On the non-Riemannian quantity $H$ of a Finsler metric

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ABSTRACT

One of fundamental problems in Finsler geometry is to establish some delicate equations between Riemannian invariants and non-Riemannian invariants. Inspired by results due to Akbar-Zadeh etc., this note establishes a new fundamental equation between non-Riemannian quantity $H$ and Riemannian quantities on a Finsler manifold. As its application, we show that all R-quadratic Finsler metrics have vanishing non-Riemannian invariant $H$ generalizing result previously only known in the case of Randers metric.

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1. Introduction

Finsler geometry is more colourful than Riemannian geometry because besides the Riemannian quantities, there are several important non-Riemannian quantities in a Finsler manifold. Non-Riemannian quantities all vanish for Riemann geometry. It is one of important problems in Finsler geometry is to study the geometric meanings of these non-Riemannian invariants, in particular, to establish some delicate relation between Riemannian invariants and non-Riemannian invariants. Many Finslerian geometers have found the intrinsic relation between non-Riemannian invariants (such as the Cartan tensor, the Landsberg curvature and the $S$-curvature) and Riemannian invariants (such as the flag curvature and the Ricci scalar) and hence have obtained a series of rigidity theorems and classification theorems. See [1,4,7,8,12] for some recent developments. Is there any other intrinsic relation between non-Riemannian invariants and Riemannian invariants? In [1], H. Akbar-Zadeh considered a non-Riemannian quantity $H$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics.

In this paper we establish a natural relation among non-Riemannian invariant $H$, the flag curvature and its averaging quantity – the Ricci curvature (see (3.12) and Corollary 3.5) generalizing results previously only known in the case of constant/scalar flag curvature Finsler metrics [1,10]. It is worth mentioning the flag curvature takes the place of the sectional curvature in the Riemannian case.

Secondly, we use this fundamental equation (3.12) to study R-quadratic Finsler metrics. To our surprise all of R-quadratic Finsler metrics have vanishing non-Riemannian invariant $H$. This result naturally extends Li–Shen’ following theorem: any R-quadratic Randers metric has constant non-Riemannian invariant $S$-curvature [6], hence it has vanishing non-Riemannian invariant $H$. Recall that Randers metrics are special Finsler metrics in the following form

$F = \alpha + \beta$

where $\alpha$ is a Riemannian metric and $\beta$ is a one form.
A Finsler metric is said to be \textit{R-quadratic} if its Riemann curvature $R_F$ is quadratic in $y \in T_x M$ (see (4.1) below). R-quadratic metric was introduced by Bácsó and Matsumoto [3]. There are many non-Riemann R-quadratic Finsler metrics. For example, all Berwald metrics are R-quadratic. Some non-Berwald R-quadratic Finsler metrics have been constructed in [2,6]. Thus R-quadratic Finsler metrics form a rich class of Finsler spaces.

2. Ricci identities

In this section we are going to use the Chern connection to give some important Ricci identities for a Finsler metric which will be used in later. Let $(M, F)$ be an $n$-dimensional Finsler manifold. Throughout the paper, our index conventions are as follows: Latin indices (expect the alphabet $n$) run from 1 to $n$.

Let $\{\omega_i \}_{0}^{n}$ be a local frame for $\pi^*TM$ where $\pi: TM \setminus \{0\} \to M$ is the natural projection, $\{\omega_i, \omega_{n+i}\}$ be the corresponding local coframe for $T^*(TM \setminus \{0\})$ and $\{\omega_i\}$ be the set of local Chern connection forms with respect to $\{\frac{\partial}{\partial y^i}\}$. Let $f: TM \setminus \{0\} \to \mathbb{R}$ be a smooth function from the slit tangent bundle. We define $f_{ij}$ and $f_{i}$ by

$$df = f_{ij} \omega^j + f_{i} \omega^{n+i}$$

(2.1)

where $|i$ (resp. $-i$) denote horizontal (resp. vertical) covariant derivative. Differentiating (2.1) and using the structure equations one deduces that

$$Df_{ij} \wedge \omega^j + Df_{i} \wedge \omega^{n+i} + f_i \Omega^j = 0$$

(2.2)

where

$$Df_{ij} = df_{ij} - f_{ij} \omega^j,$$

$$Df_{i} = df_{i} - f_{ij} \omega^j,$$

$$\Omega^j = d\omega^{n+i} - \omega^{n+j} \wedge \omega^j.$$  

Recall that $\Omega^j$ have the following structure

$$\Omega^j = \frac{1}{2} R_{kl}^i \omega^k \wedge \omega^l - L_{kl}^i \omega^k \wedge \omega^{n+i}.$$  

