A Zener–Stroh crack interacting with a coated inclusion with generalized Irwin plastic zone correction

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Abstract

The elastic–plastic fracture behavior of a Zener–Stroh crack interacting with a coated inclusion in composite materials has been investigated with crack tip plastic zone corrections. With the distributed dislocation method, the crack problem is formulated into a set of singular integral equations which are solved numerically. The plastic zone sizes at the both crack tips are determined by a generalized Irwin model where Von Mises stress yielding criterion is used. The stress intensity factor (SIF), the plastic zone size (PZS), the crack tip opening displacement (CTOD) and the effective stress intensity factor have been evaluated. In the numerical examples, the influence of the inclusion shear modulus, the coating-layer thickness and shear modulus, as well as the distance between the crack and inclusion, on the SIF, the PZS and the CTOD are discussed in detail. Numerical examples show that increasing the shear modulus or the thickness of the coating phase, the influence of the inclusion on the normalized SIF and the normalized PZS will be shielded.

1. Introduction

Fiber-reinforced composite has been increasingly used in various engineering structures as it can exploit beneficial characteristics of individual constituents to achieve desired overall properties. However, due to the mismatches of material properties between the matrix and the embedded phase (the inclusion), high stress concentration at the interface between the two dissimilar materials is often induced, which causes interface cracking and fiber–matrix de-bonding. Failure mechanisms in composites can be very complex. A lot of experiments have been conducted to study the failure model in composites. Poe (1989), Johnson (1989) and Harmon and Saff (1989) modeled the failures in the form of crack while Whitehouse and Clyne (1993) studied the failure from the view of cavitation. The coalescence of a series of cavities in composites can be simulated as a crack. Such defects directly influences the overall mechanical properties and the service life of the fiber-reinforced composites. One of the efficient approaches to avoid the properties mismatch is to add an intermediate layer of a third material with appropriate geometry and properties between the matrix and inclusion. In other words, the inclusion is covered by a coating layer. This coating phase effectively reduces the materials mismatches, remove the stress concentration and increase the interface bonding. On the other hand, due to the coating layer, the third material phase is introduced in the composite structure. The mechanical behaviors, particularly the fracture behaviors of the coated inclusion–matrix composites will be changed, and worth to be investigated.

Historically, many research work has been done on the coated inclusion problems. To name a few, Walpole (1978) investigated the problem of a coated inclusion in an elastic medium and showed how to take account of the pronounced influence. Thocaris and Demakos (1988) proved that the fracture toughness of hard plates, reinforced with brittle inclusions, can be altered by introducing an intermediate thin layer made of a deformable phase in these composites. Benveniste et al. (1989) presented a micromechanics model for the prediction of stress fields in coated fiber composites with the “average stress in the matrix” concept. A penny-shaped crack above the pole of a spherical inhomogeneity in 3D elastic solid was studied by Xiao (1997). Xian and Chen (2001a, 2001b, 2001c) investigated the interacting problem between a coated inclusion and a crack by using the distributed dislocation method. A parametrical study of the interaction between a propagating edge-crack and uncoated/coated inclusion was done by Knight et al. (2002) through using the Boundary Element (BE) technique. Dong et al. (2003) studied the interaction between a coated inclusion and a crack in an infinite isotropic elastic medium using a
domain integral equation method. Fang et al. (2005) investigated the elastic interaction problem between a screw dislocation and a circular coated inclusion. Luo and Gao (2011) presented an effective method for the plane problem of a coated inclusion with arbitrary shape embedded in an isotropic matrix. With the Green’s function technique and the concept of the interior and exterior-point Eshelby tensors for an ellipsoidal inclusion, Bonfoh et al. (2012) presented a new micromechanical model for the solution of an ellipsoidal coated inclusion embedded in an infinite homogeneous medium.

