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Convergence of generalized AOR iterative method for linear systems with strictly diagonally dominant matrices ☆

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Abstract

In this paper, some improvements on Darvishi and Hessari [On convergence of the generalized AOR method for linear systems with diagonally dominant coefficient matrices, Appl. Math. Comput. 176 (2006) 128–133] are presented for bounds of the spectral radius of $l_{\omega,r}$, which is the iterative matrix of the generalized AOR (GAOR) method. Subsequently, some new sufficient conditions for convergence of GAOR method will be given, which improve some results of Darvishi and Hessari [On convergence of the generalized AOR method for linear systems with diagonally dominant coefficient matrices, Appl. Math. Comput. 176 (2006) 128–133].

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1. Introduction

Sometimes we have to solve the following linear system:

$$Hy = f, \tag{1}$$

where

$$H = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix}$$

is invertible. For example, in the generalized least-square problem [9,10], we have to solve the generalized least-square problem

$$\min_{x \in R^n} (Ax - b)^{\mathrm{T}} W^{-1} (Ax - b),$$

where W is the variance–covariance matrix [8]. If $I - B_i$ for $i = 1, 2$ are nonsingular, we may apply the regular SOR method, or the regular AOR method [3,6] to solve (1). However, $I - B_i$ for $i = 1, 2$ sometimes are singular. In fact,

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even if $I - B_i$ for $i = 1, 2$ are nonsingular, it is also not easy to solve linear system (1) because we have to find the inverses of $I - B_i$ for $i = 1, 2$, or to solve two subsystems

$$(I - B_i)x_i = d_i, \quad i = 1, 2.$$

Hence a generalized SOR (GSOR) method was proposed by Yuan to solve linear system (1) in [9], afterwards, Yuan and Jin [10] established a generalized AOR (GAOR) method to solve linear system (1) as follows:

$$y^{(k+1)} = l_{\omega,r} y^{(k)} + \omega k, \tag{2}$$

where

$$l_{\omega,r} = (1 - \omega)I + \omega J + \omega r K, \tag{3}$$

$$k = \begin{pmatrix} I & 0 \\ -rC & I \end{pmatrix} f, \tag{4}$$

$$J = \begin{pmatrix} B_1 & -D \\ -C & B_2 \end{pmatrix}, \tag{5}$$

$$K = \begin{pmatrix} 0 & 0 \\ C(I - B_1) & CD \end{pmatrix} = \begin{pmatrix} 0 \\ C \end{pmatrix} (I - B_1 \quad D). \tag{6}$$

From (2)–(6), we know that the GAOR method does not need any inverses of $I - B_i$ for $i = 1, 2$. Obversely, the GAOR method is the GSOR method [9] when $r = \omega$; the generalized Jacobi method when $r = 0$; and the regular AOR method [3] when $B_1 = B_2 = 0$.

Throughout this paper, we shall employ the following notations. Let $N = \{1, 2, \ldots, n\}$. Let us denote by $C^{n,n}$ the class of all complex matrices, and denote $\rho(l_{\omega,r})$ by the spectral radius of iterative matrix $l_{\omega,r}$.

For $A = (a_{ij}) \in C^{n,n}$ and $B = (b_{ij}) \in C^{n,n}$, we say $A \geqslant B$ if $a_{ij} \geqslant b_{ij}$ for all $i, j \in N$ and A is nonnegative matrix if $a_{ij} \geqslant 0$ for all $i, j \in N$. The absolute matrix is defined by $|A| = (|a_{ij}|)$.

For $A = (a_{ij}) \in C^{n,n}$, let $R_i(A) = \sum_{k \neq i} |a_{ik}|$ $(i \in N)$. Recall that A is called (weakly) diagonally dominant matrix ($A \in$ WD) if

$$|a_{ii}| \geqslant R_i(A) \quad \forall i \in N. \tag{7}$$

If the inequality in (7) is strict for each $i \in N$, we say that A is strictly diagonally dominant matrix ($A \in$ SD). A is called irreducibly diagonally dominant matrix if A is irreducible and at least one of the inequalities in (7) holds strictly ($A \in$ ID) [5,7].

In Ref. [2], authors obtained upper and lower bounds of the spectral radius of iterative matrix $l_{\omega,r}$ and investigated the convergence of the GAOR method, that is, the following results are presented.

Theorem 1 (*Darvishi and Hessari [2]*). *If $H \in$ SD, then $\rho(l_{\omega,r})$ satisfies the following inequality*:

$$|1 - \omega| + \min_i\{|\omega| J_i + |\omega r| K_i\} \leqslant \rho(l_{\omega,r}) \leqslant |1 - \omega| + \max_i\{|\omega| J_i + |\omega r| K_i\}, \tag{8}$$

where J_i and K_i are the i-row sums of the modulus of the entries of J and K, respectively.