(2.3)

We put

$$Df_{ij} = f_{ijj} \omega^j + f_{ij} \omega^{n+j},$$

(2.4)

$$Df_{i} = f_{ijj} \omega^j + f_{ij} \omega^{n+j}.$$  

(2.5)

Substituting (2.3), (2.4) and (2.5) into (2.2), and taking the components of $\omega^j \wedge \omega^j$, $\omega^j \wedge \omega^{n+j}$ and $\omega^{n+i} \wedge \omega^{n+j}$ respectively, we have the following:

\textbf{Lemma 2.1.} Let $(M, F)$ be a Finsler manifold. For any smooth function $f$ on $TM \setminus \{0\}$, we have

$$f_{ijj} = f_{ij} + f_k R^k_{ij},$$

(2.6)

$$f_{ij} = f_{ijj} + f_k L^k_{ij},$$

(2.7)

$$f_{ij} = f_{ij}. $$

(2.8)

Recall that the \textit{distortion} $\tau$ is defined by

$$\tau = \ln \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_F(x)}$$

where $\sigma_F(x) := \text{Vol}(B^n)/\text{Vol}(y^i \in \mathbb{R}^n | F(x, y^i \frac{\partial}{\partial y^i}) < 1)$ and $g_{ij}(x, y) = \frac{1}{2}(F^2)_{y^i y^j}$. The mean Cartan torsion $I_i$ are defined by

$$I_i = \frac{\partial}{\partial y^i} \left[ \ln \sqrt{\det(g_{jk}(x, y))} \right].$$

\textbf{Corollary 2.2.} Let $(M, F)$ be a Finsler manifold and let $\tau$ be its distortion. Then

$$\tau_{ij} = \tau_{ijj} + I_k R^k_{ij};$$

(2.9)

$$\tau_{ij} = I_{ijj} - I_k L^k_{ij}. $$

(2.10)
Proof. It is easy to see that the mean Cartan tensor satisfies the following equation

\[ l_I = \frac{\partial \tau}{\partial y^I} = \tau_I. \]

Plugging this into (2.6) and (2.7) yields (2.9) and (2.10) respectively. \qed

We have the following Ricci identities for a one form.

**Lemma 2.3.** Let \((M, F)\) be a Finsler manifold and \(T := T_i\omega^i\) a one form on \(M\). Then

\[ T_{ij; k} = T_{i[k}^l j - T_{i[l}^j k + T_{i}^j p_{l}^k]. \]  
(2.11)

\[ T_{ij|k} = T_{[ij}^l - T_{i|k}^j l - T_{i}^l R_{i}^j k - T_{i}^l R_{i}^j |k]. \]  
(2.12)

**Proof.** We define \(T_{ij}\) and \(T_{ij}\) by

\[ dT_i - T_j \omega^j = T_{ij} \omega^i + T_{ij} \omega^{n+i}. \]  
(2.13)

Differentiating (2.13) and using the structure equations one deduces

\[ -(dT_j - T_k \omega^j k) \wedge \omega^j - T_j \Omega^j + T_{i}^j \Omega^j = (dT_{ij} - T_{ijk} \omega^k) \wedge \omega^j + (dT_{ij} - T_{ik} \omega^j k) \wedge \omega^{n+j}. \]  
(2.14)

Plugging (2.13) into (2.14) we have

\[ DT_{ij} \wedge \omega^j + DT_{ij} \wedge \omega^{n+j} = T_{ij} \Omega^j + T_{ij} \Omega^j = 0. \]  
(2.15)

where

\[ DT_{ij} = dT_{ij} - T_{kj} \omega^k - T_{ij} \omega^k, \]
\[ DT_{ij} = dT_{ij} - T_{kj} \omega^k - T_{ij} \omega^k, \]
\[ \Omega^j = d\omega^j - \omega^k \wedge \omega^j. \]

