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Nilmanifolds with a calibrated G_2-structure

Diego Conti [a], Marisa Fernández [b,*]

[a] Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via Cozzi 53, 20125 Milano, Italy

[b] Universidad del País Vasco, Facultad de Ciencia y Tecnología, Departamento de Matemáticas, Apartado 644, 48080 Bilbao, Spain

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ABSTRACT

We introduce obstructions to the existence of a calibrated G_2-structure on a Lie algebra $\mathfrak{g}$ of dimension seven, not necessarily nilpotent. In particular, we prove that if there is a Lie algebra epimorphism from $\mathfrak{g}$ to a six-dimensional Lie algebra $\mathfrak{h}$ with kernel contained in the center of $\mathfrak{g}$, then $\mathfrak{h}$ has a symplectic form. As a consequence, we obtain a classification of the nilpotent Lie algebras that admit a calibrated G_2-structure.

1. Introduction

A Riemannian 7-manifold with holonomy contained in G_2 can be characterized by the existence of an associated parallel 3-form. The first examples of complete metrics with holonomy G_2 were given by Bryant and Salamon in [4], and the first examples of compact manifolds with such a metric were constructed by Joyce in [16]. Explicit examples on solvable Lie groups were constructed in [5]; examples on nilpotent Lie groups can be obtained by taking a nilpotent six-dimensional group with a half-flat structure and solving Hitchin's evolution equations (see [9,15]). More generally, one can consider G_2-structures where the associated 3-form φ is closed: then φ defines a calibration [14], and the G_2-structure is said to be *calibrated*. An equivalent condition is that the intrinsic torsion lies in the component $\mathcal{X}_2 \cong \mathfrak{g}_2$ [10].

Compact calibrated G_2-manifolds have interesting curvature properties. It is well known that a G_2 holonomy manifold is Ricci-flat, or equivalently, both Einstein and scalar-flat. On a compact calibrated G_2-manifold, both the Einstein condition [6] and scalar-flatness [3] are equivalent to the holonomy being contained in G_2. In fact, [3] shows that the scalar curvature is always non-positive.

Constructing examples is not a straightforward task. For instance, [7] classifies calibrated G_2-manifolds on which a simple group acts with cohomogeneity one, and no compact manifold occurs in this list. On the other hand, the second author exhibited the first example of a compact calibrated G_2-manifold that does not have holonomy G_2 [11]. This example is given in terms of a nilpotent Lie algebra $\mathfrak{g}$ and an element of $\Lambda^3\mathfrak{g}^*$ that corresponds to a closed left-invariant 3-form on the associated simply-connected Lie group. Since the structure constants are rational, there exists a uniform discrete subgroup [18]; the quotient, called a nilmanifold, has an induced calibrated G_2-structure.

In this paper we pursue this approach, and classify the nilpotent 7-dimensional Lie algebras that admit a calibrated G_2-structure. Since the structure constants turn out to be rational, each Lie algebra determines a nilmanifold. So, we obtain

* Corresponding author.
E-mail addresses: diego.conti@unimib.it (D. Conti), marisa.fernandez@ehu.es (M. Fernández).

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12 compact calibrated G_2 nilmanifolds (see Theorem 4, Lemmas 5 and 6). Three of them are reducible: they are the product of a circle with a symplectic half-flat nilmanifold, the latter being classified in [8]. The remaining nine are new.

The proof is based on two necessary conditions that a Lie algebra must satisfy for a calibrated G_2-structure to exist (see Proposition 1 and Lemma 3).

Our first obstruction is related to a construction of [1]. Suppose that M is a 7-manifold with a calibrated G_2 form φ, and X is a unit Killing field, i.e. $\mathcal{L}_X\varphi = 0$. Then if $\eta = X^\flat$, we can write

$$\varphi = \omega \wedge \eta + \psi^+,$$

where ω, $\psi+$ and $d\eta$ are basic forms with respect to the 1-dimensional foliation defined by X. Suppose in addition that X is the fundamental vector field of a free S^1 action; then basic forms can be identified with forms on M/S^1. By

$$0 = \mathcal{L}_X\varphi = d(X \lrcorner \varphi) = d\omega,$$

ω is a symplectic form on M/S^1. Moreover

$$0 = \omega \wedge d\eta + d\psi^+$$

implies that $[d\eta]$ is in the kernel of the map

$$H^2\big(M/S^1\big) \to H^4\big(M/S^1\big), \quad [\beta] \to [\beta \wedge \omega].$$

If this map is an isomorphism, then the S^1-bundle is trivial: this puts topological restrictions on M, which translate to algebraic conditions in our setup. A similar method was used in [8].

In principle, these restrictions reduce our problem to the classification of symplectic nilpotent Lie algebras of dimension six for which the map $H^2 \to H^4$ is non-injective (see the remark before Lemma 3). The complexity of the required calculations, however, motivate a different approach. In analogy with [9], we introduce a second obstruction, that requires computing the space of closed 3-forms. It consists in the observation that $(X \lrcorner \varphi)^3$ must be nonzero, whenever X is a nonzero vector and φ a 3-form defining a G_2-structure.

The final ingredient is Gong's classification of 7-dimensional indecomposable nilpotent Lie algebras [12]. This list contains 140 Lie algebras and 9 one-parameter families; in addition, there are 35 decomposable nilpotent Lie algebras [17,19]. Calculations on a case-by-case basis show that our list of 12 examples is complete.

2. Calibrated G_2-structures and obstructions

In this section we show obstructions to the existence of a calibrated G_2 form on a Lie algebra (not necessarily nilpotent). First, we recall some definitions and results about G_2-structures.

Let us consider the space $\mathbb{O}$ of the Cayley numbers, which is a non-associative algebra over $\mathbb{R}$ of dimension 8. Thus, we can identify $\mathbb{R}^7$ with the subspace of $\mathbb{O}$ consisting of pure imaginary Cayley numbers. Then, the product on $\mathbb{O}$ defines on $\mathbb{R}^7$ the 3-form given by

$$e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245} \tag{1}$$

(see [10] and [13] for details), where $\{e^1, \dots, e^7\}$ is the standard basis of $(\mathbb{R}^7)^*$. Here, e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. The group G_2 is the stabilizer of (1) under the standard action of $GL(7, \mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$. G_2 is one of the exceptional Lie groups, and it is a compact, connected, simply connected simple Lie subgroup of $SO(7)$ of dimension 14.

A G_2-structure on a 7-dimensional Riemannian manifold (M, g) is a reduction of the structure group $O(7)$ of the frame bundle to G_2. Manifolds admitting a G_2-structure are called G_2-manifolds. The existence of a G_2-structure on (M, g) is determined by a global 3-form φ (the G_2 form) which can be locally written as (1) with respect to some (local) basis $\{e^1, \dots, e^7\}$ of the (local) 1-forms on M. We say that the G_2-manifold M has a *calibrated G_2-structure* if there is a G_2-structure on M such that the 3-form φ is closed, and so φ defines a calibration [14].

If G is a 7-dimensional Lie group with Lie algebra $\mathfrak{g}$, then a G_2-structure on G is left-invariant if and only if the corresponding 3-form is left-invariant. Thus, a left-invariant G_2-structure on G corresponds to an element φ of $\Lambda^3\mathfrak{g}^*$ that can be written as (1) with respect to some coframe $\{e^1, \dots, e^7\}$ on $\mathfrak{g}^*$; we shall say that φ defines a G_2-structure on $\mathfrak{g}$. We say that a G_2-structure on $\mathfrak{g}$ is *calibrated* if φ is closed, i.e.

$$d\varphi = 0,$$

where d denotes the Chevalley–Eilenberg differential on $\mathfrak{g}^*$. If Γ is a discrete subgroup of G, a G_2-structure on $\mathfrak{g}$ induces a G_2-structure on the quotient $\Gamma\backslash G$. Moreover, in [18] it is proved that if $\mathfrak{g}$ is nilpotent with rational structure constants, then the associated simply connected Lie group G admits a uniform discrete subgroup Γ. Therefore, a G_2-structure on $\mathfrak{g}$ determines a G_2-structure on the compact manifold $\Gamma\backslash G$, which is called a compact nilmanifold; and if $\mathfrak{g}$ has a calibrated G_2-structure, the G_2-structure on $\Gamma\backslash G$ is also calibrated.

In order to show obstructions to the existence of a calibrated G_2 form on a Lie algebra $\mathfrak{g}$, let us consider first a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. If φ is a G_2 form on $\mathfrak{g}$, and the decomposition is orthogonal with respect to the underlying metric, then

$$\varphi = \omega \wedge \eta + \psi^+,$$

where ω, ψ^+ are forms on $\mathfrak{h}$ and η generates the dual of the ideal $\mathbb{R}$. The pair (ω, ψ^+) defines an SU(3)-structure on $\mathfrak{h}$. The condition that φ is closed is equivalent to both ω and ψ^+ being closed; this means that the SU(3)-structure is *symplectic half-flat*. There are exactly three nilpotent Lie algebras of dimension six that admit a symplectic half-flat structure, classified in [8]. So, if we focus our attention on decomposable nilpotent Lie algebras, there are at least three 7-dimensional Lie algebras with a calibrated G_2-structure; we will see that these are all.

More generally, every 7-dimensional nilpotent Lie algebra fibres over a nilpotent Lie algebra of dimension six. In fact if ξ is in the center of $\mathfrak{g}$, then the quotient $\mathfrak{g}/\mathrm{Span}\{\xi\}$ has a unique Lie algebra structure that makes the projection map

$$\mathfrak{g} \to \frac{\mathfrak{g}}{\mathrm{Span}\{\xi\}}$$

a Lie algebra morphism. Moreover, due to the nilpotency assumption every epimorphism $\mathfrak{g} \to \mathfrak{h}$, with $\mathfrak{h}$ of dimension six, is of this form. Using the pullback, we can identify forms on the quotient with *basic forms* on $\mathfrak{g}$; in this setting, α is *basic* if $\xi \lrcorner \alpha = 0$.

