An Inhomogeneous Semilinear Equation in Entire Space

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Necessary conditions and a priori estimates are obtained for the inhomogeneous semilinear equation $Au + u^p + f(x) = 0$ in $\mathbb{R}^n$. Also, several existence theorems are proved given explicit conditions on the inhomogeneous term $f$. © 1996 Academic Press, Inc.

1. Introduction

In this paper, the question of existence of the following elliptic partial differential equation is studied,

$$Au + u^p + f(x) = 0 \quad \text{in } \mathbb{R}^n$$

$$u > 0,$$

(E)

where $A = \sum_{i=1}^{n} (\partial_i^2/\partial x_i^2)$ is the Laplacian operator, $p > 1$, $n \in \mathbb{N}$ with $n \geq 3$ and $f \in C(\mathbb{R}^n)$ with $f \geq 0$ everywhere in $\mathbb{R}^n$, $f \neq 0$. Throughout this work, this equation shall be referred to as equation (E).

Inhomogeneous second-order elliptic equations defined in entire space arise naturally in probability theory in the study of stochastic processes. Equation (E) in particular appeared recently in a paper by Tzong-Yow Lee [4] establishing limit theorems for super-Brownian motion. In that paper, existence results for Eq. (E) were obtained in the case where the inhomogeneous term is compactly supported and is dominated by a function of the form $C/(1 + |x|)^{(n-2)p}$ where $C(n, p) > 0$ is sufficiently small. In order to prove stronger limit theorems for the stochastic process, existence results need to hold for a larger class of functions for the inhomogeneous term. The intent of this paper is to address this question. In addition to Lee [4], Eq. (E) has been studied by Pokhozhaev [7] and Egnell and Kaj [1].

Pokhozhaev obtained radial solutions when the inhomogeneous term is radially symmetric about the origin and satisfies certain integrability conditions. Also, Egnell and Kaj worked on an equation similar to that of Eq. (E) but having a small positive parameter $\varepsilon$ as a coefficient to the
inhomogeneous term $f(x)$. Existence results were obtained for compactly supported functions $f$ given that the parameter $\varepsilon$ is sufficiently small.

The objective of this work is to establish the necessary conditions for the existence of solutions to Eq. (E) as well as a priori estimates of its solutions and to find sufficient conditions on the inhomogeneous term to ensure existence.

The main difficulties in establishing existence of Eq. (E), in addition to its nonlinear structure, are the noncompactness of the domain and the presence of an inhomogeneous term both of which greatly limit the use of variational methods.

Among the main results of this paper are the following two theorems. The first is an existence theorem, the second a nonexistence result. Together they attempt to reconcile the necessary and sufficient conditions for the existence of solutions to Eq. (E).

**Theorem.** Let $p > n/(n - 2)$, where $n \in \mathbb{N}$ with $n \geq 3$. Assume that

1. $f \in C^0([\mathbb{R}^n])$,
2. $f \not\equiv 0$,
3. $f \leq C/(1 + |x|^2)^{p/(p - 1)}$,

where $0 < \gamma \leq 1$ and

$$C = \frac{1}{(p - 1)^{p/(p - 1)}} \left( \frac{2}{p} \left[ n - \frac{2p}{p - 1} \right] \right)^{p/(p - 1)}.$$

Then, there exists a solution to Eq. (E).

**Theorem.** Let $p > n/(n - 2)$ and $f(x) \geq C/|x|^{|p/(p - 1)|}$ at $\infty$ where $C > 2[n - 2 - 2/(p - 1)] \lambda_1^{1/(p - 1)}/(p - 1)$ and $\lambda_1$ denotes the first eigenvalue of $-\Lambda$ in the unit ball of $\mathbb{R}^n$ with vanishing Dirichlet boundary conditions. Then, Eq. (E) has no solution.

It will be shown in Section 2 that when the exponent $p \leq n/(n - 2)$, Eq. (E) does not have a solution. Note that the ratio of the constants in the above theorems converges to one as $p$ tends to infinity.

This paper is divided into two parts. Section 2 discusses the necessary conditions for existence. The main results are the integral a priori estimates which lead to the nonexistence theorem just stated. Section 3 deals with existence theorems. Basically, it presents three different results. Using the super-subsolution method, the first states the existence of general solutions to Eq. (E) having at least slow decay (i.e., $O(|x|^{-2/(p - 1)})$). The second uses the contraction mapping theorem to show the existence of fast decaying
solutions (i.e., $O(|x|^{-(n-2)})$). In the third theorem, solutions which lie in a neighborhood of the positive radial solutions to the homogeneous Eq. (E) are shown to exist.

Most of the results of this paper can be extended to the more general equation

$$\Delta u + f(x, u) = 0 \quad \text{in } \mathbb{R}^n$$

$$u > 0$$

where $n \geq 3$, $f \in C(\mathbb{R}^{n+1})$ with $f(x, u) \geq 0$ for all $x \in \mathbb{R}^n$, and $u \geq 0$ under appropriate conditions on the term $f(x, u)$.

2. Necessary Conditions

Before establishing the necessary conditions and a priori estimates, a remark concerning the dimension of the Euclidean space in Eq. (E) must be made. This number must be limited to the range $n \geq 3$. There does not exist a solution to Eq. (E) when $n = 1$ or 2. This is due to the positiveness of the terms $u^n + f(x)$. For $n = 1$, it is clear that a continuous function cannot be concave on all of the real line and remain positive. For $n = 2$, a bounded below superharmonic function must be constant. But clearly, Eq. (E) cannot have a positive constant solution.

The following lemma due to Ramaswamy is required in the proof of Lemma 2 yielding estimates of the spherical mean of solutions to Eq. (E).

**Lemma 1 (Mythily Ramaswamy [8]).** Let $\Omega$ be a bounded smooth domain of $\mathbb{R}^n$ where $n \geq 2$. There does not exist a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying

$$\Delta u + \lambda u \leq 0 \quad \text{in } \Omega,$$

$$u > 0 \quad \text{on } \partial \Omega,$$

where $\lambda$ denotes the first eigenvalue of $-\Delta$ in $\Omega$ with vanishing Dirichlet boundary conditions.

**Proof.** Recall that the first eigenvalue of $-\Delta$ is simple and has a positive eigenfunction. Let $\phi$ be such an eigenfunction, that is

$$-\Delta \phi = \lambda \phi \quad \text{in } \Omega,$$

$$\phi > 0 \quad \text{in } \Omega,$$

$$\phi = 0 \quad \text{on } \partial \Omega.$$
Furthermore, since $\Omega$ is a bounded smooth domain, $\phi \in C^1(\bar{\Omega})$ by the elliptic regularity theory. Now suppose that there exists a function $u$ satisfying this partial differential inequality.

Multiplying this inequality by $\phi$ yields

$$\int_{\Omega} \phi \Delta u \, dx + \lambda \int_{\Omega} \phi u \, dx \leq 0.$$

By Green's second identity, this becomes

$$\int_{\Omega} u \Delta \phi \, dx - \int_{\partial \Omega} u \frac{\partial \phi}{\partial \eta} \, dS + \lambda \int_{\Omega} \phi u \, dx \leq 0.$$

Since $\Delta \phi + \lambda \phi = 0$, the above inequality reduces to

$$\int_{\partial \Omega} u \frac{\partial \phi}{\partial \eta} \, dS \geq 0.$$

But $\partial \phi/\partial \eta < 0$ on $\partial \Omega$ by Hopf's boundary point lemma. Thus, we obtain a contradiction.

The spherical mean of any function $u \in C(\mathbb{R}^n)$ shall be denoted by $\bar{u}$, i.e.,

$$\bar{u}(r) = \frac{1}{\omega_n r^n} \int_{|x|=r} u(x) \, dS \quad \text{for all} \quad r > 0,$$

$$\bar{u}(0) = u(0),$$

where $\omega_n$ is the area of the unit sphere in $\mathbb{R}^n$ and $dS$ is the surface measure.

It is easy to verify that $(\bar{u} u/dr)(0) = 0$, and that, viewed as a function of $x \in \mathbb{R}^n$, $\bar{u} \in C^2(\mathbb{R}^n)$, and $\Delta \bar{u} = \mu \bar{u}$.