Recall that \(\Omega^j\) have the following structure

\[ \Omega^j = \frac{1}{2} R^j_{kl} \omega^k \wedge \omega^l + P^j_{kl} \omega^k \wedge \omega^{n+i}. \]  
(2.16)

We put

\[ DT_{ij} = T_{ij} \omega^k + T_{ij} \omega^{n+k}, \]  
(2.17)

\[ DT_{ij} = T_{ij} \omega^k + T_{ij} \omega^{n+k}. \]  
(2.18)

Plugging (2.5), (2.16), (2.17) and (2.18) into (2.15) and taking the components of \(\omega^j \wedge \omega^{n+j}\) and \(\omega^j \wedge \omega^j\) respectively yields (2.11) and (2.12). \qed

**Corollary 2.4.** Let \((M, F)\) be a Finsler manifold and let \(J_i\) be its mean Landsberg curvature. Then

\[ J_{i,j} y^k = (J_{i,j} y^k) - J_{ij} + J_{i} L^k_{ij}. \]  
(2.19)

\[ I_{i,j} y^k = J_{ij} - I_{i} R^j_{ki} y^k + I_{j} R^k_{i}. \]  
(2.20)

**Proof.** By (2.11), we have

\[ J_{i,j} = J_{i,j} + J_{i} L^k_{ij} - J_{i} P^k_{ij}. \]  
(2.21)

It is known the Landsberg curvature \(L^i_{jk}\) and the Minkowskian curvature \(P^i_{kl}\) satisfy the following equations

\[ P^i_{kl} = P^i_{kl}, \]  
(2.22)

\[ L^i_{kl} = -y^j P^j_{i} \]  
(2.23)

\[ y^k L^i_{kl} = 0. \]  
(2.24)

See [5, p. 41] for more details. Contracting (2.21) with \(y^k\) and plugging (2.22), (2.23) and (2.24) into it yields

\[ J_{i,j} y^k = J_{i,j} y^k + J_{i} L^k_{ij} y^k - J_{i} P^k_{ij} y^k = J_{i,j} y^k + J_{k} L^k_{ij}. \]  
(2.25)
On the other hand
\[ y^i \omega^j + y^i \alpha^{n+j} = D y^i = dy^i + y^i \omega^j = \alpha^{n+j}. \]
It follows that
\[ y^i, j = \delta^i_j, \quad y^i_{,ij} = 0. \] (2.26)
This gives rise to the following equation
\[ (J_{ijk} y^k)_, j = J_{ijk} y^k + J_{ijkl} y^l + J_{ij}. \]
This is,
\[ J_{ijk} y^k = (J_{ijk} y^k)_, j - J_{ij}. \] (2.27)
Substituting (2.27) into (2.25) yields (2.19).

(ii) It follows from (2.12) that
\[ I_{ij, k} = I_{ijk} - I_{ij} R^h_{ki} - I_{ij} R^i_{jk}. \] (2.28)
On the other hand, we have (cf. [12])
\[ R^i_{kl} y^k = -R^i_{lk} y^k = -R^i_{lj}. \] (2.29)
Contracting (2.28) with \( y^k \) and plugging (2.26) and (2.29) into it yields
\[ I_{ij, k} y^k = I_{ijk} y^k - I_{ij} R^i_{kl} y^l - I_{ij} R^i_{kl} y^k = (I_{ijk} y^k)_, j + I_{ij} R^i_{kl} - I_{ij} R^i_{kl} y^k. \] (2.30)
It follows from [12, (7)] that
\[ J^i = I_{ij} y^j. \] (2.31)
Substituting (2.31) into (2.30) yields (2.20).

