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Stochastic Processes and their Applications 123 (2013) 523–560
stochastic processes and their applications
www.elsevier.com/locate/spa

Large deviations for stochastic partial differential equations driven by a Poisson random measure

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Received 19 March 2012; received in revised form 20 September 2012; accepted 21 September 2012
Available online 6 October 2012

Abstract

Stochastic partial differential equations driven by Poisson random measures (PRMs) have been proposed as models for many different physical systems, where they are viewed as a refinement of a corresponding noiseless partial differential equation (PDE). A systematic framework for the study of probabilities of deviations of the stochastic PDE from the deterministic PDE is through the theory of large deviations. The goal of this work is to develop the large deviation theory for small Poisson noise perturbations of a general class of deterministic infinite dimensional models. Although the analogous questions for finite dimensional systems have been well studied, there are currently no general results in the infinite dimensional setting. This is in part due to the fact that in this setting solutions may have little spatial regularity, and thus classical approximation methods for large deviation analysis become intractable. The approach taken here, which is based on a variational representation for nonnegative functionals of general PRMs, reduces the proof of the large deviation principle to establishing basic qualitative properties for controlled analogues of the underlying stochastic system. As an illustration of the general theory, we consider a particular system that models the spread of a pollutant in a waterway.

MSC: primary 60H15; 60F10; secondary 37L55

Keywords: Stochastic partial differential equation; Poisson random measure; Large deviations; Variational representation; Freidlin–Wentzell asymptotics; Diffusion equation with Poisson point source

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doi:10.1016/j.spa.2012.09.010

1. Introduction

Stochastic partial differential equations driven by Poisson random measures arise in many different fields. For example, they have been used to develop models for neuronal activity that account for synaptic impulses occurring randomly, both in time and at different locations of a spatially extended neuron. Other applications arise in chemical reaction–diffusion systems and stochastic turbulence models. The starting point in all these application areas are deterministic partial differential equations (PDEs) that capture the underlying physics. One then develops a stochastic evolution model driven by a suitable Poisson noise process to take into account random inputs or effects to the nominal deterministic dynamics. In typical settings the solutions of these stochastic evolution equations are not smooth. In fact in many applications of interest they are not even random fields (that is, function valued), and therefore an appropriate framework is given through the theory of generalized functions. A systematic theory of existence and uniqueness of solutions (both weak and pathwise) for such stochastic partial differential equations (SPDEs) driven by Poisson random measures has been developed in [16]. Our objective in this work is to study some large deviation problems associated with such stochastic systems.

Large deviation properties of SPDEs driven by infinite dimensional Brownian motions (e.g. Brownian sheets) have been extensively studied. In such a typical setting one considers a small parameter multiplying the noise term and is interested in asymptotic probabilities of non-nominal behavior as the parameter approaches zero. This is the classical Freidlin–Wentzell problem that has been studied in numerous papers (see the references in [5]). Earlier works on this family of problems were based on ideas of [1] and relied on discretizations and other approximations combined with 'super-exponential closeness' probability estimates. For many models of interest, particularly those arising from fluid dynamics and turbulence, developing the required exponential probability estimates is a daunting task and consequently simpler alternative methods are of interest. In recent years an approach based on certain variational representation formulas for moments of nonnegative functionals of Brownian motions [5] has been increasingly used for the study of the small noise large deviation problem for Brownian motion driven infinite dimensional systems [2,5,6,8,10,11,18–21,23,25,27,29–31]. The main appealing feature of this approach is that it completely bypasses approximation/discretization arguments and exponential probability estimates, and in their place essentially requires a basic qualitative understanding of existence, uniqueness and stability (under 'bounded' perturbations) of certain controlled analogues of the underlying stochastic dynamical system of interest.

Large deviation results for finite dimensional stochastic differential equations with a Poisson noise term has been studied by several authors [28,17,12,9]. For infinite dimensional models with jumps, very little is available. One exception is the paper [22] that obtains large deviation results for an Ornstein–Uhlenbeck type process driven by an infinite dimensional Lévy noise. One reason there is relatively little work in the Poisson noise setting is that approximation arguments that one uses for Brownian noise models become much more onerous in the Poisson setting, and for general infinite dimensional models the approach of [1] becomes intractable.

With the expectation that it would prove useful for the study of large deviations for SPDEs driven by Poisson random measures (PRMs), the paper [7] developed a variational representation, for moments of non negative functionals of PRMs, which is analogous to the representation given in [4,5] for the Brownian motion case. The paper [7] also obtained large deviation results for a basic model of a finite dimensional jump–diffusion to illustrate the applicability of this variational representation for the study of large deviation problems for models with jumps.

However the feasibility of this approach for the study of complex infinite dimensional stochastic dynamical systems driven by Poisson random measures has not been addressed to date.

The goal of this work is to demonstrate that the approach based on variational representations that has been very successful for obtaining large deviation results for system driven by Brownian noises works equally well for SPDE models driven by PRMs. As in the Brownian case we study the small noise problem, which in the Poisson setting means that the jump intensity is $O(\epsilon^{-1})$ and jump sizes are $O(\epsilon)$, where ϵ is a small parameter. We consider a rather general family of models of the form

$$X_t^\epsilon = X_0^\epsilon + \int_0^t A(s, X_s^\epsilon)ds + \epsilon \int_0^t \int_{\mathbb{X}} G(s, X_{s-}^\epsilon, v)\tilde{N}^{\epsilon^{-1}}(dsdv), \tag{1.1}$$

where $N^{\epsilon^{-1}}$ is a Poisson random measure on $[0, T] \times \mathbb{X}$ with a σ-finite mean measure $\epsilon^{-1}\lambda_T \otimes \nu$, λ_T is the Lebesgue measure on $[0, T]$ and $\tilde{N}^{\epsilon^{-1}}([0, t] \times B) = N^{\epsilon^{-1}}([0, t] \times B) - \epsilon^{-1} t\nu(B)$, $\forall B \in \mathcal{B}(\mathbb{X})$ with $\nu(B) < \infty$, is the compensated Poisson random measure.

As noted previously, a key issue with a Poisson noise model is the selection of an appropriate state space, since it is natural and often convenient for there to be little spatial regularity. However, many of these foundational issues have been satisfactorily resolved in [16], where pathwise existence and uniqueness of SPDEs of the form (1.1) are treated under rather general conditions. In the framework of [16] solutions lie in the space of RCLL trajectories that take values in the dual of a suitable nuclear space. This framework covers many specific application settings that have been studied in the literature (e.g., spatially extended neuron models, chemical reaction–diffusion systems, etc.). Parallel with the case of Brownian noise, one finds that the estimates needed for establishing the well-posedness of the equation are precisely the ones that are key for the proof of the large deviation result as well.

The paper is organized as follows. We begin in Section 2 with some background results. The variational representation from [7] is recalled and also a general large deviation result established in that paper is presented. Also summarized are basic existence and uniqueness results from [16] for SPDEs with solutions in the duals of Countably Hilbertian Nuclear Spaces (CHNS). In Section 3 we study the small noise problem and state verifiable conditions on the model data in (1.1) under which a large deviation principle holds. Section 4 considers a particular system designed to model the spread of a pollutant in a waterway, and verifies all the conditions assumed on (1.1). Finally, the Appendix collects some auxiliary results.

The following notation will be used. For a topological space $\mathcal{E}$, denote the corresponding Borel σ-field by $\mathcal{B}(\mathcal{E})$. We will use the symbol "$\Rightarrow$" to denote convergence in distribution. Let $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{R}_+, \mathbb{R}^d$ denote the set of positive integers, non-negative integers, integers, real numbers, positive real numbers, and d-dimensional real vectors respectively. For a Polish space $\mathbb{X}$, denote by $C([0, T] : \mathbb{X})$ and $D([0, T] : \mathbb{X})$ the space of continuous functions and right continuous functions with left limits from $[0, T]$ to $\mathbb{X}$, endowed with the uniform and Skorokhod topology, respectively. For a metric space $\mathcal{E}$, denote by $M_b(\mathcal{E})$ and $C_b(\mathcal{E})$ the space of real bounded $\mathcal{B}(\mathcal{E})/\mathcal{B}(\mathbb{R})$-measurable maps and real continuous bounded functions respectively. For a measure ν on $\mathcal{E}$ and a Hilbert space H, let $L^2(\mathcal{E}, \nu; H)$ denote the space of measurable functions f from $\mathcal{E}$ to H such that $\int_{\mathcal{E}} \|f(v)\|^2 \nu(dv) < \infty$, where $\|\cdot\|$ is the norm on H. For a function $x : [0, T] \to \mathcal{E}$, we use the notation x_t and $x(t)$ interchangeably for the evaluation of x at $t \in [0, T]$. A similar convention will be followed for stochastic processes. We say a collection $\{X^\epsilon\}$ of $\mathcal{E}$-valued random variables is tight if the distributions of X^ϵ are tight in $\mathcal{P}(\mathcal{E})$ (the space of probability measures on $\mathcal{E}$).

A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function on $\mathcal{E}$, if for each $M < \infty$ the level set $\{x \in \mathcal{E} : I(x) \le M\}$ is a compact subset of $\mathcal{E}$. A sequence $\{X^\epsilon\}$ of $\mathcal{E}$ valued random variables is said to satisfy the Laplace principle upper bound (respectively lower bound) on $\mathcal{E}$ with rate function I if for all $h \in C_b(\mathcal{E})$

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{E} \left\{ \exp \left[-\frac{1}{\epsilon} h(X^\epsilon) \right] \right\} \le - \inf_{x \in \mathcal{E}} \{h(x) + I(x)\},$$

and, respectively,

$$\liminf_{\epsilon \to 0} \epsilon \log \mathbb{E} \left\{ \exp \left[-\frac{1}{\epsilon} h(X^\epsilon) \right] \right\} \ge - \inf_{x \in \mathcal{E}} \{h(x) + I(x)\}.$$

The Laplace principle is said to hold for $\{X^\epsilon\}$ with rate function I if both the Laplace upper and lower bounds hold. It is well known that when $\mathcal{E}$ is a Polish space, the family $\{X^\epsilon\}$ satisfies the Laplace principle upper (respectively lower) bound with a rate function I on $\mathcal{E}$ if and only if $\{X^\epsilon\}$ satisfies the large deviation upper (respectively lower) bound for all closed sets (respectively open sets) with the rate function I. For a proof of this statement we refer the reader to Section 1.2 of [12].

2. Preliminaries

2.1. Poisson random measure and a variational representation

Let $\mathbb{X}$ be a locally compact Polish space. Let $\mathcal{M}_{FC}(\mathbb{X})$ be the space of all measures ν on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ such that $\nu(K) < \infty$ for every compact K in $\mathbb{X}$. Endow $\mathcal{M}_{FC}(\mathbb{X})$ with the weakest topology such that for every $f \in C_c(\mathbb{X})$ (the space of continuous functions with compact support), the function $\nu \mapsto \langle f, \nu \rangle = \int_{\mathbb{X}} f(u) d\nu(u)$, $\nu \in \mathcal{M}_{FC}(\mathbb{X})$ is continuous. This topology can be metrized such that $\mathcal{M}_{FC}(\mathbb{X})$ is a Polish space (see e.g. [7]). Fix $T \in (0, \infty)$ and let $\mathbb{X}_T = [0, T] \times \mathbb{X}$. Fix a measure $\nu \in \mathcal{M}_{FC}(\mathbb{X})$, and let $\nu_T = \lambda_T \otimes \nu$, where λ_T is Lebesgue measure on $[0, T]$.

We recall that a Poisson random measure $\mathbf{n}$ on $\mathbb{X}_T$ with mean measure (or intensity measure) ν_T is a $\mathcal{M}_{FC}(\mathbb{X}_T)$ valued random variable such that for each $B \in \mathcal{B}(\mathbb{X}_T)$ with $\nu_T(B) < \infty$, $\mathbf{n}(B)$ is Poisson distributed with mean $\nu_T(B)$ and for disjoint $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{X}_T)$, $\mathbf{n}(B_1), \ldots, \mathbf{n}(B_k)$ are mutually independent random variables (cf. [14]). Denote by $\mathbb{P}$ the measure induced by $\mathbf{n}$ on $(\mathcal{M}_{FC}(\mathbb{X}_T), \mathcal{B}(\mathcal{M}_{FC}(\mathbb{X}_T)))$. Then letting $\mathbb{M} = \mathcal{M}_{FC}(\mathbb{X}_T)$, $\mathbb{P}$ is the unique probability measure on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which the canonical map, $N : \mathbb{M} \to \mathbb{M}$, $N(m) \doteq m$, is a Poisson random measure with intensity measure ν_T. With applications to large deviations in mind, we also consider, for $\theta > 0$, probability measures $\mathbb{P}_\theta$ on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$ under which N is a Poisson random measure with intensity $\theta \nu_T$. The corresponding expectation operators will be denoted by $\mathbb{E}$ and $\mathbb{E}_\theta$, respectively. We now present a variational representation, obtained in [7], for $-\log \mathbb{E}_\theta(\exp[-F(N)])$, where $F \in M_b(\mathbb{M})$, in terms of a Poisson random measure constructed on a larger space. We begin by describing this construction.

The analysis of large deviation properties for a process such as (1.1) is simplified considerably by a convenient control representation for the exponential integrals appearing in the Laplace principle. In contrast with the case of Brownian motion, the formulation of a useful representation is not immediate for Poisson noise. With a Poisson random measure, one needs a control that alters the intensity at time t and for jump type x from that of the underlying PRM to essentially any value in $[0, \infty)$ in a non-anticipating fashion. To accommodate this form of control, we augment the space of jump times and jump types by a variable $r \in [0, \infty)$, and consider in place of the original PRM one whose intensity is a product of ν_T and Lebesgue measure on r. The desired jump intensities can then be obtained by "thinning" this variable.

Thus we let $\mathbb{Y} = \mathbb{X} \times [0, \infty)$ and $\mathbb{Y}_T = [0, T] \times \mathbb{Y}$. Let $\bar{\mathbb{M}} = \mathcal{M}_{FC}(\mathbb{Y}_T)$ and let $\bar{\mathbb{P}}$ be the unique probability measure on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$ under which the canonical map, $\bar{N} : \bar{\mathbb{M}} \to \bar{\mathbb{M}}$, $\bar{N}(m) \doteq m$, is a Poisson random measure with intensity measure $\bar{\nu}_T = \lambda_T \otimes \nu \otimes \lambda_\infty$, with λ_∞ Lebesgue measure on $[0, \infty)$. The corresponding expectation operator will be denoted by $\bar{\mathbb{E}}$. Let $\mathcal{F}_t \doteq \sigma\{\bar{N}((0, s] \times A) : 0 \le s \le t, A \in \mathcal{B}(\mathbb{Y})\}$, and let $\bar{\mathcal{F}}_t$ denote the completion under $\bar{\mathbb{P}}$. We denote by $\bar{\mathcal{P}}$ the predictable σ-field on $[0, T] \times \bar{\mathbb{M}}$ with the filtration $\{\bar{\mathcal{F}}_t : 0 \le t \le T\}$ on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}))$. Let $\bar{\mathcal{A}}$ be the class of all $(\bar{\mathcal{P}} \otimes \mathcal{B}(\mathbb{X}))/\mathcal{B}[0, \infty)$-measurable maps $\varphi : \mathbb{X}_T \times \bar{\mathbb{M}} \to [0, \infty)$. For $\varphi \in \bar{\mathcal{A}}$, define a counting process N^φ on $\mathbb{X}_T$ by

$$N^\varphi((0, t] \times U) = \int_{(0,t]\times U} \int_{(0,\infty)} 1_{[0,\varphi(s,x)]}(r) \bar{N}(dsdxdr), \quad t \in [0, T], U \in \mathcal{B}(\mathbb{X}). \tag{2.1}$$

N^φ is then the controlled random measure, with φ selecting the intensity for the points at location x and time s, in a possibly random but non-anticipating way. When $\varphi(s, x, \bar{m}) \equiv \theta \in (0, \infty)$, we write $N^\varphi = N^\theta$. Note that N^θ has the same distribution with respect to $\bar{\mathbb{P}}$ as N has with respect to $\mathbb{P}_\theta$. Define $l : [0, \infty) \to [0, \infty)$ by

$$l(r) = r \log r - r + 1, \quad r \in [0, \infty).$$

For any $\varphi \in \bar{\mathcal{A}}$ the quantity

$$L_T(\varphi) = \int_{\mathbb{X}_T} l(\varphi(t, x, \omega)) \nu_T(dtdx) \tag{2.2}$$

is well defined as a $[0, \infty]$-valued random variable. The following is a representation formula proved in [7].