However, all the above mentioned research work is mainly based on the linear elastic fracture mechanics, and is far from enough to meet the need of failure analysis and prevention for engineering structures made by ductile materials. Particularly for metal-matrix composites which can undergo large plastic deformation around the crack tips. As an extension of elastic investigation, plastic zone corrections for such crack problems should be considered. In our current work, we look into the elastic-plastic behavior of the Zener–Stroh crack problem by studying the plastic deformation ahead the crack tips.

Zener–Stroh crack, different to conventional Griffith crack, is another type of crack which was proposed initially by Zener (1948). It is of particular interest in solids when they are physically small; so small that these cracks, as well as Griffith cracks, should no longer be considered, for the simplest picture, to be imbedded in a continuum elastic solid (Weertman, 1986). The Zener–Stroh crack and the famous Griffith crack form a complementary pair. Actually, based on the physical mechanism of micro crack initiation in fiber-reinforced composites, cracks initiated are usually in the form of Zener–Stroh crack at the early stage. Therefore, it is important to understand the Zener–Stroh crack problems in composite materials. In this model, a pileup of edge dislocations that are stopped at an obstacle, such as a grain boundary (Fig. 1(a)), could coalesce into a crack nucleus. An experimental SEM photo of a crack initiated by this physical mechanism is shown in Fig. 2. Zener’s research work was further developed by Stroh (1954, 1955), who analyzed the amount of dislocations needed for such nucleation in the absence/presence of a slip plane. Fan and Xiao (1997) investigated the Zener–Stroh crack near an interface. Besides Zener’s mechanism of micro crack initiation, there are some other variants. One was presented by Cottrell (1958), where piled-up dislocations on two intersecting slip planes can coalesce into a micro crack as shown in Fig. 1(b). Another variant proposed by Kikuchi et al. (1981) is shown in Fig. 1(c). In this model, dislocations of one sign move away from the region, leaving stationary dislocations of the opposite sign behind to form a crack near the particle.

Different to the famous Griffith crack, the physical parameters that are symmetric for the Griffith crack are anti-symmetric for the Zener–Stroh crack and vice versa. The Griffith crack dislocation distribution along the crack plane is anti-symmetric and results to a symmetric crack plane traction stress. The Zener–Stroh crack has an anti-symmetric crack plane traction stress which arises from a symmetric crack plane dislocation distribution. To a Zener–Stroh crack, the total sum of the Burgers vectors of the dislocations does not equal zero according to the displacement loading mechanism. The crack tip where the dislocation enters the crack is a blunt tip and the other tip is a sharp tip (shown in Fig. 3(a)), the crack propagation always starts from the sharp tip.

To evaluate the plastic deformation ahead the crack tips of Zener–Stroh crack, the generalized Irwin model is introduced to determine the PZS, where the Von Mises stress yielding criterion is satisfied in the plastic zone. Comparing to the widely used Dugdale model which assumes closure stress within the plastic zone area to evaluate the plastic zone size, the Irwin model is a much convenient approach for matrix cracking problems (not applicable for interface problem due to the oscillatory singularities) by finding the relationship among the stress intensity factor, plastic zone size and CTOD, avoiding the huge calculation for iterative process. By this model, even the problem is under mixed loading condition and the stress field ahead each crack tips are different due to the
effect of nearby inclusion, we can still get quick analysis about the elastic–plastic fracture behavior of the crack. Based on this, the influence of the coating phase properties, crack-inclusion distance and other constants on the effective stress intensity factor, PZS and CTOD is emphasized. By adjusting the various parameters involved, some possible ways of enhancing the fracture toughness of the coated-inclusion composites are proposed.

