Abstract

We investigate the maximum number of elements in an optimal $t$-friendship decomposition of graphs of order $n$. Asymptotic results will be obtained for all fixed $t \geq 4$ and for $t = 2, 3$ exact results will be derived.

Keywords: Graph decomposition; Friendship

1. Introduction and terminology

For notation and terminology not discussed here the reader is referred to [1]. All graphs considered here are finite and simple, i.e., they have no loops or multiple edges. Let $G$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The set of neighbors of $v$ is denoted by $N_G(v)$ or briefly by $N(v)$ if it is clear which graph is being considered. We always have $v \notin N(v)$. Let $N_G(v) = V - (N_G(v) \cup \{v\})$. A clique is a complete graph and the complete bipartite graph with parts of size $m$ and $n$ will be denoted by $K_{m,n}$. A graph of the form $K_1,m$ with possible some isolated vertices will be called a star. The center of a star is its vertex of maximum degree.

Let $t \geq 1$ be fixed. A graph that consists of $t$ edge disjoint cliques sharing a vertex $v$ is said to be a $t$-friendship graph ($t$-fs graph) with center $v$. A $t$-friendship decomposition of a graph $G$ is a set, $\mathcal{F}$, of edge disjoint $t$-friendship subgraphs of $G$, such that any edge of $G$ is an edge of exactly one element of $\mathcal{F}$. For $t \geq 1$ let $\phi_t(G)$ denote the minimum number of $t$-friendship graphs in a $t$-friendship decomposition of the graph $G$ and $\phi_t(n) = \max \phi_t(G)$ where $G$ runs over all graphs of order $n$. In [2] Erdös, Goodman and Pósa proved that $\phi_1(n) = \lfloor n^2/4 \rfloor$. In our paper we will study the function $\phi_t(n)$ for every fixed $t \geq 2$. For $t \geq 4$ asymptotic results will be obtained and for $t = 2, 3$ the exact value will be derived.

2. $t$-Friendship decompositions

We start by studying the decomposition problem of any graph of order $n$ into $t$-friendship graphs, for all fixed $t \geq 2$. This problem will be asymptotically solved. However, for $t = 2, 3$ we will be able to improve our proof in order to obtain exact results.

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*Research done while the author was at Carnegie Mellon University and partially supported by the Portuguese Science Foundation under Grant SFRH/BD/8617/2002.

E-mail address: tmjs@fct.unl.pt.
Theorem 2.1. Any graph of order \( n \) admits a \( t \)-friendship decomposition with at most \( n^2/4t + n/4t + n \) elements, for all fixed \( t \geq 2 \).

To prove the theorem we will need the following trivial lemma:

Lemma 2.2 (Győri and Kostochka [3]). Let \( G \) be an arbitrary graph of order \( n \) such that \( \deg v \geq d \) for every vertex \( v \) of \( G \). Then there exists a partition of \( V(G) \) such that the number of the classes of the partition is \( n - d \) and such that every class of the partition spans a clique.

Proof of Theorem 2.1. By induction on the number of vertices in a graph. The theorem is obviously true for \( n = 2 \). Let \( v \) be a vertex of minimum degree and let \( \delta =: \delta(G) \).

If \( \delta \leq n/2 \) we consider the edges incident with \( v \) as \( \lceil \deg v/t \rceil \) elements of a \( t \)-fs decomposition and consider an optimal decomposition of \( G - v \). Then

\[
\phi_t(G) \leq \phi_t(G - v) + \left\lceil \frac{\delta}{t} \right\rceil.
\] (2.1)

If \( \delta > n/2 \) then \( \deg G[N(v)](u) =: \delta - 1 - (n - \delta - 1) = 2\delta - n > 0 \), for all \( u \in N(v) \). Lemma 2.2 implies that there exists a partition \( \{R_1, \ldots, R_{n-\delta}\} \) of \( N(v) \) such that \( G[R_i] \) is a clique, for \( i = 1, \ldots, n - \delta \). Then \( G[R_i \cup \{v\}] \) is also a clique for \( i = 1, \ldots, n - \delta \). Therefore by grouping all these cliques into \( t \)-tuples we obtain \( \lceil (n - \delta)/t \rceil \) \( t \)-fs graphs and combining them with an optimal \( t \)-fs decomposition of the remaining graph \( H \), where \( V(H) = V \setminus \{v\} \) and \( E(H) = E(G) - \bigcup_{i=1}^{n-\delta} E(G[R_i \cup \{v\}]) \), we obtain

\[
\phi_t(G) \leq \phi_t(H) + \left\lceil \frac{n - \delta}{t} \right\rceil.
\] (2.2)

