State-Space model of a mechanical system in MATLAB/Simulink

Peter Sivák*, Darina Hroncová

Technical University of Košice, Faculty of Mechanical Engineering, Letná 9, 042 00 Košice, Slovak republic

Abstract

This paper describes solution of the equations of motion of the mechanical system by using State-Space blocks in MATLAB/Simulink. It deals with the mechanical system with two degrees of freedom. State-Space block solution is compared with solution made by an alternative approach, using so called Transfer Fcn block. Equations are also derived by Newton’s second law, Lagrange’s equations and the Hamilton’s equations.

Keywords: State Space, Transfer Function, kinetic energy, potential energy, Matlab, Simulink.

Nomenclature

\( m_1, m_2 \) masses
\( k_1, k_2 \) stiffness of the springs
\( b_1, b_2 \) coefficients of viscous damping
\( H \) Hamilton’s function
\( E_k \) kinetic energy
\( E_p \) potential energy
\( D \) Rayleigh’s dissipative function
\( p_i \) momentum of the \( i \)-th member
\( x \) state vector
\( y \) output vector
\( u \) input vector
\( A \) state matrix
\( B \) input matrix
\( C \) output matrix
\( D \) feedforward matrix

1. Introduction

In the first part we derived the equations of motion of the mechanical system with two degrees of freedom using the...
second Newton’s law. Next the these equations are derived by Lagrange’s equations of the second kind. Finally we
determine the state equations by using Hamilton’s equations [2]. The aim is to describe the use of State-Space blocks and
Transfer Fcn of the dynamic system in Matlab/Simulink. The obtained results are compared with direct solution in Matlab.
The results are presented in graphical form.

2. Newton’s Equations

In this section we describe the compilation of equations of motion of the mechanical system with two degrees of freedom
with Lagrange equations of the second kind and the Hamilton equations. These equations are then solved in Matlab with
Runge-Kutta method. The results provide information about displacement, velocity and acceleration of individual members
of the mechanical system as a function of time. A harmonically variable force \( F(t) \) is used for the model excitation (Fig. 1).

![Damped mass-spring system with two degrees of freedom.](image)

Fig. 1. Damped mass-spring system with two degrees of freedom.

We consider a mechanical system with two degrees of freedom of movement (Fig. 1), which consists of bodies with
masses \( m_1 \) and \( m_2 \) connected with springs with stiffnesses \( k_1 \) and \( k_2 \) and dampers with linear damping coefficients \( b_1 \) and \( b_2 \)
connected to a rigid frame [5-10]. The system performs linear motion in direction of springs and dampers axes. The weights
of the springs are not considered. A harmonically variable force \( F(t) = F_0 \sin(\omega t) \) is used for the model excitation. The
respective members perform linear forced oscillating motion. Equations of the motion:

\[
m_1 \ddot{x}_1 = -F_{r1} - F_{d1} + F_{r2} + F_{d2} + F(t),
\]

\[
m_2 \ddot{x}_2 = -F_{r2} - F_{d2},
\]

reaction forces in the springs

\[
F_{r1} = k_1 x_1, \quad F_{r2} = k_2 (x_2 - x_1),
\]

damping forces

\[
F_{d1} = b_1 \dot{x}_1, \quad F_{d2} = b_2 (\dot{x}_2 - \dot{x}_1).
\]

The aim is to determine the response of the oscillating system - displacements \( x_1=x_1(t), x_2=x_2(t) \). This, however, requires
to solve a system of differential equations. For a mechanical system with two degrees of freedom it is a nonhomogeneous
system of 2nd order linear differential equations with constant coefficients.

3. Lagrange’s equations

Next we determine the equations of motion of the above mentioned system using Lagrange’s equations of the second
kind. For this we need to determine kinetic and potential energy of the system and the Rayleigh dissipative function.

If we define the generalized coordinates as \( q_1 = x_1, q_2 = x_2 \), and the generalized velocities as \( \dot{q}_1 = \dot{x}_1, \dot{q}_2 = \dot{x}_2 \), the
kinetic energy of the system is:

\[
E_k = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2,
\]

potential energy

\[
E_p = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2,
\]

Rayleigh’s dissipative function

\[
D = \frac{1}{2} b_1 \dot{x}_1^2 + \frac{1}{2} b_2 (\dot{x}_2 - \dot{x}_1)^2,
\]

generalized forces

\[
Q_{b1} = F(t), \quad Q_{b2} = 0.
\]
Substituting the generalized coordinates the potential energy of the system:

\[ E_p = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 = \frac{1}{2} k_1 x_1^2 - k_2 x_1 x_2 + \frac{1}{2} k_2 x_2^2, \]  