2. The generalized Irwin model

2.1. Plastic zone sizes

For small scale yielding, to moderate the plastic zone around crack tips in metals, simple corrections to linear elastic fracture mechanics solutions are available by using Irwin model. It was proposed by Irwin (1968), where the plastic zone size \( r_y \) is determined by setting the crack tip stress \( r_{yy}(r, \theta = 0) \) in mode I for instance to the yield stress \( r_{ys} \). This results to the following plastic zone size:

\[
r_y = \frac{1}{2\pi} \left( \frac{K_I}{r_{ys}} \right)^2.
\]

While in open literature, the Irwin model is mostly used to deal with crack problems in homogeneous materials. For cracks in multi-phase materials (composite materials), a generalized Irwin model should be developed. The generalized Irwin approach for the current problem is shown in Fig. 3. The stress fields ahead of the crack tips along the crack line are given by Anderson (2005):

\[
\sigma_{yy}^{(m)} = \sigma_{yy}^{(m)} = \frac{K_{I,yy}^{(m)}}{\sqrt{2\pi r_y^{(m)}}}, \quad \sigma_{xy}^{(m)} = \frac{K_{II,xy}^{(m)}}{\sqrt{2\pi r_y^{(m)}}},
\]

\[
\sigma_{zz}^{(m)} = \frac{2v_3 K_{III,zz}^{(m)}}{\sqrt{2\pi r_y^{(m)}}} \quad m = 1, 2.
\]

Here, \((r_1)\) and \((r_2)\) represent the left and right crack tip respectively. \(K_{I,yy}^{(m)}\) and \(K_{II,xy}^{(m)}\) are mode I and mode II stress intensity factor near \((r_m)\), \(v_3\) is the Poisson’s ratio of phase 3 (the matrix). Plane strain condition is considered in this paper. To estimate the plastic zone size around the crack tips, the Von Mises criterion is used, in which yielding occurs when the effective stress reaches the yield stress \( r_e = r_{ys} \).

\[
\sigma_e^{(m)} = \frac{1}{2} \left( \sigma_{xx}^{(m)} - \sigma_{yy}^{(m)} \right)^2 + \left( \sigma_{yy}^{(m)} - \sigma_{zz}^{(m)} \right)^2 + \left( \sigma_{zz}^{(m)} - \sigma_{xx}^{(m)} \right)^2 + 6\left( \sigma_{xy}^{(m)} \right)^2.
\]

Substituting Eq. (2) into (3), we can get:
\[ \sigma_{yx}^{(i)} = \sqrt{\frac{(1 - 2 \nu_y)^2 (K_{1y}^{(i)})^2 + 3 (K_{0y}^{(i)})^2}{2 \pi r_{cr}^{(i)}}}. \]  

Let
\[ K_{1y}^{(i)} = \sqrt{\frac{(1 - 2 \nu_y)^2 (K_{0y}^{(i)})^2 + 3 (K_{1y}^{(i)})^2}{2 \pi r_{cr}^{(i)}}}, \]

where \( K_{0y}^{(i)} \) is a constant related to the stress intensity factors \( K_{1y}^{(i)} \) and \( K_{0y}^{(i)} \) which represents the strength of the equivalent stress fields near the crack tips along the crack line. Substituting Eq. (5) into (4) and making \( \sigma_{yx}^{(i)} = \sigma_{yy} \), the plastic zone size at the \((tm)\) crack tip can be obtained:
\[ r_p^{(tm)} = \frac{1}{2 \pi} \left( \frac{K_{1y}^{(tm)}}{\sigma_{yy}} \right)^2. \]

This is the first-order estimation of plastic zone size. When the stress in the cross-hatched region (shown in Fig. 3(b)) is considered, a second-order estimation of plastic zone size is obtained by the force balance equation:
\[ \sigma_{yy} r_p^{(i)} = \int_0^{r_f} \sigma_{yx}^{(i)} dr_{cr}^{(i)} = \int_0^{r_f} \frac{K_{1y}^{(i)}}{\sqrt{2 \pi r_{cr}^{(i)}}} dr_{cr}^{(i)}, \]

which leads to:
\[ r_p^{(i)} = 2 r_{cr}^{(i)} = \frac{1}{2 \pi} \left( \frac{K_{1y}^{(i)}}{\sigma_{yy}} \right)^2. \]

