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Staircase Algorithm and Construction of Convex Spline Interpolants up to the Continuity C^3

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Abstract—This paper is concerned with the convex interpolation of data sets. Based on the staircase algorithm, several methods are presented which allow the construction of convex spline interpolants up to the continuity C^3.

Keywords—Closedness of the solvability set, Rational and lacunary splines, Splines on refined grids, Constructive methods.

1. INTRODUCTION

Because of their practical applications, as well as their theoretical attractiveness, restricted interpolations have received wide attention in the past. Depending on the background of the interpolation problem, the preservation, e.g., of convexity, monotonicity, or nonnegativity, may be essential. For recent reviews on methods in convex and other types of restricted interpolations, we refer to [1–3].

The present paper starts with negative results on convex data interpolation. If the set of interpolating functions is a finite dimensional linear subspace of C^1, the set Y of ordinates for which convex interpolation is successful turns out to be closed. This property gives rise to considerable difficulties in any numerical method for ordinates near the boundary of Y. Further, as a consequence of the closedness, convex interpolation in finite dimensional linear C^1 subspaces may fail even for data sets in strictly convex position [4].

Therefore, in convex interpolation, one should consider nonlinear approximation sets, e.g., exponential splines [5,6], lacunary splines [7,8], rational splines [9–12], or splines on refined grids with variable additional nodes [13–15]. In the present paper, for choosing the free nonlinearity parameters, we apply the staircase algorithm [16,17]. In this way, we are in the position to give computable bounds for the respective parameters such that within these bounds convexity can be preserved.

In the cited papers, mostly convex C^1 interpolation is considered. Recently, using splines on refined grids, convex interpolation of C^2 continuity was successfully treated [14,15]. We now also include convex C^3 interpolation applying quartic splines on threefold refined grids [18]. The main purpose of the present paper is, however, to give a unified representation using the staircase algorithm as a basic tool.

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2. NEGATIVE RESULTS IN CONVEX INTERPOLATION ON LINEAR SUBSPACES

Let $\Delta : a = x_0 < x_1 < \cdots < x_n = b$ be a fixed grid on the interval $I = [a, b]$. With given C^2 functions $\varphi_{i,j}$, the splines s may be defined on the subintervals $I_i = [x_{i-1}, x_i]$ by

$$s_{|I_i} \in \operatorname{span}\{\varphi_{i,0}, \varphi_{i,1}, \dots, \varphi_{i,N_i}\}, \qquad i = 1, \dots, n, \tag{2.1}$$

or, with real numbers $\lambda_{i,j}$, by

$$s(x) = \sum_{j=0}^{N_i} \lambda_{i,j}\varphi_{i,j}(x), \qquad x \in I_i, \quad i = 1, \dots, n.$$

We obtain $s \in C^1[a, b]$ if and only if

$$\sum_{j=0}^{N_i} \lambda_{i,j}\varphi_{i,j}^{(\nu)}(x_i) = \sum_{j=0}^{N_{i+1}} \lambda_{i+1,j}\varphi_{i+1,j}^{(\nu)}(x_i), \qquad i = 1, \dots, n-1, \quad \nu = 0, 1. \tag{2.2}$$

For $i = 1, \dots, n$, the systems $\{\varphi_{i,0}, \varphi_{i,1}, \dots, \varphi_{i,N_i}\}$ are assumed to satisfy a weak form of the Haar condition. That is, there are numbers $z_0 = z_{i,0}, \dots, z_{N_i} = z_{i,N_i} \in I_i$ such that the following determinants do not vanish:

$$\begin{vmatrix} \varphi_{i,0}(z_0) & \cdots & \varphi_{i,N_i}(z_0) \\ \vdots & & \vdots \\ \varphi_{i,0}(z_{N_i}) & \cdots & \varphi_{i,N_i}(z_{N_i}) \end{vmatrix} \neq 0, \qquad i = 1, \dots, n. \tag{2.3}$$

The finite dimensional linear set of these C^1 splines is abbreviated by $S^1(\Delta)$. For $s \in S^1(\Delta)$, convexity means

$$\sum_{j=0}^{N_i} \lambda_{i,j}\, \varphi_{i,j}''(x) \geq 0, \qquad x \in I_i, \quad i = 1, \dots, n. \tag{2.4}$$

Next, let $a = t_0 < t_1 < \cdots < t_m = b$ be the nodes for interpolation, and $y_0, y_1, \dots, y_m \in \mathbb{R}^1$ be the given ordinates. The interpolation requirement

$$s(t_\ell) = y_\ell, \qquad \ell = 0, \dots, m, \tag{2.5}$$

hence reads

$$\sum_{j=0}^{N_i} \lambda_{i,j}\, \varphi_{i,j}\,(t_\ell) = y_\ell, \qquad \ell = 0, \dots, m, \tag{2.6}$$

if i is chosen such that $t_\ell \in I_i$. Now we define Y to be the set of ordinates for which convex interpolation with splines from $S^1(\Delta)$ is successful; i.e.,

$$Y = \{(y_0, \dots, y_m) : \text{ there exist interpolating convex splines } s \in S^1(\Delta)\}. \tag{2.7}$$

It is obvious that $Y \neq \mathbb{R}^{m+1}$. Moreover, we obtain the following theorem.

THEOREM 1. *For $m \geq 2$, the set of ordinates Y for which the problem of convex interpolation is solvable in $S^1(\Delta)$ is closed.*

PROOF. Let $(y_0^{(k)}, \dots, y_m^{(k)}) \in Y$, $k = 1, 2, \dots$, be *a* convergent sequence of vectors from Y. The components of the limit vector are denoted by $y_i^* = \lim_{k\to\infty} y_i^{(k)}$, $i = 0, \dots, m$. We have to show that $(y_0^*, \dots, y_m^*) \in Y$.