Theorem 2.1. *Let $F \in M_b(\mathbb{M})$. Then, for $\theta > 0$,*

$$-\log \mathbb{E}_\theta(e^{-F(N)}) = -\log \bar{\mathbb{E}}(e^{-F(N^\theta)}) = \inf_{\varphi\in\bar{\mathcal{A}}} \bar{\mathbb{E}}\left[\theta L_T(\varphi) + F(N^{\theta\varphi})\right].$$

2.2. A general large deviation result

In this section, we summarize the main large deviation result of [7]. Let $\{\mathcal{G}^\epsilon\}_{\epsilon>0}$ be a family of measurable maps from $\mathbb{M}$ to $\mathbb{U}$, where $\mathbb{U}$ is some Polish space. We present below a sufficient condition for a large deviation principle to hold for the family $Z^\epsilon = \mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}})$, as $\epsilon \to 0$. Define

$$S^N = \{g : \mathbb{X}_T \to [0, \infty) : L_T(g) \le N\}. \tag{2.3}$$

A function $g \in S^N$ can be identified with a measure $\nu_T^g \in \mathbb{M}$, defined by

$$\nu_T^g(A) = \int_A g(s, x) \nu_T(dsdx), \quad A \in \mathcal{B}(\mathbb{X}_T).$$

This identification induces a topology on S^N under which S^N is a compact space. See the Appendix for a proof of this statement. Throughout we use this topology on S^N. Define

$\mathbb{S} = \cup_{N\geq 1} S^N$, and let

$$\mathcal{U}^N = \{\varphi \in \bar{\mathcal{A}} : \varphi(w) \in S^N, \bar{\mathbb{P}}\, a.e.\, w\}.$$

The following condition will be sufficient to establish an LDP for a family $\{Z^\epsilon\}_{\epsilon>0}$ defined by $Z^\epsilon = \mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}})$. When applied to the SDE (1.1) later on, $\mathcal{G}^\epsilon$ will be the mapping that takes the PRM into X^ϵ.

Condition 2.2. *There exists a measurable map $\mathcal{G}^0 : \mathbb{M} \to \mathbb{U}$ such that the following hold.*

a. *For $N \in \mathbb{N}$, let $g_n, g \in S^N$ be such that $g_n \to g$ as $n \to \infty$. Then*
$$\mathcal{G}^0\left(\nu_T^{g_n}\right) \to \mathcal{G}^0\left(\nu_T^{g}\right).$$

b. *For $N \in \mathbb{N}$, let $\varphi_\epsilon, \varphi \in \mathcal{U}^N$ be such that φ_ϵ converges in distribution to φ as $\epsilon \to 0$. Then*
$$\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}) \Rightarrow \mathcal{G}^0\left(\nu_T^{\varphi}\right).$$

The first condition requires continuity in the control for deterministic controlled systems. The second condition is a law of large numbers result for small noise controlled stochastic systems. In both cases we are allowed to assume the controls take values in a compact set.

For $\phi \in \mathbb{U}$, define $\mathbb{S}_\phi = \left\{g \in \mathbb{S} : \phi = \mathcal{G}^0(\nu_T^g)\right\}$. Let $I : \mathbb{U} \to [0, \infty]$ be defined by

$$I(\phi) = \inf_{g\in\mathbb{S}_\phi} \{L_T(g)\}, \quad \phi \in \mathbb{U}. \tag{2.4}$$

By convention, $I(\phi) = \infty$ if $\mathbb{S}_\phi = \varnothing$.

The following theorem was established in [7, Theorem 4.2].

Theorem 2.3. *For $\epsilon > 0$, let Z^ϵ be defined by $Z^\epsilon = \mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}})$, and suppose that Condition 2.2 holds. Then I defined as in (2.4) is a rate function on $\mathbb{U}$ and the family $\{Z^\epsilon\}_{\epsilon>0}$ satisfies a large deviation principle with rate function I.*

For applications, the following strengthened form of Theorem 2.3 is useful. The proof follows by straightforward modifications; for completeness we include a sketch in the Appendix.

Let $\{K_n \subset \mathbb{X}, n = 1, 2, \ldots\}$ be an increasing sequence of compact sets such that $\cup_{n=1}^\infty K_n = \mathbb{X}$. For each n let

$$\bar{\mathcal{A}}_{b,n} \doteq \{\varphi \in \bar{\mathcal{A}} : \text{ for all } (t,\omega) \in [0,T] \times \bar{\mathbb{M}}, n \geq \varphi(t,x,\omega) \geq 1/n \text{ if } x \in K_n$$
$$\text{and } \varphi(t,x,\omega) = 1 \text{ if } x \in K_n^c\},$$

and let $\bar{\mathcal{A}}_b = \cup_{n=1}^\infty \bar{\mathcal{A}}_{b,n}$. Define $\tilde{\mathcal{U}}^N = \mathcal{U}^N \cap \bar{\mathcal{A}}_b$.

Theorem 2.4. *Suppose Condition 2.2 holds with $\mathcal{U}^N$ replaced by $\tilde{\mathcal{U}}^N$. Then the conclusions of Theorem 2.3 continue to hold.*

2.3. A family of SPDEs driven by Poisson random measures

In this section we introduce the basic SPDE model that will be studied in this work. We begin by giving a precise meaning to a solution for such a SPDE and then recall a result from [16] which gives sufficient conditions on the coefficients ensuring the strong existence and pathwise uniqueness of solutions. To introduce the solution space, we start with some basic definitions (cf. [16]).

Definition 2.5. Let $\mathcal{E}$ be a vector space. A family of norms $\{\|\cdot\|_p : p \in \mathbb{N}_0\}$ on $\mathcal{E}$ is called *compatible* if for any $p, q \in \mathbb{N}_0$, whenever $\{x_n\} \subseteq \mathcal{E}$ is a Cauchy sequence with respect to both $\|\cdot\|_p$ and $\|\cdot\|_q$, and converges to 0 with respect to one norm, then it also converges to 0 with respect to the other norm. The family is said to be *increasing* if for all $x \in \mathcal{E}$, $\|x\|_p \leq \|x\|_q$ whenever $p \leq q$.

Definition 2.6. A separable Frèchet space Φ is called a *countable Hilbertian space* if its topology is given by an increasing sequence $\|\cdot\|_n$, $n \in \mathbb{N}_0$, of compatible Hilbertian norms. A countable Hilbertian space Φ is called *nuclear* if for each $n \in \mathbb{N}_0$ there exists $m > n$ such that the canonical injection from Φ_m into Φ_n is Hilbert–Schmidt, where Φ_k, for each $k \in \mathbb{N}_0$, is the completion of Φ with respect to $\|\cdot\|_k$.

If Φ, $\{\Phi_n\}_{n\in\mathbb{N}_0}$ are as above, then $\{\Phi_n\}_{n\in\mathbb{N}_0}$ is a sequence of decreasing Hilbert spaces and $\Phi = \cap_{n=0}^{\infty} \Phi_n$. Identify Φ_0' with Φ_0 using Riesz's representation theorem, and denote the space of bounded linear functionals on Φ_n by Φ_{-n}. This space has a natural inner product [and norm] which we denote by $\langle\cdot,\cdot\rangle_{-n}$ [resp. $\|\cdot\|_{-n}$], $n \in \mathbb{N}_0$ such that $\{\Phi_{-n}\}_{n\in\mathbb{N}_0}$ is a sequence of increasing Hilbert spaces and the topological dual of Φ, denoted as Φ' equals $\cup_{n=0}^{\infty} \Phi_{-n}$ (see Theorem 1.3.1 of [16]). Elements of Φ' need not have much regularity. Solutions of the SPDE considered in this paper will have sample paths in Φ'. In fact under the conditions imposed here the solutions will take values in $D([0, T] : \Phi_{-n})$ for some finite value of n.

We will assume that there is a sequence $\{\phi_j\} \subset \Phi$ such that $\{\phi_j\}$ is a complete orthonormal system (CONS) in Φ_0 and is a complete orthogonal system (COS) in each $\Phi_n, n \in \mathbb{Z}$. Then $\{\phi_j^n\} = \{\phi_j \|\phi_j\|_n^{-1}\}$ is a CONS in Φ_n for each $n \in \mathbb{Z}$. Define the map $\theta_p : \Phi_{-p} \to \Phi_p$ by $\theta_p(\phi_j^{-p}) = \phi_j^p$. It is easy to check that for all $p \in \mathbb{N}$, $\theta_p(\Phi) \subseteq \Phi$ (see Remark 6.1.1 of [16]). Also, for each $r > 0$, $\eta \in \Phi_{-r}$ and $\phi \in \Phi_r$, $\eta[\phi]$ is defined by the formula

$$\eta[\phi] = \sum_{j=1}^{\infty} \langle \eta, \phi_j \rangle_{-r} \langle \phi, \phi_j \rangle_r. \tag{2.5}$$

We refer the reader to Example 1.3.2 of [16] for a canonical example of such a Countable Hilbertian Nuclear Space (CHNS) defined using a closed densely defined self-adjoint operator on Φ_0. A similar example is considered in Section 4 of this paper.

Following [13], we introduce the following conditions on the coefficients A and G in Eq. (1.1). Let $A : [0, T] \times \Phi' \to \Phi'$, $G : [0, T] \times \Phi' \times \mathbb{X} \to \Phi'$ be maps satisfying the following condition.

Condition 2.7. *There exists $p_0 \in \mathbb{N}$ such that, for every $p \geq p_0$, there exists $q \geq p$ and a constant $K = K(p, q)$ such that the following hold.*

a. *(Continuity) For all $t \in [0, T]$ and $u \in \Phi_{-p}$, $A(t, u) \in \Phi_{-q}$ and $G(t, u, \cdot) \in L^2(\mathbb{X}, \nu; \Phi_{-p})$. The maps $u \mapsto A(t, u)$ and $u \mapsto G(t, u, \cdot)$ are continuous.*

b. *(Coercivity) For all $t \in [0, T]$, and $\phi \in \Phi$,*

$$2A(t, \phi)[\theta_p \phi] \leq K(1 + \|\phi\|_{-p}^2).$$

c. *(Growth) For all $t \in [0, T]$, and $u \in \Phi_{-p}$,*

$$\|A(t, u)\|_{-q}^2 \leq K(1 + \|u\|_{-p}^2)$$

and

$$\int_{\mathbb{X}} \|G(t, u, v)\|_{-p}^2 \nu(dv) \leq K(1 + \|u\|_{-p}^2).$$

d. *(Monotonicity) For all* $t \in [0,T]$*, and* $u_1, u_2 \in \Phi_{-p}$*,*

$$2\langle A(t,u_1) - A(t,u_2), u_1 - u_2\rangle_{-q} + \int_{\mathbb{X}} \|G(t,u_1,v) - G(t,u_2,v)\|_{-q}^2 \nu(dv) \\ \le K\|u_1 - u_2\|_{-q}^2.$$

In Section 4, we will consider a model motivated by problems in hydrology where all parts of Condition 2.7 are satisfied.

We now give a precise definition of a solution to the SDE (1.1).

Definition 2.8. Let $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}}, \{\bar{\mathcal{F}}_t\})$ be the filtered probability space from Section 2.1. Fix $p \in \mathbb{N}_0$, suppose that X_0 is a $\bar{\mathcal{F}}_0$-measurable Φ_{-p}-valued random variable such that $\mathbb{E}\|X_0\|_{-p}^2 < \infty$. A stochastic process $\{X_t^\epsilon\}_{t\in[0,T]}$ defined on $\bar{\mathbb{M}}$ is said to be a Φ_{-p}-valued strong solution to the SDE (1.1) with initial value X_0, if

(a) X_t^ϵ is a Φ_{-p}-valued $\bar{\mathcal{F}}_t$-measurable random variable for all $t \in [0,T]$;
(b) $X^\epsilon \in D([0,T] : \Phi_{-p})$ a.s.;
(c) there is a $q \ge p$ such that for all $t \in [0,T]$ and $u \in \Phi_{-p}$, $A(t,u) \in \Phi_{-q}$ and $G(t,u,\cdot) \in L^2(\mathbb{X}, \nu; \Phi_{-q})$, and there exists a sequence $\{\sigma_n\}_{n\ge1}$ of $\{\bar{\mathcal{F}}_t\}$-stopping times increasing to infinity such that for each $n \ge 1$,

$$\bar{\mathbb{E}} \int_0^{T\wedge\sigma_n} \int_{\mathbb{X}} \|G(s, X_s^\epsilon, v)\|_{-q}^2 \nu(dv) ds < \infty$$

and

$$\bar{\mathbb{E}} \int_0^{T\wedge\sigma_n} \|A(s, X_s^\epsilon)\|_{-q}^2 ds < \infty;$$

(d) for all $t \in [0,T]$, almost all $\omega \in \bar{\mathbb{M}}$, and all $\phi \in \Phi$

$$X_t^\epsilon[\phi] = X_0[\phi] + \int_0^t A(s, X_s^\epsilon)[\phi] ds + \epsilon \int_0^t \int_{\mathbb{X}} G(s, X_{s-}^\epsilon, v)[\phi] \tilde{N}^{\epsilon^{-1}}(ds dv). \quad (2.6)$$

In Definition 2.8, $\tilde{N}^{\epsilon^{-1}}$ is the compensated version of $N^{\epsilon^{-1}}$ as defined below (1.1), with $N^{\epsilon^{-1}}$ having jump rates that are scaled by $1/\epsilon$ and is constructed from $\bar{N}$, as below (2.1).

One can similarly define a Φ_{-p}-valued strong solution on an arbitrary filtered probability space supporting a suitable PRM.

Definition 2.9 (*Pathwise Uniqueness*)**.** We say that the Φ_{-p}-valued solution for the SDE (1.1) has the *pathwise uniqueness* property if the following is true. Suppose that X and X' are two Φ_{-p}-valued solutions defined on the same filtered probability space with respect to the same Poisson random measure and starting from the same initial condition X_0. Then the paths of X and X' coincide for almost all ω.

The following theorem is taken from [16] (see Theorems 6.2.2, 6.3.1 and Lemma 6.3.1 therein).

Theorem 2.10. *Suppose that Condition 2.7 holds. Let X_0 be a Φ_{-p}-valued random variable satisfying $\mathbb{E}\|X_0\|_{-p}^2 < \infty$. Then for sufficiently large $p_1 \ge p$, the canonical injection from Φ_{-p} to Φ_{-p_1} is Hilbert–Schmidt, and for all such p_1 the SDE (1.1) with initial value X_0 has a pathwise unique Φ_{-p_1}-valued strong solution.*

3. Large deviation principle

Throughout this section we will assume that Condition 2.7 holds.

Fix $p \geq p_0$ and $X_0 \in \Phi_{-p}$. Let X^ϵ be the Φ_{-p_1}-valued strong solution to the SDE (1.1) with initial value X_0. In this section, we establish an LDP for $\{X^\epsilon\}$ under suitable assumptions, by verifying the sufficient condition in Section 2.2.

We begin by introducing the map $\mathcal{G}^0$ that will be used to define the rate function and also used for verification of Condition 2.2. Recall that $\mathbb{S} = \cup_{N \geq 1} S^N$, where S^N is defined in (2.3). As a first step we show that under Conditions 3.1 and 3.5 below, for every $g \in \mathbb{S}$, the integral equation

$$\tilde{X}^g_t = X_0 + \int_0^t A(s, \tilde{X}^g_s)ds + \int_0^t \int_{\mathbb{X}} G(s, \tilde{X}^g_s, v)(g(s, v) - 1)\nu(dv)ds \tag{3.1}$$

has a unique continuous solution. Here g plays the role of a control. Keeping in mind that (2.6) is driven by the compensated measure and that equations such as (3.1) will arise as law of large number limits, g corresponds to a shift in the scaled jump rate away from that of the original model, which corresponds to $g = 1$. Let

$$\|G(t, v)\|_{0,-p} = \sup_{u \in \Phi_{-p}} \frac{\|G(t, u, v)\|_{-p}}{1 + \|u\|_{-p}}, \quad (t, v) \in [0, T] \times \mathbb{X}.$$

Condition 3.1 (*Exponential Integrability*). *There exists $\delta_1 \in (0, \infty)$ such that for all $E \in \mathcal{B}([0, T] \times \mathbb{X})$ satisfying $\nu_T(E) < \infty$,*

$$\int_E e^{\delta_1 \|G(s,v)\|^2_{0,-p}} \nu(dv)ds < \infty.$$

Remark 3.2. Under Condition 3.1, for every $\delta_2 \in (0, \infty)$ and for all $E \in \mathcal{B}([0, T] \times \mathbb{X})$ satisfying $\nu_T(E) < \infty$

$$\int_E e^{\delta_2 \|G(s,v)\|_{0,-p}} \nu(dv)ds < \infty.$$

The proof of Remark 3.2 is given in the Appendix.

Remark 3.3. The following inequalities will be used several times. Proofs are omitted.

a. For $a, b \in (0, \infty)$, $\sigma \in [1, \infty)$,

$$ab \leq e^{\sigma a} + \frac{1}{\sigma}(b \log b - b + 1) = e^{\sigma a} + \frac{1}{\sigma} l(b). \tag{3.2}$$

b. For each $\beta > 0$ there exists $c_1(\beta) > 0$, such that $c_1(\beta) \to 0$ as $\beta \to \infty$ and

$$|x - 1| \leq c_1(\beta) l(x) \quad \text{whenever } |x - 1| \geq \beta.$$

c. For each $\beta > 0$ there exists $c_2(\beta) < \infty$, such that

$$|x - 1|^2 \leq c_2(\beta) l(x) \quad \text{whenever } |x - 1| \leq \beta.$$

In particular, using the inequalities we have the following lemma.

