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A boundary value problem for nonlinear hyperbolic equations with order degeneration

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Abstract

In this paper we study the equation $L(u) := k(y)u_{xx} - \partial_y(\ell(y)u_y) + a(x,y)u_x + b(x,y)u_y = f(x,y,u)$, where $k(y) > 0$, $\ell(y) > 0$ for $y > 0$, $k(0) = \ell(0) = 0$; it is strictly hyperbolic for $y > 0$ and its order degenerates on the line $y = 0$. Consider the boundary value problem $Lu = f(x,y,u)$ in G, $u|_{AC} = 0$, where G is a simply connected domain in $\mathcal{R}^2$ with piecewise smooth boundary $\partial G = AB \cup AC \cup BC$; $AB = \{(x,0)\colon\ 0 \leqslant x \leqslant 1\}$, AC: $x = F(y) = \int_0^y (k(t)/\ell(t))^{1/2}\,dt$ and BC: $x = 1 - F(y)$ are characteristic curves. If $f(x,y,u) = g(x,y,u) - r(x,y)u|u|^\rho$, $\rho \geqslant 0$, we obtain existence of generalized solution by a finite element method. The uniqueness problem is considered under less restrictive assumptions on f. Namely, we prove that if f satisfies Carathéodory condition and $|f(x,y,z_1) - f(x,y,z_2)| \leqslant C(|z_1|^\beta + |z_2|^\beta)|z_1 - z_2|$ with some constants $C > 0$ and $\beta \geqslant 0$ then there exists at most one generalized solution.
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1. Introduction

Boundary value problems for degenerated hyperbolic equations in the plane have been studied by many authors (see [3,4,16,17] and the bibliography therein), but mainly in the case where the type (but not the order) of the corresponding differential operator degenerates. The case of order degeneration (where the entire principal part of the differential operator vanishes on the line of degeneration) is not studied so well. Bitsadze [3] observed that boundary value problems for

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hyperbolic equations with order degeneration deserve a special attention and require a special treatment. For such equations the classical boundary value problems are not well posed, and moreover the coefficients of lower order terms determine whether a given boundary value problem is well posed (see [2,7,8,14, 15] and the literature cited therein).

Consider the equation

$$L(u) := k(y)u_{xx} - \partial_y\big(\ell(y)u_y\big) + a(x,y)u_x + b(x,y)u_y = f(x,y,u), \quad (1)$$

where $k(y) > 0, \ell(y) > 0$ for $y > 0$, $k(0) = \ell(0) = 0$ and $\lim_{y\to 0} k(y)/\ell(y)$ exists. Equation (1) is strictly hyperbolic for $y > 0$ and its principal part degenerates on the line $y = 0$.

Let G be a simply connected domain on the (x, y) plane with piecewise smooth boundary $\partial G = AB \cup AC \cup BC$, where $AB = \{(x,0)\colon\ 0 \leqslant x \leqslant 1\}$, and AC: $x = F(y) = \int_0^y (k(t)/\ell(t))^{1/2}\,dt$ and BC: $x = 1 - F(y)$ are characteristics of (1) issued from the point $C(1/2, Y)$, where the constant $Y > 0$ is determined by $F(Y) = 1/2$.

We consider the following boundary value problem.

Problem B. Find in the domain G a solution of (1) satisfying the boundary condition $u = 0$ on AC.

Set

$$(u, v)_0 = \int_G u(x,y)v(x,y)\,dx\,dy, \qquad \|u\|_0 = (u,u)_0^{1/2}$$

and

$$(u, v)_1 = \int_G [u_x v_x + u_y v_y + uv]\,dx\,dy, \qquad \|u\|_1 = (u,u)_1^{1/2}.$$

Let $C^p_{AC}(\overline{G})$ and $C^p_{BC}(\overline{G})$, $p = 1, 2, \dots, \infty$, be the sets of functions $u, v \in C^p(\overline{G})$ such that, respectively, $u|_{AC} = 0$ or $v|_{BC} = 0$. Denote, respectively, by H^1, H^1_{AC}, H^1_{BC} the corresponding Sobolev spaces defined as completions of the spaces $C^\infty(\overline{G})$, $C^\infty_{AC}(\overline{G})$ and $C^\infty_{BC}(\overline{G})$ with respect to the norm $\|\cdot\|_1$.

Let

$$B[u, v] = \int_G \{-ku_x v_x + \ell u_y v_y + au_x v + bu_y v\}\,dx\,dy.$$

Definition 1. A function $u \in H^1_{AC}$ is called generalized solution of Problem B if the identity

$$B[u, v] = \int_G f(x,y,u)v\,dx\,dy \quad (2)$$

holds for every $v \in H^1_{BC}$.

Definition 2. A function $u \in H^1_{AC}$ is called strong solution of Problem B if there exists a sequence $(u_n)_{n=1}^{\infty}$, $u_n \in C^{\infty}_{AC}(\overline{G})$ such that

$$\|u_n - u\|_1 \to 0, \quad \big\|Lu_n - f\big(x, y, u_n(x, y)\big)\big\|_0 \to 0 \quad \text{as } n \to \infty.$$

The paper consists of 4 sections. We obtain by energy-integral method (see [13]) the necessary a priori estimates in Section 2. For technical reason we use weighted norms defined by the weight $\exp(-\lambda x)$, $\lambda > 0$. Although the corresponding weighted norms are equivalent, respectively, to the norms $\|\cdot\|_0$ and $\|\cdot\|_1$, they play an important role in proving our uniqueness result.

In Section 3 we assume that $f(x, y, u) = g(x, y, u) - r(x, y)u|u|^{\rho}$, $\rho \geqslant 0$, and prove by a finite element method existence of generalized solution of Problem B (Theorem 3.1).

In Section 4 we obtain under very mild restriction on f that each generalized solution of Problem B is a strong solution of the same problem (Theorem 4.2). This fact is used in Theorem 4.3 to prove that Problem B has at most one generalized solution under the assumption

$$|f(x, y, z_1) - f(x, y, z_2)| \leqslant C\big(|z_1|^{\beta} + |z_2|^{\beta}\big)|z_1 - z_2|, \quad C > 0,\ \beta \geqslant 0.$$

Observe that we do not require the constant C to be "sufficiently small"; it is an arbitrary positive constant.