**Lemma 2.** Any solution $u$ of Eq. (E) satisfies the inequality

$$\bar{u}(r) \leq \frac{\lambda_1^{1/(p-1)}}{r^{2/(p-1)}} \quad \forall r > 0,$$

where $\bar{u}$ is the spherical mean of $u$ and $\lambda_1$ is the first eigenvalue of $-\Delta$ in the unit ball of $\mathbb{R}^n$ with vanishing Dirichlet boundary conditions.

**Proof.** Let $u$ be a positive solution of Eq. (E). From the spherical mean of Eq. (E), we easily obtain using Jensen's inequality

$$\Delta \bar{u} + \bar{u}^p \leq 0 \quad \forall r \geq 0.$$
Since $\bar{u}$ is a decreasing positive function, it follows that
\[ \Delta \bar{u} + \bar{u}^{p-1}(R) \bar{u} \leq 0 \quad \text{in } B_R(0), \]
\[ \bar{u}(R) > 0. \]
Thus, by Ramaswamy's lemma we conclude that
\[ \bar{u}(R) < \lambda_R^{1/(p-1)}, \]
where $\lambda_R$ denotes the first eigenvalue of $-\Delta$ in $B_R(0)$ with vanishing Dirichlet boundary conditions.
But, it is well known that $\lambda_R = \lambda_1/R^2$. Hence, we obtain that
\[ \bar{u}(R) < \frac{\lambda_1^{1/(p-1)}}{R^{2(p-1)}} \quad \forall R > 0. \]

The following theorems are a priori estimates of the solutions to Eq. (E). Since they involve the inhomogeneous term $f$, necessary conditions for existence on this term will follow. The first theorem presents an equality relating integrals of a solution $u$ to its spherical mean.

**Theorem 1.** If Eq. (E) possesses a solution $u$, the following equality holds
\[ \int_{|x| \leq R} \frac{(u^p + f)}{|x|^{n-2}} \, dx + \frac{1}{R^n} \int_{|x| \leq R} (u^p + f) \, dx = (n-2) \omega_n \bar{u}(R) \quad \forall R > 0, \]
where $\bar{u}$ denotes the spherical mean of $u$ and $\omega_n$ the area of the unit sphere in $\mathbb{R}^n$.

**Proof.** Let $u$ be a solution of Eq. (E). Taking the spherical mean of Eq. (E), we obtain
\[ \mathcal{M}u + [u^p + f] = 0 \quad \forall r > 0. \]
Hence, we obtain
\[ \frac{d}{dr} \left( r^{n-1} \frac{d\bar{u}}{dr} \right) + r^{n-1} (u^p + f) = 0 \quad \forall r > 0, \]
\[ \bar{u}(0) = u(0), \]
\[ \frac{d\bar{u}}{dr}(0) = 0. \]
Integrating this equation from 0 to \(r\) yields

\[
\frac{du}{dr} + \frac{1}{r^{n-2}} \int_0^r s^{n-1} (u^p + f) \, ds = 0.
\]

By integrating this last equation from \(r\) to \(R\), where \(0 < r < R\), we obtain

\[
\bar{u}(R) - \bar{u}(r) + \int_r^R \frac{1}{r^{n-2}} \int_0^r s^{n-1} (u^p + f) \, ds \, dt = 0.
\]

Applying Fubini’s theorem, this equation becomes,

\[
\bar{u}(R) - \bar{u}(r) + \int_r^R \int_0^r \frac{s^{n-1}}{r^{n-1}} (u^p + f) \, ds \, dt = 0.
\]

Thus

\[
\int_0^r \frac{s^{n-1}}{(n-2)} \left( \frac{1}{r^{n-2}} - \frac{1}{s^{n-2}} \right) \, dt
\]

\[
+ \int_r^R \frac{s^{n-1}}{(n-2)} \left( \frac{1}{s^{n-2}} - \frac{1}{R^{n-2}} \right) \, ds = \bar{u}(r) - \bar{u}(R).
\]

Taking the limit as \(R \to \infty\) of this equation yields by the monotone convergence theorem

\[
\frac{1}{r^{n-2}} \int_0^r \frac{s^{n-1}}{(n-2)} (u^p + f) \, ds + \int_r^\infty \frac{s^{n-1}}{(n-2) s^{n-2}} (u^p + f) \, ds = \bar{u}(r).
\]

But, this is equivalent to

\[
\int_{|x| < r} (u^p + f) \, dx + \frac{1}{r^{n-2}} \int_{|x| < r} (u^p + f) \, dx = (n-2) \omega_n \bar{u}(r) \quad \forall r > 0,
\]

which is the required result.

The last theorem with the previous lemma immediately implies the following a priori estimates.
Theorem 2. Every solution \( u \) of Eq. (E) with \( n \geq 3 \) satisfies the following estimates for all \( R > 0 \):

\[
\int_{|x| < R} \frac{(u^n + f)}{|x|^{n-2}} \, dx < \frac{(n-2) \omega_n \lambda_1^{1/(p-1)}}{R^{2/(p-1)}},
\]

\[
\int_{|x| \leq R} (u^n + f) \, dx < (n-2) \omega_n \lambda_1^{1/(p-1)} R^{n-2-2/(p-1)}.
\]

Furthermore,

\[
u(0) = \frac{1}{(n-2) \omega_n} \int_{|x| = R} |x|^{n-2} \, dx
\]

and every solution satisfies the integral equality

\[
u(x) = \frac{1}{(n-2) \omega_n} \int_{|x| = R} \frac{(u^n + f)(y)}{|x-y|^{n-2}} \, dy.
\]

Proof. The first two inequalities follow from Theorem 1 after replacing \( \bar{u}(R) \) by its estimate of Lemma 2; i.e.,

\[
\bar{u}(R) < \frac{1}{R^{2/(p-1)}}.
\]

Letting \( R \to 0 \) in the equality of Theorem 1 yields the first equation. The second equation follows from the first, since Eq. (E) is invariant under translation.

From the above theorem, any solution \( u \) to Eq. (E) must satisfy the integral condition

\[
\int_{|x| = R} \frac{u^n}{|x|^{n-2}} \, dx < \infty.
\]

The next theorem obtains a stronger integral condition.

Theorem 3. If Eq. (E) with \( n \geq 3 \) possesses a solution \( u \), the following estimate holds for all \( x \in [0, 1) \)

\[
\frac{1}{R^{n-2-2x/(p-1)}} \int_{|x| \leq R} (u^n + f) \, dx + \int_{R < |x|} \frac{(u^n + f)}{|x|^{n-2-2x/(p-1)}} \, dx
\]

\[
\leq \omega_n \frac{(n-2-2x/(p-1))}{(1-x)} \lambda_1^{n/(p-1)} R^{x/(p-1)}
\]

for all \( R > 0 \).
Proof. The demonstration of this result is similar to that of Theorem 1. Let \( u \) be a solution of Eq. (E) and \( \tilde{u} \) its spherical mean. As shown in Theorem 1, by taking the spherical mean of Eq. (E) we obtain the differential equation

\[
\frac{d}{dr} \left( r^{n-1} \frac{d\tilde{u}}{dr} \right) + r^{n-1} (u^p + f) = 0 \quad \forall r > 0,
\]

\[
\tilde{u}(0) = u(0), \\
\frac{d\tilde{u}}{dr}(0) = 0.
\]

Integrating this equation from 0 to \( r \) and then dividing the result by \( r^{n-1} \) yields

\[
\frac{d\tilde{u}}{dr}(r) + \frac{1}{r^{n-1}} \int_0^r s^{n-1} (u^p + f) \, ds = 0.
\]

At this point the proof differs from that of Theorem 1.

Let \( \alpha \in [0, 1) \). Dividing the last equality throughout by \( \tilde{u}^\alpha \) gives

\[
\frac{1}{\tilde{u}(t)^\alpha} \frac{d\tilde{u}}{dr}(t) + \frac{1}{\tilde{u}(t)^\alpha} t^{n-1} \frac{\omega_n}{\omega_{n-1}} \int_0^t (u^p + f)(x) \, ds \, dt = 0,
\]

where \( ds \) denotes surface measure.