Let \( F \) be a Finsler metric on a manifold \( M \). The Riemann curvature of \( F \) is a family of endomorphisms
\[ R^i_{kl} dx^l \otimes \frac{\partial}{\partial x^i} : T_x M \to T_x M, \]
where \( R^i_{kl} = R^i_{jk,l} \). \( F \) is said to be of scalar flag curvature \( K \) if
\[ R^i_{kl} = K F^2 h^i_k, \]
where \( F h^i_k := F \delta^i_k - F F y^l y^l \) [5]. The trace of \( R^i_{kl} \), i.e. \( \sum_i R^i_{ii} \), is called the Ricci scalar.

3. Non-Riemannian quantity \( H \) and Riemann curvature

In this section, we will show that the non-Riemann quantity \( H \) is closely related to the Riemann curvature.
The \( S \)-curvature is defined by
\[ S = \sum_i \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial (\ln \sigma^F)}{\partial \xi}. \]
where \((x^i, y^j)\) are local coordinate systems of \( TM \), \( G^i \) are the geodesic coefficients of \( F \) and \( \sigma^F \) is given in Section 2 [12].
Then the mean Berwald curvature \( E = E_{ij} dx^i \otimes dx^j \) can be defined using the \( S \)-curvature \( S \).

\[ E_{ij} = \frac{1}{2} \frac{\partial^2 S}{\partial y^i \partial y^j}. \] (3.1)

**Lemma 3.1.** Let \((M, F)\) be a Finsler manifold. Then
\[ S_{,i} = J_{i} + \tau_{ij}, \] (3.2)
\[ E_{ij} = \frac{1}{2} J_{ij} + \frac{1}{2} I_{ij} - \frac{1}{2} l_k L^k_{ij}. \] (3.3)

**Proof.** By [5, (5.48)], we obtain (3.2). For a scalar function \( f \), we have
\[ f_{ij} = \frac{\partial^2 f}{\partial y^i \partial y^j}. \] (3.4)
One can refer to [9] for details. From (3.1), (3.2), (3.4) and (2.10), we have
\[ E_{ij} = \frac{1}{2} S_{ij} - \frac{1}{2} (J_{ij} + \tau_{ij}) = \frac{1}{2} J_{ij} + \frac{1}{2} I_{ij} - \frac{1}{2} l_k L^k_{ij}. \] □
It is worth mentioning (3.3) is equivalent to (6.38) in [13]. Shen derived (6.38) by using the Berwald connection. The quantity $H_{ij} = H_{ij}^i d\xi^i \otimes dx^j$ is defined as the covariant derivative of $E$ along geodesics. More precisely,

$$H_{ij} := E_{ij|k} y^k.$$  \hspace{1cm} (3.5)

**Lemma 3.2.** Let $(M, F)$ be a Finsler manifold. Then

$$2H_{ij} = \left( J_{ijk}y^k \right)_j - J_{i|j} + J_{j|i} - I_l R_{jk}^l y^k + I_j R^l_{i|j} - I_l L_{ij|k} y^k.$$ \hspace{1cm} (3.6)

**Proof.** From (3.3), (3.5), (2.21), (2.22) and (2.24), we obtain

$$2H_{ij} = E_{ij|k} y^k$$

$$= \left( J_{i|j} + I_{j|i} - I_l L_{ij|k} y^k \right)_k y^k$$

$$= J_{i|j} y^k + J_{j|i} y^k - I_l L_{ij|k} y^k - I_l L_{ij|k} y^k$$

$$= \left( J_{ijk} y^k \right)_j - J_{i|j} + J_{j|i} - I_l R_{jk}^l y^k + I_j R_{l|i}^l - I_l L_{ij|k} y^k$$

$$= \left( J_{ijk} y^k \right)_j - J_{i|j} + J_{j|i} - I_l R_{jk}^l y^k + I_j R_{l|i}^l - I_l L_{ij|k} y^k.$$ \hspace{1cm} \(\square\)

**Lemma 3.3.** Let $(M, F)$ be a Finsler manifold. Then

$$L_{ij|k} y^k = R_{ij|k} y^k - R_{j|k|} y^k,$$ \hspace{1cm} (3.7)

$$S_{ki|} y^l - S_{kl} = -\frac{1}{3}(2R_{k|} y^l + R_{l|k})$$ \hspace{1cm} (3.8)

$$J_{ij|} y^k = -I_{k} R_{i|} y^k - \frac{1}{3}(2R_{k|} y^l + R_{l|k})$$ \hspace{1cm} (3.9)