Results on existence and uniqueness of generalized solution of Problem B have been obtained in [15], but in the case where $b(x, y) \equiv 0$ and the right-hand side f of the corresponding equation is only "weakly nonlinear" in the sense that f is satisfying the Carathéodory condition, $|f(x, y, u)| \leqslant Q(x, y) + C|u|$ with $Q(x, y) \in L^2(G)$ and $|f(x, y, z_1) - f(x, y, z_2)| \leqslant C|z_1 - z_2|$, $C = \text{const} > 0$.

In the following lemma we formulate a partial case of the well known multiplicative inequality (see, e.g., Theorem 7.3 in Chapter 1 of [10]).

Lemma 1.1. *If $G \subset \mathcal{R}^2$ is a bounded domain with the uniform cone property then for every $p > 2$*

$$\|u\|_{L^p(G)} \leqslant C_0 \|u\|_1^{(p-2)/p} \|u\|_0^{2/p}, \quad u \in H^1(G),$$

where the constant C_0 depends only on G.

Remark. Since we assume that the limit $\lim_{y \to 0} k(y)/\ell(y)$ exists our domain G has the uniform cone property.

2. A priori estimates

Consider for any $\lambda > 0$ the following norms:

$$\|u\|_{0,\lambda} = \|e^{-(\lambda/2)x}u\|_0, \qquad \|u\|_{1,\lambda} = \left(\int_G e^{-\lambda x}\left[u_x^2 + u_y^2 + u^2\right]dx\,dy\right)^{1/2}.$$

Obviously, for every fixed λ these norms are equivalent respectively to $\|u\|_0$ and $\|u\|_1$.

Lemma 2.1. *Suppose $k(y), \ell(y) \in C^1[0,Y]$, $a(x,y), b(x,y) \in C(\overline{G})$ and*

$$\ell'(0) > 0, \quad a(x,0) > 0, \quad b(x,0) = 0 \quad \textit{for } x \in [0,1].$$

Then there exist constants $m > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$

$$m\|u\|_{1,\lambda} \leqslant \|Lu\|_{0,\lambda}, \quad \forall u \in C^\infty_{AC}\big(\overline{G}\big).$$

Proof. Let λ and μ be positive constants. By Green's formula,

$$\begin{aligned}\big(2Lu, e^{-\lambda x}(\mu u_x - u_y)\big)_0 &= \int_G (2Lu)e^{-\lambda x}(\mu u_x - u_y)\,dx\,dy\\ &= I(\lambda) + \int_{\partial G} e^{-\lambda x}\big[\big[-\big(ku_x^2 + \ell u_y^2\big) + 2\mu\ell u_x u_y\big]dx\\ &\quad + \big[\mu\big(ku_x^2 + \ell u_y^2\big) - 2ku_x u_y\big]dy\big],\end{aligned}$$

where

$$I(\lambda) = \int_G e^{-\lambda x}\left[A(x,y)u_x^2 + 2B(x,y)u_x u_y + C(x,y)u_y^2\right]dx\,dy, \tag{3}$$

with

$$A(x,y) = \lambda\mu k - k' + 2\mu a, \quad B(x,y) = -\lambda k - a + \mu b,$$
$$C(x,y) = \lambda\mu\ell + \ell' - 2b.$$

The line integral $\int_{\partial G} = \int_{AB} + \int_{BC} + \int_{CA}$ is nonnegative. Indeed, $\int_{AB} = 0$ because $k(0) = \ell(0) = 0$. On BC: $x = 1 - F(y)$ we have $dx = -\sqrt{k/\ell}\,dy$, therefore

$$\int_{BC} = \int_0^Y e^{-\lambda(1-F(y))}\big(\mu + \sqrt{k/\ell}\big)\big(\sqrt{k}u_x - \sqrt{\ell}u_y\big)^2\,dy \geqslant 0.$$

On AC: $x = F(y)$ we have $dx = \sqrt{k/\ell}\,dy$, and, in addition, $u \equiv 0$ on AC implies $\sqrt{k}u_x + \sqrt{\ell}u_y = 0$ on AC, therefore

$$\int_{CA} = \int_Y^0 e^{-\lambda F(y)}\big(\mu - \sqrt{k/\ell}\big)\big(\sqrt{k}u_x + \sqrt{\ell}u_y\big)^2\,dy = 0.$$

Hence

$$\big(2Lu, e^{-\lambda x}(\mu u_x - u_y)\big)_0 \geqslant I(\lambda).$$

Taking into account that

$$-2au_xu_y \geqslant -\frac{2a^2}{\ell'(0)}u_x^2 - \frac{\ell'(0)}{2}u_y^2,$$
$$2(-\lambda k + \mu b)u_xu_y \geqslant -(\lambda k + \mu|b|)\big(u_x^2 + u_y^2\big)$$

we obtain

$$I(\lambda) \geqslant \int_G e^{-\lambda x}\big[A_1(x,y)u_x^2 + C_1(x,y)u_y^2\big]\,dx\,dy, \tag{4}$$

where

$$A_1(x,y) = \lambda k(y)(\mu - 1) - k'(y) + \mu\big(2a(x,y) - |b(x,y)|\big) - 2a^2(x,y)/\ell'(0),$$
$$C_1(x,y) = \lambda[\mu\ell(y) - k(y)] + \ell'(y) - \ell'(0)/2 - \mu|b(x,y)|.$$

By $k(0) = \ell(0) = 0$ and $b(x,0) = 0$ for $x \in [0,1]$ we have

$$A_1(x,0) = -k'(0) + 2\mu a(x,0) - 2a^2(x,0)/\ell'(0),$$
$$C_1(x,0) = \ell'(0)/2.$$

Since the function $a(x,0)$ has a strictly positive lower bound on $[0,1]$ (because it is continuous and $a(x,0) > 0$) there exists $\mu > 0$ such that $A_1(x,0) \geqslant \ell'(0)/2$, $x \in [0,1]$. Fix the constant μ so that $\mu \geqslant 2 + \sup_G k/\ell$.