An interpolating convex spline which belongs to $(y_0^{(k)}, \dots, y_m^{(k)})$ is called s_k, and the coefficients may be $\lambda_{i,j}^{(k)}$. The convexity of s_k yields

$$s_k(x) \le \max_{i=0,\dots,m} y_i^{(k)}, \qquad x \in I.$$

Thus, the splines s_k are uniformly bounded from above. On the other hand, for $m \ge 2$ we obtain due to the convexity of s_k

$$s_k(x) \ge \min_{i=1,\dots,m-1} \min \left\{ g_i^{(k)}, f_i^{(k)} \right\}, \qquad x \in I,$$

with

$$g_i^{(k)} = y_{i-1}^{(k)} + \frac{y_i^{(k)} - y_{i-1}^{(k)}}{t_i - t_{i-1}} (x_{i+1} - x_{i-1}), \qquad f_i^{(k)} = y_{i+1}^{(k)} + \frac{y_i^{(k)} - y_{i+1}^{(k)}}{t_i - t_{i+1}} (x_{i-1} - x_{i+1}).$$

Hence, the splines s_k are also uniformly bounded from below.

Now, using (2.3), the boundedness of the sequence (s_k) implies the boundedness of the sequences of the coefficients; i.e., we obtain

$$\left| \lambda_{i,j}^{(k)} \right| \le K, \qquad k = 1, 2, \dots,$$

with a constant K. Therefore, we have subsequences $(\lambda_{i,j}^{(k_r)})$ being convergent, say $\lambda_{i,j}^* = \lim_{r\to\infty} \lambda_{i,j}^{(k_r)}$. Of course, if (2.2),(2.4),(2.6) are satisfied for $\lambda_{i,j} = \lambda_{i,j}^{(k_r)}$, $y_\ell = y_\ell^{(k_r)}$, then also for the limit values $\lambda_{i,j} = \lambda_{i,j}^*$, $y_\ell = y_\ell^*$. This means that the spline

$$s^*(x) = \sum_{j=0}^{N_i} \lambda_{i,j}^* \varphi_{i,j}(x), \qquad x \in I_i, \quad i = 1, \dots, n$$

is a convex C^1 spline and interpolates $(y_0^*, \dots, y_m^*)$; i.e., $(y_0^*, \dots, y_m^*) \in Y$. Thus, the proof of Theorem 1 is complete. ∎

The closedness of the set Y of ordinates suitable for convex C^1 interpolation causes large numerical problems if ordinates are near or on the boundary ∂Y. In every neighbourhood of vectors from ∂Y, there are vectors not belonging to Y. Therefore, in view of the unavoidable rounding errors, numerical algorithms to compute convex interpolants must fail in general for ordinates from ∂Y. In addition, it is very easy to find vectors $(y_0, \dots, y_m) \in \partial Y$. For instance, $(y_0, \dots, y_m)$ is from ∂Y if the points (t_i, y_i), $i = 0, \dots, m$ are lying on a straight line.

A widely used counterexample in convex C^1 interpolation is the function $f(x) = |x|$ combined with $t_0 = -1$, $t_1 = -1/2$, $t_2 = 0$, $t_3 = 1/2$, $t_4 = 1$; i.e., we have to set $y_0 = y_4 = 1$, $y_1 = y_3 = 1/2$, $y_2 = 0$. Obviously, interpolating convex functions s have to be identical with f. Thus, they are not from C^1. Hence, the above vector (1, 1/2, 0, 1/2, 1) belongs to the open set $\mathbb{R}^5 \setminus Y$. Therefore, there are ordinates $(y_0, \dots, y_4)$ being even in strictly convex position, i.e.,

$$\tau_1 < \tau_2 < \dots < \tau_m, \qquad \tau_i = \frac{y_i - y_{i-1}}{t_i - t_{i-1}}, \tag{2.8}$$

such that $(y_0, \dots, y_4) \notin Y$. Summarizing, we obtain the following corollary.

COROLLARY 2. *There are data sets in strictly convex position such that the problem of convex C^1 interpolation is not solvable in $S^1(\Delta)$.*

3. STAIRCASE ALGORITHM

For given nonempty sets $W_1, W_2, \dots, W_n \subset \mathbb{R}^2$, the abstract staircase algorithm [16,17] is concerned with the existence and construction of numbers $p_0, p_1, \dots, p_n \in \mathbb{R}^1$ satisfying

$$(p_{i-1}, p_i) \in W_i, \qquad i = 1, \dots, n. \tag{3.1}$$

ALGORITHM 3. Let $V_0 = \mathbb{R}^1$, and for $i = 1, 2, \dots, n$

$$V_i = \left\{ y \in \mathbb{R}^1 : \text{ there exist } x \in V_{i-1} \text{ with } (x, y) \in W_i \right\}. \tag{3.2}$$

THEOREM 4. *Problem (3.1) is solvable if and only if*

$$V_i \neq \emptyset, \qquad i = 1, 2, \dots, n. \tag{3.3}$$

All solutions can be determined as follows. Choose $p_n \in V_n$, *and for* $i = n, n-1, \dots, 1$

$$p_{i-1} \in V_{i-1} \cap \left\{ x \in \mathbb{R}^1 : (x, p_i) \in W_i \right\}. \tag{3.4}$$

For a short proof, we refer to [19].

In the following, we are interested in two special systems $W_1, \dots, W_n$. In the first case, being useful in (4.1)–(4.3), W_i is described by

$$W_i = \left\{ (x, y) \in \mathbb{R}^2 : (2 + \beta_i) x + y \leq (3 + \beta_i) \tau_i, \ (3 + \alpha_i) \tau_i \leq x + (2 + \alpha_i) y \right\}, \tag{3.5}$$

where $\alpha_i \geq 0$, $\beta_i \geq 0$, $i = 1, \dots, n$, are parameters while $\tau_1, \dots, \tau_n$ are constants with

$$\tau_1 < \tau_2 < \cdots < \tau_n. \tag{3.6}$$

The sets V_i from (3.2) now are intervals $[A_i,\ B_i]$ if $A_i \leq B_i$. Algorithm 3 leads immediately to the following method to compute these intervals.