**Proof.** (3.7) follows from the proof of Lemma 2.4 in [9]. By (5.4.2) in [9] we have (3.8). Now we show (3.9). By (2.24) and (2.28), we obtain

$$J_{ij|} y^k = \left( I_{ij|} y^l \right)^{|k} y^k = \left( I_{ij|} y^l + I_{ij|} y^l \right) y^k = I_{ij|} y^l y^k.$$ \hspace{1cm} (3.10)

On the other hand, from (2.70) of [4] (also see [5]) we have

$$I_{ij|} y^l y^k + I_{j} R_{i|} = -\frac{1}{3}(2R_{k|} y^l + R_{l|k}).$$ \hspace{1cm} (3.11)

Plugging (3.10) into (3.11) yields (3.9). \hspace{1cm} \(\square\)

In the following we are going to establish an important relation between the Riemann curvature and the non-Riemann quantity $H$.

**Proposition 3.4.** Let $F$ be a Finsler metric on a manifold. If $R_{ij}$ and $\text{Ric}$ denote the coefficients of the Riemannian curvature and the Ricci scalar respectively, then

$$6H_{ij} + \text{Ric}_{i|j} + R_{k|j} + R_{j|k} = 0.$$ \hspace{1cm} (3.12)

**Proof.** Plugging (3.9) into (3.6) and using (3.2) and (3.7) yields

$$2H_{ij} = \left[ -I_{k} R_{i|} - \frac{1}{3}(2R_{k|} + R_{l|k}) \right]_{j} - J_{i|j} + J_{j|i} + I_l (R_{j|} y^k - L_{ij|k} y^k) + I_j R_{i|}$$

$$= -I_{k} R_{i|} - I_{k} R_{i|} - \frac{2}{3} R_{k|j} + \frac{1}{3} R_{j|} + \left( S_{j} - \tau_{j|i} \right)_{i} - \left( S_{i} - \tau_{i|j} \right)_{j} + I_l (R_{j|} + R_{j|}) + I_j R_{i|}$$

$$= -\frac{2}{3} R_{k|j} - \frac{1}{3} R_{j|} + \left( I_{j} - I_{k} \right) R_{i|} + S_{j|} - S_{i|j} + \tau_{j|i} - \tau_{i|j} + I_j R_{j|}$$ \hspace{1cm} (3.13)

where we have used the equation

$$R_{j|} = -R_{j|k}.$$  

From the proof of Corollary 2.2 and (2.8) we have

$$I_{j} - I_{k} = \tau_{j|k} - \tau_{k|j}.$$ \hspace{1cm} (3.14)
By (2.9) we obtain
\[ \tau_{ij} - \tau_{iji} + l_l R^l_{ij} = \tau_{ij} - \tau_{iji} - l_l R^l_{ij} = 0. \] (3.15)
Substituting (3.14) and (3.15) into (3.13) yields
\[ 2H_{ij} = -\frac{2}{3} R^k_{ik}j - \frac{1}{3} \text{Ric}_{ij} + S_{jil} - S_{lij}. \] (3.16)
On the other hand, from (3.8), (2.7), (2.26) and (2.28), we have
\[ -\frac{1}{3} \left( 2 R^k_{ik} + R^k_{ij} \right) = S_{i|k}y^k - S_{i} = S_{j|k}y^k + S_{jL_k}y^k - S_{i} = (S_{j|k}y^k) - 2S_{i} \]
that is
\[ S_{i} = \frac{1}{2} \left( (S_{j|k}y^k) - \frac{2}{3} R^k_{ik} + \frac{1}{3} R^k_{jk} \right). \] (3.17)
By (2.7), we have
\[ S_{j|i} = S_{i|j} + S_{jL_k}. \]
Plugging this into (3.16) and using (3.17) yields
\[ 2H_{ij} = -\frac{2}{3} R^k_{ik}j - \frac{1}{3} \text{Ric}_{ij} - S_{j} + S_{jL_k} - S_{iL_k} \]
which gives (3.12).