Let \( n = 2tk + r \), with \( 0 \leq r < 2t \). If \( \delta \leq n/2 \) then from (2.1) we obtain

\[
\phi_t(G) \leq \phi_t(G - v) + k + 1.
\]

If \( \delta > n/2 \) then (2.2) implies that

\[
\phi_t(G) \leq \phi_t(H) + k + 1.
\]

Therefore,

\[
\phi_t(G) \leq \frac{(n - 1)^2}{4t} + \frac{n - 1}{4t} + n - 1 + k + 1
\]
\[
\leq \frac{(n - 1)^2}{4t} + \frac{n - 1}{4t} + n - 1 + \frac{n - r}{2t} + 1
\]
\[
\leq \frac{n^2}{4t} + \frac{n}{4t} + n. \quad \Box
\]

By considering the graph \( K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} \) we trivially have that \( \phi_t(n) \geq \phi_t(K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}) \geq \lceil n/2 \rceil \lceil n/2 \rceil/t \geq (n^2 - 1)/4t \). Therefore, for fixed \( t \geq 2 \) the decomposition problem into \( t \)-friendship graphs is asymptotically solved and we have the following theorem.

Theorem 2.3.

\[
\phi_t(n) = \left( \frac{1}{4t} + o(1) \right)n^2 \quad \text{for all fixed } t \geq 2.
\]

3. \( 2 \)-Friendship decompositions

For the special case of \( t = 2 \) the proof of Theorem 2.1 can be improved to give us the exact value of \( \phi_2(n) \), for all \( n \geq 1 \).
4.2. Lemma. Theorem types of graphs needed later.

3.1. Theorem. Any graph of order \( n \) can be decomposed into at most \( \lceil n^2/8 \rceil \) 2-friendship graphs if \( n \) is even and at most \( (n^2 - 1)/8 \) 2-friendship graphs if \( n \) is odd. Moreover, this bound is sharp for the bipartite graph \( K_{\lceil n/2 \rceil, \lceil n/2 \rceil} \).

Proof. By induction on the number of vertices in a graph. Clearly the result holds for all graphs with at most three vertices. Let \( G \) be a graph having \( n \) vertices where \( n \geq 4 \). Let \( v \) be a vertex of minimum degree and let \( \delta := \delta(G) \). Let \( n = 4k + r \), with \( 0 \leq r \leq 3 \) and \( k \geq 1 \). The theorem follows directly from (2.1) or (2.2) and the induction hypothesis for all values of \( n \) except for \( n = 4k + 3 \) and \( \delta \in \{2k + 1, 2k + 2\} \).

Let \( n = 4k + 3 \) and \( \delta = 2k + 1 \). If \( G[N(v)] \) has an edge, say \( e \), then the edge \( e \) and the edges incident with \( v \) form at most \( k \) 2-fs graphs. Hence

\[
\phi_2(G) \leq \phi_2(G - v - e) + k \leq \left\lceil \frac{(n - 1)^2}{8} \right\rceil + k = \frac{n^2 - 1}{8}.
\]

Assume that \( G[N(v)] \) has no edges and let \( u \in N(v) \). Then \( \deg \ u \leq 2k + 2 \) and we have

\[
\phi_2(G) \leq \phi_2(G - \{v, u\}) + \left\lceil \frac{\deg v - 1}{2} \right\rceil + \left\lceil \frac{\deg u}{2} \right\rceil = \frac{n^2 - 1}{8}.
\]