ALGORITHM 5. Let $A_0 = (3 + \alpha_1) \tau_1 - (2 + \alpha_1) \tau_2$, $B_0 = \tau_1$, and for $i = 1, \dots, n$

$$A_i = \max \left\{ \tau_i, \frac{(3 + \alpha_i) \tau_i - B_{i-1}}{2 + \alpha_i} \right\}, \qquad B_i = (3 + \beta_i) \tau_i - (2 + \beta_i) A_{i-1}. \tag{3.7}$$

By means of these quantities, Theorem 4 now yields the following theorem.

THEOREM 6. *Problem (3.1),(3.5) is solvable if and only if*

$$A_i \leq B_i, \qquad i = 0, 1, \dots, n. \tag{3.8}$$

The solutions can be determined by choosing $p_n \in [A_n,\ B_n]$ *and for* $i = n, n-1, \dots, 1$,

$$p_{i-1} \in \left[\max \left\{ A_{i-1}, (3 + \alpha_i) \tau_i - (2 + \alpha_i) p_i \right\}, \ \min \left\{ B_{i-1}, \frac{(3 + \beta_i) \tau_i - p_i}{2 + \beta_i} \right\} \right]. \tag{3.9}$$

Note that the complexity of this procedure is $O(n)$.

Next it will be shown that the solvability test (3.8) can always be satisfied by choosing the parameters α_i and β_i, $i = 1, \dots, n$, appropriately.

PROPOSITION 7. *Assume that (3.6) holds true. Then, system (3.1),(3.5) is solvable if*

$$\begin{aligned} &\alpha_i \geq 0, && i = 1, \dots, n, \quad \beta_1 \geq 0, \quad \beta_n \geq 0, \\ &\beta_i \geq \max \left\{ 0, \frac{\tau_{i+1} + 2\tau_{i-1} - 3\tau_i}{\tau_i - \tau_{i-1}} \right\}, && i = 2, \dots, n-1. \end{aligned} \tag{3.10}$$

PROOF. We verify (3.8). For $i = 0$, we find $A_0 = \tau_1 - (2+\alpha_1)(\tau_2 - \tau_1) < \tau_1 = B_0$, while for $i = 1$, we have $A_1 = \tau_1 < \tau_2 \le \tau_1 + (2+\alpha_1)(2+\beta_1)(\tau_2 - \tau_1) = B_1$. Next, if $A_{i-1} = \tau_{i-1} < \tau_i \le B_{i-1}$ is assumed for $i \in \{2, \dots, n-1\}$, because of (3.10), we obtain $A_i = \tau_i < \tau_{i+1} \le \tau_i + (2+\beta_i)(\tau_i - \tau_{i-1}) = B_i$. Finally, for $i = n$, we find $A_n = \tau_n < \tau_n + (2+\beta_n)(\tau_n - A_{n-1}) = B_n$. ∎

The second special case of a problem (3.1) being of interest in (5.1)–(5.3) below is defined by

$$W_i = \left\{(x,y) \in \mathbb{R}^2 : (M - L\beta_i)\, x + L\beta_i y \le M\tau_i \le (M - 1 - L\beta_i)\, x + (1 + L\beta_i)\, y\right\}, \tag{3.11}$$

where $M \ge 2$, L are integers, and β_i are parameters with $0 < \beta_i < (M-1)/L$, $i = 1, \dots, n$. Let the quantities $\tau_1, \dots, \tau_n$ again satisfy (3.6). The abstract staircase algorithm now reduces to the following algorithm.

ALGORITHM 8. Let $A_0 = (M\tau_1 - (1 + L\beta_1)\,\tau_2)/(M - 1 - L\beta_1)$, $B_0 = \tau_1$, and for $i = 1, \dots, n$

$$\begin{aligned} A_i &= \max\left\{\tau_i, \frac{M\tau_i - (M - 1 - L\beta_i)\, B_{i-1}}{1 + L\beta_i}\right\}, \\ B_i &= \frac{M\tau_i - (M - L\beta_i)\, A_{i-1}}{L\beta_i}. \end{aligned} \tag{3.12}$$

THEOREM 9. *Problem (3.1),(3.11) is solvable if and only if*

$$A_i \le B_i, \quad i = 0, 1, \dots, n, \tag{3.13}$$

for the quantities (3.12). The solutions are computed by selecting $p_n \in [A_n, B_n]$ and for $i = n$, $n-1, \dots, 1$

$$p_{i-1} \in \left[\max\left\{A_{i-1}, \frac{M\tau_i - (1 + L\beta_i)\, p_i}{M - 1 - L\beta_i}\right\}, \ \min\left\{B_{i-1}, \frac{M\tau_i - L\beta_i p_i}{M - L\beta_i}\right\}\right]. \tag{3.14}$$

Again, the parameters $\beta_1, \dots, \beta_n$ can be chosen in such a way that the solvability test (3.13) is fulfilled.

PROPOSITION 10. *If (3.6) is valid, then system (3.1),(3.11) is solvable if*

$$\begin{aligned} \beta_1, \beta_n &\in \left(0, \frac{M-1}{L}\right), \\ \beta_i &\in \left(0, \min\left\{\frac{M(\tau_i - \tau_{i-1})}{L(\tau_{i+1} - \tau_{i-1})}, \frac{M-1}{L}\right\}\right), \qquad i = 2, \dots, n-1. \end{aligned} \tag{3.15}$$

PROOF. For $i = 0$, we obtain $A_0 = \tau_1 - (1 + L\beta_1)(\tau_2 - \tau_1)/(M - 1 - L\beta_1) < \tau_1 = B_0$, and for $i = 1$, it follows $A_1 = \tau_1 < \tau_2 \le \tau_2 + M(\tau_2 - \tau_1)/(L\beta_1(M - 1 - L\beta_1)) = B_1$. Further, if $A_{i-1} = \tau_{i-1} < \tau_i \le B_{i-1}$ is assumed for $i \in \{2, \dots, n-1\}$, we get $A_i = \tau_i < \tau_{i+1} \le \tau_{i-1} + M(\tau_i - \tau_{i-1})/(L\beta_i) = B_i$, provided (3.15) is taken into account. For $i = n$, we find $A_n = \tau_n \le \tau_n + (M - L\beta_n)(\tau_n - A_{n-1})/(L\beta_n) = B_n$. Thus, the criterion (3.13) holds. ∎

Summarizing the above considerations, a solution of a system (3.1),(3.5) can be computed as follows. At first, determine the parameters $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ according to (3.10). Often it is advantageous if these parameters are taken as small as possible. Then apply the recursive formulae (3.7),(3.9) to compute a solution $(p_0, p_1, \dots, p_n)$. In general, one should prefer the midpoints of the intervals (3.9). If, on the other hand, a system (3.1),(3.11) is given, substitute (3.15) for (3.10) and (3.12),(3.14) for (3.7),(3.9). In (3.14) and (3.15), it is recommended to take the midpoints of the intervals.