Let \( 0 < r < R \). Then, integrating from \( r \) to \( R \) and using the estimate

\[
\frac{\lambda^{2/(p-1)}}{\lambda^{1/(p-1)}} \leq 1 \frac{\tilde{u}(t)}{\tilde{u}(t)}
\]

yields

\[
\frac{\tilde{u}(R)^{1-\alpha}}{(1-\alpha)} \cdot \tilde{u}(r)^{1-\alpha} + \frac{1}{\omega_{n-1}} \int_r^R \int_{|x|=r} (u^p + f)(x) \, dS \, ds \, dt \leq 0.
\]

Applying Fubini’s theorem to interchange the order of integration in these integrals, we obtain

\[
\frac{\tilde{u}(R)^{1-\alpha}}{(1-\alpha)} + \frac{1}{\omega_n \lambda^{2/(p-1)}} \int_{|x|=r} (u^p + f)(x) \, dS \int_r^R \int_{|x|=r} (u^p + f)(x) \, dS \, ds \, dt
\]

\[
+ \frac{1}{\omega_n \lambda^{2/(p-1)}} \int_{|x|=r} (u^p + f)(x) \, dS \int_r^R \int_{|x|=r} (u^p + f)(x) \, dS \, ds \, dt
\]

\[
\leq \frac{\tilde{u}(r)^{1-\alpha}}{(1-\alpha)}.
\]
Thus,
\[
\frac{\bar{u}(R)^{1-x}}{(1-x)} + \frac{1}{d} \int_0^r \frac{1}{\omega_{\nu_j} \lambda_1^{n(p-1)}} (u^p + f)(x) dS \left( \frac{1}{s^p - \frac{1}{R^p}} \right) ds \\
+ \frac{1}{d} \int_r^\infty \frac{1}{\omega_{\nu_j} \lambda_1^{n(p-1)}} (u^p + f)(x) dS \left( \frac{1}{s^p - \frac{1}{R^p}} \right) ds \leq \frac{\bar{u}(r)^{1-x}}{(1-x)},
\]
where \( d = [n - 2 - 2\alpha(p-1)] \).

Taking the limit as \( R \to \infty \) in the above equation yields by the monotone convergence theorem (recalling that \( \bar{u}(\infty) = 0 \)):
\[
\frac{1}{\omega_{\nu_j} \lambda_1^{n(p-1)}[n - 2 - 2\alpha(p-1)]} \int_0^r (u^p + f)(x) dS ds \\
+ \frac{1}{\omega_{\nu_j} \lambda_1^{n(p-1)}[n - 2 - 2\alpha(p-1)]} \int_r^\infty (u^p + f)(x) dS ds \\
\leq \frac{\bar{u}(r)^{1-x}}{1-x}.
\]

This last equation is equivalent to
\[
\frac{1}{\omega_{\nu_j} \lambda_1^{n(p-1)}[n - 2 - 2\alpha(p-1)]} \int_{|x| = r} (u^p + f) dx \\
+ \frac{1}{\omega_{\nu_j} \lambda_1^{n(p-1)}[n - 2 - 2\alpha(p-1)]} \int_{|x| \leq r} (u^p + f) dx \\
\leq \frac{\bar{u}(r)^{1-x}}{1-x},
\]
which is the required result. 

Again, from this last theorem, we easily obtain the following inequalities.

**Theorem 4.** If Eq. (E) with \( n \geq 3 \) possesses a solution \( u \), the following estimate holds for all \( x \in [0, 1) \)
\[
\int_{|x| \leq R} (u^p + f) dx \leq \frac{\omega_{\nu_j} [2 - 2 - 2\alpha(p-1)]}{R^{1-\alpha}[2(p-1)]} \frac{\lambda_1^{1(p-1)}}{(1-x)} \forall R > 0.
\]
Furthermore,
\[
\int_{|x|>1} \frac{(u^n + f)}{|x|^{n-2-2/(p-1)}} \, dx \leq \frac{\omega_n \left[ n - 2 - 2/(p-1) \right]}{1 - \alpha} \int_{|x|} \left| u(0) \right|^{1-\alpha},
\]
\[
\int_{|x-y|>1} \frac{(u^n + f)(y)}{|x-y|^{n-2-2/(p-1)}} \, dy \leq \frac{\omega_n \left[ n - 2 - 2/(p-1) \right]}{1 - \alpha} \int_{|x|} \left| u(x) \right|^{1-\alpha}.
\]

**Proof.** The first inequality is obtained from Theorem 3 by replacing \( u \) by its estimate of Lemma 2, i.e.,
\[
\bar{u}(R) \leq \frac{\omega_n \left[ n - 2 - 2/(p-1) \right]}{1 - \alpha} \int_{|x|} \left| u(0) \right|^{1-\alpha}.
\]

The second inequality follows from Theorem 3 by letting \( R \to 0 \) into the estimate of that theorem. The third inequality is a consequence of the second one since Eq. (E) is invariant under translation.

**Remark.** From Theorem 4, any solution \( u \) to Eq. (E) satisfies the integral condition
\[
\int_{|x|>1} \frac{u^n}{|x|^{n-2-2/(p-1)}} \, dx < \infty \quad \forall \alpha \in [0,1).
\]

It turns out that, without prior knowledge of the decay of the spherical mean of \( u \) at infinity, this is the best integral condition that can be said of \( u^n \), i.e., the exponent of \(|x|\) cannot be reduced to \( n-2-2/(p-1) \) or a smaller number. The following example illustrates this.

**Example.** Let \( p > n/(n-2) \) and \( f \) be the following function
\[
f(x) = \begin{cases} 
\frac{\varepsilon}{|x|^{2p/(p-1)}}, & \text{if } |x| \geq 1, \\
\varepsilon, & \text{if } |x| \leq 1,
\end{cases}
\]
where \( \varepsilon > 0 \) is sufficient small. By Theorem 7 of the following section, there exists a solution \( u \) to the Eq. (E) with this function \( f \). Furthermore, by Theorem 2 \( u(x) \geq N(x) \), where \( N \) is the Newton potential of the function \( f \), i.e.,
\[
N(x) = \frac{1}{\omega_n(n-2)} \int_{|x|} \frac{f(y)}{|x-y|^{n-2}} \, dy.
\]
Since \( f(x) \geq c|x|^{2p/(p-1)} \) at \( \infty \), by Lemma 2.6 of [5] we conclude that
\[
N(x) \geq \frac{C}{|x|^{2(p-1)}} \quad \text{at } \infty
\]
for some constant \( C > 0 \). Hence,
\[
u(x) \geq \frac{C}{|x|^{2(p-1)}} \quad \text{at } \infty,
\]
which implies that the integral
\[
\int_{|x|} |x|^{\frac{p}{n-2(p-1)}} dx \quad \text{does not converge.}
\]
The integral estimates of the previous section easily yield a necessary condition on the range of the exponent \( p \). The exponent \( p \) must lie in the range \( p > n/(n-2) \). For equations similar to Eq. (E), this is very well known but the proof given here is new and does not involve oscillation theorems (see Theorem 3.4 of [6] for such a proof).

