n)$ is a sequence in $Ba(X)$ with $Tx^n \to y$ as $n \to \infty$. Since each $T_i$ is an isomorphism and $T_i x^n_i \to y_i$, $x^n_i \to x_i$ for some $x_i$, as $n \to \infty$. Now fixing $k$; $\sum_{i=1}^k \|x^n_i\| \leq 1$; hence $\sum_{i=1}^k \|x_i\| \leq 1$. Thus $(x_i) \in X$ and of course $T((x_i)) = y$, so $T$ is a semi-embedding.

Now if all the $B_i^*$'s are separable, $Y$ is isometric to a separable dual space (namely the dual of $(\oplus B_i)_{\text{**}}$), and so $X$ semi-embeds in a separable dual. The examples in [13] are obtained with $B_i^* = l^1$ for all $i$; that is, the results of [13] yield examples of separable Banach spaces $X$ which semi-embed in $l^1$ yet do not embed in $l^1$ or in any separable dual.

The separable RNP-spaces nonembeddable in separable duals that are constructed in [3] are of a fundamentally different nature; these spaces do
not semi-embed in separable duals. Indeed, these spaces are \( \mathcal{L}_\gamma \)-spaces, and we have the following result:

**Proposition 1.5.** A separable infinite-dimensional \( \mathcal{L}_\gamma \)-space does not semi-embed in a separable dual.

**Proof.** We refer the reader to [15, 20] for standard facts about spaces and injective Banach spaces. Let \( X \) be a separable \( \mathcal{L}_\gamma \)-space and \( B^* \) a separable dual space. We first observe that any bounded linear operator \( T: X \to B^* \) is weakly compact. Indeed, suppose not. Let \( \pi: B^{***} \to B^* \) be a bounded linear projection. Then the map \( \pi T^{**} \) is also not weakly compact, since \( \pi T^{**}|X = T \). (Of course we identify a space \( Y \) with its canonical image in \( Y^{**} \).) Now \( X^{**} \) is an injective Banach space, hence by the results of [20], there exists a subspace \( Y \) of \( X^{**} \) isomorphic to \( l^\omega \) with \( \pi T^{**}|Y \) an isomorphism. That is, the nonseparable Banach space \( l^\omega \) embeds in \( B^* \) which is absurd. Now suppose \( T \) is a semi-embedding. Since \( T \) is weakly compact, \( TBa(X) \) is weakly compact, hence \( Ba(X) \) admits a weaker-than-norm separating locally convex topology in which it is compact. This implies \( X \) is isomorphic to a dual space, whence \( X \) is injective and so contains an isomorph of \( l^\omega \), contradicting the fact that \( X \) is separable.

Let \( \mathcal{R} \) denote the smallest class of separable Banach spaces having the following two properties:

1. \( l^2 \in \mathcal{R} \);
2. if \( Y \in \mathcal{R} \), \( X \) is separable and \( X \) semi-embeds in \( Y \), then \( X \in \mathcal{R} \).

It follows from Theorem 1.1 that \( X \in \mathcal{R} \) implies \( X \) has the RNP. Proposition 1.2 shows that \( \mathcal{R} \) contains all separable dual Banach spaces. We raised the question some time ago if in fact every separable RNP \( X \) belongs to \( \mathcal{R} \). Delbaen has answered this in the negative [6] by showing that in fact no separable \( \mathcal{L}_\omega \) space belongs to \( \mathcal{R} \). The argument follows along the lines of the proof of Proposition 1.5, requiring the following result due to Delbaen:

**Lemma.** Let \( X \) be a separable \( \mathcal{L}_\omega \)-space, \( Y \) and \( Z \) separable Banach spaces, \( U: X \to Y \) and \( V: Y \to Z \) given operators with \( V \) a semi-embedding and \( VU \) weakly compact. Then \( U \) is weakly compact.

To see this, suppose \( U \) is not weakly compact. Now the operator \( V^{**}U^{**} \) is weakly compact and in fact has its range contained in the image of \( V \), since \( V \) is a semi-embedding. Thus \( S = V^{-1}V^{**}U^{**} \) is a well-defined bounded linear operator and \( S|X = U \), so \( S \) is not weakly compact. Since \( X^{**} \) is injective, \( S \) fixes a copy of \( l^\omega \) as observed above, contradicting the assumption that \( Y \) is isomorphic to \( l^\omega \). To complete the proof that no separable \( \mathcal{L}_\omega \)-space belongs to \( \mathcal{R} \), suppose to the contrary that \( X_0 \in \mathcal{R} \). \( X_0 \) an \( \mathcal{L}_\gamma \)-space.
Then there exist separable Banach spaces $X_1, X_2, \ldots, X_n$ with $X_n$ isometric to a separable dual and semi-embeddings $T_i: X_{i-1} \to X_i$ for all $i$. By our proof of Proposition 1.5, the composed operator $T_n T_{n-1} \cdots T_1$ is weakly compact. Hence by iterating the above lemma, $T_1$ is weakly compact, which is impossible, again by our argument for 1.5.

Of course it follows that if $X \in \mathcal{I}$, then $X$ has no infinite dimensional $\ell^\infty$-subspace. We know of no counterexample to the converse; that is, is there a separable RNP space $X$ with no $\ell^\infty$-subspace, with $X$ not in $\mathcal{I}$?

We conclude this section with some variations on the concept of semi-embedding. We first wish to "remedy" the deficit that semi-embeddability of a Banach space $X$ in a Banach space $Y$ is not an isomorphic property of the space $X$. (For example, if $X$ semi-embeds in $l^2$, then $X$ must be isometric to a dual space. Thus, for example, if $X$ is isomorphic to $l^1$ but not isometric to a dual space, $X$ does not semi-embed in $l^2$.)

**Definition.** Let $X, Y$ be Banach spaces and $T: X \to Y$ a one-one operator. $T$ is called an $F_\alpha$-embedding if $TU$ is an $F_\alpha$ for all open $U \subset X$.

As we observed in the proof of Theorem 1.1', a semi-embedding of a separable Banach space $X$ is an $F_\alpha$-embedding, and of course $F_\alpha$-embeddability is an isomorphic invariant. In fact the converse is true; that is, an $F_\alpha$-embedding on $X$ is actually a semi-embedding on $X$ under an equivalent norm. Indeed, we have the following stronger elegant result due to Saint-Raymond:

**Proposition 1.6.** Let $X$ and $Y$ be Banach spaces and $T: X \to Y$ a one-one operator. Then $T$ is a semi-embedding of $X$ under an equivalent norm if and only if $TX$ is an $F_\alpha$.

We require a standard result in functional analysis, which we prefer to phrase in terms of semi-embeddings.

**Lemma 1.7.** Let $W$ be a closed bounded convex circled subset of a Banach space $Y$. Then there exists a Banach space $Z$ and a semi-embedding $S: Z \to Y$ with $SBa(Z) = W$.

**Proof of Proposition 1.6.** Let $TX = \bigcup_{j=1}^\infty K_j$ with each $K_j$ closed. By the Baire-category theorem, there is a $j$ so that $T^{-1}K_j$ has nonempty interior. Thus, we may choose an $x \in X$ and an $\varepsilon > 0$ so that $x + \varepsilon Ba(X) \subset T^{-1}(K_j)$. It follows that $\varepsilon TBa(X) \subset K_j - Tx$, a closed set in $TX$. Hence

$$W = TBa(X) \subset TX.$$

Now choose $Z$ and the semi-embedding $S: Z \to Y$ as in the lemma. Then in fact $TX = \text{linear span } W = SZ$, since $T^{-1}W$ contains $Ba(X)$. It follows that
S⁻¹T is an isomorphism mapping X onto Z. Indeed, since \( TBA(X) \subset W \) and \( S⁻¹W \subset BA(Z) \), \( S⁻¹T \) is a one-one onto operator of norm at most one, so \( S⁻¹T \) is an isomorphism by the open mapping theorem.

Thus defining \( \| \cdot \| \) on \( X \) by \( \|x\| = \|S⁻¹Tx\| \) for all \( x \in X \). \( \| \cdot \| \) is an equivalent norm on \( X \). If we let \( U \) be the closed unit ball of \( X \) under \( \| \cdot \| \). \( U = T⁻¹W \), so \( T \) is a semi-embedding on \( X \) under \( \| \cdot \| \).

Let us finally prove the lemma, using the concept of semi-embedding. We first observe that if \( X \) and \( Y \) are Banach spaces and \( T: X \to Y \) is an operator with \( TBA(X) \) closed, then \( \bar{T} \) is a semi-embedding. where \( \bar{T}: X/\ker T \to Y \) is the canonical operator with \( T = \bar{T}\pi, \pi: X \to X/\ker T \) the quotient map (and \( \ker T = \{ x \in X: Tx = 0 \} \)). In fact, we have that \( \bar{T}BA(X/\ker T) = TBA(X) \).

Now let \( W \) and \( Y \) be as in the statement of Lemma 1.7. Let \( Z \) equal the linear span of \( W \) and let \( \| \cdot \| \) be the norm on \( Z \) induced by the Minkowski functional corresponding to \( W \); that is, \( \|z\| = \inf\{|t > 0: z/t \in W\} \) for all \( z \in Z \). If we let \( S: Z \to Y \) be the identity injection, then \( SBA(Z) = W \). We must show that \( Z \) is complete. Now let \( X \) be the completion of \( Z \). Then \( S \) uniquely extends to an operator \( T: X \to Y \) with \( TBA(X) = W \) (since \( W \) is closed and bounded). Let \( \bar{T}: X/\ker T \to Y \) be the map described above. Since \( \bar{T}BA(X/\ker T) = W, \bar{T}(X/\ker T) = S(Z) \). The map \( S⁻¹\bar{T}: X/\ker T \to Z \) is thus a one-one map with \( S⁻¹\bar{T}(BA(X/\ker T)) = BA(Z) \). Hence \( S⁻¹\bar{T} \) is an isometry so since \( X/\ker T \) is a Banach space, \( Z \) is complete. (In other words, if a normed linear space \( Z \) admits a semi-embedding into a Banach space, then \( Z \) is a Banach space.)

Although the concept of \( F_\sigma \)-embedding gives the isomorphically invariant version of semi-embedding, this is not a hereditary concept. Thus suppose \( T: l^1 \to l^2 \) is a semi-embedding. If \( Y \) is a closed linear subspace of \( l^1 \) so that \( T|Y \) is an \( F_\sigma \)-embedding, then \( Y \) must be isomorphic to a dual space. Thus even though \( l^1 \) semi-embeds in \( l^2 \), there are subspaces of \( l^1 \) which do not admit an \( F_\sigma \)-embedding in \( l^2 \).

**Definition.** A one-one operator \( T: X \to Y \) between Banach spaces is called a \( G_\delta \)-embedding if \( TK \) is a \( G_\delta \) set in \( Y \) for all closed bounded \( K \) in \( X \).

Evidently the concept of a \( G_\delta \)-embedding is isomorphically invariant and hereditary. That is, if \( T: X \to Y \) is a \( G_\delta \)-embedding and \( S: Z \to X \) is an isomorphic embedding from \( Z \) to a subspace of \( X \), then \( TS \) is a \( G_\delta \)-embedding. We now have the following simple result:

**Proposition 1.8.** Let \( X \) and \( Y \) be Banach spaces and \( T: X \to Y \) a given operator. Then if \( T \) is an \( F_\sigma \)-embedding, \( T \) is a \( G_\delta \)-embedding. In particular, if \( X \) is separable and \( T \) is a semi-embedding, \( T \) is a \( G_\delta \)-embedding.

**Proof.** We may assume without loss of generality that \( T \) is a semi-
embedding, by Proposition 1.6. Let $K$ be a closed bounded subset; assume that $K \subset \text{Ba}(X)$. Then $T(\sim K)$ is an $F_\sigma$, hence $T(\sim K \cap \text{Ba}(X))$ is a relative $F_\sigma$-subset of $T\text{Ba}(X)$. Now $T(K)$ equals the complement of $T(\sim K \cap \text{Ba}(X))$ in $T\text{Ba}(X)$ since $T$ is one-one. Thus $T(K)$ is a relative $G_\delta$-subset of $T\text{Ba}(X)$. But since $T\text{Ba}(X)$ is closed, $T(K)$ is in fact a $G_\delta$-subset of $Y$.

The following open question is an attempt to characterize $G_\delta$-embeddings in terms of semi-embeddings.

**Question 1.** Let $X$ and $Y$ be separable Banach spaces and $T : X \to Y$ a $G_\delta$-embedding. Is there a separable Banach space $Z$, an isomorphic embedding $U : X \to Z$ and a semi-embedding $V : Z \to Y$ so that $T = VU$? (It is easily seen that the answer is yes if in fact there exist separable Banach spaces $\tilde{X}$ and $\tilde{Y}$, a semi-embedding $\tilde{T} : \tilde{X} \to \tilde{Y}$ and isomorphic embeddings $U : X \to \tilde{X}$ and $V : Y \to \tilde{Y}$ with $\tilde{T}U = VT$.)

We also do not know if $G_\delta$-embeddings preserve the RNP.

**Question 2.** Let $Y$ have the RNP and $X$ be a separable Banach space which admits a $G_\delta$-embedding into $Y$. Does $X$ have the RNP? (Evidently an affirmative answer to 1 implies an affirmative answer to 2.)

Finally, we do not know if RNP spaces admit $G_\delta$-embeddings into nice spaces.

**Question 3.** Let $X$ be a separable Banach space with the RNP. Does $X$ admit a $G_\delta$-embedding into $l^1$? (Evidently an affirmative answer to 3 implies a negative answer to 1.)

We conclude this section with some structural equivalences to $G_\delta$-embeddings, which will prove useful in the next sections.

**Proposition 1.9.** Let $X$ and $Y$ be separable Banach spaces and $T : X \to Y$ a one-one operator. Then the following assertions are equivalent:

1. $T$ is a $G_\delta$-embedding;
2. $T\text{Ba}(X)$ is a $G_\delta$ and for every closed bounded nonempty subset $K$ of $X$, $T^{-1}|TK$ has a point of continuity relative to $TK$; that is, there is a $k \in K$ so that if $(k_n)$ is a sequence in $K$ with $Tk_n \to Tk$, then $k_n \to k$;
3. $T\text{Ba}(X)$ is a $G_\delta$ and $T^{-1}|T\text{Ba}(X)$ is a map of the first Baire class.

The implications $(1) \Rightarrow (2)$ follow readily from the following known fundamental result concerning Polish spaces (i.e., topological spaces homeomorphic to complete separable metric spaces).

**Lemma 1.10.** Let $E$ be a Polish space, $F$ a separable metric space, and $\varphi : E \to F$ a given map. Then the following are equivalent:

...
(a) $\varphi^{-1}(W)$ is a $G_\delta$ for all closed $W \subset F$.