Theorem 2 (*Darvishi and Hessari [2]*). *If $H \in$ WD, then the GAOR method converges for $0 \leqslant r \leqslant 1$ and $0 < \omega \leqslant 1$.*

Theorem 3 (*Darvishi and Hessari [2]*). *Let $H \in$ SD and assume that $\omega \geqslant r \geqslant 0$. Then the sufficient condition for convergence of the GAOR method is*

$$0 < \omega < \frac{2}{1 + \max_i\{J_i + r K_i\}}. \tag{9}$$

This work is organized as follows. In Section 2, we obtain new upper and lower bounds for the spectral radius of $l_{\omega,r}$ when $H \in$ SD, which are better than ones of Theorem 1. Section 3 investigates the convergence of the GAOR method for diagonally dominant coefficient matrices and our results are better than ones of Theorem 3. In Section 4, we give two numerical examples to explain the above results and point out two inappropriate aspects in [2].

2. Bounds of the spectral radius of $l_{\omega,r}$

In this section, we obtain new upper and lower bounds of the spectral radius of iterative matrix $l_{\omega,r}$, which are better than Theorem 1 and point out an error of the lower bound of Theorem 1.

Theorem 4. *Let* $H \in \mathrm{SD}$. *Then* $\rho(l_{\omega,r})$ *satisfies the following inequality*:

$$\min_i\{|1-\omega|-|\omega|(J+rK)_i\} \leqslant \rho(l_{\omega,r}) \leqslant \max_i\{|1-\omega|+|\omega|(J+rK)_i\}, \tag{10}$$

where $(J+rK)_i$ *denotes the i-row sums of the modulus of the entries of matrix* $J+rK$.

Proof. Let λ be an arbitrary eigenvalue of iterative matrix $l_{\omega,r}$, then

$$\det(\lambda I - l_{\omega,r}) = 0, \tag{11}$$

we can show that Eq. (11) holds if and only if

$$\det((\lambda+\omega-1)I - \omega J - \omega r K) = 0. \tag{12}$$

Moreover, if $(\lambda+\omega-1)I - \omega J - \omega r K \in \mathrm{SD}$, that is,

$$|\omega|(J+rK)_i - |\omega||(J+rK)_{ii}| < |\lambda+\omega-1-\omega(J+rK)_{ii}| \quad \forall i \in N,$$

where $(J+rK)_{ii}$ denotes the diagonal element of matrix $J+rK$, then λ is not an eigenvalue of $l_{\omega,r}$. Especially, if

$$|\omega|(J+rK)_i < |\lambda+\omega-1| \quad \forall i \in N,$$

then λ is not an eigenvalue of $l_{\omega,r}$. Further, when

$$|\omega|(J+rK)_i < |\lambda| - |1-\omega| \quad \forall i \in N$$

or

$$|\omega|(J+rK)_i + |1-\omega| < |\lambda| \quad \forall i \in N,$$

where λ is not an eigenvalue of $l_{\omega,r}$, we have

$$|\lambda| > \max_i\{|1-\omega|+|\omega|(J+rK)_i\}.$$

Hence,

$$\rho(l_{\omega,r}) \leqslant \max_i\{|1-\omega|+|\omega|(J+rK)_i\}.$$

Using the same method, one may show that the lower bound of $\rho(l_{\omega,r})$ is

$$\rho(l_{\omega,r}) \geqslant \min_i\{|1-\omega|-|\omega|(J+rK)_i\}.$$

This completes the proof. $\square$

Remark 1. As it is well known, for convenient values of ω and r, the GAOR becomes the well-known iterative methods, that is,

(i) The GAOR method becomes the GSOR method when $\omega = r$, thus

$$\rho(l_{\omega,\omega}) \leqslant \max_i\{|1-\omega|+|\omega|(J+\omega K)_i\}.$$

(ii) The GAOR method is the generalized Jacobi method when $r=0$, thus

$$\rho(l_{\omega,0}) \leqslant \max_i\{|1-\omega| + |\omega| J_i\}.$$

Remark 2. The upper bound of Theorem 4 is always no larger than the upper bound of Theorem 1 because

$$\max_i\{|1-\omega| + |\omega|(J+rK)_i\} \leqslant |1-\omega| + \max_i\{|\omega| J_i + |\omega r| K_i\}.$$

If $\min_i\{|1-\omega| - |\omega|(J+rK)_i\} \leqslant 0$, we say that the lower bound of $\rho(l_{\omega,r})$ equals to zero. In addition, the lower bound of $\rho(l_{\omega,r})$ is improper in Theorem 1, the reason is explained in Section 4.