Lemma 3.4. *Under Conditions 2.7(c) and 3.1, for every $M \in \mathbb{N}$,*

$$\sup_{g\in S^M}\int_{\mathbb{X}_T}\|G(s,v)\|^2_{0,-p}(g(s,v)+1)\nu(dv)ds<\infty, \tag{3.3}$$

$$\sup_{g\in S^M}\int_{\mathbb{X}_T}\|G(s,v)\|_{0,-p}|g(s,v)-1|\nu(dv)ds<\infty, \tag{3.4}$$

and

$$\lim_{\delta\to 0}\sup_{g\in S^M}\sup_{|t-s|\le\delta}\int_{[s,t]\times\mathbb{X}}\|G(r,v)\|_{0,-p}|g(r,v)-1|\nu(dv)dr=0. \tag{3.5}$$

Proof. First notice that under Condition 2.7(c), we have

$$\int_{\mathbb{X}_T}\|G(s,v)\|^2_{0,-p}\nu(dv)ds\le KT<\infty. \tag{3.6}$$

Thus we only need to prove that

$$\sup_{g\in S^M}\int_{\mathbb{X}_T}\|G(s,v)\|^2_{0,-p}g(s,v)\nu(dv)ds<\infty.$$

If $E=\{(s,v):\|G(s,v)\|_{0,-p}\ge 1\}$, then by (3.6) we have $\nu_T(E)<\infty$. Also, from the super linear growth of the function l, we can find $\kappa_1,\kappa_2\in(0,\infty)$ such that for all $x\ge\kappa_1$, $x\le\kappa_2 l(x)$. Define $F=\{(s,v):g(s,v)\ge\kappa_1\}$. Then, from (3.2)

$$\begin{aligned}\int_{\mathbb{X}_T}\|G(s,v)\|^2_{0,-p}g(s,v)\nu(dv)ds &= \int_E\|G(s,v)\|^2_{0,-p}g(s,v)\nu(dv)ds\\ &\quad+\int_{E^c}\|G(s,v)\|^2_{0,-p}g(s,v)\nu(dv)ds\\ &\le\int_E e^{\delta_1\|G(s,v)\|^2_{0,-p}}\nu(dv)ds\\ &\quad+\int_E l\left(\frac{g(s,v)}{\delta_1}\right)\nu(dv)ds\\ &\quad+\int_{E^c\cap F}\kappa_2 l(g(s,v))\nu(dv)ds\\ &\quad+\kappa_1\int_{E^c\cap F^c}\|G(s,v)\|^2_{0,-p}\nu(dv)ds.\end{aligned}$$

Combining this estimate with Condition 3.1 and the definition of S^M, we have (3.3).

We now prove (3.4) and (3.5). Note that

$$\begin{aligned}&\int_{[s,t]\times\mathbb{X}}\|G(r,v)\|_{0,-p}|g(r,v)-1|\nu(dv)dr\\ &\quad=\int_{([s,t]\times\mathbb{X})\cap E}\|G(r,v)\|_{0,-p}|g(r,v)-1|\nu(dv)dr\\ &\qquad+\int_{([s,t]\times\mathbb{X})\cap E^c}\|G(r,v)\|_{0,-p}|g(r,v)-1|\nu(dv)dr.\end{aligned}$$

Using (3.2) twice (once with $b = g$ and once with $b = 1$), for any $M_0 \in (1, \infty)$

$$\int_{([s,t]\times\mathbb{X})\cap E} \|G(r,v)\|_{0,-p}|g(r,v)-1|\nu(dv)dr \leq 2\int_{([s,t]\times\mathbb{X})\cap E} e^{M_0\|G(r,v)\|_{0,-p}}\nu(dv)dr + \frac{M}{M_0}. \tag{3.7}$$

Recalling Remark 3.3, for any $\theta > 0$ and $g \in S^M$

$$\begin{aligned}\int_{([s,t]\times\mathbb{X})\cap E^c} &\|G(r,v)\|_{0,-p}|g(r,v)-1|\nu(dv)dr \\ &= \int_{([s,t]\times\mathbb{X})\cap E^c\cap\{|g-1|\leq\theta\}} \|G(r,v)\|_{0,-p}|g-1|\nu(dv)dr \\ &\quad + \int_{([s,t]\times\mathbb{X})\cap E^c\cap\{|g-1|>\theta\}} \|G(r,v)\|_{0,-p}|g-1|\nu(dv)dr \\ &\leq \left(\int_{[s,t]\times\mathbb{X}} \|G(r,v)\|^2_{0,-p}\nu(dv)dr\right)^{1/2}\sqrt{c_2(\theta)M} + c_1(\theta)M. \end{aligned} \tag{3.8}$$

The inequality in (3.4) now follows on setting $s = 0$, $t = T$ in (3.7) and (3.8) and using Condition 2.7(c) and Remark 3.2.

Next consider (3.5). Fix $\epsilon \in (0, \infty)$. Choose M_0 such that $\frac{M}{M_0} \leq \frac{\epsilon}{4}$. Let $\delta_1 \in (0, \infty)$ be such that

$$2\sup_{|t-s|\leq\delta_1}\int_{([s,t]\times\mathbb{X})\cap E} e^{M_0\|G(r,v)\|_{0,-p}}\nu(dv)dr \leq \frac{\epsilon}{4}.$$

Now choose $\theta \in (0, \infty)$ such that $c_1(\theta)M \leq \frac{\epsilon}{4}$. Finally, choose $\delta_2 \in (0, \infty)$ such that

$$\sup_{|t-s|\leq\delta_2}\left(\int_{[s,t]\times\mathbb{X}} \|G(r,v)\|^2_{0,-p}\nu(dv)dr\right)^{1/2}\sqrt{c_2(\theta)N} \leq \frac{\epsilon}{4}.$$

Using the above inequalities in (3.7) and (3.8), we have for all $\delta \leq \min\{\delta_1, \delta_2\}$,

$$\sup_{g\in S^M}\sup_{|t-s|\leq\delta}\int_{[s,t]\times\mathbb{X}} \|G(r,v)\|_{0,-p}|g(r,v)-1|\nu(dv)dr \leq \epsilon.$$

The result follows. □

We will need the following stronger condition on fluctuations of G than (d) of Condition 2.7. Let

$$\|G(t,v)\|_{1,-q} = \sup_{u_1,u_2\in\Phi_{-q},u_1\neq u_2} \frac{\|G(t,u_1,v)-G(t,u_2,v)\|_{-q}}{\|u_1-u_2\|_{-q}}.$$

Condition 3.5. *For q as in Condition 2.7, there exists $\delta > 0$ such that for all $E \in \mathcal{B}([0,T]\times\mathbb{X})$ satisfying $\nu_T(E) < \infty$,*

$$\int_E e^{\delta\|G(s,v)\|^2_{1,-q}}\nu(dv)ds < \infty.$$

Remark 3.6. Under Conditions 2.7(d) and 3.5, for every $M \in \mathbb{N}$,

$$\sup_{g \in S^M} \int_{\mathbb{X}_T} \|G(s,v)\|_{1,-q}^2 (g(s,v)+1)\nu(dv)ds < \infty,$$

and

$$\sup_{g \in S^M} \int_{\mathbb{X}_T} \|G(s,v)\|_{1,-q} |g(s,v)-1|\nu(dv)ds < \infty. \tag{3.9}$$

The proof of this remark is similar to that of Lemma 3.4, and thus omitted. Note that Conditions 3.1 and 3.5 hold trivially if $\|G(s,v)\|_{0,-p}$ and $\|G(s,v)\|_{1,-q}$ are bounded in (s,v).

Recall that $p_1 \geq p$ is chosen such that the canonical injection from Φ_{-p} to Φ_{-p_1} is Hilbert–Schmidt.

Theorem 3.7. *Fix $g \in \mathbb{S}$. Suppose Conditions 2.7, 3.1 and 3.5 hold, and that $X_0 \in \Phi_{-p}$. Then there exists a unique $\tilde{X}^g \in C([0,T] : \Phi_{-p_1})$ such that for every $\phi \in \Phi$,*

$$\tilde{X}_t^g[\phi] = X_0[\phi] + \int_0^t A(s, \tilde{X}_s^g)[\phi]ds + \int_0^t \int_{\mathbb{X}} G(s, \tilde{X}_s^g, v)[\phi](g(s,v)-1)\nu(dv)ds. \tag{3.10}$$

Furthermore, for $N \in \mathbb{N}$, $\sup_{t\in[0,T]} \sup_{g \in S^N} \|\tilde{X}_t^g\|_{-p} < \infty$.

We note that in the above theorem $\tilde{X}^g$ is a non-random element of $C([0,T] : \Phi_{-p_1})$. We can now present the main large deviation result. Recall that for $g \in \mathbb{S}$, $\nu_T^g(dsdv) = g(s,v)\nu(dv)ds$. Define

$$\mathcal{G}^0(\nu_T^g) = \tilde{X}^g \quad \text{for } g \in \mathbb{S}, \text{ with } \tilde{X}^g \text{ given by (3.10)}. \tag{3.11}$$

Let $I : D([0,T] : \Phi_{-p_1}) \to [0,\infty]$ be defined as in (2.4).

Theorem 3.8. *Suppose that Conditions 2.7, 3.1 and 3.5 hold. Then I is a rate function on Φ_{-p_1}, and the family $\{X^\epsilon\}_{\epsilon>0}$ satisfies a large deviation principle on $D([0,T] : \Phi_{-p_1})$ with rate function I.*

We now proceed with the proofs. In Section 3.1 we prove Theorem 3.7 and in Section 3.2, we present the proof of Theorem 3.8.

3.1. *Proof of Theorem 3.7*

The proof of the theorem is based on the following two lemmas. The first lemma is standard and so its proof is relegated to the Appendix. The norm $\|\cdot\|$ in the lemma is the Euclidean norm in $\mathbb{R}^d$.

Lemma 3.9. *Let $a, u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be measurable functions such that, for a.e. $s \in [0,T]$, the maps $y \mapsto a(s,y)$, $y \mapsto b(s,y)$ and $y \mapsto u(s,y)$ are continuous. Further suppose that for some $\kappa \in (0,\infty)$,*

$$\|a(s,y)\| + |b(s,y)| \leq \kappa(1+\|y\|), \quad \text{for all } s \in [0,T],\ y \in \mathbb{R}^d$$

$$\int_0^T \sup_{y \in \mathbb{R}^d} \|u(s,y)\|ds \leq M < \infty.$$

Fix $x_0 \in \mathbb{R}^d$. *Then there exists* $x \in C([0,T] : \mathbb{R}^d)$ *such that* x *satisfies the integral equation*

$$x(t) = x_0 + \int_0^t a(s, x(s))ds + \int_0^t b(s, x(s))u(s, x(s))ds, \tag{3.12}$$

and

$$\sup_{t \in [0,T]} \|x(t)\| \le (\|x_0\| + \kappa(M + T))e^{\kappa(M+T)}.$$

Lemma 3.10. *Let* $\{a^d, g^d\}_{d \in \mathbb{N}}$ *be a sequence of maps,* $a^d : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $g^d : [0,T] \times \mathbb{R}^d \times \mathbb{X} \to \mathbb{R}^d$, *such that the following hold.*

a. *For each* $s \in [0,T]$ *and* $y \in \mathbb{R}^d$, $g^d(s, y, \cdot) \in L^2(\mathbb{X}, \nu; \mathbb{R}^d)$ *and for each* $s \in [0,T]$, *the maps* $y \mapsto a^d(s, y)$ *and* $y \mapsto g^d(s, y, \cdot)$ *(from* $\mathbb{R}^d$ *to* $L^2(\mathbb{X}, \nu; \mathbb{R}^d)$*) are continuous.*
b. *For some* $\kappa \in (0, \infty)$ *and all* $d \in \mathbb{N}$,

$$2\langle a^d(s, y), y\rangle \le \kappa(1 + \|y\|^2), \quad \forall (s, y) \in [0,T] \times \mathbb{R}^d$$

and

$$\int_{\mathbb{X}} \|g^d(s, v)\|_0^2 \nu(dv) \le \kappa, \quad \forall s \in [0,T],$$

where $\|g^d(s, v)\|_0 = \sup_{y \in \mathbb{R}^d} \frac{\|g^d(s,y,v)\|}{1+\|y\|}$.
c. *For each* $d \in \mathbb{N}$, *there exists* $\kappa_d \in (0, \infty)$ *with*

$$\|a^d(s, y)\| \le \kappa_d(1 + \|y\|), \quad \forall (s, y) \in [0,T] \times \mathbb{R}^d.$$

d. *There is a* $\delta_0 \in (0, \infty)$ *such that for all* $E \in \mathcal{B}([0,T] \times \mathbb{X})$ *satisfying* $\nu_T(E) < \infty$,

$$\int_E e^{\delta_0 \|g^d(s,v)\|_0} \nu(dv)ds < \infty.$$

Then for any $d \in \mathbb{N}$, $\psi \in \mathbb{S}$ *and* $x_0^d \in \mathbb{R}^d$, *the equation*

$$x^d(t) = x_0^d + \int_0^t a^d(s, x^d(s))ds + \int_0^t \int_{\mathbb{X}} g^d(s, x^d(s), v)(\psi(s, v) - 1)\nu(dv)ds \tag{3.13}$$

has a solution $x^d \in C([0,T] : \mathbb{R}^d)$. *Suppose that* $\sup_{d \in \mathbb{N}} \|x_0^d\|^2 < \infty$. *Then for every* $M \in (0, \infty)$, *there exists a* $\tilde{\kappa}_M \in (0, \infty)$ *such that*

$$\sup_{d \in \mathbb{N}} \sup_{t \in [0,T]} \|x^d(t)\|^2 \le \tilde{\kappa}_M, \quad \textit{whenever } \psi \in S^M.$$

Proof. For each d fixed, Eq. (3.13) is the same as (3.12) with the following choices of a, b and u:

$$a(s, y) = a^d(s, y),$$
$$b(s, y) = 1 + \|y\|,$$

and

$$u(s, y) = \int_{\mathbb{X}} \frac{g^d(s, y, v)}{1 + \|y\|}(\psi(s, v) - 1)\nu(dv).$$

Thus in order to prove the existence of the solutions to (3.13), it suffices to verify conditions in Lemma 3.9. The continuity of a, b and first condition in Lemma 3.9 are immediate. The proof of the statement

$$y \mapsto u(s, y) \quad \text{is continuous for a.e. } s \in [0, T] \tag{3.14}$$

is given in the Appendix. Finally note that

$$\int_0^T \sup_{y \in \mathbb{R}^d} \|u(s, y)\| ds \le \int_0^T \int_{\mathbb{X}} \|g^d(s, v)\|_0 |\psi(s, v) - 1| \nu(dv) ds < \infty,$$

where the last inequality follows from conditions (b) and (d) using a similar argument as for (3.4). Thus from Lemma 3.9, for each $d \in \mathbb{N}$, there exists a $x^d \in C([0, T] : \mathbb{R}^d)$ satisfying (3.13). Next note that

$$\begin{aligned}
\|x^d(t)\|^2 &= \|x_0^d\|^2 \\
&\quad + 2\int_0^t \left\langle x^d(s), \left(a^d(s, x^d(s)) + \int_{\mathbb{X}} g^d(s, x^d(s), v)(\psi(s, v) - 1)\nu(dv)\right)\right\rangle ds \\
&\le \|x_0^d\|^2 + 2\int_0^t \left\langle x^d(s), a^d(s, x^d(s))\right\rangle ds \\
&\quad + 2\int_0^t \|x^d(s)\| \int_{\mathbb{X}} \|g^d(s, x^d(s), v)\| \, |\psi(s, v) - 1| \nu(dv) ds \\
&\le \|x_0^d\|^2 + \kappa \int_0^t (1 + \|x^d(s)\|^2) ds \\
&\quad + 2\int_0^t \|x^d(s)\|(1 + \|x^d(s)\|) \int_{\mathbb{X}} \|g^d(s, v)\|_0 |\psi(s, v) - 1| \nu(dv) ds.
\end{aligned} \tag{3.15}$$

Let

$$f^d(s) = \int_{\mathbb{X}} \|g^d(s, v)\|_0 |\psi(s, v) - 1| \nu(dv).$$

Then as before, using (b) and (d), we have that

$$\sup_{\psi \in S^M} \sup_{d \in \mathbb{N}} \int_0^T f^d(s) ds < \infty. \tag{3.16}$$

Also, from (3.15) and using that $c + c^2 \le 1 + 2c^2$ for $c \ge 0$,

$$\|x^d(t)\|^2 \le \left(\|x_0^d\|^2 + \kappa T + 2\int_0^T f^d(s) ds\right) + \int_0^t \|x^d(s)\|^2 (\kappa + 4 f^d(s)) ds.$$

Thus, by Gronwall's inequality

$$\|x^d(t)\|^2 \le \left(\|x_0^d\|^2 + \kappa T + 2\int_0^T f^d(s) ds\right) e^{\kappa t + 4\int_0^t f^d(s) ds}.$$

Hence if $\sup_{d \in \mathbb{N}} \|x_0^d\|^2 < \infty$, then by (3.16)

$$\sup_{\psi \in S^M} \sup_{d \in \mathbb{N}} \sup_{t \in [0, T]} \|x^d(t)\|^2 < \infty.$$