**Theorem 5.** Equation (E) does not have a solution if \( 1 < p \leq n/(n-2) \).

**Proof.** Assume Eq. (E) possesses a solution where \( 1 < p \leq n/(n-2) \).

First case. \( 1 < p < n/(n-2) \). Consider the second inequality of Theorem 2, i.e.,
\[
\int_{|x|=R} (u^p + f) dx < (n-2) \omega_n \lambda_1^{1/(p-1)} R^{n-2-2/(p-1)}.
\]
In this range of \( p \), the exponent of \( R \) is negative, i.e., \( n-2-2/(p-1) < 0 \). Letting \( R \to \infty \) in this estimate we obtain the contradiction
\[
\int_{|x|} (u^p + f) dx = 0.
\]
Second case. \( p = n/(n-2) \). In this case the exponent in the above estimate is zero, i.e., \( n-2-2/(p-1) = 0 \). Hence the right-hand side of this estimate is constant, i.e.,
\[
\int_{|x|=R} (u^p + f) dx \leq (n-2) \omega_n \lambda_1^{1/(p-1)} \quad \forall R > 0.
\]
Letting \( R \to \infty \) yields that
\[
\int_{\mathbb{R}^n} (u^p + f) \, dx < \infty.
\]
We know that
\[
\frac{C}{r^{n-2}} \leq u(r) \quad \forall r \geq 1
\]
for some constant \( C > 0 \) and by Jensen’s inequality \( \bar{u}^p \leq \bar{u}^r \). Therefore the following inequalities hold where \( p = n/(n-2) \),
\[
\int_{1 \leq |x|} \frac{C^p}{|x|^n} \, dx \leq \int_{1 \leq |x|} \bar{u}^p \, dx \leq \int_{1 \leq |x|} \bar{u}^r \, dx = \int_{1 \leq |x|} u^r \, dx < \infty.
\]
This implies that
\[
\int_{1 \leq |x|} \frac{1}{|x|^n} \, dx < \infty,
\]
which is false.

In the next section, existence theorems for Eq. (E) will be given where the inhomogeneous term \( f \) will need to satisfy a smallness condition of the form
\[
f(x) \leq \frac{C}{(1+|x|^2)^\gamma},
\]
where \( C > 0 \) is a sufficiently small constant and \( \gamma \) is an appropriate exponent. From the previous integral estimates, a bound for the constant \( C \) is found beyond which Eq. (E) ceased to have a solution.

**Theorem 6.** Let \( f(x) \geq C|\lambda_1 x|^{2\gamma(p-1)} \) at \( \infty \), where \( C > 2 \left[ n - 2 - 2/(p-1) \right] \lambda_1^{1/(p-1)} \). Then Eq. (E) has no solution.

**Proof.** Assume Eq. (E) possesses a solution. By Theorem 4 the inhomogeneous term \( f \) must satisfy the following necessary condition:
\[
\int_{R \leq |x|} \frac{f(x)}{|x|^{n-2-2\gamma(p-1)}} \, dx < \frac{[n - 2 - 2/(p-1)] \omega_n \lambda_1^{1/(p-1)}}{(1-\gamma) R^{2\gamma - 2\gamma(p-1)}} \quad \forall R > 0.
\]
Using the hypothesis on the function $f$, we conclude that for all sufficiently large $R > 0$

$$
\int_{R < |x|} \frac{C}{|x|^{2n/(p-1)}} \frac{1}{|x|^{n-2-2\alpha/(p-1)}} \, dx < \frac{[n - 2 - 2\alpha/(p-1)] \omega_n \lambda_1^{1/(p-1)}}{(1 - \alpha) R^{2(1 - \alpha)/(p-1)}}.
$$

Integrating the left-hand side yields

$$
\frac{\omega_n (p - 1) C}{2(1 - \alpha) R^{2(1 - \alpha)/(p-1)}} < \frac{\omega_n [n - 2 - 2\alpha/(p-1)] \lambda_1^{1/(p-1)}}{(1 - \alpha) R^{2(1 - \alpha)/(p-1)}}.
$$

After cancellations, we obtain

$$
C < \frac{2[n - 2 - 2\alpha/(p-1)] \lambda_1^{1/(p-1)}}{(p - 1)}.
$$

Letting $\alpha \to 1$ yields

$$
C < \frac{2[n - 2 - 2/(p-1)] \lambda_1^{1/(p-1)}}{(p - 1)}.
$$

Thus, we obtain a contradiction.

3. Existence Theorems

In each of these theorems, explicit conditions limit the magnitude of the inhomogeneous term $f(x)$ to ensure the existence of a solution. All but one of these theorems are obtained by the super-subsolution methods (i.e., monotone iteration methods). Thus, the solutions there obtained are dominated by the supersolutions constructed in the demonstrations.

There are four main existence theorems in this section. The first proves the existence of solutions to Eq. (E) for all values of the exponent $p > n/(n - 2)$ given smallness conditions on the inhomogeneous term. The solutions there obtained have at least slow decay, i.e., $O(1/|x|^{2/(p-1)})$.

The second theorem shows the existence of fast decaying solutions, i.e., $O(1/|x|^{n - 2})$ when the exponent is restricted to the range of $p \geq (n + 2)/(n - 2)$. It uses the super-subsolution method as in the first theorem.

The third theorem differs from the previous two in that its proof uses the contraction mapping principle. It establishes fast decaying solutions for all values of the exponent $p > n/(n - 2)$.

The last theorem shows the existence of solutions which lie in a neighborhood of the positive radial solutions $u_\alpha$ of the homogeneous equation corresponding to Eq. (E) with initial value $u_\alpha(0) = \alpha > 0$. This theorem
requires the dimension of the space to be sufficiently large (i.e., \( n \geq 12 \)) as well as the exponent (i.e., \( p > (n - 2)^2/((n - 2)^2 - 8n) \)).

3.1. General Solutions

By the results of Section 1, we know that the spherical mean \( \bar{u} \) of any solution to Eq. (E) satisfies the estimate

\[
\frac{C_1}{r^{n-2}} \leq \bar{u}(r) \leq \frac{C_2}{r^{\frac{n}{2}(p-2)}} \quad \text{at } \infty,
\]

where \( C_1 \) and \( C_2 \) are positive constants.

The decay rate on the left side of this inequality is often referred to in the literature as “fast decay” while the one on the right as “slow decay.”

The next theorem guarantees the existence of a solution to Eq. (E) for any exponent \( p > n/(n - 2) \) with the property that the solution itself has at least slow decay, i.e.,

\[
\frac{u(x)}{|x|^{\frac{n}{p} - 1}} \leq \frac{C}{|x|^{\frac{n}{p} - 1}} \quad \text{at } \infty.
\]

\textbf{Theorem 7.} Let \( n/(n - 2) < p \), where \( n \in \mathbb{N} \) with \( n \geq 3 \). Assume

\begin{enumerate}
  \item \( f \in C^{0, \gamma}([R^n]) \), where \( 0 < \gamma \leq 1 \),
  \item \( f \geq 0, f \neq 0 \),
  \item \( f \leq \frac{C}{(1 + |x|^{-\gamma})^{n/p - 1}} \),
\end{enumerate}

where \( C = \frac{1}{(p - 1)^{\frac{1}{p - 1}}} \left( \frac{n - 2p}{p - 1} \right)^{\frac{1}{n - 2p}} \).

Then there exists a solution to Eq. (E)

\[
Au + u^p + f(x) = 0 \quad \text{in } \mathbb{R}^n.
\]

\text{Furthermore, this solution satisfies the inequality}

\[
0 < u(x) \leq \frac{C_1}{(1 + |x|^{\frac{n}{p} - 1})} \quad \text{in } \mathbb{R}^n,
\]
where
\[ C_i = \left[ \frac{2(n - 2p/(p - 1))}{(p - 1)^{p-1}} \right]^{1/(p-1)}. \]

Proof. Set
\[ w(r) = \frac{\zeta}{(1 + r^2)^{\alpha}}, \]
for all \( r \geq 0, \)
where \( \zeta > 0 \) and \( \alpha > 0 \) will be specified later and
\[ S_w = \frac{d^2 w}{dr^2} + \frac{(n - 1) dw}{dr} + w^p. \]
A straightforward computation of \( S_w \) yields that
\[ S_w = \frac{4\zeta \alpha (\alpha + 1) \zeta}{(1 + r^2)^{\alpha + 2}} - \frac{2\zeta n}{(1 + r^2)^{\alpha + 1}} + \frac{\zeta^p}{(1 + r^2)^{\alpha}}, \]
We want to choose \( \alpha > 0 \) and \( \zeta > 0 \) so that
\[ S_w \leq \frac{-C}{(1 + r^2)\beta} \]
for some constant \( C > 0 \) and some exponent \( \beta > 0 \). Hence we must have \( \alpha p \geq \alpha + 1 \), i.e.,
\[ \alpha \geq \frac{1}{p - 1}. \]
With this restriction on \( \alpha \)
\[ S_w \leq \frac{4\alpha (\alpha + 1) \zeta - 2\zeta \alpha n + \zeta^p}{(1 + r^2)^{\alpha + 1}}, \]
\[ \leq \frac{-2\zeta \alpha n[1 - 2(\alpha + 1)/n] + \zeta^p}{(1 + r^2)^{\alpha + 1}}. \]
Thus we need \( 1 - 2(\alpha + 1)/n > 0 \), i.e.,
\[ \alpha < \frac{n - 2}{2}. \]
Hence $\alpha$ must satisfy the inequality

$$\frac{1}{p-1} \leq \alpha < \frac{n-2}{2}.$$ 

This is possible since

$$p > \frac{n}{n-2} \Rightarrow \frac{1}{p-1} < \frac{n-2}{2}.$$ 

Choosing $\alpha = 1/(p-1)$ in the above inequality for $Sw$ yields

$$Sw \leq \left(\frac{-2\xi n}{(p-1)} \left[ \frac{p(n-2)-n}{m(p-1)} \right] + \xi^p \right) (1+r^2)^{(p-1)}.$$ 