Taking into account that the functions k, ℓ, a, b are continuous, so uniformly continuous in $\overline{G}$, one can easily see that there exists $\delta > 0$ such that if $G_\delta^1 = \{(x,y) \in \overline{G}\colon\ 0 \leqslant y \leqslant \delta\}$ then for $\lambda > 0$ and $(x,y) \in G_\delta^1$ we have $A_1(x,y) \geqslant \ell'(0)/4$, $C_1(x,y) \geqslant \ell'(0)/4$.

Next we consider $G_\delta^2 = \{(x,y) \in \overline{G}\colon\ \delta \leqslant y\}$. By the choice of μ we have $\mu - k/\ell \geqslant 1$ and $\mu - 1 \geqslant 1$. Since the functions $k(y)$ and $\ell(y)$ are continuous and strictly positive for $y \geqslant \delta$ it is easy to see that there exists λ_0 such that for $\lambda > \lambda_0$ and $(x,y) \in G_\delta^2$ we have $A_1(x,y) \geqslant \ell'(0)/4$, $C_1(x,y) \geqslant \ell'(0)/4$. Hence for $\lambda > \lambda_0$ it holds

$$I(\lambda) \geqslant \big(\ell'(0)/4\big) \int_G e^{-\lambda x} \big(u_x^2 + u_y^2\big)\, dx\, dy.$$

On the other hand, we have

$$\begin{aligned} 0 \leqslant \int_{\partial G} e^{-\lambda x} u^2\, dy &= \int_G \partial_x \big(e^{-\lambda x} u^2\big)\, dx\, dy \\ &= \int_G \big(-\lambda e^{-\lambda x} u^2 + e^{-\lambda x} 2 u u_x\big)\, dx\, dy \\ &\leqslant \int_G e^{-\lambda x} \big(-\lambda u^2 + u^2 + u_x^2\big)\, dx\, dy. \end{aligned}$$

Therefore

$$(\lambda - 1) \int_G e^{-\lambda x} u^2\, dx\, dy \leqslant \int_G e^{-\lambda x} u_x^2\, dx\, dy,$$

so for $\lambda \geqslant \lambda_0 > 2$ it holds

$$\int_G e^{-\lambda x} u^2\, dx\, dy \leqslant \int_G e^{-\lambda x} u_x^2\, dx\, dy.$$

Hence

$$I(\lambda) \geqslant \big(\ell'(0)/8\big) \|u\|_{1,\lambda}^2 \tag{5}$$

and we obtain

$$\begin{aligned} \big(\ell'(0)/8\big) \|u\|_{1,\lambda}^2 &\leqslant \big(2Lu, e^{-\lambda x}(\mu u_x - u_y)\big)_0 \leqslant \|2Lu\|_{0,\lambda} \|\mu u_x - u_y\|_{0,\lambda} \\ &\leqslant 4\mu \|Lu\|_{0,\lambda} \|u\|_{1,\lambda}, \end{aligned}$$

which implies $m\|u\|_{1,\lambda} \leqslant \|Lu\|_{0,\lambda}$ with $m = \ell'(0)/(32\mu)$. □

The following simple observation will be useful.

Lemma 2.2. *If* $\lambda > 0$ *and* $u \in H_{AC}^1(\overline{G})$ *then* $(\lambda/2)\|u\|_{0,\lambda} \leqslant \|u\|_{1,\lambda}$.

Proof. Obviously, it is enough to prove the claim for $u \in C_{AC}^\infty(\overline{G})$. Then we have by Cauchy inequality

$$\int_G e^{-\lambda x} \partial_x \big(u^2/2\big)\, dx\, dy = \int_G e^{-\lambda x} u u_x\, dx\, dy \leqslant \|u\|_{0,\lambda} \|u\|_{1,\lambda}.$$

On the other hand, since $\int_{\partial G} e^{-\lambda x} u^2\, dy \geqslant 0$, we obtain by Green's formula

$$\int_G e^{-\lambda x}\partial_x\big(u^2/2\big)\,dx\,dy = \frac{1}{2}\int_{\partial G} e^{-\lambda x}u^2\,dy + \frac{\lambda}{2}\int_G e^{-\lambda x}u^2\,dx\,dy$$
$$\geqslant \frac{\lambda}{2}\|u\|_{0,\lambda}^2,$$

which completes the proof. □

We need also the following technical statement.

Lemma 2.3. *If $v, w \in L^s(G)$ $\forall s \geqslant 2$ and there exists a constant $C > 0$ such that $\|v\|_{L^s(G)} \leqslant C$ for all $s \geqslant s_0 \geqslant 2$, then $\|vw\|_{0,\lambda} \leqslant C\|w\|_{0,\lambda}$.*

Proof. By Cauchy inequality we obtain (using induction on k)

$$\|vw\|_{0,\lambda} \leqslant \bigg(\int_G e^{-\lambda x}v^4w^2\,dx\,dy\bigg)^{1/4}\bigg(\int_G e^{-\lambda x}w^2\,dx\,dy\bigg)^{1/4}$$
$$= \big\|v^2w\big\|_{0,\lambda}^{1/2}\|w\|_{0,\lambda}^{1/2} \leqslant \big\|v^4w\big\|_{0,\lambda}^{1/4}\|w\|_{0,\lambda}^{3/4} \leqslant \cdots$$
$$\leqslant \big\|v^{2^k}w\big\|_{0,\lambda}^{2^{-k}}\|w\|_{0,\lambda}^{1-2^{-k}} \leqslant \|v\|_{L^{2^{k+2}}(G)}\|w\|_{L^4(G)}^{2^{-k}}\|w\|_{0,\lambda}^{1-2^{-k}}.$$

Therefore, for k so large that $2^{k+2} \geqslant s_0$ we have

$$\|vw\|_{0,\lambda} \leqslant C\|w\|_{L^4(G)}^{2^{-k}}\|w\|_{0,\lambda}^{1-2^{-k}}.$$

Thus, letting $k \to \infty$ we obtain the claim. □

Lemma 2.4. *If $k, \ell \in C^1[0, Y]$ and $v \in C_{BC}^\infty(\overline{G})$ then for $\mu > 0$ the boundary problem*

$$h(u) := e^{-\lambda x}(\mu u_x - u_y) = v \quad \textit{in } G, \qquad u|_{AC} = 0,$$

has a unique solution $u \in C_{AC}^1(\overline{G}) \cap C^2(\overline{G} \setminus \{A\})$.