(9)

Rayleigh’s dissipative function

\[ D = \frac{1}{2} b_1 \dot{x}_1^2 + \frac{1}{2} b_2 (\ddot{x}_2 - \dot{x}_1)^2 = \frac{1}{2} b_1 \dot{x}_1^2 + \frac{1}{2} b_2 \dot{x}_2^2 - b_2 \dot{x}_1 \ddot{x}_2 + \frac{1}{2} b_2 \dot{x}_2^2. \]  

(10)

Lagrange’s equations of the second kind for the mechanical system (Fig. 2) are:

\[ \frac{d}{dt} \left( \frac{\partial E_k}{\partial x_i} \right) - \frac{\partial E_k}{\partial \dot{x}_i} = - \frac{\partial E_p}{\partial x_i} - \frac{\partial D}{\partial \dot{x}_i} + Q_{bi}, \]  

(11)

\[ \frac{d}{dt} \left( \frac{\partial E_k}{\partial \dot{x}_i} \right) - \frac{\partial E_k}{\partial x_i} = - \frac{\partial E_p}{\partial \dot{x}_i} - \frac{\partial D}{\partial \ddot{x}_i} + Q_{bi}. \]  

(12)

After substitution of the partial derivations we obtain equations of motion in the form:

\[ \ddot{x}_1 = - \frac{b_1}{m_1} \dot{x}_1 + \frac{b_2}{m_1} (\dot{x}_2 - \dot{x}_1) - \frac{k_1}{m_1} x_1 + \frac{k_2}{m_1} (x_2 - x_1) + \frac{1}{m_1} F(t), \]  

(13)

\[ \ddot{x}_2 = - \frac{b_2}{m_2} (\dot{x}_2 - \dot{x}_1) - \frac{k_2}{m_2} (x_2 - x_1), \]  

(14)

where \( \dot{x}_1, \dot{x}_2 \) are velocities and \( \ddot{x}_1, \ddot{x}_2 \) are accelerations of the masses \( m_1 \) and \( m_2 \). These equations are linear differential equations of second order with constant coefficients.

4. Hamilton's equations

The Hamilton’s function \( H = H(q, p) \), which is the sum of kinetic and potential energies can be used to determine the equations of motion of the mechanical system with two degrees of freedom. It has a form:

\[ H = E_k + E_p, \]  

(15)

\[ H = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 - k_2 x_1 x_2 + \frac{1}{2} k_2 x_2^2. \]  

(16)

Then Hamilton's equations:

\[ \frac{\partial \dot{x}_i}{\partial t} = \frac{\partial H}{\partial p_i}, \]  

(17)

\[ \frac{\partial p_i}{\partial t} = - \frac{\partial H}{\partial x_i} + Q_i + Q_{bi}. \]  

(18)

State variables of the mechanical system from Fig. 1 are the displacements \( x_1, x_2 \) and momentum \( p_1, p_2 \). The kinetic energy expressed by the momentum is:

\[ E_{k_i} = \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2} m_i \left( \frac{p_i}{m_i} \right)^2 = \frac{p_i^2}{2m_i}, \]  

(19)

potential energy
\[ E_p = \frac{1}{2} k_i x_i^2 . \]  

Rayleigh's dissipative function

\[ D = \frac{1}{2} b_i x_i^2 . \]

Generalized linear force for viscous damping is expressed by Rayleigh's dissipative function relationship:

\[ Q_i = -\frac{\partial D}{\partial x_i}, \quad i=1,2 \]

generalized forces

\[ Q_{b_i} = F(t), \quad Q_{b_2} = 0 . \]

Equations of the mechanical system are obtained in the form:

\[ \frac{d}{dt} x_i(t) = \frac{\partial H}{\partial p_i}, \quad i=1,2 \]

\[ \frac{d}{dt} x_1(t) = \frac{\partial H}{\partial p_1} , \]

\[ \frac{d}{dt} x_2(t) = \frac{\partial H}{\partial p_2} , \]

\[ \frac{d}{dt} p_1(t) = -\frac{\partial H}{\partial x_1} + Q_1 + Q_{b_1} , \]

\[ \frac{d}{dt} p_2(t) = -\frac{\partial H}{\partial x_2} + Q_2 + Q_{b_2} . \]