We can take further averaging on \( H \) as follows
\[ H = g^{ij} H_{ij}. \]
Then we obtain the following

**Corollary 3.5.** Let \((M, F)\) be a Finsler manifold. Then the non-Riemannian quantity \( H \) satisfy
\[ H = \frac{1}{6} g^{ij} \text{Ric}_{ij} + \frac{1}{3} g^{ij} R^k_{ikj}. \]
In the case of the scalar curvature, (3.12) is reduced to the following important relationship due to B. Najafi, Z. Shen and A. Tayebi [10]. We give an alternative proof.

**Corollary 3.6.** Let \( F \) be a Finsler metric of scalar curvature on an \( n \)-dimensional manifold \( M \). For any \( 1 \)-form \( \theta \) on \( M \), the flag curvature \( K \) and the quantity \( H \) satisfy
\[ \partial^2 (F \tilde{K}) - \tilde{K} \partial^2 F \partial y^i \partial y^j = \frac{6}{n+1} F^{-1} \left[ H_{ij} - \frac{(n+1)\theta}{2} \partial^2 F \partial y^i \partial y^j \right] \] (3.18)
where \( \tilde{K} := K - 3\theta / F. \)

**Proof.** By definition, we have
\[ R^l_{ij} = K F^l h^j. \] (3.19)
where
\[ F h^j = F \delta^i_j - g_{jk} \frac{y^k}{F} = F \delta^i_j - F y^i y^j. \] (3.20)
Substituting (3.20) into (3.19) yields
\[ R^l_{ij} = K F (F \delta^i_j - F y^i y^j). \]
It follows that
\[ R_{ijk}^l = K_k F_i^2 h_j^l + KF_k h_i^l + KF(F_k \delta_j^l - F_{jk} y^l) - F_j \delta_i^k \]

\[ = K_k F_i^2 h_j^l + 2KF_k h_i^l - KF_j F_k y^l + KFF_{jk} y^l - KF_j \delta_i^k. \]  

(3.21)

Note that both \( F_j \) and \( K \) are homogeneous degree zero with respect to \( y \). Hence we have

\[ F_j y^l = 0, \quad K y^l = 0. \]  

(3.22)

It follows from (3.21) and (3.22) that

\[ R_{ijkl} = K_k F^2 - (n-1)K F F_j. \]  

(3.23)

Differentiating (3.23) with respect to \( y^k \) gives

\[ R_{ijlk} = K_{ik} F^2 + 2K_k F F_i - (n-1)K F_k F_i - \frac{n-1}{2} K(F^2)_{ijk}. \]  

(3.24)

On the other hand, by (3.20) we get

\[ \text{Ric} = \sum_i R_{ij} = K F \sum_i (F \delta_j^i - F_i y^l) = K F(n F - F) = (n-1)K F^2. \]

It follows that

\[ \text{Ric}_{ij} = (n-1)K_{ij} F^2 + (n-1)K_i (F^2)_j + (n-1)K_j (F^2)_i + (n-1)K (F^2)_{ij}. \]  

(3.25)

By (3.24) and (3.25) we get

\[ \text{Ric}_{ij} + R_{ijkl}^k + R_{ijkl}^k = (n+1)F(K F)_{ij} + K_k F_{ij} + K_j F_i \]

\[ = (n+1)F[(F K)_{ij} - K F_{ij}]. \]

We have

\[ 0 = 6H_{ij} + \text{Ric}_{ij} + R_{ijkl}^k + R_{ijkl}^k = 6H_{ij} + (n+1)F[(F K)_{ij} - K F_{ij}]. \]  

(3.26)

This is (3.18) in the case \( \theta = 0 \). In general, \( \tilde{K} = K - 3\theta F \) where \( \theta = \theta \theta(x) y^l \) is a one form. It follows that

\( \theta_{ij} = \frac{\partial^2}{\partial y^i \partial y^j} \left[ \theta \right] = \frac{\partial}{\partial y^i} \left[ \theta \right] = 0. \)