Proof. Fix $v \in C_{BC}^\infty(\overline{G})$ and set $\tilde{v}(x, y) = -e^{\lambda x}v(x, y)$. Then we have to find $u(x, y) \in C_{AC}^1(\overline{G})$ such that

$$-\mu u_x + u_y = \tilde{v}(x, y). \tag{6}$$

Consider one parameter family of lines ℓ_c given by

$$\xi = -\mu t + c, \quad \eta = t, \quad c \in \mathcal{R}.$$

Suppose that $u(x, y)$ satisfies (6) and for some c the line ℓ_c has a nonempty intersection with G. Then

$$\frac{d}{dt}[u(-\mu t + c, t)] = -\mu u_x + u_y = \tilde{v}(-\mu t + c, t).$$

Observe that if $\varphi(s)$ is the inverse function of the function $\mu t + F(t)$ then for each point $(x, y) \in G$ the line ℓ_c with $c = x + \mu y$ passes through the point (x, y), intersects the characteristics AC into a point with a second coordinate $\varphi(c) = \varphi(x + \mu y)$ and for $t \in [y, \varphi(c)]$ the corresponding segment of the line ℓ_c lies inside $\overline{G}$. Therefore we obtain

$$u(x, y) = \int_{\varphi(x+\mu y)}^{y} \tilde{v}(-\mu t + x + \mu y, t)\, dt.$$

Conversely, if $u(x, y)$ is defined by the above formula then it is easy to see that $u \in C^1_{AC}(\overline{G}) \cap C^2(\overline{G} \setminus \{A\})$ and (6) holds. Second derivatives of u may not exist at $A = (0, 0)$ because $F''(0)$ (and as well $\varphi''(0)$) may not exist. □

Remark. Nevertheless second derivatives of u may not exist at the point $A = (0, 0)$, we can apply Green's formula in Lemma 2.5 to some expression that involves second derivatives of u. One can easily see that by applying Green's formula to domains $G_\varepsilon = G \cap \{y > \varepsilon\}$, $\varepsilon > 0$, and passing to a limit as $\varepsilon \to 0$.

Next we are going to prove a priori estimate corresponding to the nonlinear term $f(x, y, u) = -r(x, y)u|u|^\rho + g(x, y, u)$, $\rho \geqslant 0$.

Set

$$B_1[u, v] := B[u, v] - \int_G f\big(x, y, u(x, y)\big)v(x, y)\, dx\, dy.$$

Lemma 2.5. *Suppose the assumptions of Lemma* 2.1 *hold,* $r(x, y) \in C^1(\overline{G})$ *and* $g(x, y, z) \in C^1(\overline{G} \times \mathcal{R})$. *In addition, let* $g(x, y, u)$ *satisfy*

$$|g| \leqslant C + q(x, y)|u|^{p-1}, \qquad |g_x| + |g_y| \leqslant C_1 + C_2|u|^{p-1}, \tag{7}$$

where C, C_1, C_2 *are positive constants,* $p = \rho + 2$, $q(x, y) \in C(\overline{G})$ *and*

$$\inf_{\overline{G}} [r(x, y) - q(x, y)] = r_0 > 0. \tag{8}$$

Then there exist $\mu > 0$ *and* $\lambda_0 > 0$ *such that for* $\lambda \geqslant \lambda_0$ *we have for* $v \in C^\infty_{BC}(\overline{G})$ *and* u *related to* v *as in Lemma* 2.4

$$B_1[u, v] = B_1[u, h(u)] \geqslant m_0 \|u\|^2_{1,\lambda} - D, \tag{9}$$

where $m_0 = \ell'(0)/16$ *and the constant* $D > 0$ *depends on the choice of* μ *and* λ, *but does not depend on* u.

Proof. Suppose $v \in C^\infty_{BC}(\overline{G})$ and u is related to v as in Lemma 2.4, that is $v = h(u) = \exp(-\lambda x)(\mu u_x - u_y)$. We have

$$B_1[u, h(u)] = B[u, h(u)] + I_1 - I_2,$$

where

$$I_1 = \int_G e^{-\lambda x} r|u|^\rho u(\mu u_x - u_y)\,dx\,dy,$$

$$I_2 = \int_G e^{-\lambda x} g(x,y,u)(\mu u_x - u_y)\,dx\,dy.$$

By Green's formula,

$$B[u,h(u)] = \frac{1}{2} I(\lambda) + \frac{1}{2}\int_{\partial G} e^{-\lambda x}\bigl(\ell u_y^2 - k u_x^2\bigr)(\mu\,dy + dx),$$

where $I(\lambda)$ is given by (3). The linear integral $\int_{\partial G} = \int_{AB} + \int_{BC} + \int_{CA}$ is nonnegative. Indeed, $\int_{AB} = 0$ because $k(0) = \ell(0) = 0$. On BC: $x = 1 - F(y)$, so $dx = -\sqrt{k/\ell}\,dy$ and

$$\int_{BC} = \int_0^Y e^{-\lambda(1-F(y))}\bigl(\ell u_y^2 - k u_x^2\bigr)\bigl(\mu - \sqrt{k/\ell}\bigr)\,dy.$$

Since $\mu u_x - u_y = v\exp(\lambda x) \equiv 0$ on BC we have

$$\int_{BC} = \int_0^Y e^{-\lambda(1-F(y))}(\ell\mu^2 - k)\bigl(\mu - \sqrt{k/\ell}\bigr)u_x^2\,dy$$

$$= \int_0^Y e^{-\lambda(1-F(y))}\ell(\mu + \sqrt{k/\ell}\,\bigl(\mu - \sqrt{k/\ell}\bigr)^2 u_x^2\,dy \geqslant 0.$$