Finally, let \( n = 4k + 3 \) and \( \delta = 2k + 2 \). In this case \( \deg G[N(v)](u) \geq 1 \), for all \( u \in N(v) \). If \( \deg G[N(v)](u) \geq 2 \), for all \( u \in N(v) \), then by Lemma 2.2 the edges incident with \( v \) can be decomposed into at most \( k \) 2-fs graphs and the result follows by the induction hypothesis. Suppose that \( \deg G[N(v)](u) = 1 \) for some \( u \in N(v) \). In this case observe that \( \deg u = 2k + 2 \) and let \( w \in N(v) \) be adjacent to \( u \). Consider the 2-fs graph, \( F \), with edges \( uv, vw, uw \) and \( uw' \) for some \( w' \in N(u) - \{v, w\} \). Observe that such \( w' \) exists since we assumed that \( k \geq 1 \). Let \( G_1 \) be the graph obtained after deleting the edges of \( F \). Then \( \deg_{G_1}(v) = 2k \) and \( \deg_{G_1}(u) = 2k - 1 \). Therefore,

\[
\phi_2(G) \leq \phi_2(G_1 - \{v, u\}) + \left\lceil \frac{\deg_{G_1}(v)}{2} \right\rceil + \left\lceil \frac{\deg_{G_1}(u)}{2} \right\rceil + 1 = \frac{(n - 2)^2 - 1}{8} + 2k + 1 = \frac{n^2 - 1}{8}.
\]

To complete the proof it remains to see that the bound is sharp for \( K_{\lceil n/2 \rceil, \lceil n/2 \rceil} \). We trivially have \( \phi_2(K_{\lceil n/2 \rceil, \lceil n/2 \rceil}) \geq \lceil |E(K_{\lceil n/2 \rceil, \lceil n/2 \rceil})|/2 \rceil \), therefore \( \phi_2(K_{\lceil n/2 \rceil, \lceil n/2 \rceil}) \geq \lceil n^2/8 \rceil \) if \( n \) is even and \( \phi_2(K_{\lceil n/2 \rceil, \lceil n/2 \rceil}) \geq (n^2 - 1)/8 \) if \( n \) is odd, hence the equality. \( \square \)

4. 3-Friendship decompositions

As we did for 2-friendship decompositions of graphs we can also obtain exact results for 3-friendship decompositions of graphs. However, in this case, the calculations will not be so straightforward as they were for the case \( t = 2 \). Next theorem is our main result of this section.

4.1. Theorem. Any graph of order \( n \), except the 5-cycle, can be decomposed into at most \( \lceil n^2/12 \rceil \) 3-friendship graphs if \( n \) is even and at most \( \lceil (n^2 - 1)/12 \rceil \) 3-friendship graphs if \( n \) is odd. Moreover, this bound is sharp for the bipartite graph \( K_{\lceil n/2 \rceil, \lceil n/2 \rceil} \).

Before proving the general case, and for the sake of simplicity, we will see that Theorem 4.1 holds for some special types of graphs needed later.

4.2. Lemma. Theorem 4.1 holds for bipartite graphs.

Proof. Let \( G \) be a bipartite graph of order \( n \) with parts \( A \) and \( B \) of size \( s \) and \( t \), respectively. We will see that \( \phi_3(G) \leq \lceil st/3 \rceil \).

Let \( s = 3k_1 + r_1 \) and \( t = 3k_2 + r_2 \) with \( 0 \leq r_1 \leq 2 \) and \( 0 \leq r_2 \leq 2 \). Assume \( r_1 \leq r_2 \). Let \( A = A_1 \cup A_2 \) where \( |A_1| = 3k_1 \) and \( |A_2| = r_1 \) and \( B = B_1 \cup B_2 \) where \( |B_1| = 3k_2 \) and \( |B_2| = r_2 \). Consider the subgraph induced by \( A \cup B_1 \). Clearly its edges
can be decomposed into at most  sk_2 3-fs graphs. Now consider the subgraph induced by A_1 ∪ B_2 and remove all 3-fs graphs with centers in B_2, at most r_2k_1 of them. Finally, observe that the edges left have endpoints in A_2 ∪ B_2, hence it suffices to remove all 3-fs graphs with centers in A_2, at most r_1 of them. Therefore, \( \phi_3(G) \leq s k_2 + r_2 k_1 + r_1 \leq \lfloor st/3 \rfloor \), where the last inequality follows from the fact that 0 ≤ r_1 ≤ r_2 ≤ 2. Moreover, \( \lfloor st/3 \rfloor \) is at most \( \lceil (s + t)^2/12 \rceil \) if s + t is even and at most \( \lceil (s + t) - 1 \rceil/12 \) if s + t is odd.