4. CONVEX C^1 INTERPOLATION WITH SOME TYPES OF NONLINEAR SPLINES

Let $\Delta : a = x_0 < x_1 < \cdots < x_n = b$ be a grid on the interval $I = [a, b]$. We assume that the splines s considered here are defined on Δ, and that $x_0, \ldots, x_n$ are the nodes for interpolation. Thus, the interpolation condition (2.5) reads

$$s(x_i) = y_i, \qquad i = 0, \ldots, n. \tag{4.1}$$

For convex C^1 interpolation, we can refer to some kinds of nonlinear splines. The nonlinearity parameters now are determined by the staircase algorithm. In this way, only the given function values $y_0, \ldots, y_n$ are necessary.

In what follows, we denote by $u = (x - x_{i-1})/h_i$ and $v = (x_i - x)/h_i$ with $h_i = x_i - x_{i-1}$ the barycentric coordinates on the subinterval $I_i = [x_{i-1}, x_i]$, while the slopes are abbreviated by $\tau_i = (y_i - y_{i-1})/h_i$.

4.1. Rational Splines [10]

These splines are defined as follows. With rationality parameters $\alpha_1 \geq 0, \ldots, \alpha_n \geq 0$, set

$$s(x) = y_{i-1}v + y_i u + \frac{(p_{i-1} - \tau_i)\, v + (\tau_i - p_i)\, u}{1 + \alpha_i uv}\, h_i uv, \qquad x \in I_i, \quad i = 1, \ldots, n. \tag{4.2}$$

We immediately find that s is always from C^1. The interpolation condition (4.1) is satisfied, and the parameters $p_0, \ldots, p_n$ are the unknown first-order derivatives in the nodes; i.e.,

$$p_i = s'(x_i), \qquad i = 0, \ldots, n. \tag{4.3}$$

Further, after some computations, we get that

$$(2 + \alpha_i)\, p_{i-1} + p_i \leq (3 + \alpha_i)\, \tau_i \leq p_{i-1} + (2 + \alpha_i)\, p_i, \qquad i = 1, \ldots, n \tag{4.4}$$

is necessary and sufficient for convexity; see, e.g., [1]. Thus, we are led to a system (3.1),(3.5), and the results from Chapter 3 yield the following proposition.

PROPOSITION 11. *For data sets in strictly convex position, convex C^1 interpolation with rational splines (4.2) is always possible if the rationality parameters $\alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n$ are chosen according to (3.10). In this case, convex spline interpolants are given by (3.7),(3.9),(4.2).*

4.2. Rational Splines [9]

It is convenient to define these splines by

$$s(x) = y_{i-1}v + y_i u + a_i \left(\frac{v^3}{1 + \alpha_i u} - v \right) + b_i \left(\frac{u^3}{1 + \beta_i v} - u \right), \qquad x \in I_i, \quad i = 1, \ldots, n, \tag{4.5}$$

with rationality parameters $\alpha_1 \geq 0$, $\beta_1 \geq 0, \ldots, \alpha_n \geq 0$, $\beta_n \geq 0$. The splines obviously interpolate in the sense of (4.1). They are from C^1 if, using the parameters (4.3),

$$\begin{aligned} a_i &= h_i \frac{(3 + \beta_i)\tau_i - (2 + \beta_i)\, p_{i-1} - p_i}{\alpha_i\beta_i + 2\alpha_i + 2\beta_i + 3}, \\ b_i &= h_i \frac{p_{i-1} + (2 + \alpha_i)\, p_i - (3 + \alpha_i)\, \tau_i}{\alpha_i\beta_i + 2\alpha_i + 2\beta_i + 3}, \qquad i = 1, \ldots, n. \end{aligned} \tag{4.6}$$

Convexity is assured if and only if $a_i \geq 0$, $b_i \geq 0$, $i = 1, \ldots, n$, i.e., if

$$(2 + \beta_i)\, p_{i-1} + p_i \leq (3 + \beta_i)\, \tau_i, \quad (3 + \alpha_i)\, \tau_i \leq p_{i-1} + (2 + \alpha_i)\, p_i, \qquad i = 1, \ldots, n. \tag{4.7}$$

Hence, again a system (3.1),(3.5) arises, and Proposition 11 analogously holds for the rational splines (4.5),(4.6).

4.3. Lacunary Splines [8]

By means of lacunarity parameters $k_i \geq 3$, $\ell_i \geq 3$, these splines are given by

$$s(x) = y_{i-1}v + y_i u + a_i\left(v^{k_i} - v\right) + b_i\left(u^{\ell_i} - u\right), \qquad x \in I_i, \quad i = 1, \dots, n. \tag{4.8}$$

While the interpolation condition (4.1) is always satisfied, we obtain C^1 continuity if, using the parameters (4.3),

$$a_i = h_i \frac{\ell_i\tau_i - (\ell_i - 1)p_{i-1} - p_i}{k_i\ell_i - k_i - \ell_i}, \quad b_i = h_i \frac{p_{i-1} + (k_i - 1)p_i - k_i\tau_i}{k_i\ell_i - k_i - \ell_i}, \qquad i = 1, \dots, n. \tag{4.9}$$

Here, the convexity is easily seen to be equivalent to $a_i \geq 0$, $b_i \geq 0$, $i = 1, \dots, n$, i.e., to

$$(\ell_i - 1)p_{i-1} + p_i \leq \ell_i\tau_i, \quad k_i\tau_i \leq p_{i-1} + (k_i - 1)p_i, \qquad i = 1, \dots, n. \tag{4.10}$$

Again, we are led to a system of the type (3.1),(3.5), now with $\alpha_i = k_i - 3$, $\beta_i = \ell_i - 3$. Thus, Proposition 11 is analogously valid for the lacunary splines (4.8),(4.9).