(b) $\varphi(W)$ has a point of continuity relative to $W$ for all closed $W \subset E$ with $W$ nonempty.

Let us see how (1) $\Leftrightarrow$ (2) of Proposition 1.9 follows from Lemma 1.10. Suppose first that (1) holds. Let $K$ be a closed bounded nonempty subset of $X$. Without loss of generality, assume $K \subset \text{Ba}(X)$. Set $E = TK$, $F = K$ and $\varphi = T^{-1} | TK$. Then $E$ is a $G_\delta$ by assumption, hence by a standard result, $E$ is a Polish space. If $W$ is a closed subset of $F$, $\varphi^{-1}W = TW$ is a $G_\delta$, hence (a) holds and $\varphi$ has a point of continuity by the lemma. Suppose (2) holds and set $E = T\text{Ba}(X)$, $F = \text{Ba}(X)$, and $\varphi = T^{-1} | T\text{Ba}(X)$. So again $E$ is a Polish space. Suppose $W$ is a (relatively) closed subset of $E$. Then letting $K = T^{-1}W$, $K$ is closed because $T$ is continuous, so (2) implies $\varphi|W$ has a point of continuity. Thus (a) holds which of course implies (1). If (3) holds, then making the same identification as immediately above, we have that (b) holds by a standard result in analysis (due to Baire) and hence (a) holds, so (1) holds. The fact that (1)$\Rightarrow$ (3) follows from a result of Banach [1] (see also [19]). (The result asserts that Lemma 1.10(a), (b) are equivalent to $\varphi$ being of the first Baire class provided $F$ is arc-wise connected.)

For the sake of completeness we give the proof of Lemma 1.10. Let $\rho$ be a complete metric on $E$ inducing the given topology; also let $\tau$ be the metric on $F$. If $\gamma = \rho$ or $\tau$ and $A$ is a subset of $E$ or $F$, $\text{diam } K = \text{sup}\{d(a, b): a, b \in A\}$. To obtain (a) $\Rightarrow$ (b), it suffices to show that

given $\varepsilon > 0$ and $V$ a nonempty open subset of $E$.

there is a nonempty open set $U$ with $U \subset V$ and $\text{diam } \varphi U < \varepsilon$. (*)

Indeed, once (*) is established, we may choose nonempty open subsets $U_1, U_2, \ldots$ of $E$ with $\overline{U}_{n+1} \subset U_n$ and $\text{diam } U_n, \text{diam } \varphi(U_n) < 1/n$ for all $n$. (Of course the $U_n$'s may be chosen to be open spheres in $E$.) It then follows that there is an $x \in \bigcap_{n=1}^{\infty} U_n$ by the completeness of $E$; this $x$ is a point of continuity for $\varphi$. If $W$ is a closed nonempty subset of $E$, we have only to apply what we have proved to $\varphi|W$.

We now prove (*). Let $\varepsilon > 0$ and $V$ be given. Now hypothesis (a) is equivalent to the assumption that $\varphi^{-1}(W)$ is an $F_\alpha$ for all open $W$. Since $F$ is separable, there exists a countable family $F_1, F_2, \ldots$ of open subsets of $F$, each of diameter less than $\varepsilon$, with $F = \bigcup_{j=1}^{\infty} F_j$. Since $\varphi^{-1}(F_j)$ is an $F_\alpha$ for all $j$, there exists a countable family $K_1, K_2, \ldots$ of closed subsets of $E$ so that $E = \bigcup_{j=1}^{\infty} K_j$ and $\text{diam } \varphi(K_j) < \varepsilon$ for all $j$. Now $V = \bigcup_{j=1}^{\infty} K_j \cap V$. Hence by the Baire-category theorem $K_j \cap V$ has nonempty interior $U$ for some $j$, which proves (*).

Assertion (b) $\Rightarrow$ (a) requires only the separability of $F$, not its completeness. Fix $V$ an open subset of $F$ and $\varepsilon > 0$. Let $G_\varepsilon$ denote the set of
all \( x \in E \) so that \( \varphi(x) \in V \) and \( \tau(y, \varphi(x)) < \varepsilon \) implies \( y \in V \). It suffices to prove

\[
\text{there exists an } F, \text{ set } W \subset E \text{ so that } G \subset W \text{ and } \varphi(W) \subset V. \quad (***)
\]

Indeed, once this is done, we have that \( \varphi^{-1}(V) = \bigcup_{n=1}^{\infty} W_{1/n} \) which of course is an \( F, \).

We now define a transfinite descending sequence of closed subsets of \( E \) as follows: Let \( K_0 = E \). Let \( 0 < \alpha < \omega_1 \) and suppose \( K_\alpha \) has been defined for all \( \xi < \alpha \). If \( \alpha \) is a limit ordinal, set \( K_\alpha = \bigcap_{\xi < \alpha} K_\xi \), otherwise, suppose \( \alpha = \beta + 1 \). Then let \( K_\alpha \) equal the set of all \( x \in K_\beta \) such that for every open neighborhood \( V \) of \( x \), there exist \( y \) and \( z \in K_\beta \cap V \) with \( \tau(\varphi(y), \varphi(z)) > \varepsilon \). Now it is evident that \( K_\alpha \) is a closed set for all \( \alpha < \omega_1 \). Moreover if \( K_\alpha \neq \emptyset \), then \( K_{\alpha+1} \neq K_\alpha \), for otherwise \( \varphi|_{K_\alpha} \) would have no points of continuity relative to \( K_\alpha \). Since \( E \) is separable, there must be an ordinal \( \alpha_0 < \omega_1 \) with \( K_{\alpha_0} = \emptyset \). It follows then that \( E = \bigcup_{0 \leq \beta < \alpha_0} K_\beta \). Now suppose \( 0 \leq \beta < \alpha_0 \) and \( x \in G \cap K_\beta \). By definition we may choose an open set \( U_x \) with \( x \in U_x \) so that if \( y, z \in K_\beta \), then \( \tau(\varphi(y), \varphi(z)) < \varepsilon \). In particular, \( \varphi(U_x \cap K_\beta) \subset V \). Now set \( H_\beta = \bigcup_{x \in G \cap K_\beta} V_x \). Then \( H_\beta \) is a relatively open subset of the closed set \( K_\beta \), hence \( H_\beta \) is an \( F, \). Finally, \( \bigcup_{0 \leq \beta < \alpha_0} H_\beta \) is the desired \( F, \) satisfying (**), completing the proof of Lemma 1.10.

Remark 1. We do not know if the hypothesis that \( TBa(X) \) is a \( G, \) can be omitted in Proposition 1.9(2).

Remark 2. Suppose the equivalent conditions of Lemma 1.10 hold. Then for any closed \( W \subset E \), \( G \) is a dense \( G, \) subset of \( W \), where \( G = \{ x \in W : x \) is a point of continuity of \( \varphi|W \} \). This (known) result follows immediately from (*) in the proof of Lemma 1.10 and the fact that the set of points of continuity of any map is a \( G, \). Hence we obtain that if the equivalent conditions of Proposition 1.9 hold and \( K \) is a closed subset of \( X \), then letting \( W = TK \) and \( G = \{ x \in W : x \) is a point of continuity of \( T^{-1}|W \} \), then \( G \) is a dense \( G, \)-subset of \( W \).

2. Semi-embeddings of \( L^1 \) (and of \( C(K) \))

We first summarize the results of Section 1 that are needed. Let \( X \) and \( Y \) be Banach spaces and \( T: X \to Y \) a one-one (bounded linear) operator. Recall that \( T \) is defined to be a \( G, \)-embedding if \( TK \) is a \( G, \) for all closed bounded sets \( K \). By Proposition 1.8, \( T \) is a \( G, \)-embedding if \( T \) is a semi-embedding and \( X \) is separable. The following tool is the only other result required for this section:
**Lemma 2.1.** Let $X$ and $Y$ be separable Banach spaces and $T: X \rightarrow Y$ a $G_{\delta}$-embedding. Then there exists an $x \in X$ with $\|x\| = 1$ so that for any sequence $(x_n)$ in $X$ with $\|x_n\| \leq 1$ and $Tx_n \rightarrow Tx$, $x_n \rightarrow x$.

**Proof.** If $T$ is an isomorphic embedding this is obvious. So suppose $T$ is not an isomorphism. By Lemma 1.10 there is an $x \in X$ with $\|x\| \leq 1$ so that $Tx$ is a point of continuity for $T^{-1}|TBaX$. If $\|x\| < 1$, $T$ would be an isomorphism contrary to our assumption. Hence $\|x\| = 1$ so the lemma is proved.

We next briefly treat $G_{\delta}$-embeddings of $C([0, 1])$. Our result here is a light generalization of one of Lotz et al. [16] (they established it for the case of semi-embeddings, by a considerably different argument).

**Proposition 2.2.** Let $X$ and $Y$ be Banach spaces and $T: X \rightarrow Y$ a $G_{\delta}$-embedding.

(a) If $X$ is isomorphic to $c_0$, $T$ is an isomorphism.

(b) If $X$ is isomorphic to $C([0, 1])$, there exists a subspace $Y$ of $X$ isomorphic to $C([0, 1])$ with $T|Y$ an isomorphism.

**Proof.** (a) We may of course assume $X = c_0$. Choose $x \in X$ satisfying the conclusion of Lemma 2.1. Then there is a $\delta > 0$ so that if $g \in c_0$, $\|g\| \leq 1$ and $\|Tg - Tx\| < \delta$, then $\|g - x\| < \frac{1}{2}$. Next choose $N$ so that $\|x(n)\| \leq \frac{1}{2}$ for all $n \geq N$.

Suppose $\varphi \in c_0$ and $\varphi(j) = 0$, $1 \leq j < N$ with $\|\varphi\| = \frac{1}{2}$. Then $g = \varphi + x$ has norm at most one and $\|g - x\| = \frac{1}{2}$. Hence $\|Tg - Tx\| = \|T\varphi\| \geq \delta$. It follows that $\|T\varphi\| \geq 2\delta \|\varphi\|$ for all $\varphi \in c_0$ with $\varphi(j) = 0$ for all $1 \leq j < N$. Since $T$ is one one, $T$ is an isomorphism.

(b) Again, assume $X = C([0, 1])$. Let $((a_i, b_j))_{i=1}^{\infty}$ be a sequence of disjoint nonempty open intervals inside $[0, 1]$ and let $Y_j = \{f \in C[0, 1]: f(x) = 0 \text{ all } x \notin (a_j, b_j)\}$ for all $j$. Then fixing $j$, $Y_j$ is isomorphic to $C([0, 1])$ (and in fact contains a subspace isometric to $C([0, 1])$). We claim that $T|Y_j$ is an isomorphism for some $j$. Suppose this were false. Then for each $j$ we could choose $\varphi_j \in Y_j$ with $\|\varphi_j\| = 1$ and $\|T\varphi_j\| \leq 1/j$. Now it is easily seen that $Z = \{\varphi_j\}$, the closed linear span of the $\varphi_j$s, is isometric to $c_0$. Since $T$ is a $G_{\delta}$-embedding, $T|Z$ is also a $G_{\delta}$-embedding. Hence $T|Z$ is an isomorphism by part (a), contradicting the fact that $\|T\varphi_j\| \rightarrow 0$.

**Remark.** We do not know the answer to the following question: Suppose $X$ is a separable $L_{\infty}$-space and $T: X \rightarrow Y$ is a semi-embedding (or a $G_{\delta}$-embedding). Is there a subspace $Z$ of $X$ with $Z$ an infinite dimensional $L_{\infty}$ space so that $T|Z$ is an isomorphism?
For the remainder of this section, we treat the case of $G_\delta$-embeddings of $L^1$. Before passing to the elements of a general theory, we wish to draw some consequences of a special case of our results.

**Theorem 2.3.** There is no $G_\delta$-embedding of $L^1$ in $c_0$.

Before presenting the proof, we give an application in harmonic analysis. Let $\mathbb{Z}$ denote the set of integers; for $\mu$ a finite complex Borel measure on $[0, 2\pi)$, let $\hat{\mu}: \mathbb{Z} \to \mathbb{C}$ be defined by $\hat{\mu}(n) = (1/2\pi) \int_0^{2\pi} e^{inx} \, d\mu(x)$ for all $n \in \mathbb{Z}$.

**Corollary 2.4** (Menchoff [18]). There exists a Borel probability measure $\mu$ on $[0, 2\pi)$ which is singular with respect to Lebesgue measure $m$ so that $\mu \in c_0(\mathbb{Z})$.

**Proof:** Define $T: L^1([0, 2\pi]) \to c_0(\mathbb{Z})$ by $Tf = \hat{\mu}$, where $d\mu = f \, dm$. Since $T$ is one-one and not a $G_\delta$-embedding by Theorem 2.3, $T$ is not a semi-embedding. Hence $T(BaL^1)$ is not closed, so there exists a sequence $(f_n)$ in $L^1$ with $\|f_n\| \leq 1$ for all $n$ and a $g \in c_0(\mathbb{Z})$ with $g \in T(BaL^1)$ so that $Tf_n \to g$ in $c_0(\mathbb{Z})$. Now it follows that there is a complex Borel measure $\mu$ with $\|\mu\| \leq 1$ so that $f_n \to \mu$ weak*- with respect to the continuous $2\pi$ periodic functions. Thus in fact $f_n \to \hat{\mu}$ uniformly and $\hat{\mu} = g$. Now $\mu$ cannot be absolutely continuous with respect to Lebesgue measure $m$, for else $\mu \in T(BaL^1)$ since $\|\mu\| \leq 1$. Thus we may choose a singular complex measure $\nu$ with $\nu \neq 0$ and an $f \in L^1$ with $d\mu = dv + f \, dm$. Thus $\nu \in c_0(\mathbb{Z})$ and $\nu \perp m$. Now a standard argument in harmonic analysis shows that $|v| \in c_0(\mathbb{Z})$ also. Indeed, there exist trigonometric polynomials $p_n$ so that $p_n \, dv \to d|v|$ in measure norm. whence $(p_n \cdot v)^* \to |v|^*$ uniformly, but of course $(p_n \cdot v)^* \in c_0(\mathbb{Z})$ for all $n$.

Thus $\lambda = |v|/\|v\|$ has the desired properties: $\lambda \perp m$ and $\lambda \in c_0(\mathbb{Z})$.

Of course the argument for Corollary 2.4 yields considerably more than Menchoff's result.