Lemma 1 (*Berman and Plemmons [1], Horn and Johnson [5]*). *Let $A, B \in C^{n,n}$ and assume that $|B| \leqslant A$. Then $\rho(B) \leqslant \rho(|B|) \leqslant \rho(A)$.*

Lemma 2 (*Berman and Plemmons [1], Horn and Johnson [5]*). *Let $A=(a_{ij}) \in C^{n,n}$ and assume that $A \geqslant 0$. Then*

$$\min_i \sum_j a_{ij} \leqslant \rho(A) \leqslant \max_i \sum_j a_{ij}.$$

Remark 3. The upper bound of $\rho(l_{\omega,r})$ can be obtained as follows. Since

$$\begin{aligned} l_{\omega,r} = (1-\omega)I + \omega J + \omega r K &\leqslant |(1-\omega)I + \omega J + \omega r K| \\ &\leqslant |1-\omega| I + |\omega||J+rK|, \end{aligned}$$

by Lemmata 1 and 2, we obtain easily our results.

3. Convergence of the GAOR method

In this section, the convergence of the GAOR method to solve linear system (1) is investigated. We assume that H is a strictly diagonally dominant coefficient matrix and obtain some sufficient conditions for the convergence of the GAOR method.

Lemma 3. *If $H \in$ SD, then the GAOR method converges for all $0 \leqslant r \leqslant 1$ and $0 < \omega \leqslant 1$.*

Proof. Assumption that λ is an eigenvalue of $l_{\omega,r}$ and $|\lambda| \geqslant 1$. Then the following relationship holds:

$$\det(\lambda I - l_{\omega,r}) = 0,$$

or after performing a simple series of transformations

$$\det(Q) = 0, \tag{13}$$

where

$$Q = I - \frac{\omega}{\lambda+\omega-1} J - \frac{\omega r}{\lambda+\omega-1} K.$$

It follows from the proof of Theorem 2 in [2] that

$$\left|\frac{\omega}{\lambda+\omega-1}\right| \leqslant 1 \quad \text{and} \quad \left|\frac{\omega r}{\lambda+\omega-1}\right| \leqslant 1.$$

In the following, we shall show that $H-K = I-J-K \in$ SD when $H \in$ SD. From (1)–(6), we have

$$H - K = \begin{pmatrix} I - B_1 & D \\ CB_1 & I - B_2 - CD \end{pmatrix},$$

where $B_1 = (b'_{ij}) \in C^{m,m}$, $B_2 = (b''_{ij}) \in C^{n-m,n-m}$, $C = (c_{ij}) \in C^{n-m,m}$ and $D = (d_{ij}) \in C^{m,n-m}$.

Notice that the first m rows of $H-K$ are strictly diagonally dominant. We need only to show that the below $n-m$ rows are strictly diagonally dominant.

Considering, for all $i \in \{m+1,\dots,n\}$,

$$\begin{aligned}
&\left|1-b''_{ii}-\sum_{k=1}^{m} c_{ik}d_{ki}\right| - \sum_{\substack{j=m+1\\ j\neq i}}^{n}\left|-b''_{ij}-\sum_{k=1}^{m} c_{ik}d_{kj}\right| - \sum_{j=1}^{m}\left|\sum_{k=1}^{m} c_{ik}b'_{kj}\right| \\
&\geqslant |1-b''_{ii}| - \sum_{k=1}^{m}|c_{ik}d_{ki}| - \sum_{\substack{j=m+1\\ j\neq i}}^{n}\left(|b''_{ij}|+\sum_{k=1}^{m}|c_{ik}d_{kj}|\right) - \sum_{k=1}^{m}\sum_{j=1}^{m}|c_{ik}b'_{kj}| \\
&= |1-b''_{ii}| - \sum_{k=1}^{m}|c_{ik}|\left(|d_{ki}| + \sum_{\substack{j=m+1\\ j\neq i}}^{n}|d_{kj}| + \sum_{j=1}^{m}|b'_{kj}|\right) - \sum_{\substack{j=m+1\\ j\neq i}}^{n}|b''_{ij}| \\
&> |1-b''_{ii}| - \sum_{k=1}^{m}|c_{ik}| - \sum_{\substack{j=m+1\\ j\neq i}}^{n}|b''_{ij}| > 0.
\end{aligned}$$

Hence, $H-K=I-J-K\in \mathrm{SD}$. Since the coefficient of J and K are nonzero and less than one in modulus in matrix Q, thus Q is nonsingular which contradicts (13). Therefore, $\rho(l_{\omega,r})<1$. This completes the proof. $\square$

Remark 4. If $H\in \mathrm{WD}$, the above results are not true because it is possible for Q to be singular. This show that the condition of Theorem 2 seem to be unsuitable, a numerical example is given in Section 4.