Given a G_2-structure on $\mathfrak{g}$ with associated 3-form φ and a nonzero vector ξ in the center, let $\eta = \xi^\flat$; then we can write

$$\varphi = \omega \wedge \eta + \psi^+, \qquad \xi \lrcorner \omega = 0 = \xi \lrcorner \psi^+,$$

and up to a normalization coefficient the forms (ω, ψ^+) define an SU(3)-structure on the six-dimensional quotient (see also [1]). In analogy with the case of a circle bundle, we shall think of η as a connection form, and $d\eta$ as the curvature.

Proposition 1. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with a calibrated G_2-structure and a non-trivial center. If $\pi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra epimorphism with kernel contained in the center, and $\mathfrak{h}$ of dimension six, then $\mathfrak{h}$ admits a symplectic form ω, and the curvature form is in the kernel of*

$$H^2(\mathfrak{h}^*) \xrightarrow{\cdot\wedge\omega} H^4(\mathfrak{h}^*). \tag{2}$$

If the curvature form is exact on $\mathfrak{h}$, then $\mathfrak{g} \cong \mathfrak{h} \oplus \mathbb{R}$ as Lie algebras.

Proof. Write

$$\varphi = \pi^*\omega \wedge \eta + \pi^*\psi^+$$

where (ω, ψ^+) are forms on $\mathfrak{h}$. Since d commutes with the pullback,

$$0 = d\varphi = d\pi^*\omega \wedge \eta + \pi^*\omega \wedge d\eta + \pi^* d\psi^+,$$

where $\pi^* d\omega$, $d\eta$ and $\pi^* d\psi^+$ are basic. Thus ω is a symplectic form and $d\eta$ is in the kernel of (2).

Now suppose that $d\eta$ is exact on $\mathfrak{h}$. Then, the epimorphism $\pi : \mathfrak{g} \to \mathfrak{h}$ is trivial, that is $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. More precisely, we can choose a different, closed connection form $\tilde\eta$, and $\mathfrak{g} = \ker\tilde\eta \oplus \ker\pi$ is a direct sum of Lie algebras; by construction, $\ker\tilde\eta$ is isomorphic to $\mathfrak{h}$. □

Remark. In the previous proposition, we must notice that when the curvature form is zero, (ω, ψ^+) is a symplectic half-flat structure on $\mathfrak{h}$. Therefore, if $\mathfrak{h}$ is nilpotent, by [8] $\mathfrak{h}$ is one of

$$(0, 0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 12, 13), \quad (0, 0, 0, 12, 13, 23).$$

With notation from [19], $(0, 0, 0, 0, 12, 13)$ represents a the Lie algebra with a fixed basis $e^1, \dots, e^6$ of $\mathfrak{g}^*$, satisfying

$$de^1 = 0 = de^3 + de^3 = de^4, \qquad de^5 = e^{12}, \qquad de^6 = e^{13}.$$

Remark. Another obstruction to the existence of a calibrated G_2-structure on a nilpotent Lie algebra is given by the condition $b_3 > 0$. Indeed, if φ is a closed G_2 form on a nilpotent Lie algebra $\mathfrak{g}$, and X is a nonzero vector in the center of $\mathfrak{g}$, then $\mathcal{L}_X\varphi = 0$, so $X \lrcorner \varphi$ is closed. If φ were exact, say $\varphi = d\beta$, then the 7-form

$$(X \lrcorner \varphi) \wedge (X \lrcorner \varphi) \wedge \varphi = d\big((X \lrcorner \varphi) \wedge (X \lrcorner \varphi) \wedge \beta\big)$$

would also be exact, hence zero, which is absurd. On the other hand, b_3 is always positive on a nilpotent Lie algebra of dimension seven.

Proposition 1 motivates the following definition. We say that a 6-dimensional Lie algebra $\mathfrak{h}$ satisfies the 2-*Lefschetz property* if, for every symplectic structure on $\mathfrak{h}$, the map (2) is an isomorphism. This condition holds trivially when $\mathfrak{h}$ has no symplectic structure, namely when $\mathfrak{h}$ is one of

$$(0,0,0,12,23,14+35), \qquad (0,0,0,12,23,14-35),$$
$$(0,0,0,12,13,14+35), \qquad (0,0,0,0,12,15+34),$$
$$(0,0,0,0,0,12+34), \qquad (0,0,12,13,14+23,34+52),$$
$$(0,0,12,13,14,34+52), \qquad (0,0,0,12,14,24).$$

It is well known [2] that if $(\mathfrak{h},\omega)$ is a 6-dimensional, symplectic nilpotent Lie algebra, the map

$$H^1(\mathfrak{h}^*) \xrightarrow{\cdot\wedge\omega^2} H^5(\mathfrak{h}^*)$$

is not surjective. However, in the next proposition, we prove that some of those Lie algebras satisfy the 2-Lefschetz property.

Proposition 2. *Among* 6*-dimensional nilpotent Lie algebras with a symplectic structure, those that satisfy the* 2*-Lefschetz property are*

$$(0,0,0,0,0,0); \quad (0,0,12,13,23,14); \quad (0,0,12,13,23,14+25).$$

Proof. In the abelian case, the bilinear map

$$H^2 \otimes H^2 \to H^4$$

induced by the wedge product is non-degenerate, in the sense that for every nonzero $\beta \in H^2$, the induced linear map $\cdot \wedge \beta : H^2 \to H^4$ is an isomorphism.

For the second Lie algebra, the cohomology class of a generic symplectic form is represented by

$$\omega = \lambda_1 e^{16} + \lambda_2(e^{15} + e^{24}) + \lambda_3 e^{25} + \lambda_4(e^{34} - e^{26});$$

non-degeneracy implies $\lambda_4 \neq 0$. The map $H^2 \to H^4$ of (2) is represented by the matrix

$$\begin{pmatrix} \lambda_3 & 2\lambda_4 & \lambda_1 & 2\lambda_2 \\ \lambda_4 & 0 & 0 & \lambda_1 \\ 0 & 0 & 0 & -2\lambda_4 \\ 0 & 0 & \lambda_4 & \lambda_3 \end{pmatrix}$$

which is invertible by the assumption $\lambda_4 \neq 0$.

Similarly, for the last Lie algebra

$$\omega = \lambda_1 e^{14} + \lambda_2(e^{15} + e^{24}) - \lambda_3(e^{26} - e^{34}) + \lambda_4(e^{16} + e^{35}).$$

The map (2) is represented by

$$\begin{pmatrix} \lambda_3 & 2\lambda_4 & \lambda_1 & 2\lambda_2 \\ -\lambda_4 & 2\lambda_3 & 2\lambda_2 & -\lambda_1 \\ 0 & 0 & -2\lambda_3 & 2\lambda_4 \\ 0 & 0 & -\lambda_4 & -\lambda_3 \end{pmatrix},$$

which is invertible unless $\lambda_4^2 + \lambda_3^2 = 0$, which makes ω degenerate.

For all but three of the remaining Lie algebras, we observe that the bilinear map

$$H^2 \otimes H^2 \to H^4$$

is degenerate in the sense that, for every nonzero $\beta \in H^2$, the map

$$\alpha \to \alpha \wedge \beta, \qquad H^2 \to H^4$$

is non-injective. The three exceptions are

$$(0,0,12,13,23,14-25), \quad (0,0,0,12,13,23), \quad (0,0,0,0,0,12).$$

However, either Lie algebra has a symplectic form that makes the map (2) non-injective. In fact, on the Lie algebra $\mathfrak{h}$ defined by the equations $(0,0,12,13,23,14-25)$, consider the symplectic form

$$\omega = -e^{16} + e^{15} + e^{35} + e^{34} + e^{24} - e^{26}.$$

Then one can check that $e^{14} + e^{25} + e^{15} + e^{24}$ defines a non-trivial class in $H^2(\mathfrak{h}^*)$, but

$$\left(e^{14} + e^{25} + e^{15} + e^{24}\right) \wedge \omega = 2e^{1245} = 2d\left(e^{146}\right).$$

Now, on the Lie algebra $(0, 0, 0, 12, 13, 23)$ we consider the symplectic form $\omega = e^{14} + e^{26} + e^{35}$. Then,

$$\left(-e^{15} - e^{24} + e^{36}\right) \wedge \omega = d\left(e^{456}\right);$$

finally, on the Lie algebra $(0, 0, 0, 0, 0, 12)$,

$$\left(e^{16} + e^{25} + e^{34}\right) \wedge e^{13} = -de^{356}. \qquad \square$$

In principle, one could try to classify all pairs $(\mathfrak{h}, \omega)$, with $\mathfrak{h}$ nilpotent of dimension six and ω a symplectic form on $\mathfrak{h}$, for which (2) is non-injective. This means that $\omega \wedge \gamma = d\psi^+$, for some $\psi^+ \in \Lambda^3\mathfrak{h}^*$ and some closed non-exact 2-form $\gamma \in \Lambda^2\mathfrak{h}^*$. If in addition, (ω, ψ^+) are compatible in the sense that they define an SU(3)-structure, then declaring $de^7 = \gamma$ one obtains a 7-dimensional Lie algebra $\mathfrak{g}$ with a calibrated G_2-structure. By Proposition 1, all calibrated G_2-structures on indecomposable nilpotent Lie algebras are obtained in this way.

However, these calculations turn out to be difficult (although in one dimension less a similar approach was pursued succesfully in [8]), and for this reason we shall use a different method (see Section 4), starting with Gong's classification of 7-dimensional Lie algebras. In fact, given a Lie algebra, it is straightforward to compute the space of its closed 3-forms. In the spirit of [9], the existence of a calibrated G_2-structure puts restrictions on this space. Whilst straightforward, the following result turns out to give an effective obstruction.