If $\xi$ is sufficiently small the right-hand side of this inequality will be negative. The following value of $\xi$ minimizes this term

$$\xi = \left( \frac{2n}{(p-1)p} \left[ \frac{p(n-2)-n}{m(p-1)} \right] \right)^{1/(p-1)}.$$ 

Thus

$$Sw \leq \frac{-C}{(1+r^2)^{p/(p-1)}},$$

where

$$C = \frac{1}{(p-1)^{1/(p-1)}} \left( \frac{2}{p} \left[ n - \frac{2p}{p-1} \right] \right)^{p/(p-1)}.$$ 

Hence if $f$ satisfies hypothesis (2), i.e., if

$$0 \leq f(x) \leq \frac{C}{(1+r^2)^{p/(p-1)}},$$

$w$ will be a supersolution to Eq. (E). Also $v \equiv 0$ is a subsolution to Eq. (E), since $f \geq 0$. Therefore, since $v \leq w$ and $f$ is locally Hölder continuous the monotone iteration method guarantees the existence of a solution $u$ to Eq. (E) such that

$$0 \leq u(x) \leq w(x) \quad \text{for all } x \in \mathbb{R}^n.$$ 

Last, $0 < u$ by the strong maximum principle.
3.2. Fast Decaying Solutions

The next theorem ensures the existence of solutions to Eq. (E) for $p > (n + 2)/(n - 2)$ having exactly fast decay, i.e., when

$$\frac{C_1}{|x|^{n-2}} \leq u(x) \leq \frac{C_2}{|x|^{n-2}} \text{ at } \infty$$

for some positive constants $C_1$ and $C_2$.

**Theorem 8.** Let $(n + 2)/(n - 2) \leq p$, where $n \in \mathbb{N}$ with $n \geq 3$. Assume

1. $f \in C^0(\mathbb{R}^n)$, where $0 < \gamma \leq 1$,
2. $f \geq 0, f \not\equiv 0$,
3. $f \leq \frac{C}{(1 + |x|^2)^{(n+2)/2}}$, where $C = (p - 1) \left[ \frac{n(n-2)}{p} \right]^{p/(p-1)}$.

Then, there exists a solution to Eq. (E)

$$Au + u^p + f(x) = 0 \quad \text{in } \mathbb{R}^n, \quad u > 0.$$ (E)

Furthermore, this solution satisfies the inequality

$$0 < u(x) \leq \frac{\zeta}{(1 + |x|^2)^{(n-2)/2}} \quad \text{in } \mathbb{R}^n,$$

where $\zeta = (n(n-2)/p)^{1/(p-1)}$.

**Proof.** Let $w$ and $Sw$ be as in Theorem 7, i.e.,

$$w(r) = \frac{\zeta}{(1 + r^2)^{\alpha}} \quad \text{for all } r \geq 0,$$

where $\zeta > 0$ and $\alpha > 0$ are constants to be specified and

$$Sw = \frac{d^2w}{dr^2} + \frac{(n-1) dw}{dr} + w^p.$$
Computing $Sw$ yields

$$Sw = \frac{-2\alpha \xi n [r^2 (1 - (2(x + 1)/n)] + 1]}{(1 + r^2)^{n+2}} + \frac{\xi^n}{(1 + r^2)^m},$$

Setting $\alpha = (n - 2)/2$ the supersolution will have precisely fast decay and, consequently, so will the resulting solution given by the super-subsolution method. Hence, if $Sw < 0$ everywhere, we must have $p(n-2)/2 \geq (n+2)/2$ everywhere in $\mathbb{R}^n$, i.e.,

$$p \geq (n+2)/(n-2).$$

With these restrictions, $Sw$ becomes

$$Sw \leq \frac{-n(n-2)}{1 + r^2} + \xi^n.$$

The value of $\xi > 0$ must be sufficiently small so that the above numerator be negative. Choosing the following value for $\xi$ minimizes this number:

$$\xi = \left[ \frac{n(n-2)}{p} \right]^{1/(p-1)}.$$

Hence,

$$Sw \leq \frac{-C}{(1 + r^2)^{(n+2)/2}},$$

where $C = (p - 1)[n(n-2)/p]^{p/(p-1)}$. Again, if $f$ satisfies hypothesis (2), i.e., if

$$0 \leq f(x) \leq \frac{C}{(1 + r^2)^{(n+2)/2}},$$

$w$ will be a supersolution to Eq. (E). Also, $v \equiv 0$ is subsolution to Eq. (E). Therefore, since $v \leq w$ and $f$ is locally Hölder continuous, the monotone iteration method guarantees the existence of a solution $u$ to Eq. (E) such that

$$0 \leq u(x) \leq \frac{\xi}{(1 + |x|^2)^{(n-2)/2}}$$

for all $x$ in $\mathbb{R}^n$.

Furthermore, $0 < u$ by the strong maximum principle.

The last theorem demonstrates the existence of fast decaying solutions to Eq. (E) when the exponent $p \geq (n+2)/(n-2)$. This restriction on the range...
of the exponent is only due to the method of proof. In the next theorem fast decaying solutions will be obtained for any value of the exponent in its permissible range, i.e., \( p > n/(n-2) \). It will require a smallness condition on the inhomogeneous term of the form

\[
\frac{1}{(n-2) \omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy < \frac{\xi}{2(1+|x|)^{n-2}}
\]

for \( 0 < \xi \) sufficiently small.

The proof of this theorem does not use the super-solution method but rather the contraction mapping principle. By Theorem 2 of Section 2 any solution \( u \) of Eq. (E) satisfies the integral equation

\[
u(x) = \frac{1}{(n-2) \omega_n} \int_{\mathbb{R}^n} \frac{(u'' + f)(y)}{|x-y|^{n-2}} dy.
\]

This representation suggests looking for solutions as fixed points (in an appropriate function space) of the map

\[
Tu(x) = \frac{1}{(n-2) \omega_n} \int_{\mathbb{R}^n} \frac{(u'' + f)(y)}{|x-y|^{n-2}} dy.
\]

Since \( f \geq 0 \) and \( p > 1 \), it seems reasonable to expect that \( T(V) \subset V \) if \( V \) is a closed subset of nonnegative functions which are close to zero. The subset \( V \) will be of the form

\[
V_\xi = \{ u \in C(\mathbb{R}^n) \mid 0 < u \leq \frac{\xi}{(1+|x|)^{n-2}} \},
\]

where \( \xi > 0 \) is sufficiently small. This approach exploits the positivity of the functions \( u'' + f \) in Eq. (E). A similar approach to obtain fast decaying solutions to the corresponding parabolic equation to Eq. (E) can be found in the work of Lee [4] dealing with stochastic processes.