We remark that it seems to be impossible to extend these results to the convex interpolation of C^2 continuity when using the above nonlinear splines on the grid Δ.

5. CONVEX C^1, C^2, AND C^3 INTERPOLATION WITH SPLINES ON REFINED GRIDS

Another type of nonlinear splines suitable for convex interpolation is splines on grids with additional variable nodes. In this way, it is possible to preserve convexity under higher continuity than C^1.

5.1. Quadratic C^1 Splines on Refined Grids

In [13], quadratic splines are considered on grids $\tilde{\Delta}$ which originate by adding one node

$$\xi_i = \beta_i x_{i-1} + \alpha_i x_i, \qquad \alpha_i > 0, \quad \beta_i > 0, \quad \alpha_i + \beta_i = 1 \tag{5.1}$$

in each subinterval I_i, $i = 1, \dots, n$. Let $u_1 = (x - x_{i-1})/(\alpha_i h_i)$, $v_1 = (\xi_i - x)/(\alpha_i h_i)$ and $u_2 = (x - \xi_i)/(\beta_i h_i)$, $v_2 = (x_i - x)/(\beta_i h_i)$ be the barycentric coordinates on the subintervals $[x_{i-1}, \xi_i]$ and $[\xi_i, x_i]$, respectively. Then, for $i = 1, \dots, n$, we define

$$\begin{aligned} s(x) &= y_{i-1}v_1^2 + \eta_i u_1^2 + (2y_{i-1} + \alpha_i h_i p_{i-1})\, u_1 v_1, \qquad && x \in [x_{i-1}, \xi_i], \\ s(x) &= \eta_i v_2^2 + y_i u_2^2 + (2y_i - \beta_i h_i p_i)\, u_2 v_2, && x \in [\xi_i, x_i]. \end{aligned} \tag{5.2}$$

These splines satisfy the interpolation condition (4.1), and the parameters $p_0, \dots, p_n$ again have the meaning (4.3). The C^1 property is valid if we set

$$\eta_i = s(\xi_i) = \beta_i y_{i-1} + \alpha_i y_i + \frac{\alpha_i\beta_i h_i(p_{i-1} - p_i)}{2}, \qquad i = 1, \dots, n. \tag{5.3}$$

Further, the splines s are immediately seen to be convex if and only if

$$(2 - \beta_i)p_{i-1} + \beta_i p_i \leq 2\tau_i \leq (1 - \beta_i)p_{i-1} + (1 + \beta_i)p_i, \qquad i = 1, \dots, n. \tag{5.4}$$

Thus, we are led to a problem (3.1),(3.11) with $M = 2$, $L = 1$. Applying the results of Section 3, we get the following proposition.

PROPOSITION 12. *For data sets in strictly convex position, convex C^1 interpolation with quadratic splines on refined grids $\tilde{\Delta}$ is always possible provided the ratios $\beta_1, \dots, \beta_n$ are determined by (3.15). Then, convex spline interpolants are obtained via (3.12),(3.14),(5.3),(5.2).*

5.2. Cubic C^2 Splines on Twofold Refined Grids

In this section, we show how convex interpolants of C^2 continuity can be determined. To this end, we follow [15] where cubic splines on twofold refined grids are used. Analogous results are possible with quartic C^2 splines on grids with only one additional node in each subinterval [20]. Another construction was recently described in [14].

The refinement $\tilde{\Delta}$ of the original grid Δ arises by adding two nodes

$$\xi_{i0} = (\beta_i + \gamma_i)\, x_{i-1} + \alpha_i x_i, \qquad \xi_{i1} = \gamma_i x_{i-1} + (\alpha_i + \beta_i)\, x_i \tag{5.5}$$

in each subinterval I_i, with ratios $\alpha_i > 0$, $\beta_i > 0$, $\gamma_i > 0$, and $\alpha_i+\beta_i+\gamma_i = 1$, $i = 1,\dots,n$. This implies $\xi_{i0}-x_{i-1} = \alpha_i h_i$, $\xi_{i1}-\xi_{i0} = \beta_i h_i$, and $x_i-\xi_{i1} = \gamma_i h_i$. On the subintervals $[x_{i-1},\xi_{i0}]$, $[\xi_{i0},\xi_{i1}]$ and $[\xi_{i1},x_i]$, we introduce barycentric coordinates by $u_1 = (x-x_{i-1})/(\alpha_i h_i)$, $v_1 = (\xi_{i0}-x)/(\alpha_i h_i)$, $u_2 = (x-\xi_{i0})/(\beta_i h_i)$, $v_2 = (\xi_{i1}-x)/(\beta_i h_i)$, and $u_3 = (x-\xi_{i1})/(\gamma_i h_i)$, $v_3 = (x_i - x)/(\gamma_i h_i)$, respectively. Then, we can define cubic splines s on $\tilde{\Delta}$ by

$$\begin{aligned} s(x) &= y_{i-1}v_1^3 + \eta_{i0}u_1^3 + (a_i u_1 + b_i v_1)\, u_1 v_1, && x \in [x_{i-1},\xi_{i0}],\\ s(x) &= \eta_{i0}v_2^3 + \eta_{i1}u_2^3 + (c_i u_2 + d_i v_2)\, u_2 v_2, && x \in [\xi_{i0},\xi_{i1}],\\ s(x) &= \eta_{i1}v_3^3 + y_i u_3^3 + (e_i u_3 + f_i v_3)\, u_3 v_3, && x \in [\xi_{i1},x_i], \end{aligned} \tag{5.6}$$