Lemma 3. *Let $\mathfrak{g}$ be a 7-dimensional nilpotent Lie algebra. If there is a nonzero X in $\mathfrak{g}$ such that $(X \lrcorner \phi)^3 = 0$ for every closed 3-form on $\mathfrak{g}$, then $\mathfrak{g}$ has no calibrated G_2-structure.*

Proof. Obvious. $\square$

Remark. When $\mathfrak{g}$ fibers over a non-symplectic Lie algebra $\mathfrak{h}$, this obstruction is satisfied automatically. Indeed, suppose $\pi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra epimorphism; then any closed 3-form on $\mathfrak{g}$ can be written as

$$\pi^*\omega \wedge \eta + \pi^*\psi^+,$$

as in the proof of Proposition 1. So ω is a closed form on $\mathfrak{h}$; if we assume $\mathfrak{h}$ has no symplectic form, then $\omega^3 = 0$. Then the condition of Lemma 3 is satisfied with X a generator of $\ker \pi$.

3. Decomposable case

In this section we classify the decomposable nilpotent Lie algebras with a calibrated G_2-structure. Indeed, we prove:

Theorem 4. *Among the 35 decomposable nilpotent Lie algebras of dimension 7, those that have a calibrated G_2-structure are*

$$(0, 0, 0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 12, 13, 0), \quad (0, 0, 0, 12, 13, 23, 0).$$

Proof. By the remark at the beginning of Section 2, we know that these three Lie algebras have a calibrated G_2-structure (see [11] where the second of these Lie algebras was considered). In fact, on the non-abelian Lie algebras $(0, 0, 0, 0, 12, 13)$ and $(0, 0, 0, 12, 13, 23)$ we can consider the symplectic half-flat structure (ω_1, ψ_1^+) and (ω_2, ψ_2^+), respectively, defined by

$$\omega_1 = e^{14} + e^{26} + e^{35}, \qquad \psi_1^+ = e^{123} + e^{156} + e^{245} - e^{346},$$

and

$$\omega_2 = e^{16} + 2e^{25} + e^{34}, \qquad \psi_2^+ = e^{123} + e^{145} + e^{246} - e^{356}.$$

Using Lemma 3, we can see that the decomposable Lie algebra

$$0, 0, 0, 0, 12, 34, 36$$

has no calibrated G_2-structure. Indeed a basis of the space Z^3 of the closed 3-forms is given by

$$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{234}, e^{235}, e^{236},$$
$$e^{237}, e^{245}, e^{246}, -e^{126} + e^{345}, e^{346}, e^{347}, e^{127} + e^{356}, e^{367}, e^{467}.$$

Thus $e_7 \lrcorner Z^3$ is the span of e^{13}, e^{23}, e^{34}, e^{12}, e^{36}, e^{46}, which contains only degenerate forms.

Table 1
Closed 3-forms on decomposable Lie algebras.

$(0,0,12,13,14+23,24+15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{136}+e^{145}, e^{146}, e^{147}, e^{234}, e^{235}+e^{136}, e^{237}, -e^{236}+e^{245}, e^{157}+e^{247}, e^{167}+e^{257}, -\frac{1}{2}e^{156}+e^{345}-\frac{1}{2}e^{246}, e^{167}+e^{347}$
$(0,0,0,12,14,15+23,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}-e^{146}, -e^{156}+e^{236}, e^{237}, e^{245}, e^{247}, e^{156}+e^{345}, -e^{167}+e^{347}, -e^{267}+e^{457}$
$(0,0,0,12,14-23,15+34,0)$	$e^{123}, e^{124}, e^{125}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{126}+e^{145}, e^{146}, e^{147}, e^{234}, e^{235}-e^{126}, e^{237}, e^{245}, e^{247}, e^{236}+e^{345}, e^{347}+e^{157}, e^{357}+e^{167}$
$(0,0,0,12,14,15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{237}, e^{245}, e^{247}, e^{345}+e^{236}, -e^{267}+e^{457}$
$(0,0,0,12,13,14+23,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}-e^{146}, e^{237}, e^{245}+e^{146}, e^{246}, e^{247}, e^{257}+e^{167}, e^{345}+e^{156}, e^{347}-e^{167}, e^{357}$
$(0,0,0,12,13,24,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}-e^{136}, e^{246}, e^{247}, e^{267}, e^{256}+e^{346}, e^{257}+e^{347}, e^{357}$
$(0,0,0,12,14,15+24,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{136}+e^{235}, e^{237}, e^{245}, e^{246}-e^{156}, e^{247}, e^{257}+e^{167}, e^{236}+e^{345}, e^{457}-e^{267}$
$(0,0,0,12,14,15+23+24,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{136}+e^{146}, e^{147}, e^{157}, e^{234}, e^{136}+e^{235}, e^{237}, e^{245}, e^{236}+e^{246}-e^{156}, e^{247}, e^{236}+e^{345}, e^{347}-e^{257}-e^{167}, e^{457}-e^{267}$
$(0,0,12,13,14,15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{237}, e^{245}-e^{236}, e^{347}-e^{257}$
$(0,0,12,13,14,23+15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}-e^{146}, e^{236}-e^{156}, e^{237}, e^{245}-e^{156}, e^{247}+e^{167}, e^{347}-e^{257}$
$(0,0,0,12,13+42,14+23,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{136}, e^{137}, e^{145}, -e^{135}+e^{146}, e^{147}, e^{234}, e^{235}, e^{236}-e^{135}, e^{237}, e^{245}+e^{135}, e^{246}, e^{247}, e^{167}+e^{257}, -e^{157}+e^{267}, -e^{167}+e^{347}$
$(0,0,0,12,14,13+42,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}-e^{136}, e^{236}, e^{237}, e^{245}, e^{246}+e^{136}, e^{247}, e^{257}-e^{167}, e^{267}+e^{347}$
$(0,0,0,12,13+14,24,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}+e^{136}, e^{236}, e^{237}, e^{245}-e^{136}, e^{246}, e^{247}, e^{267}, e^{257}+e^{167}+e^{347}$
$(0,0,0,12,13,14,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{237}, e^{245}+e^{236}, e^{246}, e^{247}, e^{347}+e^{257}, e^{357}$
$(0,0,0,0,0,12,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{256}, e^{257}, e^{267}, e^{345}, e^{347}, e^{357}, e^{457}$
$(0,0,0,0,12,14+25,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{237}, e^{245}, e^{246}, e^{247}, -e^{146}+e^{256}, e^{257}, -e^{236}+e^{345}, e^{347}, e^{457}+e^{267}$
$(0,0,0,0,12,34,0)$	$e^{123}, e^{124}, e^{125}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{257}, e^{345}-e^{126}, e^{346}, e^{347}, e^{367}, e^{467}$
$(0,0,0,0,12,15,0)$	$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{237}, e^{245}, e^{247}, e^{256}, e^{257}, e^{347}$

Since this is the only decomposable nilpotent Lie algebra of dimension seven which does not have the form $\mathfrak{h}\oplus\mathbb{R}$, it remains to prove that if $\mathfrak{g}=\mathfrak{h}\oplus\mathbb{R}$ has a calibrated G_2 form, then $\mathfrak{g}$ must be as in the statement.

Clearly, if $\mathfrak{h}$ is one of the eight Lie algebras defined by (3), Proposition 1 implies that the Lie algebra $\mathfrak{g}=\mathfrak{h}\oplus\mathbb{R}$ has no calibrated G_2 form.

Also, one can check that none of the five Lie algebras defined by

$$(0,0,12,13,23,14,0),\quad (0,0,12,13,23,14+25,0),\quad (0,0,12,13,23,14-25,0),$$
$$(0,0,0,0,13+42,14+23,0),\quad (0,0,0,0,12,14+23,0),$$

has a calibrated G_2 form because each of these is a bundle over a non-symplectic Lie algebra of dimension six. Explicitly, the base of the bundle and curvature form are given by

$$\pi:(0,0,12,13,23,14,0)\to(0,0,12,13,23,0),\quad d\eta=e^{14},$$
$$\pi:(0,0,12,13,23,14+25,0)\to(0,0,12,13,23,0),\quad d\eta=e^{14}+e^{25},$$
$$\pi:(0,0,12,13,23,14-25,0)\to(0,0,12,13,23,0),\quad d\eta=e^{14}-e^{25},$$
$$\pi:(0,0,0,0,13+42,14+23,0)\to(0,0,0,0,13+42,0),\quad d\eta=e^{14}+e^{23},$$
$$\pi:(0,0,0,0,12,14+23,0)\to(0,0,0,0,14+23,0),\quad d\eta=e^{12}.$$

For each of the remaining 18 Lie algebras of the form $\mathfrak{g}=\mathfrak{h}\oplus\mathbb{R}$, listed in Table 1 alongside with a basis of the space of closed 3-forms, one can check that the hypothesis of Lemma 3 is satisfied with $X=e_6$. $\square$

4. Indecomposable case

In this section we complete the classification of 7-dimensional nilpotent Lie algebras with a calibrated G_2-structure. We have seen that there are exactly three decomposable Lie algebras of this type. In order to discuss the indecomposable Lie algebras, we refer to Gong's classification in [12]. This list consists of 140 Lie algebras and 9 one-parameter families.

The one-parameter families are the following:

$$\begin{aligned}
&147E = \big(0,0,0,e^{12},e^{23},-e^{13},\lambda e^{26}-e^{15}-(-1+\lambda)e^{34}\big), \quad \lambda \neq 0,1;\\
&1357M = \big(0,0,e^{12},0,e^{24}+e^{13},e^{14},-(-1+\lambda)e^{34}+e^{15}+e^{26}\lambda\big), \quad \lambda \neq 0;\\
&1357N = \big(0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23}\lambda\big);\\
&1357S = \big(0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{16}+e^{15}+\lambda e^{26}\big), \quad \lambda \neq 1;\\
&12457N = \big(0,0,e^{12},e^{13},e^{23},e^{24}+e^{15},\lambda e^{25}+e^{26}+e^{34}-e^{35}+e^{16}+e^{14}\big);\\
&123457I = \big(0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}+e^{24},\lambda e^{25}-(-1+\lambda)e^{34}+e^{16}\big);\\
&147E1 = \big(0,0,0,e^{12},e^{23},-e^{13},2e^{26}-2e^{34}-e^{16}\lambda+\lambda e^{25}\big), \quad \lambda > 1;\\
&1357QRS1 = \big(0,0,e^{12},0,e^{13}+e^{24},e^{14}-e^{23},e^{26}\lambda+e^{15}-e^{34}(-1+\lambda)\big), \quad \lambda \neq 0;\\
&12457N2 = \big(0,0,e^{12},e^{13},e^{23},-e^{14}-e^{25},e^{15}-e^{35}+e^{16}+e^{24}+e^{25}\lambda\big), \quad \lambda \geqslant 0.
\end{aligned}$$

Recall that a 3-form of type G_2 has the form (1) with respect to some coframe $e^1,\dots,e^7$; such a coframe identifies the G_2-structure.