Theorem 5. *Let $H\in \mathrm{SD}$ and assume that r and ω satisfy*

$$\max_i (J+rK)_i<1 \quad \textit{and} \quad 0<\omega<\frac{2}{1+\max_i (J+rK)_i}.$$

Then the GAOR is convergent.

Proof. From (10), we see that $\rho(l_{\omega,r})$ will be less than one if

$$|1-\omega|+|\omega|(J+rK)_i<1 \quad \forall i\in N. \tag{14}$$

Thus, ω must satisfy $0<\omega<2$. Considering the following two cases:

Case 1: If $0<\omega\leqslant 1$, then Eq. (14) is equivalent to $(J+rK)_i<1$ $(i\in N)$.

Case 2: If $1<\omega<2$, then Eq. (14) is equivalent to

$$1<\omega<\frac{2}{1+(J+rK)_i} \quad \forall i\in N,$$

which implies $(J+rK)_i<1$ $(i\in N)$. Combining Cases 1 with 2, we obtain

$$\max_i (J+rK)_i<1 \quad \text{and} \quad 0<\omega<\frac{2}{1+\max_i (J+rK)_i}.$$

This completes the proof. $\square$

Applying Lemma 3 and Theorem 5, the following Theorem 6 can be established.

Theorem 6. *If $H\in \mathrm{SD}$, then the sufficient conditions for the convergence of the GAOR method are either*

(i) $0\leqslant r\leqslant 1$ *and* $0<\omega\leqslant 1$ *or*

(ii) $|r| < \min_i\{(1-J_i)/K_i\}$ *and* $0<\omega<2/(1+\max_i(J+rK)_i)$, *where* $\{(1-J_i)/K_i\}=+\infty$ *if* $K_i=0$.

Proof. (i) This is Lemma 3.

(ii) Since $(J+rK)_i < J_i+|r|K_i \ (i \in N)$, from Theorem 5, the GAOR method is convergent if $J_i+|r|K_i<1 \ (i\in N)$. Thus $|r|<(1-J_i)/K_i \ (i \in N)$. This completes the proof. □

Now, we consider the GAOR method corresponding to $r=0$ and $\omega=r$.

Theorem 7. *If* $H \in \mathrm{SD}$, *then the sufficient condition for* $\rho(l_{\omega,0})<1$ *is*

$$0<\omega<\frac{2}{1+\max_i J_i}.$$

Proof. From Theorem 6, obviously. □

Theorem 8. *Let* $H\in\mathrm{SD}$. *Then the sufficient condition for* $\rho(l_{\omega,\omega})<1$ *is*

$$0<\omega\leqslant \max\left\{1, \min_i \frac{4}{(1+J_i)+\sqrt{(1+J_i)^2+8K_i}}\right\}.$$

Note: If the largest of the two quantities in the braces is not the 1, then the second inequality as regards ω must be a strict one.

Proof. It follows from Eq. (14) that $0<\omega<2$. If $0<\omega\leqslant 1$, by Lemma 3, we have $\rho(l_{\omega,\omega})<1$; if $1<\omega<2$, then Eq. (14) is equivalent to

$$\omega-1+\omega(J+\omega K)_i<1.$$

Especially, we have $\rho(l_{\omega,\omega})<1$ if $\omega-1+\omega J_i+\omega^2K_i<1$, that is,

$$\omega^2K_i+\omega(1+J_i)-2<0. \tag{15}$$

Let $S=\{i|K_i=0, i\in N\}$. For all $i\in S$, Eq. (15) implies

$$1<\omega<\frac{2}{1+J_i}. \tag{16}$$

For all $i \in N-S$, since the discriminant of a curve of second order $\Delta>0$, the solution of the (15) satisfies

$$1<\omega<\frac{4}{(1+J_i)+\sqrt{(1+J_i)^2+8K_i}}. \tag{17}$$

Notice that Eq. (17) becomes Eq. (16) if $i\in S$. Hence,

$$1<\omega<\min_i\frac{4}{(1+J_i)+\sqrt{(1+J_i)^2+8K_i}}.$$

Above all, ω must satisfy

$$0<\omega\leqslant \max\left\{1, \min_i \frac{4}{(1+J_i)+\sqrt{(1+J_i)^2+8K_i}}\right\}.$$

This completes the proof. □

Let us consider a first-degree linear stationary iterative method

$$x^{(k+1)}=Tx^{(k)}+c. \tag{18}$$

The following method

$$x^{(k+1)} = [(1-\omega)I + \omega T]x^{(k)} + \omega c \tag{19}$$

will be called the extrapolated method of (18) and ω is called the extrapolated parameter.