### 2.2. The crack tip opening displacement (CTOD)

As depicted in Fig. 3(c), for a Zener–Stroh crack, the plastic zone of left crack tip (the blunt tip) is caused by compression, while the plastic zone at right crack tip (the sharp tip) is caused by tension. The crack propagation always occurs from the sharp tip, which is different to the Griffith crack. Hence, only the crack tip opening displacement at the sharp tip is investigated. Based on the generalized Irwin model, the expression of CTOD can be given as:
\[ \delta_{y}^{(2)} = \frac{8}{E'} K_{1y}^{(i)} \sqrt{\frac{r_f}{2 \pi}} \]

where \( E' = E_l/(1 - \nu_l^2) \) in plane strain. Substituting Eq. (8) into (9) leads to:
\[ \delta_{y}^{(2)} = \frac{4}{E} \frac{K_{1y}^{(i)} K_{0y}^{(i)}}{E' \sigma_{yy}}. \]

### 3. Formulation of the current problem

As depicted in Fig. 3(a), the physical problem to be studied is on a Zener–Stroh crack interacting with a coated circular inclusion. Both the crack and the coated inclusion are embedded in the matrix. The left crack tip is at a distance \( t_1 \) from the inclusion center, with the plastic zone size \( r_p^{(x)} \). The right crack tip is at a distance \( t_2 \) and the plastic zone size is \( r_p^{(y)} \). The crack is oriented along the radial direction of the circular inclusion which occupies the area of \( r < a \) with material properties \( \mu_1 \) (shear modulus) and \( \nu_1 \) (Poisson’s ratio). The region \( a < r < b \) is the coating phase with material properties \( \mu_2 \) and \( \nu_2 \). The matrix occupies the region of \( r > b \) with the material properties \( \mu_3 \) and \( \nu_3 \). In order to concentrate on the effect of the net dislocations inside the Zener–Stroh crack, the problem is studied without external loading. Hence, in this paper, the external loads are zero as \( \sigma_{yy} = 0 \).

The distributed dislocation technique is adopted to simulate the crack near the inclusion. As the crack tip opening displacement (CTOD) mainly depends on the mode I loading, only the crack dislocation density \( B_y(x) \) is considered in the following derivation. Setting \( B_y(x) \) as the dislocation density in the crack zone, making use of the distributed dislocation method, the traction at \((x, 0)\) due to the dislocation distribution is:
\[ \sigma_{yy}(x, 0) = -\frac{2 \mu_2}{(K_3 + 1) \pi} \left[ t_1 \int_{r_1}^{r_2} B_y(\xi) \frac{d \xi}{\xi - x} + \int_{r_1}^{r_2} k_1(x, \xi) B_y(\xi) d \xi \right]. \]

Here, \( K_3 = 3 - 4 \nu_3 \) is a constant of material properties of the matrix. The traction component \( \sigma_{yy} = 0 \) when external loading is not considered. Then the traction free conditions on the upper and lower crack surface are written in terms of \( B_y \):
\[ \frac{1}{\pi} \left[ \int_{r_1}^{r_2} B_y(\xi) \frac{d \xi}{\xi - x} + \int_{r_1}^{r_2} k_1(x, \xi) B_y(\xi) d \xi \right] = 0, \quad t_1 \leq x \leq t_2. \]

The kernel \( k_1(x, \xi) \) is given by Xiao and Chen (2001):
\[ k_1(x, \xi) = -\frac{C_1 D_1}{2 \pi} + C_2 \frac{D_1}{\pi} \left( \frac{x^2 - \xi^2}{x - \xi} \right)^{1/2}, \]
\[ \left( \frac{x^2 - \xi^2}{x - \xi} \right)^{1/2} = \frac{C_1 D_1}{2 \pi} \left( \sqrt{(x - \xi)} + \sqrt{(x + \xi)} \right) \]
\[ + \frac{1}{2 \pi} \frac{C_1 D_1}{2 \pi} \sum_{n=1}^{\infty} n a_n (\frac{b}{\xi})^{n} - \frac{1}{2 \pi} \frac{C_1 D_1}{2 \pi} \sum_{n=1}^{\infty} n a_n (\frac{b}{x})^{n+1/2}, \]

where the detailed expression of the coefficients \( a_n \) and \( a_n' \) can be found in the work of Xiao and Chen (2001), and
\[ C = \mu_2 - \mu_3, \quad D = \frac{\mu_2 K_2 - \mu_3 K_3}{\mu_3 K_2 + \mu_2}. \]