**Theorem 9.** Let \( n \in \mathbb{N} \), with \( n \geq 3 \) and \( p > n/(n-2) \). Assume

(1) \( f \in C^{0,\gamma}(\mathbb{R}^n) \), where \( 0 < \gamma \leq 1 \),

(2) \( f \geq 0 \) and \( f \neq 0 \),

(3) \( N(x) = \frac{1}{(n-2) \omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy \leq \frac{\xi}{2(1+|x|)^{n-2}} \).
where

\[ 0 < \xi \leq \min \left\{ \frac{1}{3} K_1^{1/(p-1)}, \frac{1}{2 K_2}, \frac{1}{4 K_3} \right\}^{1/(p-1)}, \]

\[ \left\{ \frac{(n-2)}{2} \left( \frac{4}{5} \right)^{1/(p-1)} \right\} \]

with

\[ K_1 = \left\{ \frac{2((n-2)p-n)}{p(p-1)} \right\}, \]

\[ K_2 = \left\{ \frac{2(n-2)p-3}{n-2} \frac{10^{n-2}}{(n-2)[p(n-2)-n]} + \frac{3^{n-2}}{(n-2)[p(n-2)-n]} \right\} \]

\[ K_3 = \left\{ \frac{2(n-2)p-3}{(n-2)} + \frac{2n^{n-2}}{(n-2)[(n-2)p-2]} \right\}. \]

Then, there exists a solution to Eq. (E):

\[ \Delta u + u^p + f(x) = 0 \quad \text{in } \mathbb{R}^n, \quad (E) \]

\[ u > 0. \]

Furthermore, this solution \( u \) satisfies the inequality

\[ N(x) \leq u(x) \leq \frac{\xi}{(1+|x|)^{r-2}} \quad \text{for all } x \in \mathbb{R}^n. \]

**Proof.** Let \( C_d(\mathbb{R}^n) \) denote the set of bounded continuous functions on \( \mathbb{R}^n \) endowed with the standard sup-norm, i.e., \( ||v|| = \sup_{x \in \mathbb{R}^n} v(x) \) for all \( v \in C_d(\mathbb{R}^n) \).

Set \( V_\xi = \{ u \in C_d(\mathbb{R}^n) \text{ such that } 0 \leq u(x) \leq \xi/(1+|x|)^{r-2} \} \), where \( \xi > 0 \) is a constant which will be determined in the course of the proof and will need to be sufficiently small. \( V_\xi \) is clearly a closed subset of \( C_d(\mathbb{R}^n) \).

Consider the mapping \( T: V_\xi \to V_\xi \) defined by

\[ Tu(x) = \frac{1}{(n-2) \omega_n} \int_{\mathbb{R}^n} \frac{(u^p + f)(y)}{|x-y|^{n-2}} dy. \]

We need to show that \( T(V_\xi) \subseteq V_\xi \).
Claim 1. \( T(V_{\varepsilon}) \subseteq V_{\varepsilon} \) if \( \varepsilon > 0 \) is sufficiently small.

The proof of this claim is long and shall be given in the following pages. Assuming this result we first prove the following shorter claim.

Claim 2. \( T \) is a contraction if \( \varepsilon \) is sufficiently small.

Proof. Let \( u, v \in V_{\varepsilon} \). Therefore,

\[
Tu(x) - Tv(x) = \frac{1}{(n-2) \omega_n} \int_{\mathbb{R}^n} \frac{(u^p - v^p)(y)}{|x-y|^{n-p}} \ dy.
\]

By the mean value theorem, this implies

\[
|Tu(x) - Tv(x)| \leq \frac{p}{(n-2) \omega_n} \|u - v\| \int_{\mathbb{R}^n} \frac{3^{n-\varepsilon p - 1}}{|x-y|^{n-2} (1 + |y|)^{n-2n_p - 1}} \ dy.
\]

Let \( G(x) \) be the function

\[
G(x) = \frac{1}{(n-2) \omega_n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2} (1 + |y|)^{n-2n_p - 1}} \ dy.
\]

By Lemma 2.3 of Ref. [5], \( G(x) \) is bounded in \( \mathbb{R}^n \) since \( (n-2)(p-1) > 2 \). It is easy to see that \( G(0) = \sup_{\mathbb{R}^n} G(x) \). Since \( G(x) \) is radial and satisfies the equation

\[
AG + \frac{1}{(1 + |x|)^{n-2n_p - 1}} = 0 \quad \text{in} \quad \mathbb{R}^n,
\]

it follows that \( G \) is obviously a decreasing function of \( |x| \). It is straightforward to get the following bound for \( G(0) \):

\[
\sup_{\mathbb{R}^n} G(x) = G(0) < \frac{(p-1)}{2[(n-2)p-n]}
\]

Hence

\[
|Tu - Tv| \leq (p^{n-1}G(0)) \varepsilon^{n-1} \|u - v\|.
\]

If \( \varepsilon > 0 \) is such that

\[
\varepsilon < \frac{1}{4} K_1^{1/(p-1)},
\]

where \( K_1 = \{ 2[(n-2)p-n]/p(p-1) \} \), \( T \) will be a contradiction. \( \square \)
Proof of Claim 1. We need to consider two cases: $|x| \geq \frac{1}{r}$ and $|x| < \frac{1}{r}$.

Case 1. $|x| \geq \frac{1}{r}$. Set $T_1 u(x) = T_2 u(x) + T_3 u(x) + N(x)$, where

$$T_1 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |y| \leq |x|} \frac{u^n(y)}{|x-y|^{n-2}} dy,$$

$$T_2 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |y| \leq |x|} \frac{u^n(y)}{|x-y|^{n-2}} dy,$$

$$T_3 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |y| \leq |x|} \frac{u^n(y)}{|x-y|^{n-2}} dy,$$

$$N(x) = \frac{1}{(n-2) \omega_n} \int_{|y| \leq |x|} \frac{f(y)}{|x-y|^{n-2}} dy.$$

We need to consider each operator $T_i$ separately. For the first operator $T_1$, we have for all $u \in V_z$,

$$T_1 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |y| \leq |x|} \frac{u^n(y)}{|x-y|^{n-2}} dy$$

$$\leq \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |y| \leq |x|} \frac{\xi^n}{|x-y|^{n-2}} dy$$

$$\leq \frac{2^{(n-2)\rho}}{(n-2)} \int_{|x-y| \leq |y| \leq |x|} \frac{\xi^n}{|x-y|^{n-2}} dy$$

$$\leq \left( \frac{2^{(n-2)\rho - 3\xi^n}}{(n-2)} \right) \int_{|y| \leq |x|} \frac{1}{(1+|y|)^{n-2}} dy.$$  \hspace{1cm} (2)

For the second operator $T_2$, we have for all $u \in V_z$,

$$T_2 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |y| \leq |x|} \frac{u^n(y)}{|x-y|^{n-2}} dy$$

$$\leq \frac{2^{n\rho}}{(n-2) \omega_n} \int_{|x-y| \leq |y| \leq |x|} \frac{1}{|x-y|^{n-2}} dy$$

$$\leq \frac{2^{n\rho}}{(n-2) \omega_n} \frac{1}{|x|^{n-2}} \int_{|y| \leq |x|} \frac{1}{(1+|y|)^{n-2}} dy$$

$$\leq \frac{2^{n\rho}}{(n-2) \omega_n} \frac{5^{(n-2)}}{(n-2)} \int_{|y| \leq |x|} \frac{1}{(1+|y|)^{n-2}} dy$$

$$\leq \frac{10^{(n-2)}}{(n-2)[(n-2)\rho - n]} \frac{1}{(1+|x|)^{n-2}}.$$  \hspace{1cm} (3)

where the second inequality uses $|x| \geq \frac{1}{r}$. 
For the third operator $T_3$, we have for all $u \in V_\xi,$

$$T_3 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |x-y|^n \leq 2} \frac{u^p}{|x-y|^{n-2}} \, dy$$

$$\leq \frac{1}{(n-2) \omega_n} 2^n \int_{|x-y| \leq |x-y|^n} \frac{u^p}{|x-y|^{n-2}} \, dy$$

$$\leq \frac{5^{n-2} \xi^p}{(n-2) 2^{n-2} (1 + |x|)^{n-2}} \int_{|x-y| \leq |x-y|^n} \frac{u^p}{|x-y|^{n-2}} \, dy$$

$$\leq \left[ \frac{3^{n-2} \xi^p}{(n-2)[p(n-2)-n]} \right] \frac{1}{(1 + |x|)^{n-2}}.$$  \hfill (4)