To complete the proof it remains to observe that for \( s = \lfloor n/2 \rfloor \) and \( t = \lfloor n/2 \rfloor \), \( \lceil n^2/12 \rceil \) equals \( \lfloor n^2/12 \rfloor \) if n is even and \( \lfloor (n^2 - 1)/12 \rfloor \) if n is odd. □

Given a graph G and two disjoint sets A, B ⊆ V (G) we denote by G[A, B] the bipartite graph with all the edges of G with one endpoint in A and the other in B. Recall that a star with center y is a graph of the form \( K_{1,m} \) with possible some isolated vertices where y is its vertex of maximum degree.

**Lemma 4.3.** Let G be a graph with \( n = 6k + 5 \) vertices and minimum degree 3k + 2. Let v be a vertex of minimum degree and assume that G[N(v)] has no edges, G[\( \overline{N}(v) \)] is a star or an empty graph and G[N(v), \( \overline{N}(v) \)] is a complete bipartite graph. Then G admits a 3-friendship decomposition with at most \( \lceil (n^2 - 1)/12 \rceil = 3k^2 + 5k + 2 \) elements.

**Proof.** Let \( N(v) = \{u_1, \ldots, u_{3k+1}, u\} \) and \( \overline{N}(v) = \{y_1, \ldots, y_{3k+1}, y\} \) and let y be the center of the star (in case G[\( \overline{N}(v) \)] is an empty graph, y is any vertex in \( \overline{N}(v) \)). For i = 1, \ldots, 3k + 1 remove the 3-fs graph induced by the vertices \( u_i, y_i, y \). In this step we removed \( 3k + 1 \) 3-fs graphs. Now G[\( \overline{N}(v) \)] has no edges and we are left with a bipartite graph. Hence it suffices to remove all 3-fs graphs with centers in N(v). Observe that in the remaining graph \( \deg u_i = 3k \) for i = 1, \ldots, 3k + 1 and \( \deg u = 3k + 3 \) so all these edges can be decomposed into at most \( k(3k + 1) + k + 1 \) 3-fs graphs. Therefore,

\[
\phi_3(G) \leq 3k + 1 + k(3k + 1) + k + 1 = 3k^2 + 5k + 2.
\]

**Lemma 4.4.** Let G be a graph with \( n = 6k + 5 \) vertices and minimum degree 3k + 3. Let v be a vertex of minimum degree and assume that G[N(v)] = \( K_{1,3k+2} \), G[\( \overline{N}(v) \)] has no edges, and G[N(v), \( \overline{N}(v) \)] is a complete bipartite graph. Then G admits a 3-friendship decomposition with at most \( 3k^2 + 4k + 2 < \lceil (n^2 - 1)/12 \rceil \) elements.

**Proof.** Let \( N(v) = \{u_1, \ldots, u_{3k+2}, u\} \), let u be adjacent to all vertices of N(v) and let \( \overline{N}(v) = \{y_1, \ldots, y_{3k+1}\} \). The edges incident with u and the edges uw_{3k+2}, u_1 y_1, u_2 y_2, \ldots, u_{3k+1} y_{3k+1} can be decomposed using \( k + 1 \) 3-fs graphs as described by Fig. 1. Remove all edges incident with \( y_i \) except the edge \( y_i u_{3k+2} \) for i = 1, \ldots, 3k and all edges incident with \( y_{3k+1} \) except the edge \( y_{3k+1} u_1 \). In total we removed \( k(3k + 1) \) 3-fs graphs. The edges left can be decomposed using \( 2k + 1 \) 3-fs graphs. Therefore,

\[
\phi_3(G) \leq k + 1 + k(3k + 1) + 2k + 1 = 3k^2 + 4k + 2.
\]

**Lemma 4.5.** Let G be a graph with \( n = 6k + 5 \) vertices and minimum degree 3k + 3. Let v be a vertex of minimum degree. Assume that G[\( \overline{N}(v) \)] has no edges, G[N(v), \( \overline{N}(v) \)] is a complete bipartite graph and G[N(v)] looks like the graph described in Fig. 2 with the edge \( \{w_1, w_2\} \) being present or not. Then G admits a 3-friendship decomposition with at most \( \lceil (n^2 - 1)/12 \rceil = 3k^2 + 5k + 2 \) elements.