**Corollary 2.5.** Let $G$ be a compact infinite metrizable Abelian group with dual group $\Gamma$ and $\nu$ a Borel probability measure on $G$ with $\nu \in c_0(\Gamma)$ and $\hat{\nu}(\gamma) \neq 0$ for all $\gamma$. Then there is a Borel probability measure $\lambda$ with $\lambda \in c_0(\Gamma)$ so that $\lambda \perp \nu$ and $\lambda$ is in the weak*-closure of a bounded subset of $L^1(\nu)$.

Our first step in the proof of Theorem 2.3 is quite general and useful for several of our subsequent results.

**Lemma 2.6.** Let $X$ be a Banach space and $T: L^1 \to X$ be a $G_\delta$-embedding. Then there is an (into) isometry $S: L^1 \to L^1$ and a $\delta > 0$ so that

$$\|TS\phi\| \geq \delta$$

for all $\phi \in L^\infty$ with $\|\phi\|_{\infty} \leq 1$ and $\|\phi\|_1 \geq \frac{1}{2}$.
Proof. By Lemma 2.1 there is an \( f \in L^1 \) with \( \|f\| = 1 \) so that \( Tf \) is a point of continuity for \( T^{-1} \mid T(Ba(L^1)) \). By a standard result in measure theory there is an (into) isometry \( S: L^1 \to L^1 \) so that \( S1 = f \). It follows that \( TS1 \) is a point of continuity for \( (TS)^{-1} \mid TS(Ba(L^1)) \). Thus there is a \( \delta > 0 \) so that

\[
\| g - 1 \|_1 < \frac{\delta}{2} \quad \text{whenever} \quad \| g \|_1 \leq 1 \quad \text{and} \quad \| TSg - TS1 \|_1 < \delta.
\]

Now, suppose \( \varphi \) is as in the statement of Lemma 2.6 and assume first that \( \int \varphi \, dt \leq 0 \). Then \( g = 1 + \varphi \geq 0 \) and \( \int g \, dt \leq 1 \), so \( \| g \|_1 \leq 1 \). Since \( \| g - 1 \|_1 = \| \varphi \|_1 \geq \frac{1}{2} \), \( \| TS\varphi \|_1 - \| TSg - TS1 \|_1 \geq \delta \). Finally, for a general \( \varphi \), we can choose a scalar \( c \) with \( |c| = 1 \) so that \( \int c\varphi \, dt \leq 0 \). Then \( \| TS(\varphi) \| = \| TS(c\varphi) \| \geq \delta \), proving the lemma.

Remark. The ideas involved in the proof of Lemma 2.6 are similar to those in the argument of Kalton [14] showing there is no compact Hausdorff vector topology in \( L^1 \) in which the unit ball of \( L^1 \) is compact.

Before continuing with the proof of Theorem 2.3, we draw an immediate consequence of Lemma 2.6.

**Corollary 2.1.** Let \( X \) be a Banach space. If \( L^1 \) \( G\delta \)-embeds in \( X \), then \( X \) fails the Schur property. In fact if \( T: L^1 \to X \) is a \( G\delta \)-embedding, \( T \) is not a Dunford–Pettis operator.

**Proof.** Let \( T: L^1 \to X \) be a \( G\delta \)-embedding. Choose \( S \) and \( \delta \) as in Lemma 2.6. Let \( (r_j) \) denote the Rademacher functions. Then \( r_j \to 0 \) weakly, hence \( Sr_j \to 0 \) weakly, but \( \| TSr_j \| \geq \delta \) for all \( j \). Thus \( T \) is not a Dunford–Pettis operator (and of course \( TSr_j \) tends to zero weakly in \( X \), but not in norm).

**Lemma 2.8.** Let \( T: L^1 \to c_0 \) be a given operator. For all \( \varepsilon > 0 \), there exists a function \( r \) with \( |r| = 1 \) (and \( \int r \, dt = 0 \)) so that \( \| Tr \| < \varepsilon \).

Of course Theorem 2.3 follows immediately from the preceding two lemmas. Indeed, suppose \( T: L^1 \to c_0 \) were a \( G\delta \)-embedding. Choose \( \delta > 0 \) and \( S \) as in Lemma 2.6. Now applying Lemma 2.8 to the operator \( TS \), choose an \( r \) with \( |r| = 1 \) so that \( \| TSr \| < \delta \). This contradiction proves Theorem 2.3.

**Proof of 2.8.** We may assume without loss of generality that \( \| T \| \leq 1 \). Let \( (e_n) \) denote the usual basis of \( l^1 \), regarded as the dual of \( c_0 \), and let \( f_n = T^* e_n \) for all \( n \). Then of course \( f_n \to 0 \) weak* and \( \| f_n \|_\infty \leq 1 \) for all \( n \).

Now choose \( k \) so that \( 1/k < \varepsilon/2 \), and for all \( j \) with \( 1 \leq j \leq k \) set
\[ I_j = [(j - 1)/k, j/k). \]

We shall choose \( r \) of the form \( r = \sum_{j=1}^{k} r_j \), where for all \( j \),
\[
  r_j(x) = 1 \text{ or } -1, \quad \text{ if } x \in I_j, \tag{3}
\]
\[
  = 0, \quad \text{ if } x \not\in I_j, \quad \text{ and } \int r_j \, dt = 0.
\]

Now for each \( j \), \( \| r_j \|_1 = 1/k \), so \( \| Tr_j \| \leq 1/k \). We now simply choose the \( r_j \)'s so that the \( Tr_j \)'s are almost disjointly supported in \( c_0 \); thus we can insure that \( \| \sum Tr_j \| \leq 2 \max_j \| Tr_j \| < \epsilon \).

Choose \( r_1 \) arbitrarily satisfying (3). Then choose \( N_1 \) so that \( \| Tr_1(n) \| < \epsilon/4 \) for all \( n \geq N_1 \). We now claim that we can choose integers \( N_2, \ldots, N_k \) and functions \( r_2, \ldots, r_k \) so that for all \( j, 2 \leq j \leq k, \)
\[
  N_{j-1} < N_j \quad \text{and} \quad r_j \text{ satisfies Eq. (3),} \tag{4}
\]
\[
  Tr_j(n) = 0 \quad \text{if } n < N_{j-1}, \tag{5}
\]
and
\[
  \| Tr_j(n) \| < \frac{\epsilon}{2^{j+1}} \quad \text{if } n \geq N_j. \tag{6}
\]

We prove this by induction. Suppose \( 2 \leq j \leq k \) and \( N_{j-1}, r_{j-1} \) have been chosen. By the Liapunoff convexity theorem, there exists a subset \( E \) of \( I_j \) so that
\[
  \int_E f \, dt = \frac{1}{2} \int_{I_j} f \, dt \quad \text{for all } n < N_{j-1},
\]
and also
\[
  \int_E 1 \, dt = \frac{1}{2} \int_{I_j} 1 \, dt. \tag{7}
\]

Then \( r_j = \chi_E - \chi_{I_j \setminus E} \) satisfies (3) and (5). Now choose \( N_j > N_{j-1} \), so that (6) holds.

This completes the construction by induction. We now claim that \( r = \sum r_j \) has the desired properties. Evidently \( r \) is 1, \(-1\) valued almost everywhere and \( \int r \, dt = 0 \). Fix \( n \) a positive integer. If \( n \geq N_k \), then
\[
  \| (Tr)(n) \| \leq \sum_{j=1}^{k} \| Tr_j(n) \| < \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+1}} = \frac{\epsilon}{2} \quad \text{by (6)}.\]

If \( n \leq N_1 \), then \( \| Tr(n) \| = \| Tr_1(n) \| \) by (5) \( \leq \| r_1 \|_1 < \epsilon/2 \). Otherwise, choose \( 2 \leq i \leq k \) with \( N_{i-1} < n < N_i \). Then
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\[
|\text{Tr}(n)| = \left| \sum_{j=1}^{i-1} \text{Tr}_j(n) + \text{Tr}_i(n) \right| \quad \text{by (5)}
\]
\[
\leq \sum_{j=1}^{i-1} \frac{\varepsilon}{2^{j+1}} + \|r\|_1 \quad \text{by (6)}
\]
\[
\leq \frac{\varepsilon}{2} + \frac{1}{k}.
\]

Hence \( \|\text{Tr}\| \leq (\varepsilon/2) + (1/k) < \varepsilon \), proving the lemma.

Remark. We prove the considerably stronger result later that if \( T: L^1 \rightarrow X \) is a given operator and \( l^1 \) does not embed in \( X \), then the conclusion of Lemma 2.8 is satisfied (Theorem 2.12). This requires the main result of Section 3.

Recall that \( \mathcal{P} \) denotes the set of nonnegative elements of \( L^1 \) of norm one. Suppose that \( L^1 \) semi-embeds in a Banach space \( X \). It is not difficult to show then that there is a one-one operator \( T: L^1 \rightarrow X \) so that \( T\mathcal{P} \) is closed. We do not know if the converse is true. However, \( c_0 \) is not a counterexample.

Theorem 2.9. Let \( T: L^1 \rightarrow c_0 \) be a given operator such that \( T\mathcal{P} \) is one-one. Then \( T\mathcal{P} \) is not closed.

Before passing to the proof, we note that Theorem 2.9 allows us to dispense with the final piece of harmonic analysis in the proof of Corollary 2.4.

To handle Theorem 2.9, we use martingales. (For a nice treatment of martingales and some of their applications in Banach space theory, see [7]. Also see [4, 5] for further applications.) For \( \mathcal{G} \) a \( \sigma \)-subalgebra of the Lebesgue measurable subsets of \([0, 1]\), let \( \mathcal{E}_{\mathcal{G}} \) denote conditional expectation with respect to \( \mathcal{G} \). Thus if \( A \) is an atom of \( \mathcal{G} \), \( \mathcal{E}_{\mathcal{G}} f|A = (1/m(A)) \int_A f dm \).

Recall that if \( X \) is a Banach space, a sequence \( (f_n) \) of \( X \) valued functions defined on \([0, 1]\) is a martingale provided there exists an increasing sequence \( (\mathcal{G}_n) \) of sub-\( \sigma \)-algebras of the measurable subsets of \([0, 1]\) so that for all \( n \), \( f_n \) is \( \mathcal{G}_n \)-measurable and Bochner-integrable with \( \mathcal{E}_{\mathcal{G}_{n+1}} f_n = f_{n-1} \) for \( n > 1 \), where \( \mathcal{E}_n = \mathcal{E}_{\mathcal{G}_n} \) for all \( n \). We may and shall assume without loss of generality that the smallest complete \( \sigma \)-algebra containing all the \( \mathcal{G}_n \)'s coincides with the algebra of Lebesgue measurable sets.

Lemma 2.10. Let \( X \) and \( Y \) be Banach spaces with \( X \) separable and \( T: X \rightarrow Y \) a given operator. Suppose \( K \) is a closed bounded convex subset of \( X \) so that \( T(K) \) is closed and \( T|K \) is one-one. Then if \( (f_n) \) is a martingale valued in \( K \) so that \( (Tf_n) \) converges a.e., \( (f_n) \) converges a.e.
In reality, our arguments from Section 1 already establish this result. Indeed, if \((f_n)\) is a uniformly bounded \(X\)-valued martingale, then (as is well known) we may define an operator \(S : L^1 \to X\) by \(S\phi = \lim \{ \phi f_n \} dm\), the limit existing in norm. Then \(S\) is representable by \(f\), say, if and only if \((f_n)\) converges a.e., in which case \(f_n \to f\) a.e. Thus Lemma 2.10 follows from the arguments discussed in the remark following the proof of Theorem 1.2, and actually has an equivalent formulation as follows: Let \(X, Y, T, \) and \(K\) be as in Lemma 2.10. Let \(S : L^1 \to X\) be an operator with \(S \mathcal{F} \subset K\). Then \(S\) is representable if \(TS\) is representable. For the sake of completeness, we give an intrinsic argument for Lemma 2.10.

**Proof.** Let \((\mathcal{Q}_n), \mathcal{E}_n\) be the objects defined preceding the statement of the lemma. Let \(Tf_n \to g\) a.e. Since \(TK\) is closed, \(g\) is valued in \(TK\) except on a set of measure zero, so assume that \(g(t) \in TK\) for all \(t\). Now define \(f = T^{-1}g\). If \(U\) is a Borel subset of \(K\), then by Lusin's theorem, \(TU\) is a Borel set, hence \(f^{-1}(U) = g^{-1}(TU)\) is Lebesgue measurable, so \(f\) is measurable hence Bochner integrable. By the Doob martingale convergence theorem, it follows that \(\mathcal{E}_nf \to f\) a.e. To complete the proof, we only need show that

\[ \mathcal{E}_nf = f_n \quad \text{a.e. for all } n. \]  

(8)

Fix \(n\); since \(Tf_j \to g\) in \(L^1\)-norm,

\[ Tf_n = \mathcal{E}_ng. \]  

(9)

But

\[ \mathcal{E}_ng = \mathcal{E}_nTf = T\mathcal{E}_nf. \]  

(10)

Thus \(Tf_n = T\mathcal{E}_nf\) a.e., so since \(T\) is one–one, (8) is established.

**Proof of Theorem 2.9.** We shall in fact show that if \(T\) is any operator from \(L^1\) to \(c_0\), there exists a dyadic martingale \((f_n)\) valued in \(\mathcal{F}\) with \((f_n)\) divergent a.e. so that \(Tf_n\) converges a.e. Thus if \(T\mathcal{F}\) is one–one, \(T\mathcal{F}\) cannot be closed by Lemma 2.10.

For each \(n = 0, 1, 2, \ldots\) let \(\mathcal{Q}_n\) be the algebra of sets generated by \([\{(j - 1)/2^n, j/2^n\} : 1 \leq j \leq 2^n]\}. Let \(f_0 \equiv 1\). Suppose \(n \geq 1\) and \(f_{n-1}\) has been defined so that \(f_{n-1} \in \mathcal{Q}_{n-1}\) measurable and for all \(\omega \in [0, 1]\) there is a measurable set \(E_\omega\) with \(m(E_\omega) = 1/2^{n-1}\) and \(f_{n-1}(\omega) = 2^{n-1} \chi_{E_\omega}\).

Now fix \(j\) with \(1 \leq j \leq 2^{n-1}\) and set \(E = E_\omega\) where \(\omega = (j - 1)/2^{n-1}\). Applying Lemma 2.8 to \(T|L^1(E)\) where we may regard \((E, \mathcal{F} \cap E, m \cdot 2^{n-1})\) as a probability space (with \(\mathcal{F} \cap E\) the Lebesgue measurable subsets of \(E\)), we deduce that there exists a measurable function \(h = h_j\) with \(\int h dm = 0\), \(h(x) = \pm 2^{n-1}\) for all \(x \in E\), \(h(x) = 0\) for all \(x \notin E\) so that \(\|Th\| \leq 1/2^n\).
Now define \( d_n : [0, 1] \to \mathbb{R} \) by
\[
d_n = \sum_{j=1}^{2^n-1} h_j^n (\chi_{[j/2^{n}-1, 0]}(x) - \chi_{[2^{n-1}, x]}(j/2^{n-1})).
\]
Then set \( f_n = f_{n-1} + d_n \).