$i = 1,\dots,n$. Obviously, these splines are continuous and satisfy the interpolation condition (4.1). In the case

$$\begin{aligned} b_i &= 3y_{i-1} + \alpha_i h_i p_{i-1}, & e_i &= 3y_i - \gamma_i h_i p_i,\\ \eta_{i0} &= \frac{\alpha_i d_i + \beta_i a_i}{3(\alpha_i+\beta_i)}, & \eta_{i1} &= \frac{\beta_i f_i + \gamma_i c_i}{3(\beta_i+\gamma_i)}, \qquad i = 1,\dots,n, \end{aligned} \tag{5.7}$$

the splines easily turn out to be in C^1, and the parameters $p_0,\dots,p_n$ again are the first derivatives in the nodes; i.e., (4.3) holds. Further, if we set

$$\begin{aligned} a_i &= 3y_{i-1} + 2\alpha_i h_i p_{i-1} + \frac{\alpha_i^2 h_i^2}{2}\, P_{i-1}, \quad f_i = 3y_i - 2\gamma_i h_i p_i + \frac{\gamma_i^2 h_i^2}{2}\, P_i,\\ c_i &= \frac{\gamma_i}{\alpha_i}\left((1-\gamma_i)\, a_i - \beta_i b_i\right) + \frac{1-\gamma_i}{\gamma_i}\left((1-\alpha_i)\, f_i - \beta_i e_i\right),\\ d_i &= \frac{1-\alpha_i}{\alpha_i}\left((1-\gamma_i)\, a_i - \beta_i b_i\right) + \frac{\alpha_i}{\gamma_i}\left((1-\alpha_i)\, f_i - \beta_i e_i\right), \qquad i = 1,\dots,n, \end{aligned} \tag{5.8}$$

the C^2 property is directly verified when the equalities

$$\begin{aligned} (\alpha_i+\beta_i)\beta_i a_i - \beta_i^2 b_i + \alpha_i^2 c_i - \alpha_i(\alpha_i+\beta_i) d_i = 0,\\ (\beta_i+\gamma_i)\gamma_i c_i - \gamma_i^2 d_i + \beta_i^2 e_i - \beta_i(\beta_i+\gamma_i) f_i = 0 \end{aligned}$$

(being equivalent with the latter two of (5.8)) are used. The parameters $P_0,\dots,P_n$ are the second-order derivatives in the nodes

$$P_i = s''(x_i), \qquad i = 0,\dots,n. \tag{5.9}$$

The convexity of the cubic splines (5.6)–(5.8) is obviously equivalent to $s''(x_i) \ge 0$, $i = 0,\dots,n$, $s''(\xi_{i0}) \ge 0$, $s''(\xi_{i1}) \ge 0$, $i = 1,\dots,n$. Hence, we get the necessary and sufficient convexity conditions

$$\begin{aligned} & P_i \ge 0, \qquad i = 0,\dots,n,\\ & 3\tau_i - (1+2\alpha_i+\beta_i)\, p_{i-1} - (2-2\alpha_i-\beta_i)\, p_i\\ & \quad - \frac{\alpha_i(2-\gamma_i)}{2}\, h_i P_{i-1} + \frac{(1-\alpha_i)\gamma_i}{2}\, h_i P_i \ge 0,\\ & -3\tau_i + (2-2\gamma_i-\beta_i)\, p_{i-1} + (1+2\gamma_i+\beta_i)\, p_i\\ & \quad + \frac{(1-\gamma_i)\alpha_i}{2}\, h_i P_{i-1} - \frac{\gamma_i(2-\alpha_i)}{2}\, h_i P_i \ge 0, \qquad i = 1,\dots,n. \end{aligned} \tag{5.10}$$

Now, to prove the existence of convex C^2 interpolants, the choice

$$P_i = 0, \qquad i = 0, \dots, n \tag{5.11}$$

is admissible. Further, if we assume $\beta_i = \gamma_i$, $i = 1 \dots, n$, the system (5.10) reduces to

$$(3 - 3\beta_i)\, p_{i-1} + 3\beta_i p_i \le 3\tau_i \le (2 - 3\beta_i)\, p_{i-1} + (1 + 3\beta_i)\, p_i, \qquad i = 1, \dots, n. \tag{5.12}$$

Hence, we obtain a problem (3.1),(3.11) with $M = 3$, $L = 3$. The considerations of Section 3 now lead to the following proposition.

PROPOSITION 13. *Let the data set be in strictly convex position. Then, convex C^2 interpolation with cubic splines on twofold refined grids is always successful if the ratios $\beta_1 = \gamma_1, \dots, \beta_n = \gamma_n \in (0, 1/2)$ are computed by (3.15), and $\alpha_1 = 1 - 2\beta_1, \dots, \alpha_n = 1 - 2\beta_n$. Convex interpolants can then be determined by using the formulae (3.12),(3.14),(5.11),(5.6)–(5.8).*

Note that this smooth result is not possible if $\alpha_i = \gamma_i$, $i = 1, \dots, n$, is set [21].

5.3. Quartic C^3 Splines on Threefold Refined Grids

It is even possible to retain convexity under C^3 continuity [18]. This can be achieved using quartic splines on refined grids with three additional nodes

$$\begin{aligned}
\xi_{i0} &= (\beta_i + \gamma_i + \delta_i)\, x_{i-1} + \alpha_i x_i, \\
\xi_{i1} &= (\gamma_i + \delta_i)\, x_{i-1} + (\alpha_i + \beta_i)\, x_i, \\
\xi_{i2} &= \delta_i x_{i-1} + (\alpha_i + \beta_i + \gamma_i)\, x_i
\end{aligned} \tag{5.13}$$