Choosing $\xi > 0$ sufficiently small such that the sum of the constants in the estimates (2), (3), and (4) for the operators $T_i$ satisfy the inequality

$$\xi^p K_2 \leq \frac{\xi}{2^n}$$

where

$$K_2 = \left\{ \frac{2^{n-2} p - 3}{n-2} + \frac{10^{n-2}}{(n-2)[p(n-2)-n]} + \frac{3^{n-2}}{(n-2)[p(n-2)-n]} \right\}$$

which is equivalent to

$$\xi \leq \left[ \frac{1}{2K_2} \right]^{\frac{1}{p-1}}.$$  \hfill (5)

Using the hypothesis on the Newton potential $N(x)$, we obtain that

$$T_1 u(x) \leq \frac{\xi}{(1 + |x|)^{n-2}} \quad \text{for all} \quad |x| \geq \frac{1}{4}.$$  

**Case 2.** $|x| < \frac{1}{4}$. Set $T_u(x) = T_1 u(x) + T_4 u(x) + T_5 u(x) + N(x)$, where

$$T_1 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |x-y|^n} \frac{u^p(y)}{|x-y|^{n-2}} \, dy,$$

$$T_4 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x|/2 \leq |x-y| \leq 1} \frac{u^p(y)}{|x-y|^{n-2}} \, dy,$$

$$T_5 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |x-y|^n} \frac{u^p(y)}{|x-y|^{n-2}} \, dy,$$

$$N(x) = \frac{1}{(n-2) \omega_n} \int_{|x-y| \leq |x-y|^n} f(y) \, dy.$$
The first operator $T_1$ is the same as that of Case 1. The estimate (2) obtained there did not use the condition on the size of $|x|$. Hence, it is still valid for this case.

For the operator $T_4$, we shall need the smallness condition on $\xi>0$,

$$
\frac{\xi^p}{2(n-2)} \leq \frac{\xi}{4} \left(\frac{4}{3}\right)^{n-2},
$$

which is equivalent to

$$
\xi \leq \left[\frac{(n-2)}{2} \left(\frac{4}{3}\right)^{n-2}\right]^{1/(p-1)}.
$$

We have for all $u \in V_\xi$,

$$
T_4 u(x) = \frac{1}{(n-2) \omega_n} \int_{|x|/2 \leq |x-y| \leq 1} \frac{u''(y)}{|x-y|^{n-2}} dy
\leq \frac{1}{(n-2) \omega_n} \int_{|x|/2 \leq |x-y| \leq 1} \frac{\xi^p}{|x-y|^{n-2} (1 + |y|)^{n-2} p} dy
\leq \frac{\xi^p}{(n-2)} \int_{r=|x|/2}^1 r \ dr
= \frac{\xi^p}{2(n-2)} \left[1 - \frac{|x|^2}{4}\right]
\leq \frac{\xi}{4} \left(1 + |x|\right)^{n-2},
$$

where the last inequality holds by the above condition on $\xi$.

For the operator $T_5$, we have for all $u \in V_\xi$,

$$
T_5 u(x) = \frac{1}{(n-2) \omega_n} \int_{1 \leq |x-y| \leq |x|} \frac{u''(y)}{|x-y|^{n-2}} dy
\leq \frac{1}{(n-2) \omega_n} \int_{1 \leq |x-y| \leq |x|} \frac{\xi^p}{|x-y|^{n-2} (1 + |y|)^{n-2} p} dy
\leq \frac{2^{n-2} \xi^p}{(n-2) \omega_n} \int_{r=1-|x|}^{|x|} \frac{r}{(1+r)^{n-2} p} dr
$$
\[ n \leq \frac{2^{n-2} p \xi}{(n-2) \int_{|r| \leq 1-|s|} \frac{1}{(1+r)^{n-2} r^{n-p}} dr} \]

\[ \leq \frac{2^{n-2} p \xi}{(n-2) \left[ (n-2) p - 2 \right] (2 - |s|)^{(n-2)p-2}} \]

\[ \leq \left[ \frac{2^{n-2} p \xi}{(n-2) \left[ (n-2) p - 2 \right]} \right] \frac{1}{(1+|s|)^{n-2}}, \quad (8) \]

where the third inequality holds since \(|y|/2 \leq |x-y|\) when \(|x| < \frac{1}{4}\) and the last inequality holds since \(1 + |x| \leq 2 - |x|\) when \(|x| < \xi\).

Choosing \(\xi > 0\) sufficiently small so that it satisfies inequality (6) and that the sum of the constants in the estimates (2), (7), and (8) verifies the inequality

\[ \frac{2^{(n-2)p-3} \xi^p}{(n-2)} + \xi + \frac{2^{n-2} \xi^p}{(n-2) \left[ (n-2) p - 2 \right]} \leq \frac{\xi}{2} \]

Hence

\[ \xi^{p-1} K_3 \leq \xi/4, \]

where

\[ K_3 = \left\{ \frac{2^{(n-2)p-3}}{(n-2)} + \frac{2^{n-2}}{(n-2) \left[ (n-2) p - 2 \right]} \right\} \]

which is equivalent to

\[ \xi < (1/4K_3)^{1/(p-1)}, \quad (9) \]

Using the hypothesis on the Newton potential \(N(x)\), we obtain that

\[ Tu(x) \leq \frac{\xi}{(1 + |x|)^{n-2}} \quad \text{for all} \quad |x| < \frac{1}{4}. \]

Hence, with \(\xi > 0\) satisfying inequalities (5), (6), and (9) we obtain that \(T(V_\xi) \subset V_\xi\), proving the claim.

We now complete the proof of the theorem. With the constant \(\xi > 0\) satisfying the inequalities (1), (5), (6), and (9) the map \(T\) will be a contraction from the subset \(V_\xi\) into itself. By the contraction mapping principle, \(T\) has a unique fixed point in \(V_\xi\), that is,

\[ u(x) = \frac{1}{(n-2) \omega_n} \int_{|y| \leq 2} (u^p + f(y)) \frac{dy}{|x-y|^{n-2}} \]

for some function \(u \in V_\xi\). Hence \(0 \leq u(x) \leq \xi/(1 + |x|)^{n-2} \).
Since \( f \) is locally Hölder continuous in \( \mathbb{R}^n \), it is straightforward to show, using Lemmas 4.1 and 4.2 of Ref. [3] that \( u \in C^2(\mathbb{R}^n) \) and satisfies
\[
Au + u^p + f(x) = 0 \quad \text{in } \mathbb{R}^n.
\]

By the maximum principle \( u > 0 \).

3.3. Solutions in the Neighborhood of \( u_* \)

It is well known (by Theorem 4.5 of [6]) that for each \( p \geq (n+2)/(n-2) \) and each \( \alpha > 0 \), there exists a positive radial solution \( u_* \) to the equation
\[
Au_* + u_*^p = 0 \quad \text{in } \mathbb{R}^n,
\]
\[
u_*(0) = \alpha.
\]

Thus, each \( u_* \) is a solution of the homogeneous equation corresponding to Eq. (E) as is the trivial solution \( u_0 \equiv 0 \). The previous theorems of this section yield solutions to Eq. (E) which are near zero, i.e., which lie in a neighborhood of \( u_0 \). It seems natural to inquire if there exists solutions which lie in a neighborhood of \( u_* \), where \( \alpha > 0 \), when \( p \geq (n+2)/(n-2) \).

The next theorem gives an affirmative answer to this question. It yields a solution \( u \) to Eq. (E) which lies above \( u_* \). More precisely, it states that when \( n \geq 12 \) and the exponent \( p \) is very large (\( p > (n-2)^2/[(n-2)^2 - 8n] \)), there exists a solution \( u \) to Eq. (E) (for very small \( f \)) such that
\[
u_* \leq u \leq u_0 + \frac{\delta}{(1 + |x|^2)^{(n-2)/4}}
\]
for some small \( \delta > 0 \). The need for \( n \) and \( p \) to be very large is a necessary condition resulting from the following proposition which is itself an immediate corollary of Proposition 3.5 of [9].