In matrix form:

\[
\begin{bmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt} \\
\frac{dp_1(t)}{dt} \\
\frac{dp_2(t)}{dt}
\end{bmatrix}
= \begin{bmatrix}
x_1(t) \\
x_2(t) \\
p_1(t) \\
p_2(t)
\end{bmatrix}
+ \begin{bmatrix}
F(t) \\
0
\end{bmatrix},
\]

state vector is

\[ X = [x_1(t), x_2(t), p_1(t), p_2(t)]^T . \]

Equations of the mechanical system in matrix form for \( p_i(t) = m_i \cdot v_i(t) \), \( i=1,2 \) are:

\[
\begin{bmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt} \\
\frac{dv_1(t)}{dt} \\
\frac{dv_2(t)}{dt}
\end{bmatrix}
= \begin{bmatrix}
x_1(t) \\
x_2(t) \\
v_1(t) \\
v_2(t)
\end{bmatrix}
+ \begin{bmatrix}
F(t) \\
0
\end{bmatrix} \]
and state vector
\[
X = [x_1(t), x_2(t), v_1(t), v_2(t)]^T.
\]

We have determined the equations of state of a mechanical system with two degrees of freedom for the state variables \(x_1, x_2\) and \(v_1, v_2\) of respective objects with masses \(m_1, m_2\).

5. State Space equation in MATLAB/Simulink

Solution of the nonhomogenious system of differential equations of a mechanical system with two degrees of freedom is first done in Matlab/Simulink using State-Space and Transfer Fcn blocks [7], [2].

In control engineering, a state space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations. To abstract from the number of inputs, outputs and states, the variables are expressed as vectors. Additionally, if the dynamical system is linear and time invariant, the differential and algebraic equations may be written in matrix form. The state space representation (also known as the "time-domain approach") provides a convenient and compact way to model and analyze systems with multiple inputs and outputs. Unlike the frequency domain approach, the use of the state space representation is not limited to systems with linear components and zero initial conditions. "State space" refers to the space whose axes are the state variables. The state of the system can be represented as a vector within that space.

The most general state-space representation of a linear system with \(u\) inputs, \(y\) outputs and \(n\) state variables is written in the following form (Fig. 2):

\[
\begin{align*}
\dot{x}(t) &= A \cdot x(t) + B \cdot u(t), \\
y(t) &= C \cdot x(t) + D \cdot u(t).
\end{align*}
\]

We obtain:
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{v}_1 \\
\dot{v}_2
\end{bmatrix} = A \cdot \begin{bmatrix}
x_1 \\
x_2 \\
v_1 \\
v_2
\end{bmatrix} + B \cdot \begin{bmatrix}
F \\
0
\end{bmatrix},
\]
\[
y = C \cdot \begin{bmatrix}
x_1 \\
x_2 \\
v_1 \\
v_2
\end{bmatrix} + D \cdot \begin{bmatrix}
F \\
0
\end{bmatrix}.
\]

\(A, B, C, D\) are the respective matrices of the mechanical system defined as follows:
The input $u(t)$ is represented by the excitation forces:

$$u(t) = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}.$$  

(37)

And the output $y(t)$ is represented by the displacements $x_1(t)$ and $x_2(t)$:

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ v_1(t) \\ v_2(t) \end{bmatrix}.$$  

(38)

Parameters used in the process of solution (Fig. 3) of the mechanical system were: $m_1=70 \text{ kg}$, $m_2=140 \text{ kg}$, $k_1=500 \text{ N/m}$, $k_2=250 \text{ N/m}$, $b_1=10 \text{ N/(m.s)}$, $b_2=50 \text{ N/(m.s)}$, $F_0=100 \text{ N}$, $\omega=2 \text{ rad.s}^{-1}$, initial conditions $x(0)=0 \text{ m}$, $v(0)=0 \text{ m/s}$, [1-4].

6. Transfer Fcn model

The obtained State-Space representation of the mechanical system can be in the Matlab easily transformed to an equivalent Transfer Fcn model (Fig. 5) by ss2tf function. The syntax of this command in Matlab is:
Solving the mechanical system by using State-Space equations and the Transfer Function (Fig. 6) gave the same results, as we anticipated.

7. Conclusion

The paper describes the compilation of the equations of motion of a mechanical system with two degrees of freedom in Matlab/Simulink by using State - Space and Transfer function.

Results are shown graphically. The calculation is done for a model with force excitation in Matlab and Simulink, to illustrate the methodology.

The contribution of this work is primarily educational, especially in the field of Applied Mechanics and Mechatronics. The above procedure presents the possibility of practical implementation of this solution to simple equations of motion of a mechanical system in Matlab/Simulink.

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