Lemma 5. *Exactly three of the above Lie algebras admit a calibrated* G_2*-structure. Explicit examples are given in terms of a coframe by*

$$\begin{aligned}
1357N(\lambda=1):\quad &\sqrt{3}\big(2e^1-e^7-e^6-e^5\big), \sqrt{3}\big(e^4+e^3-2e^2-e^6+e^5\big), 2e^3-e^6, 2e^5,\\
&-e^3+3e^4+e^5-e^6, 2e^3-e^5-e^6+3e^7, -\sqrt{3}e^6;\\
1357S(\lambda=-3):\quad &\sqrt{7}\big(2e^1+e^2-e^5+e^6\big), 7e^2+3e^5+5e^6, \sqrt{7}\left(e^3+2e^4-\frac{3}{2}e^7\right),\\
&3e^3+\frac{7}{2}e^7, -\sqrt{70}e^6, \sqrt{10}\big(2e^5+e^6\big), -2\sqrt{10}e^3;\\
147E1(\lambda=2):\quad &\sqrt{3}\big(2e^1+e^5-e^2+e^6\big), 3e^2-e^5+e^6, e^3+2e^4, \sqrt{3}\big(e^3+e^7\big),\\
&\sqrt{2}\big(e^6-e^5\big), \sqrt{6}\big(e^5+e^6\big), 2\sqrt{2}\big(e^4-e^3\big).
\end{aligned}$$

Proof. It is straightforward to verify that each coframe in the statement determines a calibrated G_2-structure on the corresponding Lie algebra. Conversely, for each Lie algebra $\mathfrak{g}$ the vector e_7 is in the center, and determines an epimorphism on a 6-dimensional Lie algebra $\mathfrak{h}$; we view de^7 as the curvature form on $\mathfrak{h}$, and apply Proposition 1.

In the case of $1357M$, the generic element of $H^2(\mathfrak{h}^*)$ is represented by

$$\omega = \lambda_6 e^{46} + \lambda_3 e^{23} + \lambda_1 e^{13} + \lambda_5\big(e^{15}+e^{34}\big) + \lambda_2 e^{16} + \lambda_4\big(e^{15}+e^{26}\big).$$

Assume $de^7 \wedge \omega$ is exact. Then λ_3, λ_6 are zero, $\lambda_4 = -\lambda_5\lambda$ and

$$\big(\lambda - \lambda^2 - 1\big)\lambda_5 = 0.$$

Since $\lambda^2-\lambda+1$ has no real zeroes, λ_4 and λ_5 are zero as well, and therefore ω^3 is zero. So there is no symplectic form in the cohomology class of ω. By Proposition 1, if a calibrated G_2-structure exist, then $\mathfrak{g}$ would have to be decomposable, which is absurd.

The other cases are similar. □

We now turn to the rest of the list, where no parameters appear.

Lemma 6. *In Gong's list, only six Lie algebras with no parameters in their definition admit a calibrated* G_2*-structure, which can be expressed in terms of a coframe as follows*:

$0, 0, 12, 0, 0, 13+24, 15$	$e^1, e^2, e^5, e^6, e^3, e^7, e^4$
$0, 0, 12, 0, 0, 13, 14+25$	$e^1, e^3, e^5, e^7, e^2, e^6, e^4$
$0, 0, 0, 12, 13, 14, 15$	$e^1, e^2, e^4, e^7, e^5, e^6, e^3$
$0, 0, 0, 12, 13, 14+23, 15$	$e^2+e^7, e^3+e^6, e^7, e^6, e^5, e^4, e^1$
$0, 0, 12, 13, 23, 15+24, 16+34$	$e^2+e^4, e^7, e^2, e^5, e^3, e^6, e^1$
$0, 0, 12, 13, 23, 15+24, 16+25+34$	$\sqrt{3}(2e^2+e^5+e^7), 2e^4-3e^5-e^7, \sqrt{3}(e^1-e^3+2e^6), e^1+3e^3, \sqrt{6}e^7, \sqrt{2}(2e^4-e^7), 2\sqrt{2}e^1$

Theorem 7. *Up to isomorphism, there are exactly* 12 *nilpotent Lie algebras that admit a calibrated* G_2*-structure, namely those appearing in Theorem* 4*, Lemmas* 5 *and* 6*.*

Proof. We must show that the remaining Lie algebras in Gong's list satisfy one of the two obstructions of Section 2; we do so in Appendix A, where we reproduce Gong's list, and note which obstruction applies to each Lie algebra (as a preference, we try to use Proposition 1 rather than Lemma 3 whenever possible, because the former does not require computing the space of closed 3-forms). □

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Appendix A

This appendix contains a list (Tables 2–5) of all indecomposable nilpotent Lie algebras of dimension 7, taken from [12], except the 9 one-parameter families that we listed at the beginning of Section 4. Alongside each Lie algebra $\mathfrak{g}$, we give a chosen vector $\xi \in \mathfrak{g}$ which satisfies the conditions of Proposition 1 (when marked with a (P)) or Lemma 3, and the structure constants of the quotient $\mathfrak{g}/\operatorname{Span}\{\xi\}$. The word "resists" marks instead the six Lie algebras that resist the obstructions. Below each Lie algebra, we give a basis of its space of closed 3-forms, except when Proposition 1 applies.

Table 2
Step 2 nilpotent Lie algebras of dimension 7.

$0, 0, 0, 0, 12, 23, 24$	e_7	$[0, 0, 0, 0, e^{12}, e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{145}, e^{137}+e^{146}, e^{147}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{256}, e^{257}, e^{267}, e^{345}+e^{137}, e^{346}, e^{347}$		
$0, 0, 0, 0, 12, 23, 34$	e_7	$[0, 0, 0, 0, e^{12}, e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}-e^{127}, e^{147}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{256}, -e^{127}+e^{345}, e^{346}, e^{347}, e^{367}$		
$0, 0, 0, 0, 12+34, 23, 24$	e_5	$[0, 0, 0, 0, e^{23}, e^{24}]$
$e^{123}, e^{124}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}+e^{125}, e^{145}, -e^{125}+e^{146}, e^{147}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}, e^{247}, e^{267}, -e^{125}+e^{345}, e^{346}, e^{347}, -e^{256}+e^{367}, -e^{257}+e^{467}$		
$0, 0, 0, 0, 12+34, 13, 24$	e_7	$[0, 0, 0, 0, e^{12}+e^{34}, e^{13}]$
$e^{123}, e^{124}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}+e^{125}, e^{145}, e^{146}, e^{147}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, e^{246}+e^{125}, e^{247}, -e^{125}+e^{345}, e^{346}, e^{347}$		
$0, 0, 0, 0, 0, 12, 14+35$	e_6 (P)	$[0, 0, 0, 0, 0, e^{14}+e^{35}]$
$0, 0, 0, 0, 0, 12+34, 15+23$	e_7 (P)	$[0, 0, 0, 0, 0, e^{34}+e^{12}]$
$0, 0, 0, 0, 0, 0, 12+34+56$	e_7	$[0, 0, 0, 0, 0, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{256}, e^{345}, e^{346}, e^{356}, e^{456}$		
$0, 0, 0, 0, 12-34, 13+24, 14$	e_6	$[0, 0, 0, 0, e^{12}-e^{34}, e^{14}]$
$e^{123}, e^{124}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}-e^{125}, e^{137}, e^{145}, e^{146}, e^{147}, e^{234}, e^{235}, e^{236}, e^{125}+e^{237}, e^{245}, -e^{125}+e^{246}, e^{247}, e^{125}+e^{345}, e^{346}, e^{347}, e^{457}-e^{167}, e^{467}+e^{157}$		
$0, 0, 0, 0, 12-34, 13+24, 14-23$	e_7	$[0, 0, 0, 0, e^{12}-e^{34}, e^{24}+e^{13}]$
$e^{123}, e^{124}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}-e^{125}, e^{137}, e^{145}, e^{146}, -e^{125}+e^{147}, e^{234}, e^{235}, e^{236}, e^{125}+e^{237}, e^{245}, -e^{125}+e^{246}, e^{247}, e^{125}+e^{345}, e^{346}, e^{347}$		

Table 3
Step 3 nilpotent Lie algebras of dimension 7.