in each subinterval I_i, $i = 1, \dots, n$. The ratios α_i, β_i, γ_i, δ_i are assumed to be positive, and $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$. Moreover, for simplification, we set $\beta_i = \gamma_i = \delta_i$, $i = 1, \dots, n$. As before, we use barycentric coordinates u_1, v_1, u_2, v_2, u_3, v_3, and u_4, v_4 in order to describe the splines s on the subintervals $[x_{i-1}, \xi_{i0}]$, $[\xi_{i0}, \xi_{i1}]$, $[\xi_{i1}, \xi_{i2}]$, and $[\xi_{i2}, x_i]$, respectively,

$$\begin{aligned}
s(x) &= y_{i-1} v_1^4 + \eta_{i0} u_1^4 + \left(a_i u_1^2 + b_i u_1 v_1 + c_i v_1^2\right) u_1 v_1, && x \in [x_{i-1}, \xi_{i0}], \\
s(x) &= \eta_{i0} v_2^4 + \eta_{i1} u_2^4 + \left(d_i u_2^2 + e_i u_2 v_2 + f_i v_2^2\right) u_2 v_2, && x \in [\xi_{i0}, \xi_{i1}], \\
s(x) &= \eta_{i1} v_3^4 + \eta_{i2} u_3^4 + \left(g_i u_3^2 + j_i u_3 v_3 + k_i v_3^2\right) u_3 v_3, && x \in [\xi_{i1}, \xi_{i2}], \\
s(x) &= \eta_{i2} v_4^4 + y_i u_4^4 + \left(\ell_i u_4^2 + m_i u_4 v_4 + n_i v_4^2\right) u_4 v_4, && x \in [\xi_{i2}, x_i],
\end{aligned} \tag{5.14}$$

$i = 1, \dots, n$. These splines are in C^3 if we set

$$\begin{aligned}
a_i &= 4y_{i-1} + 3\alpha_i h_i p_{i-1} + \alpha_i^2 h_i^2 P_{i-1} + \frac{\alpha_i^3 h_i^3 G_{i-1}}{6}, \\
b_i &= 6y_{i-1} + 3\alpha_i h_i p_{i-1} + \frac{\alpha_i^2 h_i^2 P_{i-1}}{2}, \\
c_i &= 4y_{i-1} + \alpha_i h_i p_{i-1}, \\
\ell_i &= 4y_i - \beta_i h_i p_i, \\
m_i &= 6y_i - 3\beta_i h_i p_i + \frac{\beta_i^2 h_i^2 P_i}{2}, \\
n_i &= 4y_i - 3\beta_i h_i p_i + \beta_i^2 h_i^2 P_i - \frac{\beta_i^3 h_i^3 G_i}{6},
\end{aligned} \tag{5.15}$$

and

$$f_i = \frac{\beta_i\left(3(1-2\beta_i)(5-3\beta_i)\,a_i - 2\beta_i(11-15\beta_i)\,b_i + 18\beta_i^2 c_i\right)}{3\alpha_i(1-\beta_i)} + \frac{\alpha_i^2(6\ell_i - 14m_i + 18n_i)}{3(1-\beta_i)},$$
$$g_i = \frac{\beta_i\left(3(1-2\beta_i)(1-\beta_i)\,a_i - 2\beta_i(3-5\beta_i)\,b_i + 6\beta_i^2 c_i\right)}{9\alpha_i^2} + \frac{2(3(1-\beta_i)\,\ell_i - (9-7\beta_i)\,m_i + 3(5-3\beta_i)\,n_i)}{9},$$
$$d_i = \frac{-2\beta_i(1-2\beta_i)^2 a_i + 2\beta_i^2(1-2\beta_i)\,b_i - \beta_i^3 c_i + \alpha_i(1-2\beta_i)^2 f_i}{\alpha_i^3}, \tag{5.16}$$
$$e_i = \frac{-3\beta_i(1-2\beta_i)\,a_i + 2\beta_i^2 b_i + 3\alpha_i(1-2\beta_i)\,f_i}{2\alpha_i^3},$$
$$j_i = m_i - 3n_i + 3g_i, \quad k_i = -\ell_i + 4m_i - 8n_i + 4g_i,$$

and

$$\eta_{i0} = \frac{\beta_i a_i + \alpha_i f_i}{4(\alpha_i+\beta_i)}, \quad \eta_{i1} = \frac{d_i + k_i}{8}, \quad \eta_{i2} = \frac{g_i + n_i}{8}. \tag{5.17}$$

We verify the interpolation condition (4.1), and the parameters p_i, P_i, and G_i are the derivatives (4.3),(5.9), and

$$G_i = s'''(x_i), \qquad i = 0,\dots,n. \tag{5.18}$$

Under the assumption

$$P_i = G_i = 0, \qquad i = 0,\dots,n, \tag{5.19}$$

a condition sufficient for convexity is derived to read

$$(4-6\beta_i)\,p_{i-1} + 6\beta_i p_i \le 4\tau_i \le (3-6\beta_i)\,p_{i-1} + (1+6\beta_i)\,p_i, \qquad i = 1,\dots,n. \tag{5.20}$$

This is a system (3.1),(3.11) with $M = 4$, $L = 6$, treated in Section 3. Thus, we find the following proposition.

PROPOSITION 14. *For strictly convex data sets, convex C^3 interpolation with quartic splines on threefold refined grids is always possible if the ratios $\beta_i = \gamma_i = \delta_i \in (0, 1/3)$ are chosen according to (3.15), and $\alpha_i = 1 - 3\beta_i$, $i = 1,\dots,n$. For computing convex interpolants, we can then use the formulae (3.12),(3.14),(5.19),(5.14)–(5.17).*

6. CONCLUDING REMARKS

There are further methods for finding suitable nonlinearity parameters. One of these procedures is as follows (compare with [1,22]). At first determine values $p_0, p_1, \dots, p_n$ such that

$$p_0 < \tau_1 < p_1 < \cdots < p_{n-1} < \tau_n < p_n \tag{6.1}$$

holds. In view of (3.6), this is always possible. One particular choice is

$$p_i = \frac{h_i\tau_{i+1} + h_{i+1}\tau_i}{h_i + h_{i+1}}, \qquad i = 1,\dots,n-1, \quad p_0 = 2\tau_1 - p_1, \quad p_n = 2\tau_n - p_{n-1}. \tag{6.2}$$