The lemma follows. □

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7. We first argue the existence of the solutions to (3.10). Let $M \in \mathbb{N}$ be such that $g \in S^M$. Recall the CONS $\{\phi_k^p\}$ defined by $\phi_k^p = \phi_k \|\phi_k\|_p^{-1} \in \Phi_p$ that was introduced below Definition 2.6. Fix $d \in \mathbb{N}$ and let $\pi : \Phi_{-p} \to \mathbb{R}^d$ be the mapping given by

$$\pi(u)_k = u[\phi_k^p], \quad k = 1, 2, \dots, d$$

and denote $\pi(X_0)$ by x_0^d. Define $a^d : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g^d : [0, T] \times \mathbb{R}^d \times \mathbb{X} \to \mathbb{R}^d$ by

$$a^d(s, x)_k = A\left(s, \sum_{j=1}^d x_j \phi_j^{-p}\right)[\phi_k^p]$$

and

$$g^d(s, x, v)_k = G\left(s, \sum_{j=1}^d x_j \phi_j^{-p}, v\right)[\phi_k^p].$$

It is easy to verify that a^d and g^d satisfy the assumptions of Lemma 3.10, and therefore there exists $x^d \in C([0, T] : \mathbb{R}^d)$ which satisfies (3.13) with ψ replaced by g. Define the Φ_{-p}-valued continuous function X^d, associated with x^d, by

$$X_t^d = \sum_{k=1}^d (x_t^d)_k \phi_k^{-p}.$$

Then with $\tilde{\kappa}_M$ as in Lemma 3.10, we have

$$\sup_{d\in\mathbb{N}} \sup_{t\in[0,T]} \|X_t^d\|_{-p}^2 \le \tilde{\kappa}_M. \tag{3.17}$$

Recalling the definition of $u[\phi]$ from (2.5), let $\gamma^d : \Phi' \to \Phi'$ be a mapping given by

$$\gamma^d u = \sum_{k=1}^d u[\phi_k^p]\phi_k^{-p}.$$

Let, for $d \in \mathbb{N}$, $A^d : [0, T] \times \Phi' \to \Phi'$ and $G^d : [0, T] \times \Phi' \times \mathbb{X} \to \Phi'$ be measurable mappings given by

$$A^d(s, u) = \gamma^d A(s, \gamma^d u) \quad \text{and} \quad G^d(s, u, v) = \gamma^d G(s, \gamma^d u, v).$$

Then X^d solves

$$X_t^d[\phi] = X_0^d[\phi] + \int_0^t A^d(s, X_s^d)[\phi]ds + \int_0^t \int_{\mathbb{X}} G^d(s, X_s^d, v)[\phi](g(s, v) - 1)\nu(dv)ds, \quad \phi \in \Phi.$$

We now argue that for each $\phi \in \Phi$, the family $\{X^d[\phi]\}_{d\in\mathbb{N}}$ is pre-compact in $C([0, T] : \mathbb{R})$. From (3.17), we have

$$\sup_{d} \sup_{t\in[0,T]} |X_t^d[\phi]| \le \sup_{d} \sup_{t\in[0,T]} \|X_t^d\|_{-p}\|\phi\|_p \le \sqrt{\tilde{\kappa}_M}\|\phi\|_p < \infty. \tag{3.18}$$

Now we consider fluctuations of $X^d[\phi]$. For $0 \leq s \leq t \leq T$,

$$\begin{aligned}|X^d_t[\phi] - X^d_s[\phi]| &\leq \int_s^t |A^d(r, X^d_r)[\phi]|dr \\ &\quad + \int_s^t \int_{\mathbb{X}} |G^d(r, X^d_r, v)[\phi]|\, |g(r,v) - 1|\nu(dv)dr \\ &\leq \int_s^t \|A^d(r, X^d_r)\|_{-q}\|\phi\|_q dr \\ &\quad + \int_s^t \int_{\mathbb{X}} \|G^d(r, X^d_r, v)\|_{-p}\|\phi\|_p |g(r,v) - 1|\nu(dv)dr.\end{aligned}$$

Also, for $(s, u) \in [0, T] \times \Phi'$

$$\begin{aligned}\|A^d(s,u)\|^2_{-q} &= \left\| \sum_{k=1}^d A(s, \gamma^d u)[\phi^p_k]\phi^{-p}_k \right\|^2_{-q} \\ &= \left\| \sum_{k=1}^d A(s, \gamma^d u)[\phi^q_k]\phi^{-q}_k \right\|^2_{-q} \\ &= \sum_{k=1}^d \left(A(s, \gamma^d u)[\phi^q_k] \right)^2 \\ &\leq \|A(s, \gamma^d u)\|^2_{-q} \\ &\leq K\left(1 + \|\gamma^d u\|^2_{-p}\right) \\ &\leq K\left(1 + \|u\|^2_{-p}\right),\end{aligned}$$

where for the second equality we use the observation

$$u[\phi^q_j]\phi^{-q}_j = u[\phi^p_j]\phi^{-p}_j, \quad \forall u \in \Phi', p, q \geq 0,$$

and the last inequality follows on observing that

$$\|\gamma^d u\|^2_{-p} \leq \|u\|^2_{-p}, \quad \forall p \geq 0.$$

Similarly,

$$\begin{aligned}\|G^d(s,u,v)\|^2_{-p} &= \left\| \sum_{k=1}^d G(s, \gamma^d u, v)[\phi^p_k]\phi^{-p}_k \right\|^2_{-p} \\ &= \sum_{k=1}^d \left(G(s, \gamma^d u, v)[\phi^p_k] \right)^2 \\ &\leq \|G(s, \gamma^d u, v)\|^2_{-p}.\end{aligned}$$

Combining the above estimates we have

$$\begin{aligned}|X^d_t[\phi] - X^d_s[\phi]| &\leq \|\phi\|_q \sqrt{K}\sqrt{1 + \tilde{\kappa}_M}(t - s) \\ &\quad + \|\phi\|_p (1 + \sqrt{\tilde{\kappa}_M}) \int_s^t \int_{\mathbb{X}} \|G(r,v)\|_{0,-p} |g(r,v) - 1|\nu(dv)dr.\end{aligned}$$

By Lemma 3.4 we now see that

$$\lim_{\delta\to 0}\sup_{d\in\mathbb{N}}\sup_{|t-s|\le\delta}|X_t^d[\phi]-X_s^d[\phi]|=0. \tag{3.19}$$

Combining (3.18) and (3.19) we now have that the family $\{X^d[\phi]\}$ is pre-compact in $C([0,T]:\mathbb{R})$ for every $\phi\in\Phi$. Combining this with (3.17) we have that $\{X^d\}_{d\in\mathbb{N}}$ is pre-compact in $C([0,T]:\Phi_{-p_1})$ (cf. Theorem 2.5.2 in [16]). Let $\tilde{X}$ be any limit point. Then by the dominated convergence theorem and the definitions of A^d and G^d (see Lemma 6.1.6 and Theorem 6.2.2 of [16]), $\tilde{X}$ satisfies the integral equation (3.10). Note that the argument also shows that whenever $g\in S^M$, $\sup_{t\in[0,T]}\|\tilde{X}_t\|_{-p}^2\le\tilde{\kappa}_M$.

Next, we argue uniqueness of solutions. Suppose there are two elements $\tilde{X}$ and $\bar{X}$ of $C([0,T]:\Phi_{-p_1})$ such that both satisfy (3.10). Then, using Condition 2.7(d),

$$\begin{aligned}\|\tilde{X}_t-\bar{X}_t\|_{-q}^2&=2\int_0^t\langle A(s,\tilde{X}_s)-A(s,\bar{X}_s),\tilde{X}_s-\bar{X}_s\rangle_{-q}ds\\&\quad+2\int_0^t\int_{\mathbb{X}}\langle G(s,\tilde{X}_s,v)-G(s,\bar{X}_s,v),\tilde{X}_s-\bar{X}_s\rangle_{-q}(g(s,v)-1)\nu(dv)ds\\&\le K\int_0^t\|\tilde{X}_s-\bar{X}_s\|_{-q}^2ds\\&\quad+2\int_0^t\|\tilde{X}_s-\bar{X}_s\|_{-q}^2\int_{\mathbb{X}}\|G(s,v)\|_{1,-q}|g(s,v)-1|\nu(dv)ds.\end{aligned}$$

Also, by Remark 3.6,

$$\int_0^T\int_{\mathbb{X}}\|G(s,v)\|_{1,-q}|g(s,v)-1|\nu(dv)ds<\infty.$$

An application of Gronwall's inequality now shows that $\|\tilde{X}_t-\bar{X}_t\|_{-q}^2=0$ for all $t\in[0,T]$. Uniqueness follows. □

3.2. *Proof of Theorem 3.8*

From Theorem 2.10 and by the classical Yamada–Watanabe argument (cf. [14]), for each $\epsilon>0$, there exists a measurable map $\mathcal{G}^\epsilon:\mathbb{M}\to D([0,T]:\Phi_{-p_1})$ such that, for any PRM $\mathbf{n}^{\epsilon^{-1}}$ on $[0,T]\times\mathbb{X}$ with mean measure $\epsilon^{-1}\lambda_T\otimes\nu$ given on some filtered probability space, $\mathcal{G}^\epsilon(\epsilon\mathbf{n}^{\epsilon^{-1}})$ is the unique Φ_{-p_1} valued strong solution of (1.1) (with $\tilde{N}^{\epsilon^{-1}}$ replaced by $\tilde{\mathbf{n}}^{\epsilon^{-1}}=\mathbf{n}^{\epsilon^{-1}}-\epsilon^{-1}\lambda_T\otimes\nu$) with initial value X_0, where p_1 is as in the statement of Theorem 2.10. In particular, $X^\epsilon=\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}})$ is the strong solution of (1.1) with initial value X_0 on $(\bar{\mathbb{M}},\mathcal{B}(\bar{\mathbb{M}}),\bar{\mathbb{P}},\{\bar{\mathcal{F}}_t\})$. In view of this observation, for proof of Theorem 3.8, it suffices to verify Condition 2.2.

We begin with the following lemma.

Lemma 3.11. *Fix $N\in\mathbb{N}$, and let $g_n,g\in S^N$ be such that $g_n\to g$ as $n\to\infty$. Let $h:[0,T]\times\mathbb{X}\to\mathbb{R}$ be a measurable function such that*

$$\int_{\mathbb{X}_T}|h(s,v)|^2\nu_T(dvds)<\infty, \tag{3.20}$$

and for all $\delta_2 \in (0, \infty)$

$$\int_E e^{\delta_2 |h(s,v)|} \nu_T(dv ds) < \infty, \tag{3.21}$$

for all $E \in \mathcal{B}([0, T] \times \mathbb{X})$ *satisfying* $\nu_T(E) < \infty$. *Then*

$$\int_{\mathbb{X}_T} h(s, v)(g_n(s, v) - 1)\nu_T(dv ds) \to \int_{\mathbb{X}_T} h(s, v)(g(s, v) - 1)\nu_T(dv ds) \tag{3.22}$$

as $n \to \infty$.

Proof. We first argue that given $\epsilon > 0$, there exists a compact set $K \subset \mathbb{X}$, such that

$$\sup_n \int_{[0,T]\times K^c} |h(s, v)||g_n(s, v) - 1|\nu(dv)ds \leq \epsilon. \tag{3.23}$$

For each $\beta \in (0, \infty)$ and compact K in $\mathbb{X}$, the left side of (3.23) can be bounded by the sum of the following two terms:

$$T_1 = \sup_n \int_{([0,T]\times K^c)\cap\{|g_n-1|>\beta\}} |h(s, v)||g_n(s, v) - 1|\nu(dv)ds,$$

and

$$T_2 = \sup_n \int_{([0,T]\times K^c)\cap\{|g_n-1|\leq\beta\}} |h(s, v)||g_n(s, v) - 1|\nu(dv)ds.$$

Consider T_1 first. Then for every $L \in (1, \infty)$

$$\begin{aligned} T_1 &\leq \sup_n \int_{([0,T]\times K^c)\cap\{|g_n-1|>\beta\}\cap\{|h|<1\}} |h(s, v)||g_n(s, v) - 1|\nu(dv)ds \\ &\quad + \sup_n \int_{([0,T]\times K^c)\cap\{|g_n-1|>\beta\}\cap\{|h|\geq1\}} |h(s, v)||g_n(s, v) - 1|\nu(dv)ds \\ &\leq \sup_n \int_{([0,T]\times K^c)\cap\{|g_n-1|>\beta\}\cap\{|h|<1\}} |g_n(s, v) - 1|\nu(dv)ds \\ &\quad + 2\int_{([0,T]\times K^c)\cap\{|h|\geq1\}} e^{L|h(s,v)|}\nu(dv)ds + \frac{1}{L}\sup_n \int_{\mathbb{X}_T} l(g_n(s, v))\nu(dv)ds \end{aligned}$$

where the inequality uses (3.2) twice (with $b = g_n$ and $b = 1$). Using inequality (b) of Remark 3.3, the first term on the right side above can be bounded by

$$c_1(\beta) \sup_n \int_{\mathbb{X}_T} l(g_n(s, v))\nu(dv)ds \leq c_1(\beta)N.$$

Therefore,

$$T_1 \leq c_1(\beta)N + 2\int_{([0,T]\times K^c)\cap\{|h|\geq1\}} e^{L|h(s,v)|}\nu(dv)ds + \frac{1}{L}N.$$

Now choose β sufficiently large so that $c_1(\beta)N \leq \epsilon/6$, L be sufficiently large so that $N/L \leq \epsilon/6$. Note that from (3.20), $\nu_T\{|h| \geq 1\} < \infty$ and so by (3.21), $\int_{|h|\geq1} e^{L|h(s,v)|}\nu_T(dv ds) < \infty$. Thus we can find a compact set $K_1 \subset \mathbb{X}$ such that

$$2\int_{([0,T]\times K_1^c)\cap\{|h|\geq1\}} e^{L|h(s,v)|}\nu_T(dv ds) \leq \epsilon/6.$$

With β chosen as above, consider now the term T_2. We have, using the Cauchy–Schwarz Inequality and inequality (c) of Remark 3.3, for every compact K,

$$T_2^2 \le \int_{[0,T]\times K^c} |h(s,v)|^2 \nu(dv)ds \times c_2(\beta) \sup_n \int_{\mathbb{X}_T} l(g_n(s,v))\nu(dv)ds \\ \le \int_{[0,T]\times K^c} |h(s,v)|^2 \nu(dv)ds \times c_2(\beta) N.$$

By (3.20), we can choose a compact set K_2, such that $T_2 \le \epsilon/2$ with K replaced by K_2. Thus by taking $K = K_1 \cup K_2$, we have on combining the above estimates that $T_1 + T_2 \le \epsilon$. This proves (3.23).

In order to prove (3.22), it now suffices to show that, for every compact $K \subset \mathbb{X}$,

$$\int_{[0,T]\times K} h(s,v)(g_n(s,v)-1)\nu_T(dvds) \\ \to \int_{[0,T]\times K} h(s,v)(g(s,v)-1)\nu_T(dvds). \tag{3.24}$$

Fix a compact $K \subset \mathbb{X}$. From (3.20), we have that $\int_{[0,T]\times K} |h(s,v)|\nu_T(dvds) < \infty$. Thus to prove (3.24), it suffices to argue

$$\int_{[0,T]\times K} h(s,v)g_n(s,v)\nu_T(dvds) \to \int_{[0,T]\times K} h(s,v)g(s,v)\nu_T(dvds). \tag{3.25}$$

When h is bounded, (3.25) can be established using Lemma 2.8 in [3]. For completeness we include the proof in the Appendix. For general h (which may not be bounded), it is enough to show

$$\sup_n \int_{[0,T]\times K} |h(s,v)| 1_{\{|h|\ge M\}} g_n(s,v)\nu_T(dvds) \to 0, \tag{3.26}$$

as $M \to \infty$. We have

$$\sup_n \int_{[0,T]\times K} |h(s,v)| 1_{\{|h|\ge M\}} g_n(s,v)\nu_T(dvds) \\ \le \sup_n \int_{([0,T]\times K)\cap\{|h|\ge M\}} e^{L|h(s,v)|}\nu(dv)ds + \frac{1}{L}\sup_n \int_{\mathbb{X}_T} l(g_n(s,v))\nu(dv)ds \\ \le \int_{([0,T]\times K)\cap\{|h|\ge M\}} e^{L|h(s,v)|}\nu(dv)ds + \frac{1}{L}N.$$

Given $\epsilon > 0$, we can choose L large enough such that $N/L \le \epsilon/2$. Also, since

$$\int_{[0,T]\times K} e^{L|h(s,v)|}\nu_T(dvds) < \infty,$$

we can choose M_0 large enough such that $\int_{([0,T]\times K)\cap\{|h|\ge M\}} e^{L|h(s,v)|}\nu(dv)ds \le \epsilon/2$, for all $M \ge M_0$. Thus for all $M \ge M_0$, $\sup_n \int_{[0,T]\times K} |h(s,v)| 1_{|h|\ge M} g_n(s,v)\nu_T(dvds) \le \epsilon$. Since $\epsilon > 0$ is arbitrary, (3.26) follows. This proves the result. □

We now proceed to verify the first part of Condition 2.2. Recall the map $\mathcal{G}^0$ defined in (3.11).