$0, 0, 12, 0, 13, 24, 14$	e_5	$[0, 0, e^{12}, 0, e^{24}, e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237}+e^{136}, e^{245}-e^{136}, e^{246}, e^{247}, e^{345}-e^{156}, e^{346}+e^{267}, e^{347}+e^{167}, e^{467}$		
$0, 0, 12, 0, 13, 23, 14$	e_7 (P)	$[0, 0, e^{12}, 0, e^{13}, e^{23}]$

Table 3 (*continued*)

0, 0, 12, 0, 13 + 24, 23, 14	e_6	$[0, 0, e^{12}, 0, e^{24}+e^{13}, e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{136}, e^{137}, e^{145}, e^{146}+e^{135}, e^{147}, e^{234}, e^{235}, e^{236}, e^{237}+e^{135}, -e^{135}+e^{245}, e^{246}, e^{247}, e^{257}+e^{345}+e^{167}, e^{346}+e^{267}, e^{347}+e^{157}$		
0, 0, 12, 0, 0, 13 + 24, 15	resists	
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{245}, -e^{136}+e^{246}, -e^{156}+e^{247}, e^{256}+e^{237}, e^{257}, e^{345}-e^{156}, -e^{347}-e^{167}+e^{456}, e^{457}$		
0, 0, 12, 0, 0, 13, 14 + 25	resists	
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{245}, e^{237}+e^{246}, e^{247}, e^{256}-e^{137}, e^{257}-e^{147}, e^{345}+e^{147}, -e^{167}+e^{356}, e^{457}$		
0, 0, 12, 0, 0, 13 + 24, 25	e_7	$[0, 0, e^{12}, 0, 0, e^{13}+e^{24}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{145}, e^{146}, e^{147}+e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237}, e^{245}, -e^{136}+e^{246}, e^{247}, e^{256}-e^{137}, e^{257}, e^{345}+e^{147}, e^{457}$		
0, 0, 12, 0, 0, 13 + 24, 14 + 25	e_7	$[0, 0, e^{12}, 0, 0, e^{13}+e^{24}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{145}, e^{146}, e^{156}+e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237}+e^{136}, e^{245}, e^{246}-e^{136}, e^{247}, e^{256}-e^{137}, e^{257}-e^{147}, e^{345}+e^{147}, e^{457}$		
0, 0, 12, 0, 0, 13 + 45, 24	e_7 (P)	$[0, 0, e^{12}, 0, 0, e^{13}+e^{45}]$
0, 0, 12, 0, 0, 13 + 45, 15 + 24	e_7 (P)	$[0, 0, e^{12}, 0, 0, e^{13}+e^{45}]$
0, 0, 12, 0, 0, 13 + 24, 45	e_7	$[0, 0, e^{12}, 0, 0, e^{13}+e^{24}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{145}, e^{146}, e^{147}, -e^{127}+e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}-e^{136}, e^{247}, e^{257}, e^{345}-e^{127}, e^{456}-e^{137}, e^{457}$		
0, 0, 12, 0, 0, 13 + 14, 15 + 23	e_7	$[0, 0, e^{12}, 0, 0, e^{14}+e^{13}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235}, -e^{147}+e^{236}, e^{237}-e^{157}, e^{245}, e^{147}+e^{246}, e^{247}+e^{256}+e^{157}, e^{257}, -e^{247}+e^{345}$		
0, 0, 12, 0, 0, 13 + 24, 15 + 23	e_7	$[0, 0, e^{12}, 0, 0, e^{24}+e^{13}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}+e^{136}, e^{234}, e^{235}, e^{236}, -e^{157}+e^{237}, e^{245}, e^{246}-e^{136}, -e^{156}+e^{247}, e^{157}+e^{256}, e^{257}, -e^{156}+e^{345}$		
0, 0, 12, 0, 0, 13, 23 + 45	e_6 (P)	$[0, 0, e^{12}, 0, 0, e^{23}+e^{45}]$
0, 0, 12, 0, 0, 13 + 24, 23 + 45	e_6 (P)	$[0, 0, e^{12}, 0, 0, e^{23}+e^{45}]$
0, 0, 0, 12, 13, 14, 15	resists	
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, e^{245}+e^{236}, e^{246}, e^{345}+e^{237}, e^{346}+e^{256}+e^{247}, e^{257}+e^{347}+e^{356}, e^{357}$		
0, 0, 0, 12, 13, 14, 35	e_7	$[0, 0, 0, e^{12}, e^{13}, e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{156}, e^{157}, e^{234}, e^{235}, e^{237}, e^{245}+e^{236}, e^{246}, e^{345}+e^{127}, e^{347}+e^{257}, -e^{147}+e^{356}, e^{357}$		
0, 0, 0, 12, 13, 14 + 35, 15	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{35}+e^{14}]$
0, 0, 0, 12, 13, 14, 25 + 34	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{34}+e^{25}]$
0, 0, 0, 12, 13, 14 + 15, 25 + 34	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{25}+e^{34}]$
0, 0, 0, 12, 13, 24 + 35, 25 + 34	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{24}+e^{35}]$
0, 0, 0, 12, 13, 14 + 15 + 24 + 35, 25 + 34	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{35}+e^{24}+e^{15}+e^{14}]$
0, 0, 0, 12, 13, 14 + 24 + 35, 25 + 34	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{24}+e^{35}+e^{14}]$
0, 0, 0, 12, 13, 25 + 34, 35	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{25}+e^{34}]$
0, 0, 0, 12, 13, 15 + 35, 25 + 34	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{34}+e^{25}]$
0, 0, 0, 12, 13, 14 + 35, 25 + 34	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{14}+e^{35}]$
0, 0, 0, 12, 13, 14 + 23, 15	resists	
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}, -e^{146}+e^{236}, -e^{156}+e^{237}, e^{245}+e^{146}, e^{246}, e^{167}+e^{257}, e^{156}+e^{345}, e^{346}+e^{256}+e^{247}, -e^{167}+e^{356}+e^{347}, e^{357}$		
0, 0, 0, 12, 13, 14 + 23, 35	e_7	$[0, 0, 0, e^{12}, e^{13}, e^{23}+e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, -e^{127}+e^{156}, e^{157}, e^{234}, e^{235}, -e^{146}+e^{236}, e^{237}, e^{146}+e^{245}, e^{246}, e^{127}+e^{345}, e^{257}+e^{347}, -e^{147}+e^{356}, e^{357}$		
0, 0, 0, 12, 13, 15 + 24, 23	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{24}+e^{15}]$
0, 0, 0, 12, 13, 14 + 35, 15 + 23	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{35}+e^{14}]$
0, 0, 0, 12, 13, 23, 25 + 34	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{25}+e^{34}]$
0, 0, 0, 12, 13, 14 + 23, 25 + 34	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{25}+e^{34}]$
0, 0, 0, 12, 13, 14 + 15 + 23, 25 + 34	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{25}+e^{34}]$
0, 0, 12, 0, 0, 0, 13 + 24 + 56	e_7	$[0, 0, e^{12}, 0, 0, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{256}, -e^{157}+e^{345}, -e^{167}+e^{346}, e^{356}-e^{127}, e^{456}$		

(*continued on next page*)

Table 3 (*continued*)

0, 0, 0, 12, 13, 0, 16 + 25 + 34	e_7	$[0, 0, 0, e^{12}, e^{13}, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235}, e^{236}, e^{245} + e^{127}, e^{246}, e^{256} - e^{237}, e^{345} - e^{137}, e^{346} + e^{237}, e^{356}$		
0, 0, 0, 12, 13, 0, 14 + 26 + 35	e_7	$[0, 0, 0, e^{12}, e^{13}, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235}, e^{236}, e^{237} + e^{245}, e^{246}, e^{137} + e^{256}, e^{127} + e^{345}, -e^{137} + e^{346}, e^{356}, e^{157} + e^{456} - e^{367}$		
0, 0, 0, 12, 23, −13, 15 + 26 + 16 − 2 ∗ 34	e_7	$[0, 0, 0, e^{12}, e^{23}, -e^{13}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, \frac{1}{2}e^{127} + e^{145}, e^{146}, e^{156} - e^{137}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246} + \frac{1}{2}e^{127}, e^{256} - e^{137} - e^{237}, e^{345} + e^{137} + e^{237}, e^{346} - e^{137}, e^{356}$		
0, 0, 0, 0, 12, 34, 15 + 36	e_7	$[0, 0, 0, 0, e^{12}, e^{34}]$
$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{345} - e^{126}, e^{346}, e^{347} - e^{156}, e^{356} + e^{127}, -e^{157} + e^{367}$		
0, 0, 0, 0, 12, 34, 15 + 24 + 36	e_7	$[0, 0, 0, 0, e^{12}, e^{34}]$
$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137} + e^{126}, e^{145}, e^{146}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{345} - e^{126}, e^{346}, e^{347} - e^{156}, e^{356} + e^{127}$		
0, 0, 0, 0, 12, 14 + 23, 16 − 35	e_7	$[0, 0, 0, 0, e^{12}, e^{14} + e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{156} + e^{127}, e^{157}, e^{234}, e^{235}, -e^{146} + e^{236}, e^{245}, e^{246}, -e^{146} + e^{345}, e^{346}, e^{147} + e^{237} + e^{356}$		
0, 0, 0, 0, 12, 14 + 23, 16 + 24 − 35	e_7	$[0, 0, 0, 0, e^{12}, e^{23} + e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{137} + e^{146}, e^{156} + e^{127}, e^{234}, e^{235}, e^{236} + e^{137}, e^{245}, e^{246}, e^{157} + e^{256}, e^{345} + e^{137}, e^{346}, e^{356} + e^{237} + e^{147}$		
0, 0, 12, 0, 0, 13 + 14 + 25, 15 + 23	e_7	$[0, 0, e^{12}, 0, 0, e^{25} + e^{14} + e^{13}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{156}, e^{157} + e^{136}, e^{234}, e^{235}, -e^{147} + e^{236}, e^{136} + e^{237}, e^{245}, e^{147} + e^{246}, e^{146} + e^{247}, e^{256} - e^{146} - e^{136},$ $e^{257}, e^{345} + e^{146}$		
0, 0, 0, 12, 13, 14, 24 + 35	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{24} + e^{35}]$
0, 0, 0, 12, 13, 24 − 35, 25 + 34	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, -e^{35} + e^{24}]$
0, 0, 0, 12, 13, 14 + 24 − 35, 25 + 34	e_7 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{24} - e^{35} + e^{14}]$
0, 0, 0, 12, 13, 23, 24 + 35	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{35} + e^{24}]$
0, 0, 0, 12, 13, 14 + 23, 24 + 35	e_6 (P)	$[0, 0, 0, e^{12}, e^{13}, e^{24} + e^{35}]$
0, 0, 0, 12, 13, 0, 16 + 24 + 35	e_7	$[0, 0, 0, e^{12}, e^{13}, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{235}, e^{236}, e^{245} - e^{137}, e^{246}, e^{256} - e^{237}, e^{127} + e^{345}, e^{237} + e^{346}, e^{356}$		
0, 0, 0, 0, 13 + 24, 14 − 23, 15 + 26	e_7	$[0, 0, 0, 0, e^{13} + e^{24}, e^{14} - e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{136}, e^{145}, -e^{135} + e^{146}, e^{234}, e^{235}, e^{135} + e^{236}, e^{237} - e^{147} + e^{156}, -e^{135} + e^{245}, e^{246}, e^{137} + e^{247} + e^{256}, e^{257} - e^{167}, e^{345}, e^{346}$		
0, 0, 0, 0, 13 + 24, 14 − 23, 15 + 26 + 24	e_7	$[0, 0, 0, 0, e^{24} + e^{13}, e^{14} - e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{136}, e^{145}, e^{146} - e^{135}, e^{234}, e^{235}, e^{135} + e^{236}, e^{156} - e^{147} + e^{237}, -e^{135} + e^{245}, e^{246}, e^{256} + e^{137} - e^{135} + e^{247},$ $e^{257} - e^{167} - e^{147}, e^{345}, e^{346}$		

Table 4
Step 4 nilpotent Lie algebras of dimension 7.