Finally $\int_{CA} = 0$ because $u \equiv 0$ on CA implies $u_x\sqrt{k/\ell} + u_y = 0$ on CA. Hence from (5) it follows

$$B[u,h(u)] \geqslant m_0\|u\|_{1,\lambda}^2, \quad m_0 = \ell'(0)/16. \tag{10}$$

By Green's formula

$$\begin{aligned}
I_1 &= \int_G r(x,y)|u|^\rho u e^{-\lambda x}(\mu u_x - u_y)\,dx\,dy \\
&= \frac{1}{p}\int_{\partial G} e^{-\lambda x} r|u|^p(dx + \mu\,dy) + \frac{1}{p}\int_G e^{-\lambda x}|u|^p(\lambda\mu r - \mu r_x + r_y)\,dx\,dy \\
&\geqslant \frac{1}{p}\int_{\partial G} e^{-\lambda x} r|u|^p(dx + \mu\,dy) \\
&\quad + \frac{1}{p}\int_G e^{-\lambda x}|u|^p(\lambda\mu r - \mu|r_x| - |r_y|)\,dx\,dy.
\end{aligned}$$

Next we consider the integral I_2. Set

$$\tilde{g}(x, y, z) := \int_0^z g(x, y, t)\, dt.$$

Then we have

$$\partial_x\big[\tilde{g}\big(x, y, u(x, y)\big)\big] = \tilde{g}_x\big(x, y, u(x, y)\big) + g\big(x, y, u(x, y)\big)u_x,$$
$$\partial_y\big[\tilde{g}\big(x, y, u(x, y)\big)\big] = \tilde{g}_y\big(x, y, u(x, y)\big) + g\big(x, y, u(x, y)\big)u_y.$$

Therefore by Green's formula we obtain

$$\begin{aligned} I_2 &= \int_G g(x, y, u)e^{-\lambda x}(\mu u_x - u_y)\, dx\, dy \\ &= \int_{\partial G} e^{-\lambda x}\tilde{g}(x, y, u)(dx + \mu\, dy) + \int_G e^{-\lambda x}(\lambda\mu\tilde{g} - \mu\tilde{g}_x + \tilde{g}_y)\, dx\, dy. \end{aligned}$$

Since $\mu - k/\ell > 0$ (by the choice of μ) it is easy to see that for any function $\psi(x, y)$ with nonnegative values $\int_{\partial G} \psi(x, y)(dx + \mu\, dy) \geqslant 0$. Therefore

$$\int_{\partial G} e^{-\lambda x}\tilde{g}(x, y, u)(dx + \mu\, dy) \leqslant \int_{\partial G} e^{-\lambda x}|\tilde{g}(x, y, u)|(dx + \mu\, dy).$$

From (7) it follows (with some $\varepsilon > 0$ that will be fixed later)

$$|\tilde{g}| \leqslant C|u| + q(x, y)|u|^p/p \leqslant C^2/\varepsilon + \varepsilon|u|^2 + q(x, y)|u|^p/p,$$
$$|\tilde{g}_x| + |\tilde{g}_y| \leqslant C_1|u| + C_2|u|^p/p \leqslant C_1^2/\varepsilon + \varepsilon|u|^2 + C_2|u|^p/p,$$

thus

$$\begin{aligned} I_2 &\leqslant \int_{\partial G} e^{-\lambda x}|\tilde{g}(x, y, u)|(dx + \mu\, dy) \\ &\quad + \int_G e^{-\lambda x}\big(\lambda\mu|\tilde{g}| + \mu|\tilde{g}_x| + |\tilde{g}_y|\big)\, dx\, dy \\ &\leqslant D + \int_{\partial G} e^{-\lambda x}\big(\varepsilon|u|^2 + q(x, y)|u|^p/p\big)(dx + \mu\, dy) \\ &\quad + \int_G e^{-\lambda x}\big[\lambda\mu\big(\varepsilon|u|^2 + q(x, y)|u|^p/p\big) \\ &\quad + \mu\big(\varepsilon|u|^2 + C_2|u|^p/p\big)\big]\, dx\, dy, \end{aligned}$$

where

$$D = (\mu/\varepsilon)\int\limits_G e^{-\lambda x}\bigl(\lambda C^2 + C_1^2\bigr)\,dx\,dy + (C^2/\varepsilon)\int\limits_{\partial G} e^{-\lambda x}(dx + \mu\,dy).$$

Combining the estimates for I_1 and I_2 we obtain (due to (8))

$$I_1 - I_2 \geqslant -D - \varepsilon J_0 + (1/p)J_1 + (1/p)J_2,$$

where

$$J_0 = \int\limits_{\partial G} e^{-\lambda x}u^2(dx + \mu\,dy) + \int\limits_G e^{-\lambda x}(\lambda\mu + \mu)u^2\,dx\,dy,$$

$$J_1 = \int\limits_{\partial G} e^{-\lambda x}|u|^p[r(x,y) - q(x,y)](dx + \mu\,dy)$$

$$\geqslant \int\limits_{\partial G} e^{-\lambda x}|u|^p r_0(dx + \mu\,dy) \geqslant 0,$$

$$J_2 = \int\limits_G e^{-\lambda x}|u|^p\mu\bigl[\lambda[r(x,y) - q(x,y)] - (|r_x| + |r_y| + C_2)\bigr]\,dx\,dy \geqslant 0$$

for $\lambda \geqslant \lambda_0 \geqslant (1/r_0)\sup_G(|r_x| + |r_y| + C_2)$.