We thus obtain that \( (f_n) \) is valued in \( \mathbb{R}^\omega \) and for all \( n \geq 1 \) and almost all \( \omega \),
\[
\| f_n - f_{n-1}(\omega) \|_1 = 1 \quad \text{while} \quad \| (Tf_n - Tf_{n-1})(\omega) \| \leq 1/2^n.
\]
Hence \( (f_n) \) has the desired properties, completing the proof.

Remark. Theorem 2.9 may be generated still further as follows (yielding also a generalization of Theorem 2.3): Let \( T : L^1 \to c_0 \) be a one-one operator. Then \( T \) is not a \( G_\delta \).

As mentioned in the Introduction, we do not know whether \( L^1 \) embeds in a Banach space if \( L^1 \) semi-embeds in the space. However, we do know this in special cases. (Our arguments here are very simple deductions of some rather deep known results.)

**Theorem 2.11.** Let \( L^1 \) \( G_\delta \)-embed in the Banach space \( X \). Then \( L^1 \) embeds in \( X \) if either

(a) \( X \) is isomorphic to a dual space, or

(b) \( X \) itself embeds in \( L^1 \).

**Proof.** Consider first case (a). Assume without loss of generality that \( X = B^* \) for some Banach space \( B \) and let \( T : L^1 \to X \) be a \( G_\delta \)-embedding. Now it suffices to prove that \( l^1 \) embeds in \( B \) (cf. [21]). By Corollary 2.3, \( T \) is not a Dunford–Pettis operator. Choose \( (f_n) \) an \( L^1 \)-bounded sequence and \( \delta > 0 \) with \( f_n \to 0 \) weakly and \( \| T f_n \| > \delta \) for all \( n \). Choose then a bounded sequence \( (b_n) \) in \( B \) so that \( (T f_n)(b_n) > \delta \). Now if \( l^1 \) does not embed in \( B \), then by the basic result discussed in [21], \( (b_n) \) has a weak-Cauchy subsequence \( (b_{n_k}) \). Regarding \( (b_{n_k}) \subset B^{**} \), we have that \( (T^* b_{n_k}) \) is weak-Cauchy in \( L^1 \), hence \( (T f_{n_k})(b_{n_k}) = f_{n_k}(T^* b_{n_k}) \to 0 \) since \( L^1 \) has the Dunford–Pettis property, a contradiction. (We are simply reviewing here the argument for the result of Fakhoury [9] that if \( l^1 \) does not embed in \( B \), then every operator from \( L^1 \) to \( B^* \) is Dunford–Pettis.)

Case (b) is an immediate consequence of Lemma 2.6 and the following result due to Enflo and Starbird [8]: Let \( T : L^1 \to L^1 \) be a given operator. Suppose there exists a \( \delta > 0 \) so that for all \( n \), setting \( f_j = \sum_{j=1}^{2^n} r_j(\omega) f_j \) for all \( j \), \( 1 \leq j \leq 2^n \), then
\[
\int \left\| \sum_{j=1}^{2^n} r_j(\omega) f_j \right\| d\omega \geq \delta
\]
(where \((r_j)\) denotes the Rademacher sequence). Then there is a subspace \(Y\) of \(L^1\) with \(Y\) isometric to \(L^1\) so that \(T|Y\) is an isomorphism and \(TY\) is complemented in \(L^1\).

Thus suppose without loss of generality that \(X \subseteq L^1\) and let \(U: L^1 \to X\) be a \(G_*\)-embedding. By Lemma 2.6 we may choose an into isometry \(V: L^1 \to L^1\) and \(\delta > 0\) so that \(\|UV(r)\| \geq \delta\) for all measurable \(r\) with \(\|r\| \equiv 1\). Now setting \(T = UV\), and fixing \(n\) and \(\omega\), define \(r = \sum r_j(\omega)\mathcal{X}_{(j-1)/2^n, j/2^n}\). Then \(\|Tr\| = \|\sum r_j(\omega) TX_{(j-1)/2^n, j/2^n}\| \geq \delta\). Hence \(T\) satisfies the Enflo–Starbird criterion.

Choosing \(Y\) as above, we have that \(U|VY\) is an isomorphism and of course \(U(VY)\) is isomorphic to \(L^1\).

Remark. We also do not know the answer to the following question: Let \(X\) be a given Banach space. Suppose \(T: L^1 \to X\) is a semi-embedding. Is there a \(Y \subseteq L^1\) with \(Y\) isomorphic to \(L^1\) so that \(T|Y\) is an isomorphism? In joint work, Ghoussoub and Rosenthal [10] have answered this in the affirmative in the case where \(X\) \(G_*\)-embeds in \(L^1\). In fact in this case, we obtain that \(Y\) can be chosen in addition to satisfy: \(TY\) is complemented in \(X\). Thus it follows that if \(X\) \(G_*\)-embeds in \(L^1\) and \(L^1\) semi-embeds in \(X\), then \(L^1\) complementably embeds in \(X\).

Our next result may be used to extend considerably Theorems 2.3 and 2.9. It also shows that the conclusion of Lemma 2.8 holds for Banach spaces not containing \(l^1\).

**Theorem 2.12.** Let \(X\) be a Banach space and \(T: L^1 \to X\) an operator with the following property: there is a \(\delta > 0\) so that \(\|Tr\| \geq \delta\) for all measurable \(\pm 1\) valued \(r\) with \(\int r\,dt = 0\). Then there is a subspace \(Y\) of \(L^1\) with \(Y\) isomorphic to \(l^1\) so that \(T|Y\) is an isomorphism.

We prove this result in Section 3. We deduce here some immediate corollaries based on Theorem 2.12 and our previous results.

**Corollary 2.13.** Let \(X\) be a Banach space and \(T: L^1 \to X\) a \(G_*\)-embedding. There is a subspace \(Y\) of \(L^1\) isomorphic to \(l^1\) so that \(T|Y\) is an isomorphism.

Corollary 2.13 follows immediately from Lemma 2.6 and Theorem 2.12.

**Corollary 2.14.** Let \(X\) be a Banach space and \(T: L^1 \to X\) an operator so that \(T|\mathcal{P}\) is one-one and \(T\mathcal{P}\) is closed. Then the conclusion of the previous result holds.

Indeed, if not, then for all measurable sets \(E\) of positive measure, \(T|L^1(E)\) fixes no \(l^1\)-subspace, hence by Theorem 2.12, for all \(\varepsilon > 0\) there exists an \(r\)
with \(|r| = \chi_E, \int r \, dm = 0\) and \(\|Tr\| < \epsilon\). Our proof of Theorem 2.9 now produces the desired contradiction.

We conclude this section with a partial result concerning the following rather famous open question: If \(X\) is a closed linear subspace of \(L^1\), does \(L^1\) embed in \(X\) or in \(L^1/X\)?

**Corollary 2.15.** Let \(X\) be a closed linear subspace of \(L^1\). Then either \(L^1\) embeds in \(X\) or \(L^1/X\) embeds in \(L^1/X\).

**Proof.** We recall first the definition of the \(\mathcal{L}^x\)-normalized mean zero Haar functions. For \(k = 0, 1, 2, \ldots, \) and \(1 \leq j \leq 2^k\), define

\[
\hat{h}^k_j = \chi_{(j-1)/2^k, (j+1)/2^k} - \chi_{[j/2^k, 1/2^k)}.
\]

and set \(\tilde{h}^k = 2^k \hat{h}^k\). If \((\hat{h}^k_j)\) is enumerated in the “natural” way, \(\tilde{h}^1, \tilde{h}^2, \ldots, \tilde{h}^2, \ldots, \tilde{h}^4, \ldots\) we obtain a monotone basis for the mean-zero \(L^1\)-functions, a space isomorphic to \(L^1\).

Now let \(X\) be as in Corollary 2.15 and let \(T: L^1 \rightarrow L^1/X\) be the quotient map. If there exists a set \(E\) of positive measure and a \(\delta > 0\) so that \(\|Tr\| > \delta |E|\) for all measurable \(r\) with \(|r| = \chi_E\) and \(\int r \, dt = 0\), then Theorem 2.12 shows there is a subspace of \(L^1/X\) isomorphic to \(L^1\). So suppose there are no such \(E\) and \(\delta > 0\). Let \(\epsilon > 0\). We may then construct by induction a sequence of measurable sets \((E^k_j)\) with the following properties for all \(k = 0, 1, 2, \ldots, \) and \(1 \leq j \leq 2^k:\)

(a) \(|E^k_j| = 1/2^k\).

(b) \(E^k_j = E_{2^k-j-1}^1 \cup E_{2^k-j}^{k+1}\) and \(E_{2^k-j}^{k+1} \cap E_{2^k-j-1}^1 = \emptyset\).

(c) \(2^k \|Tr^k_j\| < \epsilon / 2^{2k}\), where \(r^k_j = \chi_{E^k_j} - \chi_{E_{2^k-j-1}^1}\).

Indeed, let \(E^0_1 = [0, 1]\). Fix \(k \geq 0\) and suppose disjoint sets \((E^k_j)_{j=1}^{2^k}\) have been constructed satisfying (a). For each \(j; 1 \leq j \leq 2^k\), choose \(r^k_j\) with \(|r^k_j| = \chi_E\), \(\int r^k_j \, dt = 0\) and \(2^k \|Tr^k_j\| < \epsilon / 2^{2k}\). Now let \(E_{2^k-j}^{k+1} = \{\omega: r^k_j(\omega) = 1\}\) and \(E_{2^k-j-1}^1 = \{\omega: r^k_j(\omega) = -1\}\).

Then setting \(\tilde{r}^k_j = 2^k r^k_j\) for all \(k\) and \(j\), it follows that the \(\hat{r}^k_j\)’s have the same distribution as the \(\tilde{r}^k_j\)’s, and hence \((\tilde{r}^k_j)\) is isometrically equivalent to \((\hat{r}^k_j)\) arranged in its natural order; in particular, the closed linear span of the \(\tilde{r}^k_j\)’s is isomorphic to \(L^1\). Finally, by the definition of the quotient map and (c), choose for all \(k\) and \(j\). \(x^k_j \in X\) with \(\|x^k_j - \tilde{r}^k_j\| < \epsilon / 2^{2k}\). It follows that

\[
\sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \|x^k_j - \tilde{r}^k_j\| < \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \epsilon / 2^{2k} = 2\epsilon.
\]

Thus by a standard perturbation argument, for \(\epsilon < \frac{1}{4}\), \((x^k_j)\) is also equivalent to the basis \((\tilde{h}^k_j)\), hence \(L^1\) is isomorphic to a subspace of \(X\).
3. A Class of Banach Spaces Containing $l^1$

This section is devoted to the proof of Theorem 2.12, which shows in particular that $l^1$ embeds in a Banach space $X$ provided $L^1$ semi-embeds in $X$.

We first require some preliminary definitions and notation. We let $2$ denote the set of all finite sequences of 0's and 1's. If $\alpha = \langle \alpha_1, \ldots, \alpha_k \rangle$ is in $2$, let $|\alpha| = k$. If $\alpha = \emptyset$, the empty sequence, $|\alpha| = 0$. For $\alpha, \beta \in 2$, define $\beta \geq \alpha$ if $|\beta| \geq |\alpha|$ and $\beta_i = \alpha_i$ for all $1 \leq i \leq |\alpha|$. Given a measurable real-valued function $f$ defined on $[0, 1]$, supp $f = \text{support } f = \{x : f(x) \neq 0\}$. By a tree $\mathcal{T}$ we mean a family $(h_{\alpha})_{\alpha \in \mathcal{G}}$ of $1, 0, -1$-valued measurable functions defined on $[0, 1]$ whose supports $(\text{supp } h_{\alpha})_{\alpha \in \mathcal{G}}$ have the following properties for all $\alpha \in \mathcal{G}$:

(i) $\text{supp } h_{\alpha} = \mathcal{S}$.

(ii) $\text{supp } h_{\alpha} \cap \text{supp } h_\beta = \emptyset$.

(iii) $|\text{supp } h_{\alpha}| = \frac{1}{2} |\text{supp } h_{\alpha}| > 0$ (where $|S|$ denotes the Lebesgue measure of $S$).

We refer to $\text{supp } h_{\alpha}$ as the support of $\mathcal{G}$, denoted supp $\mathcal{T}$. A function $g$ is called an elementary $\mathcal{T}$-function if $g = \sum_{i=1}^{k} \pm h_{\alpha_i}$ for some $k$, some choice of $\pm$, and choice $h_{\alpha_1}, \ldots, h_{\alpha_k}$ of disjointly supported members of $\mathcal{G}$. We say that $\mathcal{T}$ is a standard dyadic tree if the supports of the members of $\mathcal{T}$ are just the usual dyadic intervals; $[0, 1), [0, \frac{1}{2}), [\frac{1}{2}, 1), [0, \frac{1}{4}), \ldots$ The reader should note that the $L^\infty$-normalized Haar functions themselves form a tree, really our "canonical" example.

Given an $f \in L^1$ with $f \neq 0$, we set $f = f/\|f\|_1$. The main result of this section may be formulated as follows:

**Main Theorem.** Let $B$ be a Banach space, $T : L^1 \to B$ a bounded linear operator and $\mathcal{T}$ a tree. Assume there is a $\delta > 0$ so that $\|Tf\| \geq \delta$ for all elementary $\mathcal{T}$-functions $f$ with supp $f = \text{supp } \mathcal{T}$. Then there is a sequence $(g_j)$ of $1, 0, -1$-valued measurable functions with $\|g_j\| > 0$ for all $j$ so that $(Tg_j)$ is equivalent to the usual $l^1$-basis.

If we let $\mathcal{T}$ be the tree of $L^\infty$-normalized mean-zero Haar functions, we immediately obtain Theorem 2.12.

We now continue with more notation and definitions, then pass to a reformulation of the Main Theorem. Given a tree $\mathcal{T}$ and an $\epsilon > 0$, a 1, 0, $-1$-valued measurable function $g$ is called an $\epsilon$-elementary $\mathcal{T}$-function if there is an elementary $\mathcal{T}$-function $h$ with $\|h - g\| < \epsilon$. (From now on, we denote the $L^1$-norm $\|\cdot\|_1$ by $\|\cdot\|$.) $\mathcal{T}' = \{h_{\alpha} : \alpha \in \mathcal{G}\}$ is said to be related to a tree $\mathcal{T}$ if for
every \( \varepsilon > 0 \), there is a \( k \) so that if \( |\alpha| \geq k \), then \( h_\alpha \) is an \( \varepsilon \)-elementary \( \mathcal{F} \)-function. We say that \( \mathcal{F}' \) is *closely related to* \( \mathcal{F} \) if every \( h_\alpha \) is an elementary \( \mathcal{F}' \)-function. Finally we say that \( \mathcal{F}' \) is a *piece of* \( \mathcal{F} \) if there is an \( \alpha \in \mathcal{F} \) so that \( h_\beta = h_\alpha \beta \) for all \( \beta \in \mathcal{F} \).