Proposition 3.12. *Fix $N \in \mathbb{N}$, and let $g_n, g \in S^N$ be such that $g_n \to g$ as $n \to \infty$. Then*

$$\mathcal{G}^0\left(\nu_T^{g_n}\right) \to \mathcal{G}^0\left(\nu_T^{g}\right).$$

Proof. Let $\tilde{X}^n = \mathcal{G}^0\left(\nu_T^{g_n}\right)$. By Theorem 3.7, there exists a constant $\tilde{\kappa} \in (0, \infty)$ such that

$$\sup_n \sup_{t\in[0,T]} \|\tilde{X}_t^n\|_{-p} \le \tilde{\kappa}. \tag{3.27}$$

Using similar arguments as in the proof of Theorem 3.7 (cf. (3.18) and (3.19)), we have, for any $\phi \in \Phi$,

$$\sup_n \sup_{t\in[0,T]} |\tilde{X}_t^n[\phi]| < \infty.$$

Also,

$$|\tilde{X}_t^n[\phi] - \tilde{X}_s^n[\phi]| \le \|\phi\|_q \sqrt{K}\sqrt{1+\tilde{\kappa}}(t-s) + \|\phi\|_p(1+\sqrt{\tilde{\kappa}}) \int_s^t \int_{\mathbb{X}} \|G(r,v)\|_{0,-p} |g_n(r,v) - 1| \nu(dv)dr.$$

Using (3.5) in Lemma 3.4 we now have that

$$\lim_{\delta\to 0} \sup_n \sup_{|t-s|\le\delta} |\tilde{X}_t^n[\phi] - \tilde{X}_s^n[\phi]| = 0.$$

This proves that the family $\{\tilde{X}_t^n[\phi]\}$ is pre-compact in $C([0,T]:\mathbb{R})$ for every $\phi \in \Phi$.

Combining this with (3.27), we have that $\{\tilde{X}^n\}_{n\in\mathbb{N}}$ is pre-compact in $C([0,T] : \Phi_{-p_1})$ (see Theorem 2.5.2 in [16]). Let $\tilde{X}$ be any limit point. An application of the dominated convergence theorem shows that, along the convergent subsequence,

$$\int_0^t A(s, \tilde{X}_s^n)[\phi]ds \to \int_0^t A(s, \tilde{X}_s)[\phi]ds \tag{3.28}$$

as $n \to \infty$. Furthermore, using the convergence of $\tilde{X}^n$ to $\tilde{X}$, Condition 2.7(d) and (3.9), we have that

$$\int_0^t \int_{\mathbb{X}} G(s, \tilde{X}_s^n, v)[\phi](g_n(s,v) - 1)\nu(dv)ds - \int_0^t \int_{\mathbb{X}} G(s, \tilde{X}_s, v)[\phi](g_n(s,v) - 1)\nu(dv)ds \to 0. \tag{3.29}$$

Here we have used the inequality

$$\left| G(s, \tilde{X}_s^n, v)[\phi] - G(s, \tilde{X}_s, v)[\phi] \right| \le \|G(s,v)\|_{1,-q} \sup_{t\in[0,T]} \|\tilde{X}_s^n - \tilde{X}_s\|_{-q}$$

along with inequality (3.9) in Remark 3.6.

Also, from (3.27), we have that for some $\kappa_1 \in (0, \infty)$

$$|G(s, \tilde{X}_s, v)[\phi]| \le \kappa_1 \|G(s,v)\|_{0,-p}, \quad \forall (s,v) \in \mathbb{X}_T.$$

Combining this with Condition 2.7(c) and Remark 3.2, we now get from Lemma 3.11 that, as $n \to \infty$,

$$\int_0^t \int_{\mathbb{X}} G(s, \tilde{X}_s, v)[\phi](g_n(s, v) - 1)\nu(dv)ds \\ \to \int_0^t \int_{\mathbb{X}} G(s, \tilde{X}_s, v)[\phi](g(s, v) - 1)\nu(dv)ds. \tag{3.30}$$

Combining (3.28), (3.29) and (3.30) we now see that $\tilde{X}$ must satisfy the integral equation (3.10) for all $\phi \in \Phi$. In view of unique solvability of (3.10) (Theorem 3.7), it now follows that $\tilde{X} = \mathcal{G}^0\left(\nu_T^g\right)$. The result follows. □

We now proceed to the second part of Condition 2.2. As noted in Theorem 2.4, it suffices to verify this condition with $\mathcal{U}^M$ replaced with $\tilde{\mathcal{U}}^M$.

Recall from the beginning of this section that $X^\epsilon = \mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}})$ is the strong solution of (1.1) with initial value X_0 on $(\bar{\mathbb{M}}, \mathcal{B}(\bar{\mathbb{M}}), \bar{\mathbb{P}}, \{\bar{\mathcal{F}}_t\})$. Let $\varphi_\epsilon \in \tilde{\mathcal{U}}^M$, define $\psi_\epsilon = 1/\varphi_\epsilon$, and recall the definitions of $\bar{N}$ and $\bar{\nu}_T$ from Section 2.1. Then it is easy to check (see Theorem III.3.24 of [15], see also Lemma 2.3 of [7]) that

$$\mathcal{E}_t^\epsilon(\psi_\epsilon) = \exp\left\{\int_{(0,t]\times\mathbb{X}\times[0,\epsilon^{-1}]} \log(\psi_\epsilon(s, x))\bar{N}(ds\,dx\,dr) \right. \\ \left. + \int_{(0,t]\times\mathbb{X}\times[0,\epsilon^{-1}]} (-\psi_\epsilon(s, x) + 1)\,\bar{\nu}_T(ds\,dx\,dr)\right\}$$

is an $\left\{\bar{\mathcal{F}}_t\right\}$-martingale. Consequently

$$\mathbb{Q}_T^\epsilon(G) = \int_G \mathcal{E}_t^\epsilon(\psi_\epsilon)d\bar{\mathbb{P}}, \quad \text{for } G \in \mathcal{B}(\bar{\mathbb{M}})$$

defines a probability measure on $\bar{\mathbb{M}}$, and furthermore $\bar{\mathbb{P}}$ and $\mathbb{Q}_T^\epsilon$ are mutually absolutely continuous. Also it can be verified that under $\mathbb{Q}_T^\epsilon$, $\epsilon N^{\epsilon^{-1}\varphi_\epsilon}$ has the same law as that of $\epsilon N^{\epsilon^{-1}}$ under $\bar{\mathbb{P}}$. Thus it follows that $\tilde{X}^\epsilon = \mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi_\epsilon})$ is the unique solution of the following controlled stochastic differential equation:

$$\tilde{X}_t^\epsilon = X_0 + \int_0^t A(s, \tilde{X}_s^\epsilon)ds + \int_0^t \int_{\mathbb{X}} G(s, \tilde{X}_{s-}^\epsilon, v)\left(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}(dsdv) - \nu(dv)ds\right). \tag{3.31}$$

Proposition 3.13. *Fix $M \in \mathbb{N}$. Let $\varphi_\epsilon, \varphi \in \tilde{\mathcal{U}}^M$ be such that φ_ϵ converges in distribution to φ, under $\bar{\mathbb{P}}$, as $\epsilon \to 0$. Then $\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}) \Rightarrow \mathcal{G}^0(\nu^\varphi)$.*

Proof. If $\tilde{X}^\epsilon = \mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi_\epsilon})$, then as just noted, $\tilde{X}^\epsilon$ is the unique solution of (3.31). We now show that the family $\{\tilde{X}^\epsilon\}_{\epsilon>0}$ of $D([0, T] : \Phi_{-p_1})$ valued random variables is tight.

We begin by showing that for some $\epsilon_0 \in (0, \infty)$

$$\sup_{0<\epsilon<\epsilon_0} \mathbb{E} \sup_{0\le t\le T} \|\tilde{X}_t^\epsilon\|_{-p}^2 < \infty. \tag{3.32}$$

Recall that θ_p is defined by $\theta_p(\phi_j^{-p}) = \phi_j^p$ for the CONS $\{\phi_j^{-p}, j \in \mathbb{Z}\}$. By Itô's formula,

$$\begin{aligned}
\|\tilde{X}_t^\epsilon\|_{-p}^2 &= \|X_0\|_{-p}^2 + 2\int_0^t A(s, \tilde{X}_s^\epsilon)[\theta_p \tilde{X}_s^\epsilon] ds \\
&\quad + 2\int_0^t \int_{\mathbb{X}} \langle G(s, \tilde{X}_s^\epsilon, v), \tilde{X}_s^\epsilon \rangle_{-p} (\varphi_\epsilon - 1)\nu(dv)ds \\
&\quad + \int_0^t \int_{\mathbb{X}} \left(\|\epsilon G(s, \tilde{X}_{s-}^\epsilon, v)\|_{-p}^2 + 2\langle \epsilon G(s, \tilde{X}_{s-}^\epsilon, v), \tilde{X}_{s-}^\epsilon \rangle_{-p} \right) \\
&\quad \times \left(N^{\epsilon^{-1}\varphi_\epsilon}(dsdv) - \epsilon^{-1}\varphi_\epsilon \nu(dv)ds \right) \\
&\quad + \epsilon \int_0^t \int_{\mathbb{X}} \|G(s, \tilde{X}_s^\epsilon, v)\|_{-p}^2 \varphi_\epsilon \nu(dv)ds. \qquad (3.33)
\end{aligned}$$

For completeness we include the proof of (3.33) in the Appendix.

For the second term in (3.33), we have by Condition 2.7(b) that

$$2\int_0^t A(s, \tilde{X}_s^\epsilon)[\theta_p \tilde{X}_s^\epsilon] ds \le K \int_0^t (1 + \|\tilde{X}_s^\epsilon\|_{-p}^2) ds. \qquad (3.34)$$

Also, using $a + a^2 \le 1 + 2a^2$ for $a \ge 0$

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{X}} \langle G(s, \tilde{X}_s^\epsilon, v), \tilde{X}_s^\epsilon \rangle_{-p} (\varphi_\epsilon - 1)\nu(dv)ds \right| \\
&\quad \le \int_0^t \int_{\mathbb{X}} \frac{\|G(s, \tilde{X}_s^\epsilon, v)\|_{-p}}{1 + \|\tilde{X}_s^\epsilon\|_{-p}} (1 + \|\tilde{X}_s^\epsilon\|_{-p}) \|\tilde{X}_s^\epsilon\|_{-p} |\varphi_\epsilon - 1| \nu(dv)ds \\
&\quad \le \int_0^t (1 + 2\|\tilde{X}_s^\epsilon\|_{-p}^2) \left(\int_{\mathbb{X}} \|G(s, v)\|_{0,-p} |\varphi_\epsilon - 1| \nu(dv) \right) ds \\
&\quad \le L_1 + 2\int_0^t \|\tilde{X}_s^\epsilon\|_{-p}^2 \left(\int_{\mathbb{X}} \|G(s, v)\|_{0,-p} |\varphi_\epsilon - 1| \nu(dv) \right) ds,
\end{aligned}$$

where $L_1 = \sup_{\varphi \in S^M} \int_{\mathbb{X}_T} \|G(s, v)\|_{0,-p} |\varphi - 1| \nu(dv)ds < \infty$, from (3.4).

For the last term in (3.33), we have

$$\begin{aligned}
&\epsilon \int_0^t \int_{\mathbb{X}} \|G(s, \tilde{X}_s^\epsilon, v)\|_{-p}^2 \varphi_\epsilon \nu(dv)ds \\
&\quad = \epsilon \int_0^t \int_{\mathbb{X}} \frac{\|G(s, \tilde{X}_s^\epsilon, v)\|_{-p}^2}{(1 + \|\tilde{X}_s^\epsilon\|_{-p})^2} (1 + \|\tilde{X}_s^\epsilon\|_{-p})^2 \varphi_\epsilon \nu(dv)ds \\
&\quad \le 2\epsilon \int_0^t (1 + \|\tilde{X}_s^\epsilon\|_{-p}^2) \left(\int_{\mathbb{X}} \|G(s, v)\|_{0,-p}^2 \varphi_\epsilon \nu(dv) \right) ds \\
&\quad \le 2\epsilon L_2 + 2\epsilon \int_0^t \|\tilde{X}_s^\epsilon\|_{-p}^2 \left(\int_{\mathbb{X}} \|G(s, v)\|_{0,-p}^2 \varphi_\epsilon \nu(dv) \right) ds,
\end{aligned}$$

where $L_2 = \sup_{\varphi \in S^M} \int_{\mathbb{X}_T} \|G(s, v)\|_{0,-p}^2 \varphi \nu(dv)ds < \infty$, from (3.3).

We split the martingale term as $M_t = M_t^1 + M_t^2$, where

$$M_t^1 = \int_0^t \int_{\mathbb{X}} \|\epsilon G(s, \tilde{X}_{s-}^\epsilon, v)\|_{-p}^2 \left(N^{\epsilon^{-1}\varphi_\epsilon}(dsdv) - \epsilon^{-1}\varphi_\epsilon \nu(dv)ds \right),$$

and

$$M_t^2 = \int_0^t \int_{\mathbb{X}} 2\langle \epsilon G(s, \tilde{X}_{s-}^\epsilon, v), \tilde{X}_{s-}^\epsilon \rangle_{-p} \left(N^{\epsilon^{-1}\varphi_\epsilon}(dsdv) - \epsilon^{-1}\varphi_\epsilon \nu(dv)ds \right).$$

We now use the following Gronwall inequality:

If η and $\psi \geq 0$ satisfy $\eta(s) \leq a + \int_0^s \eta(r)\psi(r)dr$ for all $s \in [0, t]$, then $\eta(t) \leq a e^{\int_0^t \psi(s)ds}$.

Using this inequality, the above estimates, and Lemma 3.4, we have that for some constants $L_3, L_4 \in (1, \infty)$,

$$\sup_{0\leq s\leq t} \|\tilde{X}_s^\epsilon\|_{-p}^2 \leq L_3 \left(L_4 + \sup_{0\leq s\leq t} |M_s^1| + \sup_{0\leq s\leq t} |M_s^2| \right), \tag{3.35}$$

for all $\epsilon \in (0, 1)$ and $t \in [0, T]$.

For the term M_t^1, we have, for $\epsilon \in (0, 1)$

$$\begin{aligned}
\mathbb{E} \sup_{0\leq s\leq T} |M_s^1| &\leq \mathbb{E} \left| \int_0^T \int_{\mathbb{X}} \|\epsilon G(s, \tilde{X}_{s-}^\epsilon, v)\|_{-p}^2 N^{\epsilon^{-1}\varphi_\epsilon}(dsdv) \right| \\
&\quad + \mathbb{E} \left| \int_0^T \int_{\mathbb{X}} \|\epsilon G(s, \tilde{X}_{s-}^\epsilon, v)\|_{-p}^2 \epsilon^{-1}\varphi_\epsilon \nu(dv)ds \right| \\
&\leq 2\mathbb{E} \int_0^T \int_{\mathbb{X}} \|\epsilon G(s, \tilde{X}_s^\epsilon, v)\|_{-p}^2 \epsilon^{-1}\varphi_\epsilon \nu(dv)ds \\
&\leq 4\epsilon\mathbb{E} \int_0^T (1 + \|\tilde{X}_s^\epsilon\|_{-p}^2) \left(\int_{\mathbb{X}} \|G(s, v)\|_{0,-p}^2 \varphi_\epsilon \nu(dv) \right) ds \\
&\leq 4\epsilon\mathbb{E} \int_{\mathbb{X}_T} \|G(s, v)\|_{0,-p}^2 \varphi_\epsilon \nu(dv)ds \\
&\quad + 4\epsilon\mathbb{E} \sup_{0\leq s\leq T} \|\tilde{X}_s^\epsilon\|_{-p}^2 \int_{\mathbb{X}_T} \|G(s, v)\|_{0,-p}^2 \varphi_\epsilon \nu(dv)ds \\
&\leq 4\epsilon L_2 \left(1 + \mathbb{E} \sup_{0\leq s\leq T} \|\tilde{X}_s^\epsilon\|_{-p}^2 \right).
\end{aligned} \tag{3.36}$$