Then solve the separable inequalities (3.1),(3.5) for α_i, β_i or (3.1),(3.11) for β_i, $i = 1,\dots,n$. In the first case, the result is

$$\alpha_i \ge \max\left\{0, \frac{3\tau_i - p_{i-1} - 2p_i}{p_i - \tau_i}\right\}, \qquad \beta_i \ge \max\left\{0, \frac{2p_{i-1} + p_i - 3\tau_i}{\tau_i - p_{i-1}}\right\}, \tag{6.3}$$

and in the second case, we have

$$\beta_i \in \left(\max\left\{0, \frac{M(\tau_i - p_i) + p_{i-1} - p_i}{L(p_i - p_{i-1})}\right\}, \min\left\{\frac{M-1}{L}, \frac{M(\tau_i - p_i)}{L(p_i - p_{i-1})}\right\}\right). \tag{6.4}$$

In the staircase algorithm (3.2), the recursion runs forward from $i = 1$ to $i = n$. It is possible to organize this algorithm also in a backward form, or in mixed forms. In these ways, algorithms arise being somewhat different from the algorithms (3.7),(3.9) and (3.12),(3.14). From a numerical point of view, it seems not to be essential which form is used.

In the next remark, we assume that the nonlinearity parameters are fixed if once determined by one of the methods described above. Then the formulae (3.9) or (3.14) can be applied in order to choose the first derivatives $p_0, \dots, p_n$, while the higher order derivatives, if needed, can be taken equal to zero. However, visually more pleasing interpolants in general are obtained by varying the derivatives $p_0, P_0, G_0, \dots, p_n, P_n, G_n$ within the convexity constraints. In our computational tests, we preferred an automatic choice based on the minimization of an objective function like the Holladay functional

$$\int_a^b s''(x)^2 \, dx \tag{6.5}$$

(minimization of the mean curvature). The constraints are the convexity conditions, for instance (5.10), if cubic C^2 splines on twofold refined grids are applied.

One of the data sets used for test purposes is

$$x_i = i, \quad i = 0, \dots, n, \qquad y_i = \frac{i(i+1)}{2}, \quad i = 0, \dots, n-1, \qquad y_n = \frac{(n-1)(n+2)}{2} + M, \tag{6.6}$$

with $M \geq 0$ (Example 1). When considering the spline curves in Figures 1 and 2, it is difficult to observe differences. However, the curves of the second order derivatives show the usefulness of the optimization approach.

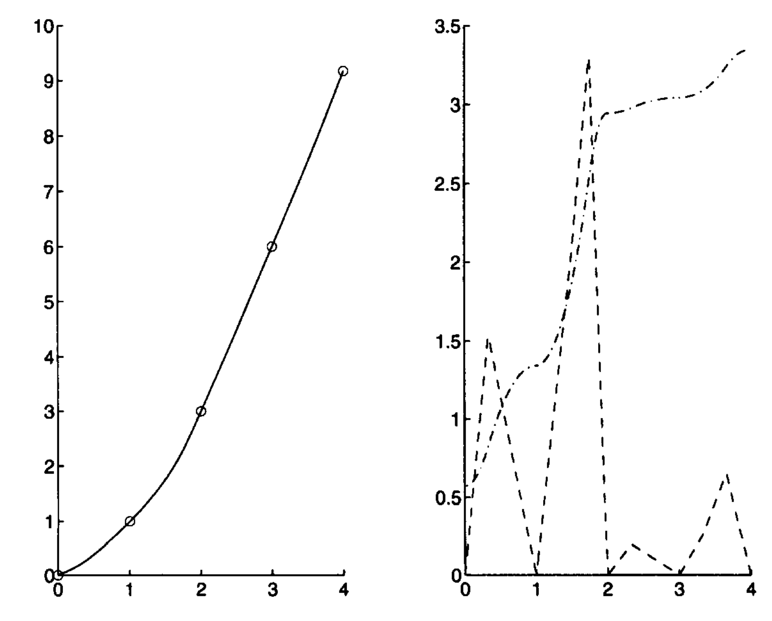

Figure 1. Example 1 with $M = 0.175$; p_i according to (3.9), $P_i = 0$ ($i = 0, \dots, 4$); spline (solid), first derivative (dashdot), second derivative (dashed).

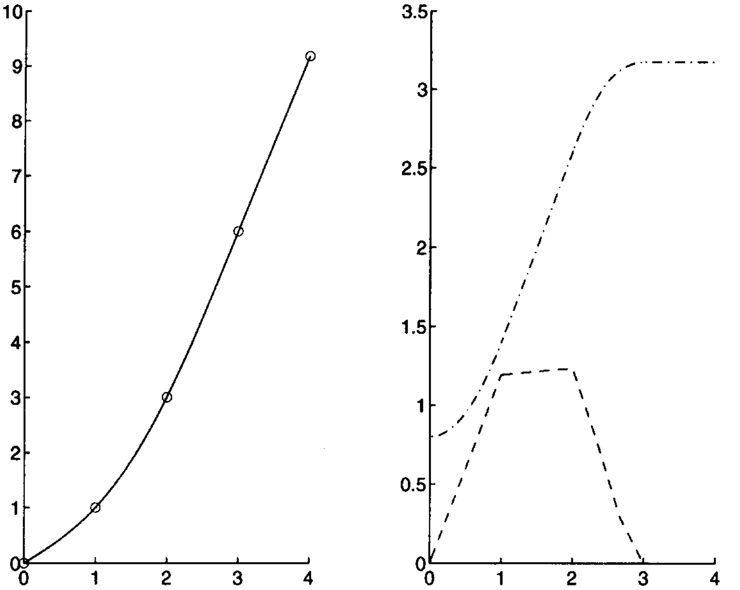

Figure 2. Example 1 with $M = 0.175$; p_i, P_i $(i = 0, \dots, 4)$ by curvature minimization according to (6.5); spline (solid), first derivative (dashdot), second derivative (dashed).

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