Given \( \varphi \in L^\infty \) and \( f \in L^1 \), set \( \langle \varphi, f \rangle = \int \varphi f \, dt \). Given a nonempty subset \( M \) of the unit ball of \( L^\infty \) and \( f \in L^1 \), set \( M(f) = \sup_{m \in M} |\langle m, f \rangle| \). If \( b > 0 \), say that \( M \) *b-norms* \( f \) if \( M(f) \geq b \) (assuming \( f \neq 0 \)). We say that \( M \) *b-norms* a tree \( \mathcal{F} \) if \( M \) b-norms every elementary \( \mathcal{F} \)-function. If \( M \) b-norms \( \mathcal{F} \), then evidently \( M \) \((b - \varepsilon)\)-norms every \( \varepsilon \)-elementary \( \mathcal{F} \)-function. We say that \( M \) \( b \)-norms \( \mathcal{F} \) if there is an \( \varepsilon > 0 \) so that \( M \) \( b + \varepsilon \)-norms \( \mathcal{F} \).

Throughout our discussion, \( M \) (resp. \( \mathcal{F} \)) with or without sub or superscripts denotes a subset of the unit ball of \( L^\infty \) (resp. a tree).

It is fairly easy to show that our Main Theorem follows from the following result: If \( M \) b-norms a tree \( \mathcal{F} \) for some \( b > 0 \), then \( M \) norms an \( l^1 \)-sequence. We concentrate now on proving this result; the Main Theorem will be deduced from it at the end of the section. We have broken the proof down into several steps, and so we wish to provide the reader with a “road-map” before proceeding. The first four results are intuitively evident results concerning our definitions and normalizations. For example, Corollary 3.2 shows that relatedness of trees is a transitive concept. while Corollary 3.4 shows that the concept preserves normability by a certain set \( M \). The main result is deduced from basic known characterizations of \( l^1 \)-sequences (as discussed in [21]); the fundamental criterion is presented in Theorem 3.5. Theorem 3.6 presents the main reduction and the entire remainder of the section is devoted to its proof. Theorem 3.6 follows easily from Theorem 3.13 and the next two lemmas; the reader may prefer to jump ahead to this argument, presented after Lemma 3.15, before going back over the preliminary steps. Theorem 3.13 is in turn deduced from Corollary 3.11 and Lemma 3.12: the preceding four results are designed to set up these two steps.

We begin the proof with an intuitively clear but somewhat technical result.

**Lemma 3.1.** For every \( \varepsilon > 0 \), there is a \( \delta = \delta(\varepsilon) > 0 \) so that for any tree \( \mathcal{F} \) and \( 1, 0, -1 \)-valued measurable function \( g \), if \( g \) is a (finite or infinite) disjoint sum of \( \delta \)-elementary \( \mathcal{F} \)-functions, then \( g \) is an \( \varepsilon \)-elementary \( \mathcal{F} \)-function.

We delay the proof and pass to an important consequence: “relatedness” is transitive.

**Corollary 3.2.** Let \( \mathcal{F}, \mathcal{F}', \mathcal{F}'' \) be trees with \( \mathcal{F}' \) related to \( \mathcal{F} \) and \( \mathcal{F}'' \) related to \( \mathcal{F}' \). Then \( \mathcal{F}'' \) is related to \( \mathcal{F} \).

We shall prove 3.2, employing the following elementary fact:
PROPOSITION 3.3. Let \( f \) and \( g \) be \( 1, 0, -1 \)-valued nonzero-a.e. measurable functions and \( 0 < \epsilon < 1 \).

(a) If \( \| f - g \| < \epsilon \), then

\[
\| f - g \| < 2 \epsilon \min\{\| f \|, \| g \|\}.
\]

(b) If \( \| f - g \| < \epsilon \| f \| \), then \( \| f - g \| < 2 \epsilon \).

We shall prove Lemma 3.1 and Proposition 3.3 together, towards the end of the section. (Proposition 3.3 is deduced from Proposition 3.16; then Lemma 3.1 is proved using Lemmas 3.17 and 3.18, the final results of the section.)

**Proof of 3.2.** Let \( \mathcal{F}' = \{ h'_\alpha \}_{\alpha \in \mathcal{A}} \) and \( \mathcal{F}'' = \{ h''_\alpha \}_{\alpha \in \mathcal{A}} \). Let \( \epsilon > 0 \) be given and let \( \delta = \delta(\epsilon/2) \) from Lemma 3.1. Choose \( k \) so that

\[
h'_\alpha \text{ is } \delta - \mathcal{F}' \text{-elementary if } |\beta| \geq k.
\]  

(11)

Now choose positive constants \( c' \) and \( c'' \) so that

\[
\| h'_\alpha \| = \frac{c'}{2^{|\alpha|}} \quad \text{and} \quad \| h''_\alpha \| = \frac{c''}{2^{|\alpha|}} \quad \text{for all } \alpha.
\]  

(12)

Next choose \( m \) so that

\[
\frac{c'}{2^m} \leq \frac{(1 + \epsilon)c''}{2^m} \quad \text{implies } l \geq k.
\]  

(13)

Now choose \( n \geq m \) so that if \( |\alpha| \geq n \), then \( h''_\alpha \) is \( (\epsilon/2) - \mathcal{F}' \)-elementary. Fix \( \alpha \) with \( |\alpha| \geq n \). We wish to show that \( h''_\alpha \) is \( \epsilon - \mathcal{F}'' \)-elementary. We may choose \( h' = \Sigma \pm h'_{\beta_i} \) with the \( h'_{\beta_i} \)'s disjoinply supported so that

\[
\| h''_\alpha - h' \| \leq \epsilon/2.
\]  

(14)

By Proposition 3.3 we have that \( \| h''_\alpha - h' \| < \epsilon \| h''_\alpha \| \) or \( \| h' \| < (1 + \epsilon) \| h''_\alpha \| \).

Letting \( \beta = \beta_i \) for some \( i \), it follows by (12) that

\[
\frac{c'}{2^{|\beta|}} < (1 + \epsilon) \frac{c''}{2^{|\alpha|}} \leq \frac{(1 + \epsilon)c''}{2^m},
\]

hence \( |\beta_i| \geq k \) for all \( i \) by (13). It follows that each \( \pm h'_{\beta_i} \) is \( \delta - \mathcal{F}' \)-elementary by (11). Hence by Lemma 3.1, \( h' \) is \( (\epsilon/2) - \mathcal{F}' \)-elementary. Thus by (14), \( h''_\alpha \) is \( \epsilon - \mathcal{F}'' \)-elementary, completing the proof.

The following result is another useful consequence of Lemma 3.1.
COROLLARY 3.4. Let $M, \mathcal{E}', \mathcal{E}$, and $b > 0$ be given with $\mathcal{E}'$ related to $\mathcal{E}$. If $M b^+$-norms $\mathcal{E}$, then $M b^+$-norms a piece of $\mathcal{E}'$.

Proof. Choose $\varepsilon > 0$ so that $M b + 2\varepsilon$-norms $\mathcal{E}$. Let $\delta = \delta(\varepsilon)$ as in Lemma 3.1 and choose $k$ so that $h''_a$ is a $\delta$-elementary $\mathcal{E}$-function if $|a| \geq k$ (where $\mathcal{E}' = \{h''_a\}_{a \in \mathcal{A}}$). Fix $\alpha$ with $|\alpha| \geq k$ and let $h''_b = h''_{\alpha}$ for all $\beta \in \mathcal{A}$; thus $\mathcal{E}'' = \{h''_b\}_{b \in \mathcal{B}}$ is a piece of $\mathcal{E}'$. Now suppose $g$ is $\mathcal{E}''$-elementary. Then $g$ is $\varepsilon - \mathcal{E}$-elementary by Lemma 3.1. Thus there is an elementary $\mathcal{E}$-function $h$ with $\|h - g\| < \varepsilon$. Since $M(h) \geq b + 2\varepsilon$, $M(g) \geq b + \varepsilon$, proving Corollary 3.4.

Remark. The proof of 3.4 yields immediately the following: Suppose $M, \ldots, M, \mathcal{E}', \mathcal{E}$, and $k > 0$ are such that $\mathcal{E}'$ is related to $\mathcal{E}$ and $M b^+$-norms $\mathcal{E}$ for all $i$. Then there is a piece $\mathcal{E}''$ of $\mathcal{E}'$ such that $M_i b^+$-norms $\mathcal{E}''$ for all $i$.

We wish now to indicate the "location" of the $l^1$-sequence named by an $M$ which norms a tree. We first cite the following result, where proof follows easily from the results of [21, Sect. 2]:

THEOREM 3.5. Let $S$ be a set, $(f_n)$ a sequence of real valued uniformly bounded functions defined on $S$, $0 < a < b$ real numbers. For each $n$, set $A_n = \{s \in S : |f_n(s)| < a\}$ and $B_n = \{s \in S : |f_n(s)| > b\}$. Assume that $(A_n, B_n)$ is Boolean independent. Then $(f_n)$ has a subsequence $(f'_n)$ so that $\sup_{s \in S} \left| \sum_{j=1}^{n} c_j f'_j(s) \right| > ((b - a)/2) \sum_{j=1}^{n} |c_j|$ for all $n$ and scalars $c_1, \ldots, c_n$.

Recall that $(A_n, B_n)$ is Boolean independent provided for all $n$, setting $+B_n = B_n$ and $-B_n = A_n$, then $\cap_{i=1}^{n} A_i$ is nonempty for all choices $e_1, \ldots, e_n$ of $\pm 1$.

Now suppose that $b > 0$, $M$, $\mathcal{E}$ are such that $M b^+$-norms $\mathcal{E}$. Let $0 < a < b$. We shall prove that there exists a sequence $g_1, g_2, \ldots, g_i$ of $1, 0, -1$-valued measurable functions with the following property: For all $n$ set

$B_n = \{m \in M : |\langle m, g_n \rangle| > b\} \quad \text{and} \quad A_n = \{m \in M : |\langle m, g_n \rangle| < a\}.$

As above, set $+B_n = B_n$, $-B_n = A_n$ for all $n$. Then for all $n$.

there exists a tree $\mathcal{E}_n$ so that $\bigcap_{j=1}^{n} e_j B_j$ holds $b^+$-norms $\mathcal{E}_n$ for all choices of $e_j = \pm 1$, $1 \leq j \leq n$.

It follows immediately that $(A_n, B_n)$ is Boolean independent, hence by Theorem 3.5, some subsequence $(g'_n)$ of $(g_n)$ is equivalent to the $l^1$-basis and is normed by $M$.
We now fix $0 < a < b$. For any subset $M$ of the unit ball of $L^p$, any nonzero $g \in L^q$, set

$$M^a = \{ m \in M : |\langle m, g \rangle| > b \} \quad \text{and} \quad M_g = \{ m \in M : |\langle m, g \rangle| < a \}.$$ 

The following result is the main reduction which produces a sequence $(g_n)$ satisfying (15) for all $n$.

**Theorem 3.6.** Let $k, M_1, \ldots, M_k$ and $\mathcal{E}$ be given. Assume that $M_i b^+$-norms $\mathcal{E}$ for all $1 \leq i \leq k$. Then there exist a $1, -1, 0$-valued measurable $g$ and a tree $\mathcal{E}'$ related to $\mathcal{E}$ so that for all $i, 1 \leq i \leq k,$

$$M_i^a \quad \text{and} \quad M_i g, b^+ \text{-norm } \mathcal{E}'.$$

Let us see how Theorem 3.6 yields the desired sequence $(g_n)$. First apply 3.6 for $k = 1$ to $M$ itself, and choose $\theta$ a tree and $g_1$ so that $M g_1$ and $M, b^+ \text{-norm } \mathcal{E}_1$. (This step itself is distinctly nontrivial, as will be seen.) Suppose then $n \geq 1$, and $g_1, \ldots, g_n$ have been chosen satisfying (15). Letting $\mathcal{E}_n$ be as in (15), let $k = 2^n$ and $M_1, \ldots, M_{2^n}$ be an enumeration of the sets $\bigcap_{j=1}^{n} c_j B_j$ over all choices of $c_j = \pm 1, 1 \leq j \leq n$. Now simply choose $g_{n+1}, \mathcal{E}_{n+1}$ related to $\mathcal{E}_n$ so that $M_i g_{n+1}$ and $M_i, b^+ \text{-norm } \mathcal{E}_{n+1}$ for all $i, 1 \leq i \leq k$.

It follows that $\bigcap_{j=1}^{n+1} \epsilon_j B_j$ $b^+ \text{-norms } \mathcal{E}_{n+1}$ for all choices of $\epsilon_j = \pm 1, 1 \leq j \leq n$, completing the construction.

We now pass to the work of the proof, establishing the main reduction, Theorem 3.6. We shall delay the proofs of Lemma 3.1 and Proposition 3.3 until the end of the discussion, using the results meanwhile.

For the next result, let $\mathcal{E} = (h_n)_{a \in \mathcal{E}}$ be a tree and set $H_a = \text{supp } h_a$ for all $a$. Recall that $\text{supp } \mathcal{E} = H_\emptyset$.

**Lemma 3.7.** Let $\mathcal{E} = (h_n)_{a \in \mathcal{E}}$ and $(H_a)_{a \in \mathcal{E}}$ be as above and $E$ be a set of positive measure belonging to the $\sigma$-ring generated by the $H_a$'s. There exists a tree $\mathcal{E}'$ related to $\mathcal{E}$ with $E = \text{supp } \mathcal{E}'$.