Next consider the term M_t^2. From the Burkholder–Davis–Gundy inequality, we have that

$$\begin{aligned}
\mathbb{E} \sup_{0\leq s\leq T} |M_s^2| &\leq 4\mathbb{E}[M^2]_T^{1/2} \\
&\leq 4\mathbb{E} \left\{ \int_0^T \int_{\mathbb{X}} 4\epsilon^2 \langle G(s, \tilde{X}_{s-}^\epsilon, v), \tilde{X}_{s-}^\epsilon \rangle_{-p}^2 N^{\epsilon^{-1}\varphi_\epsilon}(dsdv) \right\}^{1/2} \\
&\leq 4\mathbb{E} \left\{ \int_0^T \int_{\mathbb{X}} 4\epsilon^2 \|G(s, \tilde{X}_{s-}^\epsilon, v)\|_{-p}^2 \|\tilde{X}_{s-}^\epsilon\|_{-p}^2 N^{\epsilon^{-1}\varphi_\epsilon}(dsdv) \right\}^{1/2} \\
&\leq 8\mathbb{E} \left\{ \sup_{0\leq s\leq T} \|\tilde{X}_s^\epsilon\|_{-p}^2 \int_0^T \int_{\mathbb{X}} \epsilon^2 \|G(s, \tilde{X}_{s-}^\epsilon, v)\|_{-p}^2 N^{\epsilon^{-1}\varphi_\epsilon}(dsdv) \right\}^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{8L_3}\mathbb{E}\sup_{0\leq s\leq T}\|\tilde{X}^\epsilon_s\|^2_{-p} \\
&\quad + 128\epsilon^2 L_3\mathbb{E}\left(\int_0^T\int_{\mathbb{X}}\|G(s,\tilde{X}^\epsilon_{s-},v)\|^2_{-p}N^{\epsilon^{-1}\varphi_\epsilon}(dsdv)\right) \\
&= \frac{1}{8L_3}\mathbb{E}\sup_{0\leq s\leq T}\|\tilde{X}^\epsilon_s\|^2_{-p} \\
&\quad + 128\epsilon L_3\mathbb{E}\left(\int_0^T\int_{\mathbb{X}}\|G(s,\tilde{X}^\epsilon_s,v)\|^2_{-p}\varphi_\epsilon\nu(dv)ds\right) \\
&\leq \frac{1}{8L_3}\mathbb{E}\sup_{0\leq s\leq T}\|\tilde{X}^\epsilon_s\|^2_{-p} + 256\epsilon L_2L_3\left(1+\mathbb{E}\sup_{0\leq s\leq T}\|\tilde{X}^\epsilon_s\|^2_{-p}\right). \qquad (3.37)
\end{aligned}
$$

For the fifth inequality, we have used the inequality $\sqrt{ab}\leq\frac{a}{2}+\frac{b}{2}$ with $a=\frac{1}{32L_3}\sup_{0\leq s\leq T}\|\tilde{X}^\epsilon_s\|^2_{-p}$ and $b=32L_3\epsilon^2\int_0^T\int_{\mathbb{X}}\|G(s,\tilde{X}^\epsilon_{s-},v)\|^2_{-p}N^{\epsilon^{-1}\varphi_\epsilon}(dsdv)$. Combining (3.35)–(3.37) we now have

$$
\left(\mathbb{E}\sup_{0\leq s\leq T}\|\tilde{X}^\epsilon_s\|^2_{-p}\right)\left(1-4\epsilon L_2L_3-256\epsilon L_2L_3^2-\frac{1}{8}\right)\leq L_3L_4+4L_2L_3+256L_2L_3^2.
$$

Choose ϵ_0 small enough so that $\max\{4\epsilon_0L_2L_3, 256\epsilon_0L_2L_3^2\}\leq\frac{1}{8}$. Then for $\epsilon\leq\epsilon_0$, we have that

$$
\mathbb{E}\sup_{0\leq s\leq T}\|\tilde{X}^\epsilon_s\|^2_{-p}\leq\frac{8}{5}(L_3L_4+4L_2L_3+256L_2L_3^2).
$$

This proves (3.32).

In view of the estimate in (3.32), to prove tightness of $\{\tilde{X}^\epsilon\}_{\epsilon\leq\epsilon_0}$ in $D([0,T]:\Phi_{-p_1})$, it suffices to show that for all $\phi\in\Phi$, $\{\tilde{X}^\epsilon[\phi]\}_{\epsilon\leq\epsilon_0}$ is tight in $D([0,T]:\mathbb{R})$. For the rest of the proof we will only consider $\epsilon\leq\epsilon_0$, however we will suppress ϵ_0 from the notation. Fix $\phi\in\Phi$. Let

$$
C^\epsilon_t=\int_0^t A(s,\tilde{X}^\epsilon_s)[\phi]ds+\int_0^t\int_{\mathbb{X}}G(s,\tilde{X}^\epsilon_s,v)[\phi](\varphi_\epsilon-1)\nu(dv)ds
$$

and

$$
M^\epsilon_t=\epsilon\int_0^t\int_{\mathbb{X}}G(s,\tilde{X}^\epsilon_{s-},v)[\phi]\tilde{N}^{\epsilon^{-1}\varphi_\epsilon}(dsdv).
$$

To argue tightness of C^ϵ in $C([0,T]:\mathbb{R})$, it suffices to show (cf. Lemma 6.1.2 of [16]) that for all $\tau>0$, there exists $\delta=\delta_\tau>0$ such that

$$
\sup_{0\leq\epsilon\leq\epsilon_0}\mathbb{P}\left(\sup_{0<\beta-\alpha<\delta}|C^\epsilon_\alpha-C^\epsilon_\beta|>\tau\right)<\tau. \qquad (3.38)
$$

Fix $\tau>0$. Then for arbitrary $\delta>0$,

$$
\begin{aligned}
&\sup_\epsilon\mathbb{P}\left(\sup_{0<\beta-\alpha<\delta}|C^\epsilon_\alpha-C^\epsilon_\beta|>\tau\right) \\
&\quad=\sup_\epsilon\mathbb{P}\left(\sup_{0<\beta-\alpha<\delta}\left|\int_\alpha^\beta A(s,\tilde{X}^\epsilon_s)[\phi]ds\right.\right.
\end{aligned}
$$

$$+ \int_\alpha^\beta \int_{\mathbb{X}} G(s, \tilde{X}_s^\epsilon, v)[\phi](\varphi_\epsilon - 1)\nu(dv)ds \Big| > \tau \Big)$$

$$\leq \sup_\epsilon \mathbb{P}\left(\sup_{0<\beta-\alpha<\delta} \left| \int_\alpha^\beta A(s, \tilde{X}_s^\epsilon)[\phi]ds \right| > \frac{\tau}{2} \right)$$

$$+ \sup_\epsilon \mathbb{P}\left(\sup_{0<\beta-\alpha<\delta} \left| \int_\alpha^\beta \int_{\mathbb{X}} G(s, \tilde{X}_s^\epsilon, v)[\phi](\varphi_\epsilon - 1)\nu(dv)ds \right| > \frac{\tau}{2} \right)$$

$$\leq \sup_\epsilon \frac{4}{\tau^2} \mathbb{E}\left(\delta^2 \sup_{0\leq s\leq T} \left| A(s, \tilde{X}_s^\epsilon)[\phi] \right|^2 \right)$$

$$+ \sup_\epsilon \frac{2}{\tau} \mathbb{E}\left(\sup_{0<\beta-\alpha<\delta} \left| \int_\alpha^\beta \int_{\mathbb{X}} G(s, \tilde{X}_s^\epsilon, v)[\phi](\varphi_\epsilon - 1)\nu(dv)ds \right| \right). \tag{3.39}$$

From (3.32) and Condition 2.7(c), it follows that

$$\sup_\epsilon \mathbb{E}\left(\sup_{0\leq s\leq T} \left| A(s, \tilde{X}_s^\epsilon)[\phi] \right|^2 \right) < \infty.$$

Thus we can find $\delta_1 > 0$ such that for all $\delta \leq \delta_1$, the first term on the last line of (3.39) is bounded by $\tau/2$.

Now we consider the second term:

$$\left| \int_{[\alpha,\beta]\times\mathbb{X}} G(s, \tilde{X}_s^\epsilon, v)[\phi](\varphi_\epsilon - 1)\nu(dv)ds \right|$$

$$\leq \|\phi\|_p \left(1 + \sup_{0\leq s\leq T} \|\tilde{X}_s^\epsilon\|_{-p} \right) \int_{[\alpha,\beta]\times\mathbb{X}} \|G(s, v)\|_{0,-p} |\varphi_\epsilon - 1| \nu(dv)ds$$

$$\leq \|\phi\|_p \left(1 + \sup_{0\leq s\leq T} \|\tilde{X}_s^\epsilon\|_{-p} \right) \sup_{g\in S^M} \sup_{|t-s|\leq\delta} \int_{[s,t]\times\mathbb{X}} \|G(s, v)\|_{0,-p} |g - 1| \nu(dv)ds.$$

Then from (3.5) in Lemma 3.4 and (3.32), we can find $\delta_2 > 0$ such that for all $\delta \leq \delta_2$, the second term on the last line of (3.39) is bounded by $\tau/2$. By taking $\delta = \min(\delta_1, \delta_2)$, (3.38) holds and the tightness of $\{C^\epsilon\}_{\epsilon\leq\epsilon_0}$ follows.

Next consider M^ϵ. We have

$$\mathbb{E}\langle M^\epsilon \rangle_T = \epsilon \mathbb{E} \int_0^T \int_{\mathbb{X}} (G(s, \tilde{X}_s^\epsilon, v)[\phi])^2 \varphi_\epsilon \nu(dv)ds$$

$$\leq 2\epsilon \|\phi\|_p \left(1 + \mathbb{E} \sup_{0\leq s\leq T} \|\tilde{X}_s^\epsilon\|_{-p}^2 \right) \sup_{\varphi\in S^M} \int_{\mathbb{X}_T} \|G(s, v)\|_{0,-p}^2 \varphi \nu(dv)ds. \tag{3.40}$$

Using Lemma 3.4, we have $\mathbb{E}\sup_{0\leq s\leq T} \langle M^\epsilon \rangle_s$ goes to 0 as $\epsilon \to 0$. Then by Theorem 6.1.1 in [16], for any $\phi \in \Phi$, the sequence of semimartingales $\tilde{X}_t^\epsilon[\phi] = X_0[\phi] + C_t^\epsilon + M_t^\epsilon$ is tight in $D([0, T] : \mathbb{R})$. It then follows from (3.32) and Theorem 2.5.2 in [16] that $\{\tilde{X}^\epsilon\}_{\epsilon\leq\epsilon_0}$ is tight in $D([0, T] : \Phi_{-p_1})$.

Choose a subsequence along which $(\tilde{X}^\epsilon, \varphi_\epsilon, M^\epsilon)$ converges in distribution to $(\tilde{X}, \tilde{\varphi}, 0)$. Without loss of generality, we can assume the convergence is almost sure by using the Skorokhod representation theorem. Note that $\tilde{X}^\epsilon$ satisfies the following integral equation

$$\tilde{X}^\epsilon_t[\phi] = X_0[\phi] + \int_0^t A(s, \tilde{X}^\epsilon_s)[\phi]ds + \int_0^t \int_{\mathbb{X}} G(s, \tilde{X}^\epsilon_s, v)[\phi](\varphi_\epsilon - 1)\nu(dv)ds + M^\epsilon.$$

Along the lines of Theorem 3.7 and Proposition 3.12 (see (3.28)–(3.30)), we see that $\tilde{X}$ must solve

$$\tilde{X}_t[\phi] = X_0[\phi] + \int_0^t A(s, \tilde{X}_s)[\phi]ds + \int_0^t \int_{\mathbb{X}} G(s, \tilde{X}_s, v)[\phi](\tilde{\varphi} - 1)\nu(dv)ds.$$

The unique solvability of the above integral equation gives that $\tilde{X} = \mathcal{G}^0\left(\nu^{\tilde{\varphi}}\right)$, thus we have proved part 2 of Condition 2.2, i.e., $\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}) \Rightarrow \mathcal{G}^0(\nu^\varphi)$. □

We are now ready to prove the main theorem.

Proof of Theorem 3.8. Using Propositions 3.12 and 3.13, Theorem 3.8 is an immediate consequence of Theorem 2.4. □

4. A one dimensional model for spread of a chemical agent

In the hydrology literature (see [26] for example), partial differential equations of the following type are often used to model the spread of a pollutant in a reservoir, river or air:

$$D\Delta\phi - V \cdot \nabla\phi - \alpha\phi + Q = 0. \tag{4.1}$$

Here $\phi(x)$ represents the water quality or pollutant concentration at location x; Δ is the Laplacian operator modeling the diffusion of the chemical; D is the coefficient capturing the strength of the diffusion effect. The term $V \cdot \nabla\phi$ models the convection term, here ∇ is the gradient operator and V is the velocity vector. The scalar $\alpha \geq 0$ can be interpreted as the rate of dissipation of the chemical and $Q \geq 0$ is the "load" or pollutant issued from outside. Pollutants take various forms, such as nutrients (e.g., runoff fertilizer), microbiological, and chemical (e.g., pesticides).

The deterministic (4.1) models the steady state density profile of the pollutant and does not take into account any temporal or stochastic variability. A dynamic stochastic model for pollutant spread described through a stochastic partial differential equation (SPDE) driven by a PRM was studied in [16]. We begin by describing this model in a one dimensional setting, where it describes the evolution of a pollutant deposited at different sites along a reservoir. Our goal is to study probabilities of deviations from the nominal behavior by establishing a suitable large deviation principle.

4.1. Dynamic SPDE model

The model considered here describes the spread of a chemical agent which is released by several different sources along a one-dimensional reservoir. Suppose that there are r such sources located at different sites $\kappa_1, \ldots, \kappa_r \in [0, l]$, where the interval $[0, l]$ represents the reservoir. These sources release pollutants according to independent Poisson streams $N_i(t)$, with rate f_i, $i = 1, \ldots, r$, and with random magnitudes $A_i^j(\omega)$, $j \in \mathbb{N}$, $i = 1, \ldots, r$, which are mutually independent with magnitudes in the ith stream having common distribution $F_i(da)$.

Formally, the model describing the evolution of concentration is written as follows:

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) - V\frac{\partial}{\partial x}u(t,x) - \alpha u(t,x) + \sum_{i=1}^r \sum_j A_i^j(\omega)\delta_{\kappa_i}(x)1_{\left\{t=\tau_i^j(\omega)\right\}} \tag{4.2}$$

where $\tau_i^j(\omega)$, $j \in \mathbb{N}$ are the jump times of N_i, and $\delta_a(x)$ is the Dirac delta measure with unit mass at a. The equation is considered with a Neumann boundary condition on $[0, l]$. A Neumann boundary condition is reasonable as a model for a reservoir, though one would expect in this case that at the boundary the component of the velocity orthogonal to the boundary would be zero, which in the current setting would mean $V = 0$. However, the example is for illustrative purposes only, and the domain, boundary conditions and differential operator may be made much more general, though one will not always obtain expressions as explicit as those given below.

Eq. (4.2) can be regarded as a stochastic partial differential equation driven by a Poisson random measure. The Poisson random measure N driving the equation is a random measure on the space $\mathbb{R}_+ \times \mathbb{X}$ with $\mathbb{X} = \mathbb{J} \times \mathbb{R}_+$ and $\mathbb{J} = \{1, 2, \dots, r\}$, and can be represented as

$$N([0,t] \times A \times B) = \sum_{i=1}^{r} 1_A(i) \sum_{j=1}^{N_i(t)} 1_B(A_i^j(\omega)), \quad t \geq 0, A \subseteq \mathbb{J}, B \in \mathcal{B}(\mathbb{R}_+).$$

The intensity measure of N is given by $\nu_0 = \lambda \otimes \nu$, where λ is the Lebesgue measure on $\mathbb{R}_+$ and

$$\nu(A \times B) = \sum_{i=1}^{r} 1_A(i) f_i F_i(B), \quad A \subseteq \mathbb{J}, B \in \mathcal{B}(\mathbb{R}_+). \tag{4.3}$$

We now introduce a natural CHNS associated with Eq. (4.2) (see [16]). Let $\rho \in \mathcal{M}_F[0, l]$ be defined by

$$\rho(A) = \int_A e^{-2cx} dx; \quad A \in \mathcal{B}[0, l],$$

where $c = \frac{V}{2D}$. Let $H = L^2([0, l], \rho)$. Then $\{\phi_j\}_{j \in \mathbb{N}_0}$ defined below is a complete orthonormal system on H of eigen-functions of the operator L defined by

$$L\phi = D \frac{\partial^2}{\partial x^2} \phi - V \frac{\partial}{\partial x} \phi, \tag{4.4}$$

with Neumann boundary $\phi'(0) = \phi'(l) = 0$.

$$\phi_0(x) = \sqrt{\frac{2c}{1 - e^{-2cl}}}, \qquad \phi_j(x) = \sqrt{\frac{2}{l}} e^{cx} \sin\left(\frac{j\pi}{l} x + \alpha_j\right);$$
$$\alpha_j = \tan^{-1}\left(-\frac{j\pi}{lc}\right), \quad j = 1, 2, \dots.$$

The corresponding eigenvalues, denoted by $\{-\lambda_j\}_{j \in \mathbb{N}_0}$, are given as

$$\lambda_0 = 0, \qquad \lambda_j = D\left(c^2 + \left(\frac{j\pi}{l}\right)^2\right).$$

For $\phi \in H$ and $n \in \mathbb{Z}$ let

$$\|\phi\|_n^2 = \sum_{j=0}^{\infty} \langle \phi, \phi_j \rangle^2 (1 + \lambda_j)^{2n},$$

where $\langle \phi, \psi \rangle$ is the inner product on H. Define

$$\Phi = \{\phi \in H : \|\phi\|_n < \infty, \forall n \in \mathbb{Z}\} \tag{4.5}$$

and let Φ_n be the completion of Φ with respect to the norm $\|\cdot\|_n$. Note that $\Phi_0 = H$, and it can be checked that Φ is a CHNS.