In order to estimate J_0 observe that

$$\int\limits_{\partial G} e^{-\lambda x}u^2(dx + \mu\,dy) = \int\limits_G e^{-\lambda x}\bigl[-\lambda\mu u^2 + 2\mu u u_x - 2u u_y\bigr]\,dx\,dy$$

$$\leqslant \int\limits_G e^{-\lambda x}\bigl[-\lambda\mu u^2 + \mu^2 u^2 + u_x^2 + u^2 + u_y^2\bigr]\,dx\,dy,$$

therefore

$$J_0 \leqslant \int\limits_G e^{-\lambda x}\bigl[(\mu^2 + \mu + 1)u^2 + u_x^2 + u_y^2\bigr]\,dx\,dy \leqslant (\mu^2 + \mu + 1)\|u\|_{1,\lambda}^2.$$

Hence by (10) we obtain $B_1(u,v) \geqslant [m_0 - \varepsilon(\mu^2 + \mu + 1)]\|u\|_{1,\lambda}^2 - D$, so the claim holds with $m = m_0/2$ if ε is fixed so that $\varepsilon(\mu^2 + \mu + 1) < m_0/2$. □

3. Existence of generalized solution

Theorem 3.1. *Suppose*

1. $k(y), \ell(y) \in C^1[0,Y], a(x,y), b(x,y) \in C(\overline{G})$ *and*

$$\ell'(0) > 0, \quad a(x,0) > 0, \quad b(x,0) = 0 \quad \textit{for } x \in [0,1];$$

2. $r(x,y) \in C^1(\overline{G})$, $g(x,y,z) \in C^1(\overline{G} \times \mathcal{R})$ *and*

$$|g| \leqslant C + q(x,y)|u|^{p-1}, \qquad |g_x| + |g_y| \leqslant C_1 + C_2|u|^{p-1}, \tag{11}$$

where C, C_1, C_2 *are positive constants,* $p = \rho + 2$, $q(x,y) \in C(\overline{G})$ *and*

$$\inf_{\overline{G}} [r(x,y) - q(x,y)] = r_0 > 0. \tag{12}$$

Then there exists a generalized solution of Problem B.

Proof. For convenience we divide the proof into several steps.

Step 1. Since the space H^1_{BC} is separable we can choose a linearly independent sequence of functions $(v_j)_{j=1}^{\infty}$, $v_j \in C^{\infty}_{BC}(\overline{G})$ which linear span is dense in H^1_{BC}.

Let $u_j \in C^1_{AC}(\overline{G}) \cap C^2_{AC}(\overline{G} \setminus \{A\})$, $j = 1, 2, \ldots,$ corresponds to v_j as in Lemma 2.4, that is $h(u_j) = e^{-\lambda x}(\mu \partial_x u_j - \partial_y u_j) = v_j$, $j = 1, 2, \ldots.$

We claim that for each $n \in \mathcal{N}$ there exist constants c_j^n, $j = 1, \ldots, n$, such that

$$u^n = \sum_{j=1}^{n} c_j^n u_j \tag{13}$$

satisfies the system of equations

$$B_1[u^n, v_i] = B_1\left[\sum_{j=1}^{n} c_j^n u_j, v_i\right] = 0, \quad i = 1, \ldots, n. \tag{14}$$

Indeed, fix $n \in \mathcal{N}$ and set for any $\bar{c} = (c_1, \ldots, c_n) \in \mathcal{R}^n$

$$u(\bar{c}) = \sum_{j=1}^{n} c_j u_j;$$

then obviously $h(u(\bar{c})) = \sum_{j=1}^{n} c_j h(u_j) = \sum_{j=1}^{n} c_j v_j$. Consider a mapping $P : \mathcal{R}^n \to \mathcal{R}^n$ defined by $P(\bar{c}) = (B_1[u(\bar{c}), v_i])_{i=1}^n$. Obviously P is continuous.

Fix constants μ, λ, D as in Lemma 2.5. Then, by a priori estimate (9) proved in Lemma 2.5 it follows

$$\sum_{i=1}^{n} B_1[u(\bar{c}), v_i] \cdot c_i = B_1\big[u(\bar{c}), h\big(u(\bar{c})\big)\big] \geqslant m\|u(\bar{c})\|^2_{1,\lambda} - D. \tag{15}$$

The norms $\|\bar{c}\|_{\lambda} := \|u(\bar{c})\|_{1,\lambda}$ and $\|\bar{c}\| := (\sum_{j=1}^{n} c_j^2)^{1/2}$ are equivalent (since any two norms in $\mathcal{R}^n$ are equivalent), so from (15) we obtain

$$\big(P(\bar{c}), \bar{c}\big) = \sum_{i=1}^{n} B_1[u(\bar{c}), v_i] \cdot c_i \geqslant 0 \quad \text{if } \|\bar{c}\| \geqslant \text{const} > 0.$$

Hence, by well-known Sharp-Angle Lemma (see, e.g., Lemma 4.3 in Chapter 1 of [12]) there exists $\bar{c} \in \mathcal{R}$ such that $P(\bar{c}) = 0$.

Step 2. The sequence $(u^n)_{n=1}^{\infty}$ from Step 1 is bounded in the space H^1_{AC}. Indeed, from $B_1[u^n, h(u^n)] = \sum_{i=1}^{n} B_1[u^n, v_i] \cdot c_i^n = 0$, and a priori estimate (9) it follows

$$m\|u^n\|^2_{1,\lambda} \leqslant D, \quad n = 1, 2, \ldots.$$

Since every bounded set in the (Hilbert) space H^1_{AC} is weakly compact there exists a subsequence (u^{n_k}) and an element $u \in H^1_{AC}$ such that

$$u^{n_k} \to u \quad \text{weakly in } H^1_{AC}. \tag{16}$$

On the other hand, by Relich's Theorem about compact embedding of the Sobolev space $W^{1,2}(G)$ into $L^2(G)$ (see [1]), we can assume without loss of generality that the subsequence (u^{n_k}) converges in $L^2(G)$ to some function $w \in L^2(G)$. Moreover, due to a well-known property of strong convergence in $L^2(G)$ we can assume that the subsequence (u^{n_k}) converges to w almost everywhere in G.

Since weak convergence in H^1 implies weak convergence in $L^2(G)$, the subsequence (u^{n_k}) converges to u weakly in $L^2(G)$, thus $w = u$ and

$$u^{n_k} \to u \quad \text{almost everywhere in } G. \tag{17}$$

Observe also that in view of Lemma 1.1 the subsequence (u^{n_k}) is bounded in the space $L^p(G)$, that is there exists a constant $C_p > 0$ such that

$$\|u^{n_k}\|_p = \left(\int_G |u^{n_k}(x,y)|^p \, dx\, dy\right)^{1/p} \leqslant C_p. \tag{18}$$

Step 3. We shall prove that the function u is a generalized solution of Problem B.