**Proof.** Let $\mathcal{R}$ denote the $\sigma$-ring generated by the $H_a$'s. We first note that for any $\epsilon > 0$ and $F \in \mathcal{R}$ with $|F| > 0$, there exists an $\epsilon$-elementary $\mathcal{E}$-function $h$ with $\text{supp } h = F$. Indeed, we may choose $\alpha_1, \ldots, \alpha_k$ with $H_{\alpha_i} \cap H_{\alpha_j} = \emptyset$ for all $i \neq j$ and $|\bigcup_{i=1}^{k} H_{\alpha_i}| \Delta F < (\epsilon/2)|F|$. (The proof is the same as for the case of a standard dyadic tree, in which case the $H_{\alpha_i}$'s are simply disjoint dyadic intervals.) Now define $h = \sum_{i=1}^{k} h_{\alpha_i} \cdot \chi_i + \chi_G$, where $G = F \sim \bigcup_{i=1}^{k} H_{\alpha_i}$. Then

$$\left\| h - \sum_{i=1}^{k} h_{\alpha_i} \right\| = \left\| \left( \bigcup_{i=1}^{k} H_{\alpha_i} \right) \Delta F \right\| < \frac{\epsilon}{2}|F|.$$
hence \[ \| \hat{h} - \Sigma \hat{h}_a \| < 2(\varepsilon/2) = \varepsilon \] by Proposition 3.3(b). (For any sets \( E \) and \( F \). 
\( E \Delta F = (E \sim F) \cup (F \sim E) \).)

Now choose sets \( \{E_a\}_{a \in \mathcal{S}} \) belonging to \( \mathcal{S} \) so that \( E_\emptyset = E \) and for all \( a \), \( E_a = E_{a_0} \cup E_{a_1} \), \( E_{a_0} \cap E_{a_1} = \emptyset \) and \( |E_{a_0}| = \frac{1}{2} |E_a| \). By our initial observation, for each \( a \in \mathcal{S} \), we may choose a \((1/2^|a|)\)-elementary \( \mathcal{Z} \)-function \( h'_a \) with \( \text{supp } h'_a = E_a \). It follows that \( \mathcal{Z}' = \{h'_a\}_{a \in \mathcal{S}} \) is the desired tree.

Our next result shows that for example if \( M b + \varepsilon \)-norms all sums of disjointly supported Haar-functions \( h \) with \( \text{supp } h = [0, 1) \), then \( M b^- \)-norms some tree \( \mathcal{Z}' \). It will be used in several subsequent reductions.

**Lemma 3.8.** Let \( M, \mathcal{F} \) be given. Suppose for some \( \varepsilon > 0 \), that \( M b + \varepsilon \)-norms all elementary \( \mathcal{Z} \)-functions \( \varphi \) with \( \text{supp } \varphi = \text{supp } \mathcal{Z} \). Then there is a tree \( \mathcal{Z}' \) related to \( \mathcal{Z} \) so that \( M b^- \)-norms \( \mathcal{Z}' \).

**Proof.** Let \( E = \text{supp } \mathcal{Z} \): choose \( \varepsilon > 0 \) as in the statement of Lemma 3.8. We first note that

\[
M(\varphi) \geq b + \varepsilon/2 \text{ if } \varphi \text{ is an elementary } \mathcal{Z} \text{-function}
\]

with \( |\text{supp } \varphi| \geq (1 - (\varepsilon/2)) |E| \).

Indeed, let \( A = \text{supp } \varphi \); choose \( g \) an elementary \( \mathcal{Z} \)-function with \( \text{supp } g = E \sim A \). Then \( M(\varphi + g) \geq (b + \varepsilon) |E| \). But by assumption, \( \|g\| \leq (\varepsilon/2) |E| \), hence

\[
M(g) \leq \frac{\varepsilon}{2} |E|, \quad \text{so } M(\varphi) \geq \frac{M(\varphi)}{|E|} \geq b + \frac{\varepsilon}{2}.
\]

Now let \( \delta = \delta(\varepsilon/4), \delta(\varepsilon) \) being the function of Lemma 3.1 and let \( \mathcal{S} \) be the \( \sigma \)-ring generated by the supports of the members of \( \mathcal{Z} \).

By virtue of Lemma 3.7, it suffices to prove that there is an \( F \in \mathcal{S} \) of positive measure so that \( M b + (\varepsilon/4) \)-norms every \( \delta \)-elementary \( \mathcal{Z} \)-function \( g \) with \( \text{supp } g \subset F \). Indeed, once this is established, we obtain by Lemma 3.7 that there exists a tree \( \mathcal{Z}' \) related to \( \mathcal{Z} \) with \( \text{supp } \mathcal{Z}' = F \); we may of course assume that every member of \( \mathcal{Z}' \) is a \( \delta \)-elementary \( \mathcal{Z} \)-function, hence \( M b + (\varepsilon/4) \)-norms \( \mathcal{Z}' \).

Suppose there is no such \( F \). Using Zorn's lemma or a simple measure exhaustion argument, we obtain disjointly supported \( \delta \)-elementary \( \mathcal{Z} \)-functions \( g_i \) so that

\[
M(g_i) < (b + (\varepsilon/4)) \|g_i\| \text{ and supp } g_i \in \mathcal{S} \text{ for all } i.
\]

with \( \Sigma \|g_i\| = |E| \).
Thus setting \( g = \Sigma g_i \), we obtain that \( E = \text{supp} \ g \) (a.e.) and

\[
M(g) < \left( b + \frac{\varepsilon}{4} \right).
\]  

By Lemma 3.1, we may choose an elementary \( \mathcal{E} \)-function \( \varphi \) with \( \| \varphi - g \| < \varepsilon/4 \). Thus by (18), \( M(\varphi) < b + (\varepsilon/2) \). By Proposition 3.3, \( \| \varphi - g \| < (\varepsilon/2) \| g \| = (\varepsilon/2) |E| \), hence \( \| \varphi \| > (1 - (\varepsilon/2)) |E| \), so (16) is contradicted.

Our next result yields a crucial permanence property of norming sets.

**Lemma 3.9.** Let \( M \) \( b^+ \)-norm a tree \( \mathcal{E} \) and suppose \( M = \bigcup_{i=1}^{k} M_i \). Then there is a tree \( \mathcal{E}' \) related to \( \mathcal{E} \) and an \( i, 1 \leq i \leq k \) so that \( M_i \) \( b^+ \)-norms \( \mathcal{E}' \).

**Proof.** It suffices to show this for \( k = 2 \). Suppose this has been done. We then establish the result by induction on \( k \): suppose proved for \( k \geq 2 \) and \( M = \bigcup_{i=1}^{k+1} M_i \). Then there is a tree \( \mathcal{E}' \) related to \( \mathcal{E} \) so that either \( M_{k+1} \) \( b^+ \)-norms \( \mathcal{E}' \) (in which case we are done) or \( \bigcup_{i=1}^{k} M_i \) \( b^+ \)-norms \( \mathcal{E}' \). In the latter case, there is a \( \mathcal{E}'' \) related to \( \mathcal{E}' \) and an \( i, 1 \leq i \leq k \) so that \( M_i \) \( b^+ \)-norms \( \mathcal{E}'' \). By Corollary 3.2, \( \mathcal{E}'' \) is related to \( \mathcal{E} \), completing the induction step.

We now consider the case \( k = 2 \). Choose \( \varepsilon > 0 \) so that \( M \) \( b + \varepsilon \)-norms \( \mathcal{E} \). Let \( \mathcal{E} = (h_\alpha)_{\alpha \in \mathcal{E}} \) with supports \( (H_\alpha)_{\alpha \in \mathcal{E}} \). Suppose there exists an \( \alpha \) so that \( M_i \) \( b + (\varepsilon/2) \)-norms all elementary \( \mathcal{E} \)-functions \( \varphi \) with \( \text{supp} \ \varphi = H_\alpha \). Thus the hypotheses of Lemma 3.8 apply to the \( \alpha \)-th piece of \( \mathcal{E} \), hence \( M_i \) \( b^+ \)-norms some \( \mathcal{E}' \) related to \( \mathcal{E} \). If there is no such \( \alpha \), then for each \( \alpha \) we may choose an elementary \( \mathcal{E} \)-function \( h'_\alpha \) with \( M_i(h'_\alpha) < b + (\varepsilon/2) \) and \( \text{supp} \ h'_\alpha = H_\alpha \). Then in fact \( M_2 \) must \( b + \varepsilon \)-norm \( \mathcal{E}' = (h'_\alpha)_{\alpha \in \mathcal{E}} \) and of course \( \mathcal{E}' \) is closely related to \( \mathcal{E} \). Indeed, if \( \varphi \) is an elementary \( \mathcal{E}' \)-function, \( M_1(\varphi) < b + (\varepsilon/2) \), hence \( M_2(\varphi) > b + \varepsilon \) since \( (M_1 \cup M_2)(\varphi) > b + \varepsilon \).

The next result is a kind of stability result for norming sets \( M \). Essentially, by refining \( \mathcal{E} \) to \( \mathcal{E}' \) we obtain that \( M \) acts almost as a multiple of an isometry on elementary \( \mathcal{E}' \)-functions.

**Lemma 3.10.** Let \( M \) \( b^+ \)-norm \( \mathcal{E} \). There is an \( \eta > b \) so that for every \( \varepsilon > 0 \), there is a \( \mathcal{E}' \) related to \( \mathcal{E} \) so that

\[
|M(\varphi) - \eta| < \varepsilon \quad \text{for all} \quad \mathcal{E}' \text{-elementary} \ \varphi.
\]  

**Proof.** Let \( (H_\alpha)_{\alpha \in \mathcal{E}} \) be the supports of the members of \( \mathcal{E} \). For each \( \alpha \), set \( \eta_\alpha = \inf \{ M(\varphi) : \varphi \text{ is an elementary } \mathcal{E} \text{-function with } H_\alpha = \text{supp } \varphi \} \). Finally,
set $\eta = \sup_{\alpha \in \mathcal{A}} \eta_{\alpha}$. Of course $\eta > b$. Now let $\varepsilon > 0$ and set $\tau = \varepsilon/2$. Choose $a_0 \in \mathcal{A}$ with

$$\eta_{a_0} > \eta - \tau. \quad (20)$$

For each $\alpha \geq a_0$, choose $u_\alpha$ an elementary $\mathcal{F}$-function with $H_\alpha = \text{supp } u_\alpha$ and $M(u_\alpha) < \eta_\alpha + \tau$. By the definition of $\eta$, we have that

$$M(u_\alpha) < \eta + \tau. \quad (21)$$

Now let $\mathcal{H} = \{u_\alpha : \alpha \geq a_0\}$. Thus $\mathcal{H}$ is a tree: it follows from (21) that

$$M(G) < \eta + \tau \quad \text{for every } \mathcal{H}\text{-elementary } \varphi. \quad (22)$$

Now it follows from (20) that $M(\varphi) \geq \eta - \tau = (\eta - 2\tau) + \tau$ for every elementary $\mathcal{H}$-function $\varphi$ with $\text{supp } \varphi = \text{supp } \varphi'$. Hence by Lemma 3.8, there exists a tree $\mathcal{H}'$ related to $\mathcal{H}$ (and hence to $\mathcal{F}$) so that $M(\eta - 2\tau)^+\text{-norms } \mathcal{F}'$. By passing to a piece of $\mathcal{F}'$ if necessary, in virtue of Lemma 3.1 we may assume without loss of generality that every $\mathcal{F}'$-elementary function is a $\tau$-elementary $\mathcal{H}$-function. Hence by (22), $M(\varphi) < \eta + 2\tau$ for every $\mathcal{F}'$-elementary $\varphi$, completing the proof.

The next result generalizes the preceding stability lemma to the case of $k$ sets all norming a tree.

**Corollary 3.11.** Let $\mathcal{F}, M_1, \ldots, M_k$ be given so that $M_1, \ldots, M_k$ all $b^+$-norm $\mathcal{F}$. There is a $\tau > 0$ so that for all $\varepsilon > 0$ there exist a tree $\mathcal{F}'_i$ related to $\mathcal{F}$ and for all $i$, an $\eta_i > b + \tau$ so that

$$|M_i(\varphi) - \eta_i| < \varepsilon \quad \text{for all } \mathcal{F}'_i\text{-elementary } \varphi. \quad (23)$$

**Proof.** Choose $\beta > 0$ so that $M_i b + \beta$-norms $\mathcal{F}$ for all $i$. Set $\tau = \beta/2$ and let $\varepsilon > 0$. We first choose trees $\mathcal{F} = \mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k$ by induction and numbers $\eta_1, \ldots, \eta_k$ so that for all $i \geq 1$, $\eta_i > b + (\beta/2)$, $\mathcal{F}_i$ is related to $\mathcal{F}_{i-1}$, and

$$|M_i(\varphi) - \eta_i| < \frac{\varepsilon}{2} \quad \text{for all } \mathcal{F}'_i\text{-elementary } \varphi. \quad (24)$$

To see that this is possible, let $0 < i < k$ and suppose $\mathcal{F}'_i$ has been chosen. Thus by Corollary 3.2, $\mathcal{F}'_i$ is related to $\mathcal{F}$, so by Corollary 3.4 we may choose a piece $\mathcal{F}'_i'$ of $\mathcal{F}'_i$ so that $M_{i+1} (b + (\beta/2))^+\text{-norms } \mathcal{F}'_i$. By Lemma 3.10 we may now choose $\mathcal{F}'_{i+1}$ related to $\mathcal{F}'_i$ and $\eta_i$ with the desired properties.

The induction completed, we have that $\mathcal{F}'_k$ is related to $\mathcal{F}_i$ for all $i$ (by Corollary 3.2). Hence by Lemma 3.1 we may choose a piece $\mathcal{F}'$ of $\mathcal{F}_k$ so that every elementary $\mathcal{F}'$ function $\varphi$ is an $(\varepsilon/2)$-elementary $\mathcal{F}_i$-function for all $i$. Equation (23) now follows from this and (24).
The next result is a kind of generalization of Lemma 3.8 to the case of \( k \) sets all norming functions with full support. We do require an additional "stability" hypothesis which in application is fulfilled by using Corollary 3.11.

**Lemma 3.12.** Let \( r, \tau > 0 \) be given with \( \tau < 1 \). Let \( \epsilon = \tau/(2(4r + 1)) \). Assume that \( M_1, \ldots, M_r \) and \( \mathcal{F} \) are given satisfying the following hypothesis for all \( i, 1 \leq i \leq k \): there is an \( \eta_i \) with \( \eta_i \geq b + \tau \) so that for all elementary \( \mathcal{F} \)-functions \( \varphi \),

(a) \( M_i(\varphi) < \eta_i + \epsilon \),

and

(b) \( M_i(\varphi) > \eta_i - \epsilon \) provided \( \text{supp } \varphi = \text{supp } \mathcal{F} \).

Then there exists a tree \( \mathcal{F}' \) related to \( \mathcal{F} \) so that \( M_i \) \( b^+ \)-norms \( \mathcal{F}' \) for all \( i \).