Now, we recall the Theorem of extrapolation [4].

Theorem (*Theorem of extrapolation*). *The sufficient conditions for the convergence of* (19) *are*:

(1) *The original* (18) *is convergent*,
(2) $0 < \omega < 2/(1+\rho(T))$.

Theorem 9. *If* $H \in \mathrm{SD}$, *the sufficient condition for* $\rho(l_{\omega,0}) < 1$ *is*

$$0 < \omega < \frac{2}{1+\rho(J)}.$$

Proof. Since $H \in \mathrm{SD}$, then the Jacobi method is convergent. We know that the generalized Jacobi method is an extrapolation of the Jacobi method with extrapolated parameter ω. By the Theorem of extrapolation, we obtain $0 < \omega < 2/(1+\rho(J))$. □

Remark 5. Since $\rho(J) \leqslant \max_i J_i$, the result of Theorem 9 is better than one of Theorem 7.

Since the GAOR method is an extrapolation of the GSOR method with extrapolated parameter ω/r. By Theorem 8 and Theorem of extrapolation, we have the following Theorem 10.

Theorem 10. *Let* $H \in \mathrm{SD}$. *The sufficient condition for* $\rho(l_{\omega,r}) < 1$ *are*

$$0 < r \leqslant \max\left\{1, \min_i \frac{4}{(1+J_i)+\sqrt{(1+J_i)^2+8K_i}}\right\} \quad \text{and} \quad 0 < \omega < \frac{2r}{1+\rho(l_{r,r})}.$$

Note: If the largest of the two quantities in the braces is not the 1, then the second inequality as regards ω must be a strict one.

Proof. From Theorem 8 and Theorem of extrapolation, obviously. □

Remark 6. It is obvious that the results of Theorem 10 are better than ones of Theorem 5 in [2].

4. Examples

The following examples show that the results of Theorems 4 and 6 are better than ones of Theorems 1 and 3, respectively. Example 2 shows that the condition of Theorem 2 is inappropriate.

Example 1. Let

$$H = \left(\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} \\ \hline \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix}.$$

Obviously, $H \in \mathrm{SD}$. For convenient, supposing that $\omega = r = 1$. By Theorem 4, we get $0 \leqslant \rho(l_{\omega,r}) \leqslant \frac{4}{9}$, but we have, from Theorem 1, $\frac{2}{3} \leqslant \rho(l_{\omega,r}) \leqslant \frac{16}{9}$. In fact, $\rho(l_{\omega,r}) = \frac{1}{3}$, while $\frac{2}{3} \nleqslant \frac{1}{3}$ and $\frac{4}{9} < \frac{16}{9}$. These show that our result is better and the lower bound of $\rho(l_{\omega,r})$ in Theorem 1 is not true.

By Theorem 6, the following regions of convergence are obtained:

(1) $0 \leqslant r \leqslant 1$ and $0 < \omega \leqslant 1$;
(2) $0 \leqslant r < 0.3$ and $0 < \omega < 1.2$;
(3) $-0.3 < r < 0$ and $0 < \omega < 2/(1 + \frac{2}{3} - \frac{10}{9}r)$.

Now applying Theorem 3 into it, we have $\omega \geqslant r \geqslant 0$ and $0 < \omega < 2/(1 + \frac{2}{3} + \frac{10}{9}r)$. It is easy to show that our results are better than ones of Theorem 3.

Example 2. Let

$$H = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \hline 0 & \frac{1}{2} & 1 \end{array}\right) = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix}.$$

Obviously, $H \in \mathrm{WD}$. By Theorem 2, the GAOR method is convergent if $0 \leqslant r \leqslant 1$ and $0 < \omega \leqslant 1$. But we get $\rho(l_{\omega,r}) = 1$ if $\omega = r = 1$, a contradiction.

Remark 7. If $H \in \mathrm{ID}$, then $H - K = I - J - K$ may be reducible matrix when $B_1 = 0$. Thus it is likely that Q is singular, that is, Lemma 3 may be not true if $H \in \mathrm{ID}$. Unfortunately, we have not find such an example.

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