0, 0, 12, 13, 0, 14, 15	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{167}, e^{234}, e^{235}, -e^{237} + e^{245}, e^{257}, e^{256} + e^{345}$		
0, 0, 12, 13, 0, 25, 14	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{25}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{136} + e^{245}, e^{256}, -e^{146} + e^{257}, e^{146} + e^{345}$		
0, 0, 12, 13, 0, 14 + 25, 15	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{25} + e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{136} + e^{237}, e^{136} + e^{245}, e^{256} - e^{146}, e^{257}, e^{345} + e^{146}, e^{357} + e^{167}$		
0, 0, 12, 13, 0, 14 + 23 + 25, 15	e_6	$[0, 0, e^{12}, e^{13}, 0, e^{15}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{147}, e^{136} + e^{156}, e^{157}, e^{234}, e^{235}, e^{237} + e^{136}, e^{245} + e^{136}, e^{256} + e^{236} - e^{146}, e^{257}, e^{345} - e^{236} + e^{146},$ $e^{247} - e^{236} + e^{357} + e^{167} + e^{146}$		
0, 0, 12, 13, 0, 23 + 25, 14	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{23} + e^{25}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{147}, e^{156} + e^{136}, e^{157}, e^{234}, e^{235}, e^{236}, e^{136} + e^{245}, e^{256}, e^{257} - e^{146} + e^{237}, e^{345} + e^{146} - e^{237}$		
0, 0, 12, 13, 0, 14 + 23, 15	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{23} + e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}, -e^{146} + e^{236}, -e^{156} + e^{237}, e^{245} - e^{156}, e^{247} + e^{256} + e^{167}, e^{257}, e^{345} - e^{247} - e^{167}$		
0, 0, 12, 13, 0, 15 + 23, 14	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{15} + e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{147}, e^{157}, e^{234}, e^{235}, -e^{156} + e^{236}, -e^{146} + e^{237}, -e^{156} + e^{245}, e^{167} + e^{247}, e^{256}, e^{345} + e^{257}$		
0, 0, 12, 13, 0, 23, 14 + 25	e_7 (P)	$[0, 0, e^{12}, e^{13}, 0, e^{23}]$
0, 0, 12, 13, 0, 14 + 23, 25	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{14} + e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{145}, e^{137} + e^{156}, e^{157}, e^{234}, e^{235}, -e^{146} + e^{236}, e^{237}, e^{137} + e^{245}, -e^{147} + e^{256}, e^{257}, e^{147} + e^{345}$		
0, 0, 12, 13, 0, 14 + 23, 23 + 25	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{23} + e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{145}, e^{156} + e^{137}, e^{157} + e^{137}, e^{234}, e^{235}, -e^{146} + e^{236}, e^{237}, e^{137} + e^{245}, e^{146} - e^{147} + e^{256}, e^{257}, -e^{146} + e^{147} + e^{345}$		

Table 4 (*continued*)

$0,0,12,13,0,15+23,14+25$	e_7	$[0,0,e^{12},e^{13},0,e^{15}+e^{23}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{145},e^{156}+e^{137},e^{157},e^{234},e^{235},e^{236}+e^{137},-e^{146}+e^{237},e^{245}+e^{137},e^{256},-e^{147}+e^{257},e^{345}+e^{147},e^{356}-e^{167}-e^{247}$		
$0,0,12,13,23,14+25,15+24$	e_7 (P)	$[0,0,e^{12},e^{13},e^{23},e^{14}+e^{25}]$
$0,0,12,13,23,24+15,14$	e_6 (P)	$[0,0,e^{12},e^{13},e^{23},e^{14}]$
$0,0,0,12,14+23,23,15-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},e^{23}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{145}+e^{127},e^{146}+e^{127},e^{147},e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246},e^{345}+e^{237},-e^{156}+e^{346}+e^{237},e^{167}-\frac{1}{2}e^{157}-\frac{1}{2}e^{347}+e^{356}$		
$0,0,0,12,14+23,13,15-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},e^{13}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{127}+e^{145},e^{146},e^{147},e^{234},e^{127}+e^{235},e^{236},e^{245},-e^{127}+e^{246},e^{237}+e^{345},-e^{156}+e^{346},e^{167}+e^{356}$		
$0,0,0,12,14+23,24,15-34$	e_7	$[0,0,0,e^{12},e^{14}+e^{23},e^{24}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}-e^{127},e^{137},e^{145}+e^{127},e^{146},e^{147},e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246},e^{345}+e^{237},e^{247}+e^{346}-e^{156}$		
$0,0,0,12,14+23,13+24,15-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},e^{24}+e^{13}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}-e^{127},e^{137},e^{145}+e^{127},e^{146},e^{147},e^{234},e^{127}+e^{235},e^{236},e^{245},e^{246}-e^{127},e^{345}+e^{237},e^{346}-e^{156}+e^{247}$		
$0,0,12,13,0,0,14+56$	e_7	$[0,0,e^{12},e^{13},0,0]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{157},e^{167},e^{234},e^{235},e^{236},e^{256},e^{257}+e^{345},e^{346}+e^{267},-e^{127}+e^{356},e^{456}-e^{137},e^{567}-e^{147}$		
$0,0,12,13,0,0,23+14+56$	e_7	$[0,0,e^{12},e^{13},0,0]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145},e^{146},e^{156},e^{234},e^{235},e^{236},-e^{157}+e^{245},-e^{167}+e^{246},e^{256},e^{257}+e^{345},e^{346}+e^{267},-e^{127}+e^{356},-e^{137}+e^{456}$		
$0,0,0,12,14+23,0,15+26-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},0]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145}+e^{127},e^{146},-e^{137}+e^{156},e^{234},e^{235}+e^{127},e^{236},e^{245},e^{246},e^{256}+e^{147},e^{345}+e^{237},e^{346}-e^{137},e^{356}+e^{167}$		
$0,0,0,12,14+23,0,15+36-34$	e_7	$[0,0,0,e^{12},e^{23}+e^{14},0]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{137},e^{146},e^{127}+e^{156}+e^{145},-e^{147}+e^{167},e^{234},e^{235}-e^{145},e^{236},e^{245},e^{246},e^{237}+e^{345},e^{346}+e^{127}+e^{145},e^{147}+e^{356}$		
$0,0,0,12,14+23,0,15+24+36-34$	e_7	$[0,0,0,e^{12},e^{14}+e^{23},0]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{145}+e^{137},e^{146},e^{156}+e^{127}-e^{137},e^{234},e^{235}+e^{137},e^{236},e^{245},e^{246},-e^{167}+e^{147}+e^{256},e^{345}+e^{237},e^{346}+e^{127}-e^{137},e^{147}+e^{356}$		
$0,0,12,0,23,24,16+25+34$	e_7	$[0,0,e^{12},0,e^{23},e^{24}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{127},e^{145}-e^{127},e^{146},e^{234},e^{235},e^{236},e^{147}+e^{237}+e^{156},e^{245},e^{246},e^{256},e^{147}+e^{345},-e^{247}+e^{346},e^{456}+e^{267}$		
$0,0,12,0,23,24,25+46$	e_7	$[0,0,e^{12},0,e^{23},e^{24}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{145}+e^{136},e^{146},e^{234},e^{235},e^{236},e^{245},e^{246},e^{247},e^{256},e^{267},e^{147}+e^{345},-e^{127}+e^{346},e^{237}+e^{456},e^{467}-e^{257}$		
$0,0,12,0,23,24,13+25-46$	e_7	$[0,0,e^{12},0,e^{23},e^{24}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136}+e^{145},e^{146},e^{234},e^{235},e^{236},e^{245},e^{246},e^{247}-e^{136},e^{256},-e^{147}+e^{267}-e^{156},e^{147}+e^{345},e^{127}+e^{346},-e^{237}+e^{456}$		
$0,0,12,0,23,14,16+25$	e_7	$[0,0,e^{12},0,e^{23},e^{14}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},e^{156}+e^{237},e^{245},e^{246},-e^{147}+e^{256},-e^{167}+e^{257},e^{147}+e^{345},e^{346}-e^{247}$		
$0,0,12,0,23,14,16+25+26-34$	e_7	$[0,0,e^{12},0,e^{23},e^{14}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{127}+e^{145},e^{146},e^{234},e^{235},e^{127}+e^{236},e^{156}+e^{237},e^{245},e^{246},e^{256}-e^{247}-e^{147},e^{345}+e^{247}+e^{147},e^{346}-e^{247}$		
$0,0,12,0,23,14,25+46$	e_7	$[0,0,e^{12},0,e^{23},e^{14}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},e^{245},e^{246},e^{247},e^{256}-e^{147},e^{345}+e^{147},e^{346}-e^{127},e^{237}+e^{456},-e^{257}+e^{467}$		
$0,0,12,0,23,14,13+25+46$	e_7	$[0,0,e^{12},0,e^{23},e^{14}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},e^{135},e^{136},e^{146},e^{234},e^{235},e^{236}-e^{145},e^{245},e^{246},e^{145}+e^{247},e^{256}-e^{147},e^{147}+e^{345},-e^{127}+e^{346},e^{237}+e^{456}$		
$0,0,12,0,13+24,14,15+23+1/2*(26+34)$	e_7	$[0,0,e^{12},0,e^{13}+e^{24},e^{14}]$
$e^{123},e^{124},e^{125},e^{126},e^{134},2e^{127}+e^{135},e^{136},e^{145},e^{146},4e^{127}-2e^{147}+e^{156},e^{234},e^{235},-2e^{127}+e^{236},2e^{127}+e^{245},e^{246},2e^{137}+e^{247}+e^{256},-e^{247}+e^{345},-4e^{127}+e^{346}+2e^{147},e^{456}+4e^{137}+2e^{167}$		
$0,0,12,0,13+24,23,16+25$	e_7	$[0,0,e^{12},0,e^{24}+e^{13},e^{23}]$
$e^{123},e^{124},e^{125},e^{126},e^{127},e^{134},e^{136},e^{145},e^{146}+e^{135},e^{234},e^{235},e^{236},-e^{135}+e^{245},e^{246},e^{156}+e^{137}+e^{247},e^{256}-e^{237},e^{267},e^{345}+e^{147},e^{346}+e^{156}+e^{137}$		