**Proof.** Let \( N(v) = \{u_1, \ldots, u_{3k+1}, w_1, w_2\} \), \( \overline{N}(v) = \{y_1, \ldots, y_{3k+1}\} \) and \( \deg G[N(v)] u_i = 2 \) for all i = 1, \ldots, 3k + 1.
We first consider the case $3k + 1$ even. We start our 3-fs decomposition by removing the 3-fs graphs with edge sets $y_iu_i, y_iw_1, uy_iw_2, y_iu_{i+1}, u_{i+1}w_2$, and $y_{i+1}u_i, y_{i+1}w_2, uy_{i+1}w_1, y_{i+1}u_{i+1}, u_{i+1}w_1$, where $1 \leq i \leq 3k + 1$ and $i$ is odd (See Fig. 3). We remove $3k + 1$ 3-fs graphs. After this first step the edges left are incident with the vertices $v, y_1, \ldots, y_{3k+1}$ and we might also have the edge $w_1w_2$. Furthermore, in the graph left $v$ has degree $3k + 3$ and the vertices $y_1, \ldots, y_{3k+1}$ have degree $3k − 1$. Therefore, the edges incident with $v$ and the edge $w_1w_2$ if it exists, can be decomposed using at most $k + 1$ 3-fs graphs and for $1 \leq i \leq 3k + 1$ the edges incident with $y_i$ can be decomposed using at most $k$ 3-fs graphs.

Therefore,

$$\phi_3(G) \leq 3k + 1 + k + 1 + (3k + 1)k = 3k^2 + 5k + 2.$$

Suppose that $3k + 1$ is odd. We repeat the procedure described before to decompose all the edges incident $y_1, \ldots, y_{3k}$. In total we remove $3k + 3k^2$ 3-fs graphs. If the edge $w_1w_2$ exists then the vertices $w_1, w_2$ and $u_{3k+1}$ induce a triangle, so the edges incident with $v$ plus the above triangle can be decomposed into $k + 1$ 3-fs graphs. The edges left are incident with $y_{3k+1}$ and can be decomposed using $k + 1$ 3-fs graphs. If the edge $w_1w_2$ is not in the graph then the edge $w_1u_{3k+1}$ and the edges incident with $v$ can be decomposed into $k + 1$ 3-fs graphs and the edge $w_2u_{3k+1}$ and the edges incident with $y_{3k+1}$ can be decomposed into $k + 1$ 3-fs graphs and this completes the proof. □

We are now able to prove Theorem 4.1.

**Proof of Theorem 4.1.** By induction on the number of vertices in a graph. By inspection, and using Harary’s [4] atlas of all graphs of order at most 5, we can see that the result holds for all graphs with at most five vertices. Let $G$ be a graph having $n$ vertices, where $n \geq 6$. Let $v$ be a vertex of minimum degree and let $\delta =: \delta(G)$.

Let $n = 6k + r$, with $0 \leq r \leq 5$ and $k \geq 1$. The result follows directly from (2.1) or (2.2) and the induction hypothesis for all values of $n$ except for $n = 6k + 3$ and $\delta \in \{3k + 1, 3k + 2\}$ and for $n = 6k + 5$ and $\delta \in \{3k + 1, 3k + 2, 3k + 3, 3k + 4\}$.

Consider the case $n = 6k + 3$ and $\delta = 3k + 1$. If $G[N(v)]$ has an edge, say $e$, then the edges incident with $v$ and the edge $e$ form at most $k$ 3-fs graphs, hence

$$\phi_3(G) \leq \phi_3(G - v - e) + k \leq \left\lceil \frac{(n - 1)^2}{12} \right\rceil + k = \left\lceil \frac{n^2 - 1}{12} \right\rceil.$$
Assume that $G[N(v)]$ has no edges and let $u \in N(v)$. Then $\deg u \leq 3k + 2$ and we have

$$\phi_3(G) \leq \phi_3(G - \{v, u\}) + \left\lceil \frac{\deg v - 1}{3} \right\rceil + \left\lceil \frac{\deg u}{3} \right\rceil = \left\lceil \frac{n^2 - 1}{12} \right\rceil.$$ 