**Proof.** For the sake of simplicity, assume that \( 1 = |\text{supp } \mathcal{F}| \). In analogy to the proof of Lemma 3.8, first fix \( j \), let \( M = M_j \), \( \eta = \eta_j \); also set \( \gamma = 1/2r \). We then have that

\[
M(\varphi) > \eta - (2\epsilon/\gamma) \quad \text{provided } g
\]

is an elementary \( \mathcal{F} \)-function with \( |\text{supp } g| > \gamma \) (23)

To see this, let \( g \) be an elementary \( \mathcal{F} \)-function with \( u = |\text{supp } g| > \gamma \) and now choose an elementary \( \mathcal{F} \)-function \( h \) with support disjoint from \( g \) with \( \|g\| + \|h\| = 1 \). Thus by assertion (b) applied to \( \varphi = g + h = \tilde{\varphi} \),

\[
\eta - \epsilon < M(g) + M(h)
\]

\[
= M(\tilde{\varphi}) \cdot u + M(\tilde{\varphi}) \cdot (1 - u)
\]

\[
\leq M(\tilde{\varphi}) \cdot u + (\eta + \epsilon)(1 - u) \quad \text{(by assumption (a)).}
\]

That is,

\[
\eta - \epsilon - (\eta + \epsilon)(1 - u) < M(\tilde{\varphi}) \cdot u.
\] (26)

Since the left side of (26) equals \( (\eta + \epsilon)u - 2\epsilon \), we obtain that \( M(\tilde{\varphi}) > \eta + \epsilon - (2\epsilon/\gamma) > \eta + \epsilon - (2\epsilon/\gamma) \) proving (25).

Let now \( \mathcal{R} \) be the \( \sigma \)-algebra generated by the supports of the members of \( \mathcal{F} \). Let \( \delta = \delta(\epsilon) \) from Lemma 3.1. We claim that there exists an \( E \in \mathcal{R} \) of positive measure so that for all \( i, M_i \) \( b + (\tau/2) \)-norms every \( \delta \)-elementary \( \mathcal{F} \)-function with support contained in \( E \). This will complete the proof, since in virtue of Lemmas 3.1 and 3.7 there exists a tree \( \mathcal{F}' \) related to \( \mathcal{F} \) with support contained in \( E \) so that every elementary \( \mathcal{F}' \)-function is a \( \delta \)-elementary \( \mathcal{F} \)-function.
Assuming the claim is false, by measure exhaustion we may choose a sequence \( h_1, h_2, \ldots \), of disjointly supported \( \delta \)-elementary \( \mathcal{F} \)-functions so that
\[
\sum \| h_i \| = 1
\]  
(27)
and so that for each \( i \), there is a \( j \) with
\[
M_j(h_i) < \| h_i \| (b + \tau/2).
\]  
(28)

For each \( j \), let \( G_j \) be the set of \( i \) satisfying (28). Since \( \bigcup_{j=1}^\infty G_j = N \), the set of positive integers, by (27) we have that
\[
\sum_{i \in \bigcup_{j=1}^\infty G_j} \| h_i \| = 1.
\]
hence we may choose a \( j \) so that
\[
\sum_{i \in G_j} \| h_i \| \geq \frac{1}{r}.
\]  
(29)

Now set \( h = \sum_{i \in G_j} h_i \). By Lemma 3.1, \( h \) is an \( \varepsilon \)-elementary \( \mathcal{F} \)-function and by (28) and (29),
\[
M_j(h) < \| h \| \left( b + \frac{\tau}{2} \right) \quad \text{and} \quad \| h \| \geq \frac{1}{r}.
\]  
(30)

Now choose \( g \) an elementary \( \mathcal{F} \)-function with \( \| g - h \| < \varepsilon \); so by Proposition 3.3, \( \| g - h \| < 2\varepsilon \| h \| \) or
\[
\| g \| > \| h \| (1 - 2\varepsilon) \geq \frac{1}{r} (1 - 2\varepsilon).
\]  
(31)

Now let \( M = M_j \) and \( \eta = \eta_j \); since \( (1/r)(1 - 2\varepsilon) \geq 1/2r \), we have by (25) that
\[
M(g) > \eta - 4\varepsilon \geq b + \tau - 4\varepsilon.
\]  
(32)

But by (30), \( M(g) < b + (\tau/2) + \varepsilon \). Hence \( b + \tau - 4\varepsilon < b + (\tau/2) + \varepsilon \) which implies that \( \tau < 2(4r + 1)\varepsilon \), a contradiction.

We have now arrived at a crucial stage of our discussion. Assuming the hypotheses of Theorem 3.6, we are prepared to produce a \( g \) and a \( \mathcal{F}' \) so that \( M_i^f b^+\)-norms \( \mathcal{F}' \) for all \( i \). In order to prepare the way for also obtaining that \( M_{lx} b^+\)-norms an appropriate tree, we need additional information. We label our next step a theorem rather than a lemma; its proof will require everything developed so far.
THEOREM 3.13.  Let $\mathcal{F}, M_1, ..., M_k,$ and $n$ a positive integer be given and suppose that $M_i b^+$-norms $\mathcal{F}$ for all $i$. There exists a tree $\mathcal{F}_0'$ related to $\mathcal{F}$ with the following remarkable property: Given any $n$ elementary $\mathcal{F}_0'$-functions $g_1, ..., g_n$ with supp $g_i = \text{supp } \mathcal{F}_0'$ for all $i$, there is a tree $\mathcal{F}'$ related to $\mathcal{F}$ so that $M_j^f b^+$-norms $\mathcal{F}'$ for all $i$ and $j$.

Proof: Choose $\tau > 0$ as in Corollary 3.11; assume that $r \leq 1$. Now define $\varepsilon$ by

$$\varepsilon = \frac{\tau}{6(4nk + 1)}. \tag{33}$$

Now choose $\mathcal{F}'$ and $\eta_1, ..., \eta_k$ satisfying the conclusion of Corollary 3.11. Letting $(E_\alpha)_{\alpha \in \mathcal{Q}}$ be the supports of the members of $\mathcal{F}'$, let $\mathcal{F}'_\alpha$ be the piece of $\mathcal{F}'$ corresponding to $\alpha$ for $\alpha = 0$ or $\alpha = 1$. That is, $\mathcal{F}'_\alpha = \{ h \in \mathcal{F}' : \text{supp } h \subseteq E_\alpha \}$.

Now fix $j$, $1 \leq j \leq k$, let $M = M_j$ and $\eta = \eta_j$. The following observation is absolutely crucial (but very simple to prove): Let $g$ be an elementary $\mathcal{F}_0'$ function with supp $g = E_0$. Let $h$ be an elementary $\mathcal{F}_1'$ function with supp $h = E_1$. Then there exists an $m \in M$ with $|\langle m, g \rangle | > \eta - 6 \epsilon$ and $|\langle m, h \rangle | > \eta - 6 \epsilon$.

To see this, suppose for simplicity that $|E_0| = 1$. So $|E_0| = |E_1| = \frac{1}{2}$. By (23) it follows that there is an $m \in M$ with

$$|\langle m, g + h \rangle | > \eta - \epsilon. \tag{34}$$

Again by (23), we have that

$$|\langle m, \varphi \rangle | < \frac{\eta + \varepsilon}{2} \quad \text{if } \varphi = g \text{ or } h. \tag{35}$$

Thus for example, if we had that $|\langle m, g \rangle | \leq (\eta - 3\epsilon)/2$, we would have that $|\langle m, g + h \rangle | \leq ((\eta - 3\epsilon)/2) + ((\eta + \epsilon)/2) = \eta - \epsilon$, contradicting (34).

Now suppose $g$ is an elementary $\mathcal{F}_0'$-function with supp $g = \mathcal{F}_0'$. Then if $h$ is an elementary $\mathcal{F}_1'$-function;

$$M^f(h) \leq M(h) < \eta + \epsilon \quad \text{(by (23))}$$

and

$$M^f(h) > \eta - 3\epsilon \quad \text{provided } \text{supp } h = E_1.$$  

Indeed, to see the latter inequality, choose $m \in M$ as in the observation. Since $\eta - 3\epsilon \geq b$, $m \in M^f$. Now letting $r = n \cdot k$, we have that the hypotheses of Lemma 3.12 are satisfied for the $r$-sets $M_i^f, 1 \leq i \leq n, 1 \leq j \leq k$; using $\mathcal{F}_i'$ for the "$\mathcal{F}'"$ of 3.12 and noting that $3\epsilon = r/(2(4r + 1))$. 
The conclusion of Lemma 3.12 now yields the existence of a tree $T'$ related to $P'$ so that $M_i^b$-norms $T'$ for all $i$ and $j$. Since $P'$ is related to $P$, so is $P'$, completing the proof.

The next two results yield the $g$ also working for $M_{ik}$ in Theorem 3.6.

**Lemma 3.14.** Let $n$ be a positive integer and $P$ a given tree. There exist $n$ elementary $P$-functions $g_1, \ldots, g_n$ with $\text{supp } g_i = \text{supp } P$ for all $i$ and $(g_i, g_j) = 0$ for all $j$ with $i \neq j$ (i.e., the $g_i$'s are orthogonal).

**Proof.** Choose $2^n$ disjointly supported members of $P$, $h_1, \ldots, h_{2^n}$ with $\text{supp } P = \text{supp } \sum_{i=1}^{2^n} h_i$, and $\|h_i\| = c/2^n$ for all $i$, where $\|\text{supp } P\| = c$. Let $r_1, \ldots, r_n$ be the first $n$ Rademacher functions defined on $[0, 1]$. Thus $r_i$ assumes the values $1, -1$ and if $i \neq j$, $1 \leq i, j \leq n$,

$$\frac{1}{2^n} \sum_{k=1}^{2^n} r_i \left( \frac{k}{2^n} \right) r_j \left( \frac{k}{2^n} \right) = 0 = \langle r_i, r_j \rangle. \quad (36)$$

Now define $g_j$ by $g_j = \sum_{k=1}^{2^n} r_i(k/2^n)h_k$ for all $j$. Then if $i \neq j$,

$$\int g_i g_j \, dt = \sum_k \int r_i \left( \frac{k}{2^n} \right) r_j \left( \frac{k}{2^n} \right) h_k^2 \, dt$$

$$= \frac{c}{2^n} \sum_k r_i \left( \frac{k}{2^n} \right) r_j \left( \frac{k}{2^n} \right) = 0 \quad \text{by (36).}$$

We have only one final step before being able to complete the proof of Theorem 3.6.

**Lemma 3.15.** Let $P, M_1, \ldots, M_k$ be given so that $M_i$ $b^+$-norms $P$ for all $i$. Suppose that $n \geq 1 + (k/a')$ and $g_1, \ldots, g_n$ are orthogonal $1, 0, -1$-valued functions all with the same support. There exists an $i$ and a tree $T'$ related to $P$ so that $M_i^b$-norms $P'$ for all $j$.

**Proof.** Assume without loss of generality that $|\text{supp } g_i| = 1$ for all $i$. We first do some elementary counting. Suppose $M$ is arbitrary and $G$ is a nonempty subset of $\{g_1, \ldots, g_n\}$. Now if $\phi \in M$, then

$$\sum_{\phi \in G} |\langle \phi, g \rangle|^2 \leq \|\phi\|_2^2 \leq 1 \quad (37)$$

by Bessel's inequality. Hence if $B_\phi = \{g \in G : |\langle \phi, g \rangle| \geq a\}$, then $\#B_\phi \cdot a^2 \leq 1$, so

$$\# \{g \in G : |\langle \phi, g \rangle| < a\} \geq \#G - \frac{1}{a^2}. \quad (38)$$
Now if \( H \subset \{ g_1, \ldots, g_n \} \), set

\[
M_H = \{ m \in M : \langle m, g \rangle < a \text{ for all } g \in H \}.
\]

Assuming now that \( \#G - (1/a^2) \geq 1 \), we obtain by (38) that

\[
M = \bigcup \left\{ M_H : H \subset G \text{ and } \#H \geq \#G - \frac{1}{a^2} \right\}.
\]  

(39)

Thus by Lemma 3.9, if \( \mathcal{U} \) is a tree such that \( M \) \( b^+ \)-norms \( \mathcal{U} \), \( G \) as above, there exists a tree \( \theta \) related to \( \mathcal{U} \) and an \( H \subset G \) with \( \#H \geq \#G - (1/a^2) \) so that \( M_H \) \( b^+ \)-norms \( \theta \).

We now choose by induction subsets \( G_1, \ldots, G_k \) of \( \{ g_1, \ldots, g_n \} = G_0 \) and trees \( \mathcal{E} = \mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_k \) so that for all \( i \),

\[
G_i \subset G_{i-1} \text{ and } \#G_i \geq n - \frac{i}{a^2}
\]  

(40)

\( M_{IG_i} \) \( b^+ \)-norms \( \mathcal{E}_i \) and \( \mathcal{E}_i \) is related to \( \mathcal{E}_{i-1} \).  

(41)

Indeed, suppose \( 0 < i < n \) and suppose \( G_i \) and \( \mathcal{E}_i \) chosen. By our observation following (39), we choose \( \mathcal{E}_{i+1} \) related to \( \mathcal{E}_i \) and \( G_{i+1} \subset G_i \), so that \( M_{IG_{i+1}} \) \( b^+ \)-norms \( \mathcal{E}_{i+1} \) with \( \#G_{i+1} \geq \#G_i - (1/a^2) \geq n - ((i + 1)/a^2) \) by (40).

Finally \( \#G_k \geq n - (k/a^2) \geq 1 \), so choose \( g \in G_k \). By (41), if \( 1 \leq i \leq k \), since \( M_{IG_i} \subset M_{IG_k} \), we have that \( M_{IG_k} \) \( b^+ \)-norms \( \mathcal{E}_i \). Now \( \mathcal{E}_k \) is related to \( \mathcal{E}_i \).

By Corollaries 3.2 and 3.4, we may finally choose a piece \( \mathcal{E}' \) of \( \mathcal{E}_k \) so that \( M_{IG_k} \) \( b^+ \)-norms \( \mathcal{E}' \) for all \( i \); since \( \mathcal{E}_k \) is related to \( \mathcal{E} \), so is \( \mathcal{E}' \).

We are at last prepared for the proof of the main reduction, Theorem 3.6. Fix \( n \) a positive integer with \( n \geq 1 + (k/a^2) \). Now choose \( \mathcal{E}' \) satisfying the conclusion of Theorem 3.13. By Lemma 3.14, choose \( g_1, \ldots, g_n \) orthogonal elementary \( \mathcal{F} \)-functions with \( \operatorname{supp} g_i = \operatorname{supp} \mathcal{F}_0 \) for all \( i \). Applying Theorem 3.13, choose \( \mathcal{E}'' \) so that \( M_{IG_j} \) \( b^+ \)-norms \( \mathcal{E}'' \) for all \( i \) and \( j \), and \( \mathcal{E}'' \) is related to \( \mathcal{E} \). Of course then also \( M_{IG_j} \) \( b^+ \)-norms \( \mathcal{E}'' \) for all \( j \); we apply Lemma 3.15 to obtain that for some \( i \), setting \( g = g_i \), there is a tree \( \mathcal{E}_i \) related to \( \mathcal{E}'' \) so that \( M_{IG_j} \) \( b^+ \)-norms \( \mathcal{E}_i \) for all \( j \). Finally by Corollary 3.4, we may choose a piece \( \mathcal{E}' \) of \( \mathcal{E}_i \) so that \( M_{IG_j} \) \( b^+ \)-norms \( \mathcal{E}' \) for all \( j \), completing the proof.