(*continued on next page*)

Table 4 (*continued*)

0, 0, 12, 0, 13 + 24, 23, 15 + 26 + 34	e_7	$[0, 0, e^{12}, 0, e^{24}+e^{13}, e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{127}+e^{135}, e^{136}, e^{145}, e^{146}-e^{127}, e^{234}, e^{235}, e^{236}, e^{237}-e^{156}+e^{147}, e^{127}+e^{245}, e^{246}, e^{256}+e^{137}, -e^{247}+e^{345}, e^{346}+e^{147}$		
0, 0, 12, 0, 13, 23 + 24, 15 + 26	e_7	$[0, 0, e^{12}, 0, e^{13}, e^{23}+e^{24}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{145}, e^{136}+e^{146}, e^{234}, e^{235}, e^{236}, -e^{136}+e^{245}, e^{246}, e^{247}+e^{237}-e^{156}, e^{256}+e^{137}, -e^{157}+e^{267}, e^{237}-e^{156}+e^{345}, e^{147}+e^{346}$		
0, 0, 12, 0, 13, 23 + 24, 16 + 25 + 34	e_7	$[0, 0, e^{12}, 0, e^{13}, e^{24}+e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{127}+e^{136}, e^{145}, -e^{127}+e^{146}, e^{137}+e^{147}+e^{156}, e^{234}, e^{235}, e^{236}, e^{127}+e^{245}, e^{246}, e^{256}-e^{237}, e^{345}+e^{147}, e^{346}-e^{247}$		
0, 0, 12, 13, 23, 14 − 25, 15 + 24	e_7	$[0, 0, e^{12}, e^{13}, e^{23}, -e^{25}+e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{145}+e^{137}, e^{147}, e^{234}, e^{235}, e^{236}+e^{137}, -e^{136}+e^{237}, -e^{136}+e^{245}, e^{156}+e^{246}, e^{247}+e^{157}-2e^{146}, e^{256}+e^{146}, e^{257}, -e^{146}+e^{345}$		
0, 0, 0, 12, 14 + 23, 13 − 24, 15 − 34	e_7	$[0, 0, 0, e^{12}, e^{14}+e^{23}, e^{13}-e^{24}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}+e^{127}, e^{137}, e^{145}+e^{127}, e^{146}, e^{147}, e^{234}, e^{235}+e^{127}, e^{236}, e^{245}, e^{246}-e^{127}, e^{345}+e^{237}, -e^{247}+e^{346}-e^{156}$		
0, 0, 12, 0, 23, 24, 13 + 25 + 46	e_7	$[0, 0, e^{12}, 0, e^{23}, e^{24}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{145}+e^{136}, e^{146}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{247}-e^{136}, e^{256}, e^{267}-e^{147}-e^{156}, e^{345}+e^{147}, e^{346}-e^{127}, e^{237}+e^{456}$		
0, 0, 12, 0, 13 + 24, 23, 15 + 34 − 26	e_7	$[0, 0, e^{12}, 0, e^{24}+e^{13}, e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}+e^{127}, e^{136}, e^{145}, e^{146}-e^{127}, e^{234}, e^{235}, e^{236}, -e^{156}-e^{147}+e^{237}, e^{245}+e^{127}, e^{246}, e^{256}-e^{137}, e^{345}-e^{247}, e^{346}-e^{147}$		
0, 0, 12, 0, 13, 23 + 24, 15 − 26	e_7	$[0, 0, e^{12}, 0, e^{13}, e^{24}+e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{145}, e^{146}+e^{136}, e^{234}, e^{235}, e^{236}, e^{245}-e^{136}, e^{246}, -e^{156}+e^{237}+e^{247}, -e^{137}+e^{256}, e^{157}+e^{267}, -e^{156}+e^{345}+e^{237}, -e^{147}+e^{346}$		

Table 5
Step 5 nilpotent Lie algebras of dimension 7.

0, 0, 12, 13, 14, 15, 23	e_6 (P)	$[0, 0, e^{12}, e^{13}, e^{14}, e^{23}]$
0, 0, 12, 13, 14, 25 − 34, 23	e_7 (P)	$[0, 0, e^{12}, e^{13}, e^{14}, e^{25}-e^{34}]$
0, 0, 12, 13, 14, 15, 25 − 34	e_6 (P)	$[0, 0, e^{12}, e^{13}, e^{14}, e^{25}-e^{34}]$
0, 0, 12, 13, 14, 15 + 23, 25 − 34	e_6 (P)	$[0, 0, e^{12}, e^{13}, e^{14}, -e^{34}+e^{25}]$
0, 0, 12, 13, 14 + 23, 15 + 24, 23	e_6 (P)	$[0, 0, e^{12}, e^{13}, e^{23}+e^{14}, e^{23}]$
0, 0, 12, 13, 14 + 23, 25 − 34, 23	e_7 (P)	$[0, 0, e^{12}, e^{13}, e^{14}+e^{23}, e^{25}-e^{34}]$
0, 0, 12, 13, 14 + 23, 15 + 24, 25 − 34	e_6 (P)	$[0, 0, e^{12}, e^{13}, e^{23}+e^{14}, -e^{34}+e^{25}]$
0, 0, 12, 13, 14, 0, 15 + 26	e_7	$[0, 0, e^{12}, e^{13}, e^{14}, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{167}, e^{234}, e^{236}, -e^{237}+e^{245}, e^{246}+e^{137}, e^{256}+e^{147}, -e^{157}+e^{267}, e^{147}+e^{346}$		
0, 0, 12, 13, 14, 0, 15 + 23 + 26	e_7	$[0, 0, e^{12}, e^{13}, e^{14}, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{137}+e^{167}, e^{234}, e^{236}, -e^{237}+e^{245}, e^{137}+e^{246}, e^{147}+e^{256}-e^{235}, -e^{157}+e^{267}+e^{237}, e^{147}-e^{235}+e^{346}$		
0, 0, 12, 13, 14, 0, 16 + 25 − 34	e_7	$[0, 0, e^{12}, e^{13}, e^{14}, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}, e^{146}, e^{156}, e^{234}, e^{127}+e^{235}, e^{236}, e^{137}+e^{245}, -e^{237}+e^{246}, e^{147}+e^{345}, e^{346}-e^{256}$		
0, 0, 12, 13, 14 + 23, 0, 15 + 24 + 26	e_7	$[0, 0, e^{12}, e^{13}, e^{14}+e^{23}, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{136}, e^{146}, -e^{137}+e^{156}-e^{145}, e^{167}+e^{147}, e^{234}, e^{235}-e^{145}, e^{236}, -e^{237}+e^{245}, e^{137}+e^{145}+e^{246}, e^{147}+e^{256}, -\frac{1}{2}e^{157}-\frac{1}{2}e^{247}+\frac{1}{2}e^{267}+e^{345}, e^{147}+e^{346}$		
0, 0, 12, 13, 14 + 23, 0, 16 + 25 − 34	e_7	$[0, 0, e^{12}, e^{13}, e^{23}+e^{14}, 0]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}, e^{145}+e^{127}, e^{146}, e^{234}, e^{127}+e^{235}, e^{236}, e^{237}+e^{156}, e^{137}+e^{245}, e^{246}+e^{156}, e^{147}+e^{345}, e^{346}-e^{256}$		
0, 0, 12, 13, 14, 23, 15 + 26	e_7 (P)	$[0, 0, e^{12}, e^{13}, e^{14}, e^{23}]$
0, 0, 12, 13, 14, 23, 16 + 24 + 25 − 34	e_7 (P)	$[0, 0, e^{12}, e^{13}, e^{14}, e^{23}]$
0, 0, 12, 13, 14, 23, 15 + 25 + 26 − 34	e_7 (P)	$[0, 0, e^{12}, e^{13}, e^{14}, e^{23}]$
0, 0, 12, 13, 0, 14 + 25, 16 + 35	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{25}+e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{127}+e^{136}, e^{137}, e^{145}, e^{156}, e^{157}, e^{234}, e^{235}, e^{245}-e^{127}, -e^{237}+e^{246}, e^{256}-e^{146}, e^{345}+e^{146}, -e^{257}+e^{356}-e^{147}, -\frac{1}{2}e^{167}-\frac{1}{2}e^{357}+e^{456}$		

Table 5 (*continued*)