Let $n = 6k + 3$ and $\delta = 3k + 2$. Then $\deg G[N(v)](u) \geq 1$ for all $u \in N(v)$. If $\deg G[N(v)](u) \geq 2$, for all $u \in N(v)$, then using Lemma 2.2 we can decompose the edges incident with $v$ and some edges in $G[N(v)]$ into at most $k$ 3-fs graphs and the result follows by induction. Suppose that $\deg G[N(v)](u) = 1$ for some $u \in N(v)$. In this case observe that $\deg u = 3k + 2$ and let $w \in N(v)$ be adjacent to $u$. Consider the 3-fs graph, $F$, with edges $vu, vw, uw, vx$ and $vy$ for some $x, y \in N(v) - \{u, w\}$. Observe that such $x, y$ exist since we assumed that $k \geq 1$. Let $G_1$ be the graph obtained after deleting the edges of $F$. Then $\deg G_1(v) = 3k - 2$ and $\deg G_1(u) = 3k$. Therefore,

$$\phi_3(G) \leq \phi_3(G_1 - \{v, u\}) + \left\lceil \frac{\deg G_1(v)}{3} \right\rceil + \left\lceil \frac{\deg G_1(u)}{3} \right\rceil + 1 = \left\lceil \frac{n^2 - 1}{12} \right\rceil.$$ 

From now and until the end of the proof let $n = 6k + 5$. Observe that

$$\left\lceil \frac{n^2 - 1}{12} \right\rceil = \left\lceil \frac{(n - 1)^2}{12} \right\rceil + k = \left\lceil \frac{(n - 2)^2 - 1}{12} \right\rceil + 2k + 1.$$ 

Let $\delta = 3k + 1$. If $G[N(v)]$ has an edge $e$ then the $e$ and the edges incident with $v$ form at most $k$ 3-fs graphs and we are done. Assume that $G[N(v)]$ has no edges and let $u \in N(v)$. Then $\deg u \leq 3k + 4$. By Lemma 4.2 we can assume that $G$ is not bipartite, so $G[N(v)]$ must have at least one edge. Therefore the edges incident with $v$ and $u$ form at most $2k + 1$ 3-fs graphs and the result holds.

Let $\delta = 3k + 2$. If $\deg G[N(v)](u) \geq 2$, for all $u \in N(v)$, then using Lemma 2.2 we can decompose the edges incident with $v$ into at most $k$ 3-fs graphs. Suppose first that exists $u \in N(v)$ such that $\deg G[N(v)](u) = 1$, then $\deg u \leq 3k + 4$. Therefore, the edges incident with $v$ and $u$ can be decomposed into at most $2k + 1$ 3-fs graphs.

Now suppose that for all $u \in N(v)$ we have either $\deg G[N(v)](u) = 0$ or $\deg G[N(v)](u) \geq 2$. If the latter condition happens then it is not hard to see that $N(v)$ must contain at least 2 independent edges or a triangle. Thus, the edges incident with $v$ and $u$ can be decomposed into at most $k$ 3-fs graphs and the result follows by induction. Therefore it remains to consider the case when $\deg G[N(v)](u) = 0$ for all $u \in N(v)$.

(a) If $\deg u = 3k + 3$ for all $u \in N(v)$ then $G[N(v), \overline{N(v)}]$ is a complete bipartite graph. Let $u \in N(v)$. By Lemma 4.2 we can assume that $G$ is not bipartite, that is, $G[N(v)]$ must have at least one edge. If $G[N(v)]$ has two independent edges or a triangle then the edges incident with $v$ and $u$ can be decomposed into at most $k + 1$ and $k$ 3-fs graphs, respectively, and the result follows by induction. If $G[N(v)]$ has at most one independent edge and no triangles, then $G[N(v)]$ is a star and our graph is as in Lemma 4.3 so the result holds.