With Φ defined by (4.5), the equation in (4.2) can be written rigorously as a SPDE in Φ' as follows. Define $A : \Phi' \to \Phi'$ and $G : \mathbb{X} \to \Phi'$ by

$$A(u)[\phi] = u[L\phi] - \alpha u[\phi] + \sum_{i=1}^{r} a_i f_i \phi(\kappa_i)\rho(\kappa_i), \quad \phi \in \Phi, u \in \Phi' \tag{4.6}$$

$$G(i,a)[\phi] = a\phi(\kappa_i)\rho(\kappa_i), \quad (i,a) \in \mathbb{J} \times \mathbb{R}_+, \phi \in \Phi \tag{4.7}$$

where $a_i = \int_{\mathbb{R}_+} aF_i(da)$ and L is defined as in (4.4).

Let $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be a filtered probability space on which is given a Poison random measure N with intensity measure $\lambda \otimes \nu$, with ν as in (4.3), such that $N([0,t] \times A \times B) - t\nu(A \times B)$ is a $\{\mathcal{F}_t\}$ martingale for all $A \subseteq \mathbb{J}$, $B \in \mathcal{B}(\mathbb{R}_+)$ satisfying $\nu(A \times B) < \infty$, and let u_0 be a $\mathcal{F}_0$-measurable random variable with values in Φ'. In order to formulate the SPDE, we will need square integrability assumptions on F_i, but with large deviation questions in mind, we impose the following stronger integrability requirement.

Condition 4.1. *There exists $\delta > 0$ such that*

$$\int_0^\infty e^{\delta a^2} F_i(da) < \infty, \quad \forall i = 1, \ldots, r.$$

Let $\tilde{N}(dsdv)$ be the compensated random measure of N, i.e.

$$\tilde{N}([0,t] \times B) = N([0,t] \times B) - t\nu(B),$$

$\forall B \in \mathcal{B}(\mathbb{X})$ with $\nu(B) < \infty$. Note that the operator $-L$ on H is positive definite and self-adjoint, and thus the following definition of a solution of (4.2) is natural.

Definition 4.2. Fix $p \geq 0$, suppose that $\mathbb{E}\|u_0\|_{-p}^2 < \infty$. A stochastic process $\{u_t\}_{t\in[0,\infty)}$ defined on $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be a Φ_{-p}-valued strong solution to the SPDE (4.2) with initial value u_0, if

(a) u_t is a Φ_{-p}-valued $\mathcal{F}_t$-measurable random variable, for all $t \in [0,\infty)$;
(b) $u \in D([0,\infty) : \Phi_{-p})$ a.s.;
(c) For all $t \in [0,\infty)$ and a.e. ω

$$u_t[\phi] = u_0[\phi] + \int_0^t A(u_s)[\phi]ds + \int_0^t \int_{\mathbb{X}} G(v)[\phi]\tilde{N}(dsdv), \quad \forall \phi \in \Phi.$$

We are interested in the behavior of the solution when the Poisson noise is small, namely the case where ν_0 is replaced with $\epsilon^{-1}\nu_0$ and G with ϵG, and ϵ is a small parameter. More precisely, the goal is to study the large deviation behavior of $\{u_t^\epsilon\}_{0\le t\le T}$ in $D([0,T] : \Phi_{-p})$, as $\epsilon \to 0$, where u^ϵ solves the integral equation

$$u_t^\epsilon = u_0 + \int_0^t A(u_s^\epsilon)ds + \epsilon \int_0^t \int_{\mathbb{X}} G(v)\tilde{N}^{\epsilon^{-1}}(dsdv),$$

where $\tilde{N}^{\epsilon^{-1}}$ is the compensated version of $N^{\epsilon^{-1}}$ as introduced below (1.1) and $N^{\epsilon^{-1}}$ is constructed using $\bar{N}$ as in (2.1). Here $\bar{N}$, as in Section 2, is once more a Poisson random measure on $[0,T] \times \mathbb{X} \times [0,\infty)$ with intensity $\bar{\nu}_T = \lambda_T \otimes \nu \otimes \lambda_\infty$. In particular, we are assuming (without loss of generality) that the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathcal{F}_t\})$ introduced below (4.7) is large enough to support the Poisson random measure $\bar{N}$ that has the usual martingale properties with respect to the filtration $\{\mathcal{F}_t\}$.

It can be easily checked that the functions A and G satisfy Condition 2.7 with $p_0 = 1$. Moreover in the setting of this section, for any $p_1 \geq 2$, the canonical injection from Φ_{-1} to Φ_{-p_1} is Hilbert–Schmidt. Recall the space $\mathbb{M}$ and $\mathbb{S}$ from Section 2. Recall that $\nu_T^g(dsdv) = g(s,v)\nu(dv)ds$.

For $p_1 \geq 2$ fixed, define the map $\mathcal{G}^0 : \mathbb{M} \to \mathbb{U} = D([0,T] : \Phi_{-p_1})$ as follows.

$$\mathcal{G}^0(\nu_T^g) = \tilde{u}^g \quad \text{for } g \in \mathbb{S}, \text{ with } \tilde{u}^g \text{given by (4.8)}.$$
$$\tilde{u}_t^g[\phi] = u_0[\phi] + \int_0^t A(\tilde{u}_s^g)[\phi]ds + \int_0^t \int_{\mathbb{X}} G(v)[\phi](g(s,v)-1)\nu_T(dvds),$$
$$\forall \phi \in \Phi. \tag{4.8}$$

From Theorem 3.7, we have that there is a unique $\tilde{u}^g \in D([0,T] : \Phi_{-p_1})$ that solves (4.8).

Define I through (2.4), where L_T is as in (2.2). It can be checked that Conditions 3.1 and 3.5 are satisfied under Condition 4.1. Thus, as an immediate consequence of Theorem 3.8, we have the following large deviation principle for u^ϵ.

Theorem 4.3. *Suppose Condition 4.1 holds. Fix $p_1 \geq 2$. Then I is a rate function on $\mathbb{U}$ and the family $\{u^\epsilon\}_{\epsilon>0}$ satisfies a large deviation principle, as $\epsilon \to 0$, on $D([0,T] : \Phi_{-p_1})$, with rate function I.*

Note that as $\epsilon \to 0$, u^ϵ converges in $D([0,T] : \Phi_{-p_1})$ to u^0 that solves the integral equation

$$u_t^0[\phi] = u_0[\phi] + \int_0^t A(u_s^0)[\phi]ds, \quad \forall \phi \in \Phi.$$

In particular, if u_0 solves the stationary equation

$$D\frac{d^2u_0(x)}{dx^2} - V\frac{du_0(x)}{dx} - \alpha u_0(x) + Q(x) = 0, \tag{4.9}$$

where

$$Q(x) = \sum_{i=1}^r a_i f_i \delta_{\kappa_i}(x),$$

then $u_t^0 = u_0$ for all $t \geq 0$. It is easily verified that there is a unique Φ_{-1} valued solution to (4.9) which can be explicitly characterized by

$$u_0[\phi] = \sum_{i=1}^r \sum_{j=1}^\infty \frac{a_i f_i}{\alpha + \lambda_j} \langle \phi, \phi_j \rangle \phi_j(\kappa_i)\rho(\kappa_i), \quad \forall \phi \in \Phi.$$

Eq. (4.9) should be compared with the stationary deterministic equation (4.1). This equation, which appears in [26], has been proposed as a model for the long time concentration profile when there is a constant rate, non random, source term given by $Q(x)$. Theorem 4.3 provides probabilities of large deviations from the steady state nominal values given by (4.1) when the true source term is a small noise perturbation of Q. We remark that in this case the solution to the integral equation for $\tilde{u}^g$ (i.e., (4.8)) that is used to define the map $\mathcal{G}^0$ appearing in the formula for the rate function, can be explicitly written as

$$\tilde{u}_t^g[\phi] = \sum_{j=0}^{\infty}\sum_{i=1}^{r} e^{-(\alpha+\lambda_j)t} f_i \phi_j(\kappa_i)\rho(\kappa_i)\langle\phi,\phi_j\rangle \times \left[\int_0^t \int_0^\infty e^{(\alpha+\lambda_j)s} a g(s,i,a) F_i(da) ds + \frac{a_i}{\alpha+\lambda_j}\right].$$

Acknowledgments

The first author's research was supported in part by the National Science Foundation (DMS-1004418, DMS-1016441), the Army Research Office (W911NF-10-1-0158) and the US–Israel Binational Science Foundation (Grant 2008466).

The third author's research was supported in part by the National Science Foundation (DMS-1008331), and the Air Force Office of Scientific Research (FA9550-09-1-0378, FA9550-12-1-0399), and the Army Research Office (W911NF-09-1-0155).

Appendix

A.1. *Proof of compactness of S^N*

Lemma A.1. *For every $N \in \mathbb{N}$, $\{\nu_T^g : g \in S^N\}$ is a compact subset of $\mathbb{M}$.*

Proof. The topology on $\mathbb{M}$, which was described in Section 2.1, can be metrized as follows. Consider a sequence of open sets $\{O_j, j \in \mathbb{N}\}$ such that $\bar{O}_j \subset O_{j+1}$, each $\bar{O}_j$ is compact, and $\cup_{j=1}^{\infty} O_j = \mathbb{X}_T$ (cf. Theorem 9.5.21 of [24]). Let $\phi_j(x) = [1 - d(x, O_j)] \vee 0$, where d denotes the metric on $\mathbb{X}_T$. Given any $\mu \in \mathbb{M}$, let $\mu^{(j)} \in \mathbb{M}$ be defined by $[d\mu^{(j)}/d\mu](x) = \phi_j(x)$. Given $\mu, \nu \in \mathbb{M}$, let

$$\bar{d}(\mu,\nu) = \sum_{j=1}^{\infty} 2^{-j} \left\|\mu^{(j)} - \nu^{(j)}\right\|_{BL},$$

where $\|\cdot\|_{BL}$ denotes the bounded, Lipschitz norm on $\mathcal{M}_F(\mathbb{X}_T)$:

$$\left\|\mu^{(j)} - \nu^{(j)}\right\|_{BL} = \sup\left\{\int_{\mathbb{X}_T} f d\mu^{(j)} - \int_{\mathbb{X}_T} f d\nu^{(j)} : |f|_\infty \le 1, |f(x) - f(y)| \le d(x,y) \text{ for all } x, y \in \mathbb{X}_T\right\}.$$

It is straightforward to check that $\bar{d}(\mu, \nu)$ defines a metric under which $\mathbb{M}$ is a Polish space, and that convergence in this metric is essentially equivalent to weak convergence on each compact subset of $\mathbb{X}_T$. Specifically, $\bar{d}(\mu_n, \mu) \to 0$ if and only if for each $j \in \mathbb{N}$, $\mu_n^{(j)} \to \mu^{(j)}$ in the weak topology as finite nonnegative measures, i.e., for all $f \in C_b(\mathbb{X}_T)$

$$\int_{\mathbb{X}_T} f d\mu_n^{(j)} \to \int_{\mathbb{X}_T} f d\mu^{(j)}.$$

Let $\mu_n = \nu_T^{g_n}$. We first show that $\{\mu_n\} \subset \mathbb{M}$ is relatively compact for any sequence $\{g_n\} \subset S^N$. For this, by using a diagonalization method, it suffices to show that $\{\mu_n^{(j)}\} \subset \mathbb{M}$

is relatively compact for every j. Next, since $\mu_n^{(j)}$ are supported on the compact subset of $\mathbb{X}_T$ given by $K^j = \overline{\{x|\phi_j(x) \neq 0\}}$, to show $\{\mu_n^{(j)}\} \subset \mathbb{M}$ is relatively compact it suffices to show $\sup_n \mu_n^{(j)}(\mathbb{X}_T) < \infty$. The last property will follow from the fact that $L_T(g_n) \leq N$ for all n, and the super-linear growth of l. Specifically, let $c \in (0, \infty)$ be such that $z \leq c(l(z) + 1)$ for all $z \in [0, \infty)$. Then

$$\begin{aligned}\sup_n \mu_n^{(j)}(\mathbb{X}_T) &= \sup_n \int_{\mathbb{X}_T} \phi_j(x) g_n(x) \nu_T(dx) \\ &\leq \sup_n \int_{K^j} g_n(x) \nu_T(dx) \leq c(N + \nu_T(K^j)) < \infty.\end{aligned}$$

Next, suppose that along a subsequence (without loss of generality, also denoted by $\{\mu_n\}$), $\mu_n \to \mu$. We would like to show that μ is of the form ν_T^g, where $g \in S^N$. For this we will use the lower semi-continuity property of relative entropy. The result holds trivially if $\mu = 0$. Suppose now $\mu \neq 0$. Then there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, $\inf_{n \in \mathbb{N}} \nu_T^{g_n}(\bar{O}_j) > 0$. For $j \geq j_0$, define

$$\begin{aligned} c^j &= \nu_T^{(j)}(\mathbb{X}_T), & \bar{\nu}_T^j &= \nu_T^{(j)}/c^j; \\ c_n^j &= \mu_n^{(j)}(\mathbb{X}_T), & \bar{\mu}_n^j &= \mu_n^{(j)}/c_n^j; \\ c_\mu^j &= \mu^{(j)}(\mathbb{X}_T), & \bar{\mu}^j &= \mu^{(j)}/c_\mu^j. \end{aligned}$$

Then $\bar{\nu}_T^j$, $\bar{\mu}_n^j$ and $\bar{\mu}^j$ are probability measures, and

$$\begin{aligned} R(\bar{\mu}_n^j \parallel \bar{\nu}_T^j) &= \frac{1}{c_n^j} \int_{\mathbb{X}_T} \left[\log(g_n(x)) + \log\left(\frac{c^j}{c_n^j}\right) \right] g_n(x) \phi_j(x) \nu_T(dx) \\ &= \frac{1}{c_n^j} \int_{\mathbb{X}_T} [l(g_n(x)) + g_n(x) - 1]\, \phi_j(x) \nu_T(dx) + \log\left(\frac{c^j}{c_n^j}\right) \\ &\leq \frac{1}{c_n^j} N + 1 - \frac{c^j}{c_n^j} + \log\left(\frac{c^j}{c_n^j}\right). \end{aligned}$$

Since $\mu_n^{(j)} \to \mu^{(j)}$, we have $c_n^j \to c_\mu^j$. Thus by the lower semi-continuity property of relative entropy,

$$\begin{aligned} R(\bar{\mu}^j \parallel \bar{\nu}_T^j) &\leq \liminf_{n\to\infty} R(\bar{\mu}_n^j \parallel \bar{\nu}_T^j) \\ &\leq \liminf_{n\to\infty} \left[\frac{1}{c_n^j} N + 1 - \frac{c^j}{c_n^j} + \log\left(\frac{c^j}{c_n^j}\right) \right] \\ &\leq \frac{1}{c_\mu^j} N + 1 - \frac{c^j}{c_\mu^j} + \log\left(\frac{c^j}{c_\mu^j}\right) \\ &< \infty. \end{aligned} \tag{A.1}$$

Thus $\mu^{(j)}$ is absolutely continuous with respect to $\nu_T^{(j)}$. Define $g^j = d\mu^{(j)}/d\nu_T^{(j)}$, and $g = g^j$ on $\bar{O}_j$. It is easily checked that g is defined consistently, and that $\mu = \nu_T^g$. Also by a direct calculation,

$$R(\bar{\mu}^j \parallel \bar{\nu}_T^j) = \frac{1}{c_\mu^j} \int_{\mathbb{X}_T} l(g(v))\phi_j(v)\nu_T(dv) + 1 - \frac{c^j}{c_\mu^j} + \log\left(\frac{c^j}{c_\mu^j}\right).$$

Combining the last display with (A.1), we have $\int_{\mathbb{X}_T} l(g(v))\phi_j(v)\nu_T(dv) \le N$, for all j. Sending $j \to \infty$, we see that $g \in S^N$. The result follows. □

A.2. Proof of Theorem 2.4

Proof. The proof follows by modifying arguments for the lower bound and upper bound in the proof of Theorem 4.2 of [7].

Lower bound. Following the proof of Theorem 2.8 in [7], it is easy to see that $-\epsilon \log \bar{\mathbb{E}} \left(e^{-\epsilon^{-1} F(Z^\epsilon)}\right)$ is bounded below (actually equal to)

$$\inf_{\varphi \in \tilde{\mathcal{U}}} \bar{\mathbb{E}} \left[L_T(\varphi) + F \circ \mathcal{G}^\epsilon \left(\epsilon N^{\epsilon^{-1}\varphi}\right)\right], \tag{A.2}$$

where $\tilde{\mathcal{U}} = \cup_{N \ge 1} \tilde{\mathcal{U}}^N$. The rest of the proof for the lower bound is as in Theorem 4.2 of [7].