$0, 0, 12, 13, 0, 14+25, 16+25+35$	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{25}+e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}+e^{127}, e^{137}+e^{127}, e^{145}, e^{156}, e^{157}, e^{234}, e^{235}, e^{245}-e^{127}, e^{246}-e^{237}, e^{256}-e^{146}, e^{345}+e^{146}, -e^{257}+e^{356}+e^{146}-e^{147}, \frac{1}{2}e^{257}-\frac{1}{2}e^{357}+e^{456}-\frac{1}{2}e^{167}$		
$0, 0, 12, 13, 0, 14+25, 26-34$	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{25}+e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{145}, e^{156}, e^{234}, e^{235}, e^{236}+e^{127}, e^{237}, e^{245}+e^{136}, e^{246}+e^{137}, e^{256}-e^{146}, e^{146}+e^{345}, e^{147}+e^{257}+e^{346}$		
$0, 0, 12, 13, 0, 14+25, 15+26-34$	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{14}+e^{25}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{145}, e^{156}, e^{234}, e^{235}, e^{127}+e^{236}, e^{136}+e^{237}, e^{136}+e^{245}, e^{137}+e^{246}, e^{256}-e^{146}, e^{345}+e^{146}, e^{257}+e^{346}+e^{147}$		
$0, 0, 12, 13, 0, 14+23+25, 16+24+35$	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{14}+e^{25}+e^{23}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}+e^{127}, e^{145}, e^{156}-e^{127}, e^{157}+e^{137}+e^{146}, e^{234}, e^{235}, e^{137}+e^{236}, e^{245}-e^{127}, -e^{237}+e^{246}, e^{256}-e^{137}-e^{146}, e^{345}+e^{137}+e^{146}, -e^{257}-e^{147}+e^{356}$		
$0, 0, 12, 13, 0, 14+23+25, 26-34$	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{23}+e^{14}+e^{25}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{145}, e^{156}+e^{136}, e^{234}, e^{235}, e^{236}+e^{127}, e^{237}, e^{245}+e^{136}, e^{246}+e^{137}, e^{256}-e^{146}-e^{127}, e^{345}+e^{146}+e^{127}, e^{346}+e^{257}+e^{147}$		
$0, 0, 12, 13, 0, 14+23+25, 15+26-34$	e_7	$[0, 0, e^{12}, e^{13}, 0, e^{23}+e^{25}+e^{14}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{145}, e^{136}+e^{156}, e^{234}, e^{235}, e^{127}+e^{236}, e^{136}+e^{237}, e^{136}+e^{245}, e^{137}+e^{246}, -e^{127}+e^{256}-e^{146}, e^{345}+e^{127}+e^{146}, e^{346}+e^{147}+e^{257}$		
$0, 0, 12, 13, 23, 15+24, 16+34$	resists	
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{127}+e^{136}, e^{137}, -e^{127}+e^{145}, e^{146}, e^{147}, e^{234}, e^{235}, e^{245}-e^{236}, e^{246}-e^{156}-2e^{237}, e^{256}, e^{345}-e^{156}-e^{237}, e^{346}-e^{247}-e^{157}, e^{267}-e^{357}+e^{456}$		
$0, 0, 12, 13, 23, 15+24, 16+25+34$	resists	
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}+e^{127}, e^{145}-e^{127}, e^{146}, e^{234}, e^{235}, e^{236}+e^{137}, e^{237}+e^{147}+e^{156}, e^{137}+e^{245}, 2e^{147}+e^{156}+e^{246}, e^{256}, e^{345}+e^{147}, e^{346}-e^{247}-e^{157}, e^{456}-e^{357}+e^{267}$		
$0, 0, 12, 13, 23, 15+24, 16+14+25+34$	e_7	$[0, 0, e^{12}, e^{13}, e^{23}, e^{24}+e^{15}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}+e^{127}, e^{145}-e^{127}, e^{146}, e^{234}, e^{235}, e^{236}+e^{137}, e^{237}+e^{156}+e^{147}-e^{127}, e^{245}+e^{137}, e^{246}+e^{156}+2e^{147}, e^{256}, e^{345}+e^{147}, e^{346}-e^{247}-e^{157}$		
$0, 0, 12, 13, 23, 15+24, 16+14+34$	e_7	$[0, 0, e^{12}, e^{13}, e^{23}, e^{24}+e^{15}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{127}+e^{136}, e^{137}, -e^{127}+e^{145}, e^{146}, e^{147}, e^{234}, e^{235}, -e^{236}+e^{245}, 2e^{127}-2e^{237}-e^{156}+e^{246}, e^{256}, e^{127}-e^{237}-e^{156}+e^{345}, -e^{247}-e^{157}+e^{346}$		
$0, 0, 12, 13, 23, 15+24, 16+26+34-35$	e_7	$[0, 0, e^{12}, e^{13}, e^{23}, e^{24}+e^{15}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{136}+e^{145}, e^{146}, e^{234}, e^{235}, e^{127}+e^{136}+e^{236}, e^{137}+e^{237}, e^{127}+e^{136}+e^{245}, e^{246}+2e^{137}-e^{156}, e^{256}, e^{345}+e^{137}-e^{156}, -e^{346}+e^{356}+e^{247}+e^{157}$		
$0, 0, 0, 12, 14+23, 15-34, 16-35$	e_7	$[0, 0, 0, e^{12}, e^{23}+e^{14}, -e^{34}+e^{15}]$
$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{126}+e^{145}, e^{146}, e^{234}, e^{126}+e^{235}, e^{127}+e^{236}, e^{237}+e^{147}+e^{156}, e^{245}, -e^{127}+e^{345}, -2e^{147}+e^{346}-e^{156}, -e^{347}+e^{356}+e^{157}$		
$0, 0, 0, 12, 14+23, 15-34, 16+23-35$	e_7	$[0, 0, 0, e^{12}, e^{23}+e^{14}, e^{15}-e^{34}]$
$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}+e^{126}, e^{146}, e^{234}, e^{235}+e^{126}, e^{236}+e^{127}, e^{237}+e^{126}+e^{156}+e^{147}, e^{245}, e^{345}-e^{127}, e^{346}-2e^{126}-e^{156}-2e^{147}, e^{356}-e^{347}+e^{127}+e^{157}$		
$0, 0, 0, 12, 14+23, 15-34, 16+24-35$	e_7	$[0, 0, 0, e^{12}, e^{14}+e^{23}, e^{15}-e^{34}]$
$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, -e^{126}+e^{137}, e^{126}+e^{145}, e^{146}, e^{234}, e^{126}+e^{235}, e^{236}+e^{127}, e^{156}+e^{147}+e^{237}, e^{245}, e^{345}-e^{127}, -e^{156}+e^{346}-2e^{147}, -e^{246}+e^{356}-e^{347}+e^{157}$		
$0, 0, 12, 13, 23, 24+15, 16+14-25+34$	e_7	$[0, 0, e^{12}, e^{13}, e^{23}, e^{24}+e^{15}]$
$e^{123}, e^{124}, e^{125}, e^{126}, e^{134}, e^{135}, e^{127}+e^{136}, -e^{127}+e^{145}, e^{146}, e^{234}, e^{235}, e^{236}-e^{137}, -e^{127}+e^{237}-e^{147}+e^{156}, -e^{137}+e^{245}, -2e^{147}+e^{156}+e^{246}, e^{256}, -e^{147}+e^{345}, -e^{247}-e^{157}+e^{346}$		
$0, 0, 12, 13, 23, -14-25, 16-35$	e_7 (P)	$[0, 0, e^{12}, e^{13}, e^{23}, -e^{14}-e^{25}]$
$0, 0, 12, 13, 23, -14-25, 16-35+25$	e_7 (P)	$[0, 0, e^{12}, e^{13}, e^{23}, -e^{25}-e^{14}]$
$0, 0, 0, 12, 14+23, 15-34, 16-23-35$	e_7	$[0, 0, 0, e^{12}, e^{23}+e^{14}, e^{15}-e^{34}]$
$e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}+e^{126}, e^{146}, e^{234}, e^{235}+e^{126}, e^{236}+e^{127}, e^{237}-e^{126}+e^{156}+e^{147}, e^{245}, e^{345}-e^{127}, e^{346}+2e^{126}-e^{156}-2e^{147}, e^{356}-e^{347}-e^{127}+e^{157}$		

References

[1] V. Apostolov, S. Salamon, Kähler reduction of metrics with holonomy G_2, Comm. Math. Phys. 246 (2004) 43–61.

[2] C. Benson, C. Gordon, Kähler and symplectic structures on nilmanifolds, Topology 27 (1988) 513–518.

[3] R.L. Bryant, Some remarks on G_2-structures, in: Proceedings of Gökova Geometry–Topology Conference 2005, Gökova Geometry/Topology Conference (GGT), Gökova, 2006, pp. 75–109.
[4] R. Bryant, S. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (1989) 829–850.
[5] S. Chiossi, A. Fino, Conformally parallel G_2 structures on a class of solvmanifolds, Math. Z. 252 (2006) 825–848.
[6] R. Cleyton, S. Ivanov, On the geometry of closed G_2-structures, Comm. Math. Phys. 270 (2007) 53–67.
[7] R. Cleyton, A. Swann, Cohomogeneity-one G_2-structures, J. Geom. Phys. 44 (2002) 202–220.
[8] D. Conti, A. Tomassini, Special symplectic six-manifolds, Q. J. Math. 58 (2007) 297–311.
[9] D. Conti, Half-flat nilmanifolds, Math. Ann. 350 (2011) 155–168.
[10] M. Fernández, A. Gray, Riemannian manifolds with structure group G_2, Ann. Mat. Pura Appl. 32 (1982) 19–45.
[11] M. Fernández, An example of a compact calibrated manifold associated with the exceptional Lie group G_2, J. Differential Geom. 26 (1987) 367–370.
[12] M.-P. Gong, Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and $\mathbb{R}$), PhD thesis, University of Waterloo, Ontario, Canada, 1998.
[13] A. Gray, Vector cross products on manifolds, Trans. Amer. Math. Soc. 141 (1969) 463–504.
[14] R. Harvey, H.B. Lawson, Calibrated geometries, Acta Math. 148 (3) (1982) 47–157.
[15] N. Hitchin, Stable forms and special metrics, in: Global Differential Geometry: The Mathematical Legacy of Alfred Gray, in: Contemp. Math., vol. 288, Amer. Math. Soc., 2001, pp. 70–89.
[16] D.D. Joyce, Compact Riemannian 7-manifolds with holonomy G_2, I and II, J. Differential Geom. 43 (1996) 291–328, and 329–375.
[17] L. Magnin, Sur les algèbres de Lie nilpotentes de dimension $\leqslant 7$, J. Geom. Phys. 3 (1986) 119–144.
[18] A.I. Malcev, On a class of homogeneous spaces, Amer. Math. Soc. Transl. Ser. 1 (9) (1962) 276–307.
[19] S. Salamon, Complex structures on nilpotent Lie algebras, J. Pure Appl. Algebra 157 (2001) 311–333.