To complete the proof of Theorem 3.6, it remains to prove Lemma 3.1 and Proposition 3.3. The latter is a simple consequence of the following elementary result:
**Proposition 3.16.** Let $f$, $g$ be as in 3.3 and set $H = \{t: f(t) = g(t) \neq 0\}$, $F = \text{supp } f$, $G = \text{supp } g$. Then

$$\| \tilde{f} - \tilde{g} \| = 2 \left(1 - \frac{|H|}{\max\{|F|, |G|\}} \right).$$

Consequently for any $\lambda > 0$,

$$\| \tilde{f} - \tilde{g} \| < \lambda \quad \text{if and only if } |H| > (1 - (\lambda/2)) \max\{|F|, |G|\}.$$

Let us first deduce Proposition 3.3 from 3.16. Suppose first that $\| \tilde{f} - \tilde{g} \| < \epsilon$. Assume without loss of generality that $|F| \leq |G|$. Now $|H| \leq |F|$ of course, hence by the final assertion of 3.16, $|G|(1 - (\epsilon/2)) < |F|$, which yields

$$|G| - |F| \leq (1 - (\epsilon/2))^{-1} - 1) |F| \leq \epsilon |F|. \quad (42)$$

Now $\| \tilde{f} - \tilde{g} \| = \| (f/|F|) - (g/|G|) \|$ so

$$\left\| \frac{f}{|F|} - \frac{g}{|G|} \right\| < |F| \quad (43)$$

and $\| g(|F|/|G|) - g \| = |G| - |F|$, thus (42) and (43) yield that $\| f - g \| < 2\epsilon |F|$, proving Proposition 3.3(a).

Suppose now that $\| f - g \| < \epsilon \|f\|$. (We no longer assume that $|F| \leq |G|$.) Thus

$$\left\| \frac{f}{|F|} - \frac{g}{|G|} \right\| < \epsilon. \quad (44)$$

Hence $\| (f/|F|) - (g/|G|) \| = 1 - (|G|/|F|) < \epsilon$. Now $\|(g/|F|) - (g/|G|)\| = (|G|/|F|) - 1$. Combining with (44), we obtain $\| (f/|F|) - (g/|G|) \| < 2\epsilon$, proving 3.3(b).

To prove 3.16, it suffices of course to prove the equality. Suppose now that $|G| \leq |F|$ and let $W = (F \cap G) \sim H$ (so $f = -g \neq 0$ on $W$). Then

$$\| \tilde{f} - \tilde{g} \| = \int_{F \setminus G} |\tilde{f} - \tilde{g}| + \int_{G \setminus F} |\tilde{f} - \tilde{g}| + \int_{H} |\tilde{f} - \tilde{g}| + \int_{W} |\tilde{f} - \tilde{g}|$$

$$= \frac{|F \setminus G|}{|F|} + \frac{|G \setminus F|}{|G|} + |H| \left(\frac{1}{|G|} - \frac{1}{|F|}\right) + |W| \left(\frac{1}{|G|} + \frac{1}{|F|}\right)$$

$$= \frac{|F| - |W| - |H|}{|F|} + \frac{|G| - |W| - |H|}{|G|} + \frac{|H|}{|F|} + \frac{|H|}{|G|} + \frac{|W|}{|G|} + \frac{|W|}{|F|}$$

$$= 2 \left(1 - \frac{|H|}{|F|}\right),$$

proving Proposition 3.16.
We pass finally to the proof of Lemma 3.1. We fix our tree \( \mathcal{F} \). Let us say that a function \( g \) is a \( \mathcal{F} \)-function if \( g = \pm h \) for some \( h \in \mathcal{F} \). (Thus the elementary \( \mathcal{F} \)-functions are simply the sums of disjointly supported \( \mathcal{F} \)-functions.) Given \( \varepsilon > 0 \), say that a \( 1, 0, -1 \)-valued function \( g \) is an \( \varepsilon \)-function if \( \| g \| \neq 0 \) and there is a \( \mathcal{F} \)-function \( h \) with \( \| g - h \| < \varepsilon \). (So trivially an \( \varepsilon \)-function is an \( \varepsilon \)-elementary \( \mathcal{F} \)-function.)

To obtain Lemma 3.1, we require two preliminary results. The first shows that given \( \eta > 0 \), there is a \( \delta > 0 \) so that a \( \delta \)-elementary \( \mathcal{F} \)-function \( g \) can be closely approximated by sums of \( \eta \)-tree functions \( h \) with \( \text{supp} \, h \subset \text{supp} \, g \).

**Lemma 3.17.** Let \( \eta > 0 \) and let \( \delta = \eta^2/16 \). If \( g \) is a \( \delta \)-elementary \( \mathcal{F} \)-function, there exist \( k \) and disjointly supported \( \eta \)-functions \( h_1, \ldots, h_k \) with \( \text{supp} \, h_i \subset \text{supp} \, g \) for all \( i \) and \( \| g - \sum h_i \| < \eta \), where \( h = \sum_{i=1}^k h_i \).

**Proof:** Let \( G = \text{supp} \, g \) and choose disjointly supported tree functions \( f_1, \ldots, f_n \) with
\[
\| \bar{g} - f \| < \delta, \quad \text{where} \quad f = \sum f_i.
\]  
Let \( F_i = \text{supp} \, f_i \) for all \( i \), set \( F = \bigcup F_i \) and set \( I = \{ i : |F_i \cap G| > (1 - \sqrt{\delta}) |F_i| \} \). Then we claim that
\[
\sum_{i \in I} |F_i| < \frac{\sqrt{\delta}}{2} |F|.
\]  
To see this, note by Proposition 3.16 and (45) that
\[
|F \cap G| > \left( 1 - \frac{\delta}{2} \right) |F|.
\]  
For convenience, let \( |F| = c = \sum |F_i| \) and \( b = \sum_{i \in I} |F_i| \). So \( c - b = \sum_{i \in I} |F_i| \). By the definition of \( I \) we have that
\[
(1 - \sqrt{\delta}) b = (1 - \sqrt{\delta}) \sum_{i \notin I} |F_i| \geq \sum_{i \notin I} |F_i \cap G|.
\]  
By (47) we have that
\[
\sum |F_i \cap G| > \left( 1 - \frac{\delta}{2} \right) \sum |F_i| = \left( 1 - \frac{\delta}{2} \right) c.
\]  
Hence
\[
(1 - \sqrt{\delta}) b + c - b \geq \sum_{i \notin I} |F_i \cap G| + \sum_{i \in I} |F_i| \\
\geq \sum |F_i \cap G| > \left( 1 - \frac{\delta}{2} \right) c \quad \text{by (49)}
\]  
which gives \((\sqrt{\delta}/2)c > b\), proving (46).
Now let $h_i = f_i \cdot \chi_{F_i \cap G}$ for all $i \in I$. Then for $i \in I$, $\|h_i - f_i\| < \sqrt{\delta} \|f_i\|$; evidently $h_i$ is an $(\eta/2) \cdot \varepsilon$-function by Proposition 3.3. Then

$$\left\| \sum_{i \in I} h_i - \sum_{i \in I} f_i \right\| < \sqrt{\delta} \sum_{i \in I} \|f_i\| \leq \sqrt{\delta} \|f\|. \quad (50)$$

Applying (46) and setting $h = \sum_{i \in I} h_i$ we thus obtain that $\|h - f\| \leq 3(\sqrt{\delta}/2) \|f\|$, hence $\|h - f\| < 3\sqrt{\delta}$, so $\|h - g\| < \delta + 3\sqrt{\delta} < 4\sqrt{\delta} = \eta$.

To complete the proof of 3.1, we first prove the special case of disjointly supported $\delta$-tree functions.

**Lemma 3.18.** Let $1 > \varepsilon > 0$ and set $\delta = \varepsilon/9$. If $g_1, g_2, \ldots$, are disjointly supported $\delta - \varepsilon$-functions, then $\sum g_i$ is an $\varepsilon$-elementary $\varepsilon$-function.

**Proof.** We first note the following crucial combinatorial fact: Suppose $T_1, \ldots, T_n$ are supports of members of $\varepsilon$. Then for any $i$ and $j$, either $T_i \subset T_j$, $T_j \subset T_i$, or $T_i \cap T_j = \emptyset$. It follows that there is a subset $W$ of $\{1, \ldots, n\}$ with

$$\left| \bigcup_{i \in W} T_i \right| = \left| \bigcup_{i=1}^n T_i \right| \quad \text{and} \quad T_i \cap T_j = \emptyset \quad \text{all} \quad i \neq j, i, j \in W. \quad (51)$$

Now if there are infinitely many $g_i$'s, first choose an $n$ so that $\|\sum_{i=1}^n g_i - \sum g_i\| < \delta$. (This is possible since if $f_n \to f \neq 0$ in $L^1$, $f_n \to f$ in $L^1$.) Now let $g = \sum_{i=1}^n g_i$, $G_i = \text{supp } g_i$ for all $i$, and let $a = \|g\| = \sum_{i=1}^n |G_i|$. For all $i$, choose $\varepsilon$-functions $t_i$ with $\|t_i - g_i\| < \delta$ and let $T_i = \text{supp } t_i$. Then by Proposition 3.16, $|T_i \cap G_i| > (1 - (\delta/2)) |G_i|$ for all $i$. Hence

$$\left| \bigcup_{i=1}^n T_i \right| \geq \left| \bigcup_{i=1}^n T_i \cap G_i \right| = \sum_{i=1}^n |T_i \cap G_i| > \left(1 - \frac{\delta}{2}\right) \sum_{i=1}^n |G_i| = \left(1 - \frac{\delta}{2}\right) a. \quad (52)$$

Again by Proposition 3.16, $(1 - (\delta/2)) |T_i| < |G_i|$ for all $i$, hence

$$\sum_{i=1}^n |T_i| < \left(\frac{1}{1 - (\delta/2)}\right) \sum_{i=1}^n |G_i| \leq (1 + \delta)a. \quad (53)$$

Now choose $W$ a subset of $\{1, \ldots, n\}$ satisfying (51). It follows from (51)–(53) that then

$$\sum_{i \in W} |T_i| < \left[ (1 + \delta) - \left(1 - \frac{\delta}{2}\right) a = \frac{3}{2} \delta a. \quad (54)$$
Hence since also $|G_i| < (1/(1 - (\delta/2))) |T_i| \leq (1 + \delta) |T_i|$ for all $i$,

$$\sum_{i \in W} |G_i| < (1 + \delta) \sum_{i \in W} |T_i| < (1 + \delta) \cdot \frac{1}{2} \delta a < 2\delta a. \quad (55)$$

Again by Proposition 3.3, $\|t_i - g_i\| < 2\delta |G_i|$ for all $i$.

Combining this with (55), we obtain

$$\left\| \sum_{i \in W} t_i - g \right\| \leq \sum_{i \in W} \|t_i - g_i\| + \sum_{i \in W} |G_i| < 4\delta a. \quad (56)$$

Hence $\left\| \sum_{i \in W} t_i - g \right\| < 8\delta$, so $\left\| \sum_{i \in W} t_i - \sum_i g_i \right\| < 9\delta$, completing the proof.

We are finally prepared for the completion of the proof of Lemma 3.1. Let $\varepsilon > 0$. Set $\eta = \varepsilon/13$ and let $\delta = \eta^2/16$. Let $g_1, g_2, \ldots$, be disjointly supported $\delta$-elementary $\mathcal{F}$-functions and let $g = \sum g_i$. By Lemma 3.17, for each $i$ we may choose disjointly supported $\eta - \mathcal{F}$-functions $h_{ij}$ with $\text{supp } h_{ij} \subset \text{supp } g_i$ for all $j$ and

$$\left\| g_i - \sum_{j} h_{ij} \right\| < \eta. \quad (57)$$

Now it follows that the $h_{ij}$'s are disjointly supported over all $i$ and $j$. Hence by Lemma 3.18, there is an elementary $\mathcal{F}$-function $h$ with

$$\left\| h - \sum_{i,j} h_{ij} \right\| < 9\eta. \quad (58)$$

Now by (57) and Proposition 3.3, $\| g_i - \sum_j h_{ij} \| < 2\eta \|g_i\|$ for all $i$, hence $\| \sum_i g_i - \sum_{i,j} h_{ij} \| < 2\eta \sum \|g_i\| - 2\eta \|g\|$, so again by Proposition 3.3,

$$\left\| g - \sum_{i,j} h_{ij} \right\| < 4\eta. \quad (59)$$

Thus by (58) and (59), $g$ is a $13\eta$-elementary $\mathcal{F}$-function, completing the proof.

Remark. The proof shows that there is an absolute constant $c(c^{-1} \leq 16(13)^2)$ so that $\delta(e) = c\varepsilon^2$, $\delta(e)$ the function of Lemma 3.1.

We conclude this section with a proof of the Main Theorem stated at the beginning. Let $T, B, \mathcal{F}$, and $\delta$ be as in its hypotheses. We may assume without loss of generality that $\|T\| = 1$. Identifying $(L^1)^*$ with $L^\infty$, let $U$ be the unit ball of $B^*$ and set $M = T^*U$. It follows immediately that $M \delta$-norms $\phi$ for every $\mathcal{F}$-elementary function $\phi$ with $\text{supp } \phi = \text{supp } \mathcal{F}$. Now fix $0 < b < \delta$. We obtain by Lemma 3.8 that there exists a tree $\mathcal{F}'$ so that $M b^+$-
norms $F'$. Now fix $a < b$. As shown above, it follows by Theorems 3.5 and 3.6 that there exists a sequence $(g_j)$ of 1, 0, -1-valued measurable functions so that for all $n$ and choices $c_1, \ldots, c_n$ of scalars,

$$M\left(\sum_{j=1}^{n} c_j g_j\right) \geq \frac{b-a}{2} \sum_{j=1}^{n} |c_j|.$$  

Thus we obtain immediately that $(Tg_j)$ is equivalent to the usual $l^1$-basis. (Of course, the proof thus shows that assuming $\|T\| = 1$, given $\delta' < \delta/2$, $(g_i)$ may be chosen with $(Tg_j)$ $(\delta')^{-1}$-equivalent to the usual $l^1$-basis.)

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