Since the linear span of the system v_i, $i = 1, 2, \ldots$, is dense in H^1_{BC} and $B_1[u, v]$ is linear with respect to its second argument it is enough to show that $B_1[u, v_i] = 0$ for $i = 1, 2, \ldots$. By Step 1 we have $B_1[u^{n_k}, v_i] = 0$ for $n_k > i$, so it is enough to prove that

$$B_1[u^{n_k}, v_i] \to B_1[u, v_i] \quad \text{as } k \to \infty. \tag{19}$$

For every fixed v_i the linear functional $B[\cdot, v_i]$ is continuous in the space H^1, thus $B[u^{n_k}, v_i] \to B[u, v_i]$ as $k \to \infty$.

It remains to consider the nonlinear terms in $B_1[u^{n_k}, v_i]$. Put

$$w_k(x,y) := g\big(x, y, u^{n_k}(x,y)\big) - r(x,y)|u^{n_k}|^{\rho} u^{n_k},$$
$$w(x,y) = g\big(x, y, u(x,y)\big) - r(x,y)|u|^{\rho} u.$$

We are going to show for every $i = 1, 2, \ldots$ that

$$\int_G w_k(x,y) v_i(x,y)\, dx\, dy \to \int_G w(x,y) v_i(x,y)\, dx\, dy. \tag{20}$$

From (17) it follows

$$w_k(x, y) \to w(x, y) \quad \text{almost everywhere in } G. \tag{21}$$

In addition, in view of (11) and (18) we have for $s = p/(p-1)$

$$|w_k(x, y)|^s \leqslant C_3 + C_4|u^{n_k}(x, y)|^p \leqslant C_3 + C_4C_p^p,$$

where the constants C_3 and C_4 depend on C and $\sup_G q(x, y)$, so the sequence (w_k) is bounded in the space $L^s(G)$. Now (20) follows from Lemma 1.3 in Chapter 1 of [12], which says that if the sequence (w_k) is bounded in $L^s(G)$ then (21) implies that $w_k \to w$ weakly in $L^s(G)$. □

4. Strong solutions and uniqueness theorem

After Friedrichs [5,6] coincidence of weak (or generalized) and strong solutions has been proven (under some restrictions on the corresponding domain) for various linear boundary value problems. In the case of our linear Problem B the following statement holds.

Proposition 4.1. *If $k(y) \in C[0, Y]$, $\ell(y) \in C^1[0, Y]$ and $a(x, y), b(x, y) \in C^1(\overline{G})$ then every generalized solution of linear Problem* B *is also a strong solution of linear Problem* B.

We omit the proof because the same statement (but with $b(x, y) \equiv 0$) has been proven in [15, Proposition 1] (see also Lemmas 5 and 6 there).

In the next theorem we prove under very mild restriction on the right-hand side $f(x, y, u)$ that each generalized solution of nonlinear Problem B is a strong solution of the same problem. Similar result has been obtained in [11] for a nonlinear Tricomi Problem and the related evolution problem.

Recall that a function $f(x, y, z): G \times \mathcal{R} \to \mathcal{R}$ satisfies *Carathéodory condition* (e.g., [9]) if for any fixed $z \in \mathcal{R}$ it is measurable function on G and for almost all $(x, y) \in G$ the function f is continuous with respect to $z \in \mathcal{R}$.

Theorem 4.2. *Suppose $k(y), \ell(y) \in C^1[0, Y]$ and $a(x, y), b(x, y) \in C^1(\overline{G})$. If the function $f(x, y, z)$ satisfies Carathéodory condition and*

$$|f(x, y, z)| \leqslant Q(x, y) + \alpha|z|^\gamma, \tag{22}$$

where $Q(x, y) \in L^2(G)$ and α, γ are nonnegative constants, then every generalized solution of Problem B *is a strong solution of Problem* B.

Proof. Suppose $u \in H^1_{AC}$ is a generalized solution of Problem B. Then u is a generalized solution of the following linear problem:

$$Lu = \hat{f}(x, y) := f\bigl(x, y, u(x, y)\bigr), \qquad u|_{AC} = 0.$$

Thus by Proposition 4.1 there exists a sequence $(u_n)_{n=1}^{\infty}$, $u_n \in C_{AC}^{\infty}(\overline{G})$ such that $\|u_n - u\|_1 + \|Lu_n - \hat{f}\|_0 \to 0$ as $n \to \infty$.

Obviously, to prove the theorem it is enough to show that there exists a subsequence $(u_{n_k})_{k=1}^{\infty}$ such that $\|f(x,y,u_{n_k}(x,y)) - \hat{f}(x,y)\|_0 \to 0$ as $k \to \infty$. Since $\|u_n - u\|_1 \to 0$ one can choose a subsequence $(u_{n_k})_{k=1}^{\infty}$ such that $\sum_k \|u_{n_{k+1}} - u_{n_k}\|_1 < \infty$. In view of Lemma 1.1 for every $s \geqslant 2$ we have $\sum_k \|u_{n_{k+1}} - u_{n_k}\|_{L^s(G)} < \infty$. So from well known properties of the spaces $L^s(G)$, $s \geqslant 2$ it follows:

(i) the series

$$\Phi(x,y) := |u_{n_1}(x,y)| + \sum_{k=1}^{\infty} |u_{n_{k+1}}(x,y) - u_{n_k}(x,y)|$$

is convergent almost everywhere in G and $\Phi \in L^s(G)$ for every $s \geqslant 2$;

(ii) $u_{n_k}(x,y) \to u(x,y)$ as $k \to \infty$ almost everywhere in G;

(iii) $|u_{n_k}(x,y)| \leqslant |\Phi(x,y)|$ almost everywhere in G.