(b) Suppose now that exists $u \in N(v)$ such that $\deg u = 3k + 2$. Then $\exists y \in \overline{N(v)}$ such that $u$ is not adjacent to $y$. If $G[N(u)]$ has an edge then the edges incident with $v$ and $u$ can be decomposed into at most $2k + 1$ 3-fs graphs. If $G[N(u)]$ has no edges then all edges in $G[N(v)]$ are incident with $y$, i.e., $G[N(v)]$ is a star with center $y$. Assume first that $G[N(v)]$ has exactly one edge, say $uu'$. Then $y$ and $y'$ must have at least one common neighbor in $N(v)$, say $u'$. Remove the 3-fs graphs with edges $u'y, u'y', yy', uu'y_1, uu'y_2$, for some $y_1, y_2 \in N(v) - \{y, y'\}$. The graph left is bipartite, hence it suffices to remove all 3-fs graphs with centers in $N(v)$. Let $u_1, \ldots, u_{3k}$ be the remaining vertices of $N(v)$. Observe that $\deg u_i = 3k + 3$, for $i = 1, \ldots, 3k$. Then,

$$\phi_3(G) \leq 1 + \left\lceil \frac{\deg u}{3} \right\rceil + \left\lceil \frac{\deg u' - 4}{3} \right\rceil + 3k \left\lceil \frac{\deg u_1}{3} \right\rceil \leq 1 + k + 1 + k + 3k(k + 1) = \left\lceil \frac{n^2 - 1}{12} \right\rceil.$$
Now, assume that $G[\overline{N}(v)]$ has at least two edges. If $G[N(u')]$ has an edge for some $u' \in N(v)$, say $yy_1$, we remove the 3-fs graph with edges $u' y, u' y_1, y_1 y, y_1 u, y_1 u''$, for some $u'' \in N(v) - \{u, u'\}$. Then,

$$\phi_3(G) \leq \phi_3(G - \{v, u, u', y_1\}) + \left[\frac{\deg v}{3}\right] + \left[\frac{\deg u - 2}{3}\right] + \left[\frac{\deg u' - 3}{3}\right] + \left[\frac{\deg y_1 - 4}{3}\right] + 1$$

$$\leq \left\lceil \frac{(n - 4)^2 - 1}{12} \right\rceil + \left\lceil \frac{3k}{3} \right\rceil + \left\lceil \frac{3k}{3} \right\rceil + \left\lceil \frac{3k}{3} \right\rceil + \left\lceil \frac{3k - 1}{3} \right\rceil + 1$$

$$\leq 3k^2 + k + 4k + 2 = \frac{n^2 - 1}{12}.$$  

If $N(u')$ has no edge for all $u' \in N(v)$ then $G \subseteq K_{3k + 3, 3k + 2}$ and the result follows from Lemma 4.2.

Let $\delta = 3k + 3$. Then $\deg G[N(v)] \geq 1$, for all $u \in N(v)$. If $\deg G[N(v)] \geq 3$, for all $u \in N(v)$, then by Lemma 2.2 we can decompose the edges incident with $v$ and some other edges in $G[N(v)]$ into at most $k$ 3-fs graphs.

(a) Assume first that exists $u \in N(v)$ such that $\deg G[N(v)]u = 1$, then $\deg u = 3k + 3$. If $G[N(v)]$ has an edge, say $e$, then the edge $e$ and the edges incident with $u$ and $v$ can be decomposed into at most $2k + 1$ 3-fs graphs (see Fig. 4(i)). If $G[\overline{N}(v)]$ has no edges then $G[N(v), \overline{N}(v)]$ is a complete bipartite graph. So, if $G[N(v)]$ has two independent edges then the edges incident with $u$ and $y$, for some $y \in \overline{N}(v)$ can be decomposed into at most $2k + 1$ 3-fs graphs and the result follows (see Fig. 4(ii)). If $G[N(v)]$ has only one independent edge then $G[N(v)] = K_{1, 3k + 2}$ and Lemma 4.4 applies.

(b) Assume that for all $u \in N(v)$, $\deg G[N(v)]u \geq 2$ and that exists $u \in N(v)$ such that $\deg G[N(v)]u = 2$. Let $u$ be adjacent to $w_1$ and $w_2$. Observe that $G[N(v)]$ has at least two independent edges since $\deg G[N(v)]u \geq 2$ for all $u \in N(v)$. If there exist three independent edges in $G[N(v)]$ then the three independent edges and the edges incident with $v$ can be decomposed into $k$ 3-fs graphs. So assume that $G[N(v)]$ has exactly two independent edges. In this case $G[N(v)]$ looks like Fig. 2, unless $k = 1$ and $N(v)$ consists of two vertex disjoint triangles and in this case the result holds. Assume first that $w_1$ and $w_2$ are adjacent. If exists $u' \in N(v) - \{w_1, w_2\}$ such that $\deg u' = 3k + 3$ then the edges incident with $v$ and $u'$ form at most $2k + 1$ 3-fs graphs and we are done. Assume that $\deg u' = 3k + 4$ for all $u' \in N(v) - \{w_1, w_2\}$. If $G[\overline{N}(v)]$ has no edges then Lemma 4.5 applies. If $G[\overline{N}(v)]$ has an edge then we are able to decompose the edges incident with $v$ and $u'$ into at most $2k + 1$ 3-fs graphs (Fig. 5).