Hence

\[ (F \tilde{K})_{ij} = (F K - 3\theta)_{ij} = (F K)_{ij}. \]  

(3.27)

On the other hand, we have

\[ K F_{ij} = \tilde{K} F_{ij} + 3\theta F_{ij} / F. \]  

(3.28)

Plugging (3.27) and (3.28) into (3.26) yields

\[ 0 = (n+1)F \left\{ (F \tilde{K})_{ij} - \tilde{K} F_{ij} + \frac{6}{n+1} F^{-1} \left[ H_{ij} - \frac{(n+1)}{2} F_{ij} \right] \right\}, \]

thus we obtain (3.18).

\[ \text{Corollary 3.7. (See [1,10].) Let } F \text{ be a Finsler metric of scalar curvature on an } n (> 2) \text{ -dimensional manifold. Then the non-quantity } H \text{ takes a special form } (n^2 - 1)\theta^2 / 2F \text{ if and only if the flag curvature takes a special form } 3\theta F + \sigma \text{ where } \theta \text{ is a 1-form on } M \text{ and } \sigma = \sigma(x) \text{ is a scalar function on } M. \text{ In particular, the flag curvature is a scalar function on the manifold if and only if } H = 0. \]

\[ \text{4. R-quadratic Finsler metrics} \]

A Finsler metric is said to be \( R \)-quadratic if its Riemann curvature \( R_y \) (see Section 2 for definition) is quadratic in \( y \in T_x M \). In terms of local coordinates, the \( R \)-quadratic metric is characterized by (4.1). There are many non-Riemann \( R \)-quadratic Finsler metrics. For example, all Berwald metrics are \( R \)-quadratic. Thus \( R \)-quadratic Finsler metrics form a rich class of Finsler spaces. In this section we show the following

\[ \text{Theorem 4.1. Let } (M, F) \text{ be a } R \text{-quadratic Finsler manifold. Then } H = 0. \]
Proof. Similar to (3.4), we have
\[ R_{i j k} = \frac{\partial R^l_{i j}}{\partial y^k}, \quad R^l_{i j k} = \frac{\partial R^l_{i j k}}{\partial y^l} = \frac{\partial^2 R^l_{i j}}{\partial y^k \partial y^l}. \]
Suppose that \( F \) is \( R \)-quadratic. From the Remark 3.1 in [11] or [13] we get
\[ R^l_{i j} = R_k^l (x) y^j y^l; \quad R^l_{i j} = -R_k^l y^j. \] (4.1)
It follows that
\[ \text{Ric} = \frac{\partial}{\partial y^l} \left( \sum R^l_{i j} \right) = \frac{\partial}{\partial y^l} \left( \sum R^l_{i j} + \sum R^l_{i j} y^{l} \right), \]
and hence
\[ \text{Ric}_{i j} = \frac{\partial}{\partial y^l} \left( \sum R_{i j} + \sum R_{i j} y^{l} \right) = \sum R_{i j} y^k + \sum R_{i j} y^k. \] (4.2)
On the other hand, by (4.1) we have
\[ R_{i k} = \frac{\partial}{\partial y^l} \left( R_{i j} + R_{i j} y^{l} \right) = R_{i j} + R_{i j} y^l, \]
and hence
\[ R_{i k j} = \frac{\partial}{\partial y^l} \left( R_{i k} + R_{i j} y^l \right) = R_{i k} + R_{i j} y^l. \] (4.3)
From (3.12), (4.2) and (4.3) we have
\[ -6H_{i j} = \text{Ric}_{i j} + R_{i k j} + R_{i k j} - \sum R_{i j} + \sum R_{i j} + \sum R_{i j} + \sum R_{i j} + \sum R_{i j} = 0. \]
It follows that
\[ H = g^{ij} H_{ij} = 0. \]

Theorem 4.1 shows that the rate of changes of the mean Berwald curvature along geodesic is constant for any \( R \)-quadratic Finsler metric. A natural question arises: for a \( R \)-quadratic Finsler metric, is the mean Berwald curvature constant? This is verified for Randers metrics [6]. But it is still unknown for general Finsler metrics.

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References