**Proposition 1.** Let \( n \in \mathbb{N} \) with \( n \geq 3 \) and let
\[
p_* = \begin{cases} \infty & \text{if } 3 \leq n \leq 10, \\ \frac{(n-2)^2 - 4n + 8}{\sqrt{n-1}} & \text{if } n > 10. \end{cases}
\]

Then, if \( (n+2)/(n-2) \leq p < p_* \), there does not exist a solution \( u \) to Eq. (E) such that \( u_* \leq u \) for any \( \alpha > 0 \).

**Proof.** Assume there exists a solution \( u \) to Eq. (E) with \( (n+2)/(n-2) \leq p < p_* \) such that \( u_* \leq u \) in \( \mathbb{R}^n \) for some \( \alpha > 0 \). This implies that
\[
u_* \leq u,
\]
where $\bar{u}$ is the spherical mean of $u$. Clearly, $\bar{u}$ is a supersolution to the homogeneous Eq. (E), i.e.,

$$A\bar{u} + \bar{u}^p \leq 0.$$ 

By Proposition 3.5 of [9], this implies that $u_s \equiv \bar{u}$. This implies that $\bar{f} \equiv 0$, a contradiction.

**Theorem 11.** Let $n \in \mathbb{N}$ with $n \geq 12$ and $p > (n - 2)^2/[ (n - 2)^2 - 8n ]$.

Assume

1. $f \in C^{0, \gamma}(\mathbb{R}^n)$, where $0 < \gamma \leq 1$,
2. $f \geq 0$, $f \neq 0$,
3. $f \leq \frac{C}{(1 + \beta |x|^2)^{(n + 2)/4}}$,

where

$$C = \frac{\pi^n}{M \left\{ \left( \frac{p-1}{8n} \right)^2 - p \left( 1 + \frac{1}{M} \right)^{p-1} \right\}},$$

$$\beta = \frac{(p-1) \pi - 1}{2n},$$

$$M = 2 \left\{ \left[ \left( \frac{p-1}{8np} \right)^{1/(p-1)} - 1 \right] \right\}.$$ 

Then, there exists a solution to Eq. (E)

$$A\bar{u} + \bar{u}^p + f(x) = 0 \quad \text{in } \mathbb{R}^n,$$

$$u > 0,$$

satisfying the inequality

$$u_s \leq u \leq u_s + \frac{x}{M(1 + \beta r^2)^{(n - 2)/4}}.$$ 

**Proof.** Set

$$v(r) = \frac{e}{(1 + \beta r^2)^{n/2}}.$$
where \( \varepsilon > 0 \), \( m > 0 \) and \( \beta = (p - 1) n^{p-1}/2n \). The coefficient \( \varepsilon \) and the exponent \( m \) will be chosen so that the following function \( w \) is a supersolution:

\[
  w = u_x + v.
\]

As in the previous theorems, \( S \) denotes the operator

\[
  Sw \equiv Aw + w^p.
\]

Thus

\[
  Sw = (u_x + v)^p - u_x^p + Av.
\]

Computing \( Av \) and using the mean value theorem,

\[
  Sw = p \left( u_x + \frac{\varepsilon}{(1 + \beta r^2)^m} \right)^{p-1} \left( \frac{\varepsilon}{(1 + \beta r^2)^m} + \frac{2m\beta \left[ (2(m+1) - n) r^2 - n \right]}{(1 + \beta r^2)^{m+2}} \right),
\]

where \( 0 < \theta < 1 \). By a result of Gui [2], \( u_x \) satisfies the estimate

\[
  u(x) < \frac{\alpha}{(1 + \beta r^2)^{\frac{1}{2}}},
\]

where \( \beta \) is given at the beginning of the proof. Hence,

\[
  Sw \leq p \left[ \left( \frac{\alpha}{(1 + \beta r^2)^{\frac{1}{2}}} + \frac{\varepsilon}{(1 + \beta r^2)^m} \right)^{p-1} \left( \frac{\varepsilon}{(1 + \beta r^2)^m} \right) \right] + \frac{2m\beta \left[ (1 - (2(m+1)/n) r^2 + 1 \right]}{(1 + \beta r^2)^{m+2}}.
\]

In order to obtain \( Sw < 0 \) everywhere in \( \mathbb{R}^n \), we must have

\[
  1 - 2(m+1)/n > 0, \quad m(p - 1) \geq 1.
\]

Thus, we must have

\[
  \frac{1}{p - 1} \leq m < \frac{n - 2}{2}.
\]

This is possible since \( p > n/(n - 2) \). Thus, with this restriction on \( m \), \( Sw \) becomes

\[
  Sw \leq \frac{-C}{(1 + \beta r^2)^{m+1}}.
\]
where
\[
C = 2m\beta n \left[ 1 - \frac{2(m + 1)}{n} \right] - \varepsilon p(x + \varepsilon)^{p-1}.
\]

We need to choose \( \varepsilon > 0 \), \((p - 1) \leq m < (n - 2)/2\) and restrict the ranges of \( n \) and \( p \) so that \( C > 0 \). Set
\[
\varepsilon = \frac{x}{M},
\]
where \( M > 0 \) will depend on \( p \) and \( n \).

Since \( \beta = (p - 1)x^{p-1}/2n \), \( C \) becomes
\[
C = \frac{x^p}{M} \left[ m(p - 1) \left[ 1 - \frac{2(m + 1)}{n} \right] - p \left( 1 + \frac{1}{M} \right)^{p-1} \right].
\]

The expression in braces must be positive if \( C > 0 \). This is equivalent to
\[
2m^2 - (n - 2) m + \frac{np}{(p - 1)} \left( 1 + \frac{1}{M} \right)^{p-1} < 0.
\]

This inequality involves the variables \( n, p, m, \) and \( M \) (i.e., indirectly \( \varepsilon \), since \( \varepsilon = x/M \)). If this inequality is to hold for some value of \( m \), by the quadratic formula, the following inequality should be verified:
\[
(n - 2)^2 - 8np(1 + 1/M)^{p-1} > 0.
\]

Note that this inequality does not involve \( m \). To achieve the widest possible range for the dimension of the space \( n \) and the exponent \( p \), we assume \( M \) is very large (i.e., \( \varepsilon \) is very small). More precisely, we assume that \( p \) and \( n \) satisfy the following inequality, where \( M \) is absent:
\[
(n - 2)^2 - 8np > 0.
\]

We shall then choose \( M \) so large that the former inequality is positive. Since \( p > 1 \) and \( n \geq 3 \), this last inequality will hold if and only if
\[
n \geq 12, \quad p > \frac{(n - 2)^2}{(n - 2)^2 - 8n}.
\]
It is trivial to verify that for $n \geq 3$,
\[
\frac{n + 2}{n - 2} < \frac{(n - 2)^2}{(n - 2)^2 - 8n}.
\]
We need to choose $M > 0$ sufficiently large so that constant $C$ is positive. It is easy to compute that $M$ must satisfy the inequality
\[
M > 1 \left\{ \left[ \frac{(p - 1)(n - 2)^2}{8np} \right]^{1/(p-1)} - 1 \right\}.
\]
Therefore, we set
\[
M = 2 \left\{ \left[ \frac{(p - 1)(n - 2)^2}{8np} \right]^{1/(p-1)} - 1 \right\}.
\]
We need now to specify the exponent $m$ which must satisfy the above inequalities. Choosing $m = (n - 2)/4$ will satisfy these inequalities and maximize the constant $C$. Thus, for $n \geq 12$ and $p > (n - 2)^2/[8(n - 2)^2 - 8n]$, if $f(x)$ satisfies the estimate
\[
f(x) \leq \frac{C}{(1 + \beta |x|^2)^{(n + 4)/4}},
\]
where
\[
C = \frac{2^n}{M} \left\{ \left[ \frac{(p - 1)(n - 2)^2}{8np} \right]^{1/(p-1)} - p \left(1 + \frac{1}{M}\right)^{(p-1)} \right\},
\]
w will be a supersolution to Eq. (E). But clearly $u_s$ is a subsolution to Eq. (E). Since $u_s \leq w$ and $f$ is locally Hölder continuous, the monotone iteration method guarantees the existence of a solution $u$ to Eq. (E) such that
\[
u_s \leq u \leq w,
\]
which completes the proof.

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