Now suppose that $w_1$ and $w_2$ are not adjacent. Then $w_2$ must have at least one neighbor in $\overline{N}(v)$. If $w_2$ has exactly one neighbor in $\overline{N}(v)$, say $y$ then $\deg w_2 = 3k + 3$. Furthermore, $y$ must be adjacent to some $u_1 \in N(v) - \{w_1, w_2\}$. Thus we can decompose the edges incident with $v$ and $w_2$ using at most $2k + 1$ 3-fs graphs (Fig. 6(i)).

Suppose $w_2$ has at least two neighbors in $\overline{N}(v)$ and let $y$ and $y'$ be two of them. If exists $u_1 \in N(v)$ such that $\deg u_1 = 3k + 3$ then $u_1$ is adjacent to at least one of $y$ or $y'$, say $y$ and we apply induction to $G - \{v, u_1\}$. (see Fig. 6(ii)) Assume that $\deg x = 3k + 4$ for all $x \in N(v) - \{w_1, w_2\}$ and let $u_1 \in N(v)$. If $\overline{N}(v)$ has no edges then the result follows from Lemma 4.5. Let $\overline{N}(v)$ have at least one edge, say $e$. If $e = \{y, y'\}$ then there exits a $K_4$ incident with $u_1$, otherwise there exist two triangles incident with $u_1$. In both cases we can decompose the edges incident with $v$ and $u_1$, using at most $2k + 1$ 3-fs graphs.

Let $\delta = 3k + 4$. Then $\deg G[N(v)]u \geq 3$, for all $u \in N(v)$.  

Fig. 4. $\delta = 3k + 3$ case (a).
(a) If \( \text{deg } G[N(v)](u) \geq 4 \) for all \( u \in N(v) \) the result follows by Lemma 2.2 and the induction hypothesis.

(b) Suppose there exists \( u \in N(v) \) such that \( \text{deg } G[N(v)](u) = 3 \). Then \( \text{deg } G(u) = 3k + 4 \). Observe that \( u \) is adjacent to all elements of \( \overline{N}(v) \). Let \( u \) be adjacent to \( u_1, u_2, u_3 \) in \( N(v) \). Because of degree constraints, \( u_3 \) is adjacent to both \( u_1 \) and \( u_2 \) or has a neighbor in \( \overline{N}(v) \), say \( w \). We have to consider three distinct cases.

(i) \( G[N(v)] \) has an edge, say \( xy \) and \( u_3 \) is adjacent to \( u_1 \) or \( u_2 \), say \( u_2 \);
(ii) \( G[N(v)] \) has an edge, say \( xy \) and \( u_3 \) is not adjacent to \( u_1 \) and \( u_2 \). In this case \( u_3 \) has a neighbor in \( \overline{N}(v) - \{x, y\} \), say \( w \);
(iii) \( G[N(v)] \) has no edges. In this case observe that all vertices in \( \overline{N}(v) \) are adjacent to all vertices in \( N(v) \).

In all these cases Fig. 7 shows that we can always decompose the edges incident with \( v \) and \( u \) using at most \( 2k + 1 \) 3-sf graphs. This completes the proof. □
Open questions: It remains an interesting problem to better estimate the function $\phi_t(n)$ for a fixed $t \geq 4$, now known to satisfy $n^2/4t \leq \phi_t(n) \leq n^2/4t + n/4t + n$. The first open instance of this problem is the case $t = 4$ and in this case we conjecture that $\phi_4(n)$ equals $\lceil n^2/16 \rceil$ if $n$ is even and $\lceil (n^2 - 1)/16 \rceil$ if $n$ is odd.

Acknowledgements

The author would like to thank Oleg Pikhurko for helpful discussions and comments and the anonymous referee for the careful reading of this paper.

References