Now by Carathéodory condition

$$f\bigl(x,y,u_{n_k}(x,y)\bigr) \to f\bigl(x,y,u(x,y)\bigr) = \hat{f}(x,y) \quad \text{almost everywhere in } G,$$

thus $|f(x,y,u_{n_k}(x,y)) - \hat{f}(x,y)|^2 \to 0$ as $k \to \infty$ almost everywhere in G.

On the other hand (iii) and (22) imply

$$\bigl|f\bigl(x,y,u_{n_k}(x,y)\bigr)\bigr| \leqslant Q(x,y) + \alpha\Phi^{\gamma}(x,y)$$

almost everywhere in G. Letting $k \to \infty$ we obtain

$$\bigl|\hat{f}(x,y)\bigr| \leqslant Q(x,y) + \alpha\Phi^{\gamma}(x,y),$$

thus

$$\bigl|f\bigl(x,y,u_{n_k}(x,y)\bigr) - \hat{f}(x,y)\bigr|^2 \leqslant 8Q^2(x,y) + 8\alpha\Phi^{2\gamma}(x,y) \in L^1(G)$$

almost everywhere in G. Hence by Lebesgue Theorem

$$\bigl\|f\bigl(x,y,u_{n_k}(x,y)\bigr) - \hat{f}(x,y)\bigr\|_0^2 = \int_G \bigl|f\bigl(x,y,u_{n_k}(x,y)\bigr) - \hat{f}(x,y)\bigr|^2\,dx\,dy \to 0 \quad \text{as } k \to \infty,$$

which proves the claim. □

Theorem 4.3. *Suppose* $k(y), \ell(y) \in C^1[0,Y]$, $a(x,y), b(x,y) \in C^1(\overline{G})$ *and* $\ell'(0) > 0$, $a(x,0) > 0$, $b(x,0) = 0$ *for* $x \in [0,1]$.

If the function $f(x,y,z)$ *satisfies Carathéodory condition and there exist constants* $C>0$ *and* $\beta\geqslant 0$ *such that for all* $z_1,z_2\in\mathcal{R}$

$$|f(x,y,z_1)-f(x,y,z_2)|\leqslant C\big(|z_1|^\beta+|z_2|^\beta\big)|z_1-z_2|,\quad (x,y)\in G,\tag{23}$$

then Problem B *has at most one generalized solution.*

Proof. It is easy to see that the assumptions of Theorem 4.2 hold. Suppose $u^{(1)}$ and $u^{(2)}$ are two generalized solutions of Problem B. By Theorem 4.2 they are also strong solutions. Let $(u_n^{(1)})$ and $(u_n^{(2)})$ be corresponding sequences of functions in $C^\infty_{AC}(\overline{G})$ such that

$$\|u_n^{(i)}-u^{(i)}\|_1+\|Lu_n^{(i)}-f\big(x,y,u_n^{(i)}\big)\|_0\to 0\quad\text{as } n\to\infty,\quad i=1,2.$$

From (23) it follows with $v_n=|u_n^{(1)}|^\beta+|u_n^{(2)}|^\beta$ and $w_n=|u_n^{(2)}-u_n^{(1)}|$

$$\|f\big(x,y,u_n^{(2)}\big)-f\big(x,y,u_n^{(1)}\big)\|_{0,\lambda}\leqslant C\|v_nw_n\|_{0,\lambda}.$$

The sequences $(u_n^{(1)})$ and $(u_n^{(2)})$ are convergent in H^1_{AC}, therefore there exists a constant $C_1>0$ such that

$$\|u_n^{(1)}\|_1\leqslant C_1,\quad \|u_n^{(2)}\|_1\leqslant C_1,\quad n=1,2,\ldots.$$

Thus in view of Lemma 1.1 we have

$$\|u_n^{(1)}\|_{L^s(G)}\leqslant C_0C_1,\quad \|u_n^{(2)}\|_{L^s(G)}\leqslant C_0C_1\quad \forall s\geqslant 2.$$

Therefore, in case $\beta>0$ we obtain for $s\geqslant 2/\beta$

$$\|v_n\|_{L^s(G)}\leqslant \big\||u_n^{(1)}|^\beta\big\|_{L^s(G)}+\big\||u_n^{(2)}|^\beta\big\|_{L^s(G)}\leqslant 2(C_0C_1)^\beta,\quad \forall n.$$

Thus by Lemma 2.3

$$\|f\big(x,y,u_n^{(2)}\big)-f\big(x,y,u_n^{(1)}\big)\|_{0,\lambda}\leqslant 2C(C_0C_1)^\beta\|u_n^{(2)}-u_n^{(1)}\|_{0,\lambda}.\tag{24}$$

Obviously the same estimate holds for $\beta=0$.

By Lemma 2.1 there exist constants $m>0$ and $\lambda_0>0$ such that

$$m\|u_n^{(2)}-u_n^{(1)}\|_{1,\lambda}\leqslant\|Lu_n^{(2)}-Lu_n^{(1)}\|_{0,\lambda},\quad \forall\lambda\geqslant\lambda_0.$$

Therefore from (24) it follows

$$\begin{aligned}m\|u_n^{(2)}-u_n^{(1)}\|_{1,\lambda}\leqslant{}&\|Lu_n^{(2)}-f\big(x,y,u_n^{(2)}\big)\|_{0,\lambda}\\&+2C(C_0C_1)^\beta\|u_n^{(2)}-u_n^{(1)}\|_{0,\lambda}\\&+\|f\big(x,y,u_n^{(1)}\big)-Lu_n^{(1)}\|_{0,\lambda}.\end{aligned}$$

Letting $n\to\infty$ we obtain

$$m\|u^{(2)}-u^{(1)}\|_{1,\lambda}\leqslant 2C(C_0C_1)^\beta\|u^{(2)}-u^{(1)}\|_{0,\lambda}.$$

Hence, by Lemma 2.2,

$$m(\lambda/2)\|u^{(2)}-u^{(1)}\|_{0,\lambda}\leqslant 2C(C_0C_1)^\beta\|u^{(2)}-u^{(1)}\|_{0,\lambda},$$

so choosing $\lambda>(4/m)C(C_0C_1)^\beta$ we prove that $u^{(2)}=u^{(1)}$. □

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