Upper bound. Fix $\delta \in (0, 1)$ and $\phi_0 \in \mathbb{U}$ such that

$$I(\phi_0) + F(\phi_0) \le \inf_{\phi \in \mathbb{U}} (I(\phi) + F(\phi)) + \delta.$$

Choose $g \in \mathbb{S}_{\phi_0}$ such that $L_T(g) \le I(\phi_0) + \delta$. Note that $g \in \mathbb{S}_{\phi_0}$ implies $\phi_0 = \mathcal{G}^0\left(\nu_T^g\right)$. Define

$$g_n(t, x) = \begin{cases} \left[g(t, x) \vee \frac{1}{n}\right] \wedge n & \text{for } x \in K_n, \\ 1 & \text{else.} \end{cases}$$

Then $g_n \in \bar{\mathcal{A}}_{b,n} \subset \bar{\mathcal{A}}_b$. By the monotone convergence theorem, $L_T(g_n) \uparrow L_T(g)$.

Recalling from the proof of the lower bound that $-\epsilon \log \bar{\mathbb{E}} \left(\exp\left(-\epsilon^{-1} F(Z^\epsilon)\right)\right)$ equals the expression in (A.2),

$$\begin{aligned}\limsup_{\epsilon \to 0} -\epsilon \log \bar{\mathbb{E}} \left(e^{-\epsilon^{-1} F(Z^\epsilon)}\right) &\le L_T(g_n) + \limsup_{\epsilon \to 0} \bar{\mathbb{E}} \left[F \circ \mathcal{G}^\epsilon \left(\epsilon N^{\epsilon^{-1} g_n}\right)\right] \\ &\le L_T(g_n) + F \circ \mathcal{G}^0 \left(\nu_T^{g_n}\right),\end{aligned}$$

where the last inequality follows on observing that since $g_n \in \tilde{\mathcal{U}}^N$ for some N, we have by assumption that, for each fixed n, $\mathcal{G}^\epsilon(\epsilon N^{\epsilon^{-1} g_n}) \Rightarrow \mathcal{G}^0\left(\nu_T^{g_n}\right)$, as $\epsilon \to 0$. Sending $n \to \infty$, we have

$$\begin{aligned}\limsup_{\epsilon \to 0} -\epsilon \log \bar{\mathbb{E}} \left(e^{-\epsilon^{-1} F(Z^\epsilon)}\right) &\le L_T(g) + F \circ \mathcal{G}^0 \left(\nu_T^g\right) \\ &\le I(\phi_0) + \delta + F \circ \mathcal{G}^0 \left(\nu_T^g\right) \\ &= I(\phi_0) + F(\phi_0) + \delta \\ &\le \inf_{\phi \in \mathbb{U}} (I(\phi) + F(\phi)) + 2\delta.\end{aligned}$$

Since $\delta \in (0, 1)$ is arbitrary the desired upper bound follows. This completes the proof of the theorem. □

A.3. Proof of Remark 3.2

Proof. Let $E \in \mathcal{B}(\mathbb{X}_T)$ be such that $\nu_T(E) < \infty$. Fix $\delta_2 \in (0, \infty)$, and define $F = \{(s, v) \in \mathbb{X}_T : \|G(s, v)\|_{0,-p} > \delta_2/\delta_1\}$. Then

$$\begin{aligned}\int_E e^{\delta_2 \|G(s,v)\|_{0,-p}} \nu(dv)ds &= \int_{E\cap F} e^{\delta_2 \|G(s,v)\|_{0,-p}} \nu(dv)ds \\ &\quad + \int_{E\cap F^c} e^{\delta_2 \|G(s,v)\|_{0,-p}} \nu(dv)ds \\ &\le \int_{E\cap F} e^{\delta_1 \|G(s,v)\|^2_{0,-p}} \nu(dv)ds + e^{\delta_2^2/\delta_1} \int_{E\cap F^c} \nu(dv)ds \\ &\le \int_E e^{\delta_1 \|G(s,v)\|^2_{0,-p}} \nu(dv)ds + e^{\delta_2^2/\delta_1} \nu_T(E) < \infty.\end{aligned}$$

The remark follows. □

A.4. Proof of Lemma 3.9

Proof. The proof proceeds through a standard Picard iteration argument. Define $x^0(t) = x_0$ for all $t \in [0, T]$. Define $x^n(t)$ iteratively as

$$x^n(t) = x_0 + \int_0^t a(s, x^{n-1}(s))ds + \int_0^t b(s, x^{n-1}(s))u(s, x^{n-1}(s))ds, \quad t \in [0, T].$$

Then

$$\begin{aligned}\|x^n(t)\| &\le \|x_0\| + \int_0^t \|a(s, x^{n-1}(s))\|ds + \int_0^t \|b(s, x^{n-1}(s))u(s, x^{n-1}(s))\|ds \\ &\le \|x_0\| + \int_0^t \kappa(1 + \|x^{n-1}(s)\|)ds + \int_0^t \kappa(1 + \|x^{n-1}(s)\|) \sup_y \|u(s, y)\|ds \\ &\le \|x_0\| + \kappa(M + T) + \kappa \int_0^t \|x^{n-1}(s)\| \left(1 + \sup_y \|u(s, y)\|\right) ds.\end{aligned}$$

Let $L = \|x_0\| + \kappa(M + T)$, $\alpha(s) = 1 + \sup_y \|u(s, x)\|$, and $\beta(t) = \int_0^t \alpha(s)ds$. Then a recursive argument shows that for all $t \in [0, T]$,

$$\|x^n(t)\| \le L + \kappa L\beta(t) + \frac{\kappa^2 L}{2}\beta(t)^2 + \cdots + \frac{\kappa^n L}{n!}\beta(t)^n,$$

and thus

$$\sup_n \sup_{t\in[0,T]} \|x^n(t)\| \le Le^{\kappa\beta(T)} \le Le^{\kappa(M+T)}. \tag{A.3}$$

Similarly

$$\begin{aligned}\|x^n(t) - x^n(s)\| &\le \int_s^t \|a(s, x^{n-1}(r))\|dr + \int_s^t \|b(r, x^{n-1}(r))u(r, x^{n-1}(r))\|dr \\ &\le \kappa(1 + Le^{\kappa(M+T)})(t - s) + \kappa(1 + Le^{\kappa(M+T)}) \int_s^t \sup_y \|u(r, y)\|dr,\end{aligned}$$

and therefore

$$\lim_{\delta\to 0}\sup_n \sup_{|t-s|\le\delta} \|x^n(t)-x^n(s)\| = 0.$$

Together with (A.3) shows that the sequence $\{x^n\}$ is pre-compact in $C([0,T]:\mathbb{R}^d)$. Let x be a limit point of some subsequence of $\{x^n\}$. Then using the continuity properties of the functions a, b and u with respect to x and the dominated convergence theorem, it is easy to check that x satisfies (3.12). The lemma follows. □

A.5. Proof of (3.14)

Proof. Let $y_n \to y$, $y_n, y \in \mathbb{R}^d$. We will like to show that $u(s,y_n)\to u(s,y)$ for a.e. $s\in[0,T]$. Note that, since $\psi \in S^M$, $\int_{[0,T]\times\mathbb{X}} l(\psi(s,v))\nu(dv)ds \le M$. Thus there exists $\mathbb{T}_1 \subset [0,T]$, with $\lambda_T(\mathbb{T}_1^c)=0$ and such that

$$\int_{\mathbb{X}} l(\psi(s,v))\nu(dv) < \infty, \quad \forall s\in\mathbb{T}_1.$$

Also, from arguments similar to those in the proof of Lemma 3.4,

$$\int_{\mathbb{X}_T} \|g^d(s,v)\|_0|\psi(s,v)-1|\nu(dv)ds < \infty.$$

Consequently, there exists $\mathbb{T}_2\subset[0,T]$, with $\lambda_T(\mathbb{T}_2^c)=0$ and such that

$$\int_{\mathbb{X}} \|g^d(s,v)\|_0|\psi(s,v)-1|\nu(dv) < \infty, \quad \forall s\in\mathbb{T}_2. \tag{A.4}$$

Let $\mathbb{T}=\mathbb{T}_1\cap\mathbb{T}_2$ and fix $s\in\mathbb{T}$. Define $F_\beta(s)=\{v\in\mathbb{X}: |\psi(s,v)-1|\le\beta\}$ for $\beta\in(0,\infty)$. Then

$$\begin{aligned} u(s,y_n) &= \int_{\mathbb{X}\cap F_\beta} \frac{g^d(s,y_n,v)}{1+\|y_n\|}(\psi(s,v)-1)\nu(dv) \\ &\quad + \int_{\mathbb{X}\cap F_\beta^c} \frac{g^d(s,y_n,v)}{1+\|y_n\|}(\psi(s,v)-1)\nu(dv) \\ &= u_1(s,y_n)+u_2(s,y_n). \end{aligned}$$

From part (c) of Remark 3.3, for all $v\in F_\beta(s)$,

$$|\psi(s,v)-1|^2 \le c_2(\beta)l(\psi(s,v)).$$

Thus $[\psi(s,\cdot)-1]1_{F_\beta(s)}(\cdot)\in L^2(\mathbb{X},\nu;\mathbb{R})$. From assumption (a) in Lemma 3.10 we now see that, for all such s, $u_1(s,y_n)\to u_1(s,y)$, as $n\to\infty$.

For $u_2(s,y_n)$, we have

$$\left|\frac{g^d(s,y_n,v)}{1+\|y_n\|}(\psi(s,v)-1)\right| \le \|g^d(s,v)\|_0|\psi(s,v)-1|.$$

From (A.4), the term on the right hand side is ν-integrable. Furthermore, $\nu(F_\beta^c)\to 0$ from the super linear growth of l. Thus $u_2(s,y_n)$ converges to 0, uniformly in n, as β goes to ∞. The term $u_2(s,y)$ can be treated in a similar manner. Thus we have shown that, for all $s\in\mathbb{T}$, $u(s,y_n)\to u(s,y)$. Since $\lambda_T(\mathbb{T}^c)=0$, the result follows. □

A.6. Proof of (3.25) when h is a bounded and measurable function

Proof. We can assume without loss of generality that $\int_K g\nu_T(dsdv) \neq 0$ and $\int_K g_n\nu_T(dsdv) \neq 0$, for all $n \geq 1$. Define probability measures $\tilde{\nu}^n$ and $\tilde{\nu}$ as follows:

$$\tilde{\nu}^n(\cdot) = \frac{\nu_T^{g^n}(\cdot \cap K)}{m_n}, \qquad \tilde{\nu}(\cdot) = \frac{\nu_T^g(\cdot \cap K)}{m}$$

where $m_n = \int_K g_n\nu_T(dsdv)$ and $m = \int_K g\nu_T(dsdv)$. If $\theta(\cdot) = \frac{\nu_T(\cdot \cap K)}{\nu_T(K)}$, then θ is also a probability measure. We have

$$\begin{aligned} R(\tilde{\nu}^n \parallel \theta) &= \int_K \log\left(\frac{\nu_T(K)}{m_n} g_n\right) \frac{1}{m_n} g_n \nu_T(dsdv) \\ &= \frac{1}{m_n} \int_K (l(g_n) + g_n - 1)\nu_T(dsdv) + \log \frac{\nu_T(K)}{m_n} \\ &\leq \frac{N}{m_n} + 1 - \frac{\nu_T(K)}{m_n} + \log \frac{\nu_T(K)}{m_n}. \end{aligned}$$

Noting that $m_n \to m$, we have that there exists constant α such that $\sup_{n\in\mathbb{N}} R(\tilde{\nu}^n \parallel \theta) \leq \alpha < \infty$. Also note that $\tilde{\nu}^n$ converges weakly to $\tilde{\nu}$. From Lemma 2.8 of [3], we have

$$\frac{1}{m_n} \int_{[0,T]\times K} h(s,v)g_n(s,v)\nu_T(dvds) \to \frac{1}{m} \int_{[0,T]\times K} h(s,v)g(s,v)\nu_T(dvds),$$

which proves (3.25). □

A.7. Proof of Itô's formula in (3.33)

Proof. Here we will give the proof for a simpler case when X_t satisfies the following integral equation, the proof of (3.33) being very similar to this case:

$$X_t = X_0 + \int_0^t A(s, X_s)ds + \int_0^t \int_{\mathbb{X}} G(s, X_{s-}, v)\tilde{N}(dsdv).$$

For $j \in \mathbb{N}$,

$$X_t[\theta_p\phi_j] = X_0[\theta_p\phi_j] + \int_0^t A(s, X_s)[\theta_p\phi_j]ds + \int_0^t \int_{\mathbb{X}} G(s, X_{s-}, v)[\theta_p\phi_j]\tilde{N}(dsdv).$$

Note that

$$X_t[\theta_p\phi_j] = \langle X_t, \phi_j\rangle_{-p} = \|\phi_j\|_{-p}\langle X_t, \phi_j^{-p}\rangle_{-p},$$

so

$$\sum_{j=1}^{\infty} \|\phi_j\|_p^2 (X_t[\theta_p\phi_j])^2 = \sum_{j=1}^{\infty} \langle X_t, \phi_j^{-p}\rangle_{-p}^2 = \|X_t\|_{-p}^2.$$

If $\xi_j(t) = X_t[\theta_p\phi_j]$, then $\xi_j(t)$ satisfies

$$\xi_j(t) = \xi_j(0) + \int_0^t a^j(s)ds + \int_0^t \int_{\mathbb{X}} b^j(s,v)\tilde{N}(dsdv)$$

where $a^j(s) = A(s, X_s)[\theta_p\phi_j]$ and $b^j(s, v) = G(s, X_{s-}, v)[\theta_p\phi_j]$. Applying Itô's formula (cf. Theorem 2.5.1 of [14]) to the real valued semimartingale $\xi_j(t)$, we have

$$\xi_j^2(t) = \xi_j^2(0) + 2\int_0^t a^j(s)\xi_j(s)ds + 2\int_0^t \int_{\mathbb{X}} b^j(s, v)\xi_j(s-)\tilde{N}(dsdv) + \int_0^t \int_{\mathbb{X}} [b^j(s, v)]^2\tilde{N}(dsdv) + \int_0^t \int_{\mathbb{X}} [b^j(s, v)]^2\nu(dv)ds. \tag{A.5}$$

Note that $\|X_t\|_{-p}^2 = \sum_{j=1}^{\infty} \|\phi_j\|_p^2\xi_j^2(t)$. So for the second term in (A.5), we have

$$\begin{aligned}\sum_{j=1}^{\infty} \|\phi_j\|_p^2 a^j(s)\xi_j(s) &= \sum_{j=1}^{\infty} \|\phi_j\|_p^2 A(s, X_s)[\theta_p\phi_j]X_s[\theta_p\phi_j] \\ &= A(s, X_s)\left[\sum_{j=1}^{\infty} \|\phi_j\|_p^2 X_s[\theta_p\phi_j]\theta_p\phi_j\right] \\ &= A(s, X_s)\left[\sum_{j=1}^{\infty} \|\phi_j\|_p^2 \langle X_s, \phi_j\rangle_{-p}\|\phi_j\|_{-p}^2\phi_j\right] \\ &= A(s, X_s)\left[\sum_{j=1}^{\infty} \langle X_s, \phi_j^{-p}\rangle_{-p}\phi_j^p\right] \\ &= A(s, X_s)[\theta_p X_s].\end{aligned}$$

Also, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \|\phi_j\|_p^2 b^j(s, v)\xi_j(s-) &= \sum_{j=1}^{\infty} \|\phi_j\|_p^2 G(s, X_{s-}, v)[\theta_p\phi_j]X_{s-}[\theta_p\phi_j] \\ &= \sum_{j=1}^{\infty} \|\phi_j\|_p^2 \langle G(s, X_{s-}, v), \phi_j\rangle_{-p}\langle X_{s-}, \phi_j\rangle_{-p} \\ &= \sum_{j=1}^{\infty} \langle G(s, X_{s-}, v), \phi_j^{-p}\rangle_{-p}\langle X_{s-}, \phi_j^{-p}\rangle_{-p} \\ &= \langle G(s, X_{s-}, v), X_{s-}\rangle_{-p}.\end{aligned}$$

Finally, notice that

$$\begin{aligned}\sum_{j=1}^{\infty} \|\phi_j\|_p^2 [b^j(s, v)]^2 &= \sum_{j=1}^{\infty} \|\phi_j\|_p^2 \left(G(s, X_{s-}, v)[\theta_p\phi_j]\right)^2 \\ &= \sum_{j=1}^{\infty} \|\phi_j\|_p^2 \left(\langle G(s, X_{s-}, v), \phi_j\rangle_{-p}\right)^2 \\ &= \sum_{j=1}^{\infty} \left(\langle G(s, X_{s-}, v), \phi_j^{-p}\rangle_{-p}\right)^2 \\ &= \|G(s, X_{s-}, v)\|_{-p}^2.\end{aligned}$$

Combining the above equalities with (A.5), we have

$$\begin{aligned}\|X_t\|_{-p}^2 = \|X_0\|_{-p}^2 + 2\int_0^t A(s, X_s)[\theta_p X_s]ds \\ + 2\int_0^t \int_{\mathbb{X}} \langle G(s, X_{s-}, v), X_{s-}\rangle_{-p} \tilde{N}(dsdv) \\ + \int_0^t \int_{\mathbb{X}} \|G(s, X_{s-}, v)\|_{-p}^2 \tilde{N}(dsdv) \\ + \int_0^t \int_{\mathbb{X}} \|G(s, X_{s-}, v)\|_{-p}^2 \nu(dv)ds. \quad \square\end{aligned}$$

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