Similarly, \( K_1 = 3 - 4 \nu_1 \), \( i = 1, 2, 3 \), is material constant of phase \( "p" \), \( \nu_i \), \( i = 1, 2, 3 \), is the Poisson’s ratio of phase \( "p" \).

Moreover, for the Zener–Stroh crack, the dislocation density \( B_y(x) \) must satisfy:
\[ \int_{r_1}^{r_2} B_y(\xi) d \xi = b_T^{(y)}, \]

where \( b_T^{(y)} \) is the total sum of Burgers vectors of the net dislocation inside the Zener–Stroh crack.

### 3.1. The stress intensity factors

Eq. (12) is the standard singular integral equation with Cauchy type regular kernels. Once the dislocation density \( B_y \) is solved from Eqs. (12) and (15), the stress fields in the matrix phase can be obtained from Eq. (11). The singularity on both the crack tips should be inverse square root since the whole crack is located in the pure matrix material. As a result, the dislocation density function can be assumed as:
\[ B_y(x) = \omega(x) F_y(x), \]

where \( F_y(x) \) is non-singular smooth function in \( t_1 \leq x \leq t_2 \), and
\[ \omega(x) = (x - t_1)^{-1/2} (t_2 - x)^{-1/2}, \]

is the fundamental function of the integral equation.

The numerical procedure for solving the equations is given in Appendix by the method developed by Erdogan and Gupta (1972). With the numerical solution of the dislocation density
function, the mode I stress intensity factor on the left (the blunt) and right (the sharp) crack tips are given by:

\[
K_I^{(1)} = \lim_{\xi \to t_1} \frac{-2\mu_3 \sqrt{2\pi}}{K_3 + 1} \left( \sqrt{t_2 - \xi} B_y(x) \right) = \frac{2\mu_3}{K_3 + 1} \frac{b_y'}{\sqrt{(t_2 - t_1)/2}} F_y(-1),
\]

\[
K_I^{(2)} = \lim_{\xi \to t_2} \frac{2\mu_3 \sqrt{2\pi}}{K_3 + 1} \left( \sqrt{t_2 - \xi} B_y(x) \right) = \frac{2\mu_3}{K_3 + 1} \frac{b_y'}{\sqrt{(t_2 - t_1)/2}} F_y(1). \tag{18}
\]

Here, \((t_1)\) and \((t_2)\) represent the left and right crack tips, respectively.

### 3.2. The effective stress intensity factors

For the small scale yielding, the effective half crack length can be approximated as:

\[
a_{eff}^{(i)}(t_m) = a_i + r_{eff}^{(i)}, \tag{19}
\]

where, \(a_{eff}^{(i)}(t_m)\) is the effective half crack length from the \((t_m)\) tip, \(a_i\) is the initial half crack length as shown in Fig. 3, \(r_{eff}^{(i)}\) is the first order estimation of plastic zone size gotten from Eq. (6). Eqs. (12) and (15) can be rewritten as:

\[
1 = \frac{1}{\pi} \int_{t_1}^{t_2} \frac{B_y(\xi)}{x} d\xi + \int_{t_1}^{t_2} k_i(x, \xi) B_y(\xi) d\xi = 0, \quad t_1 \leq x \leq t_2, \tag{20}
\]

\[
\int_{t_1}^{t_2} B_y(\xi) d\xi = b_y', \tag{21}
\]

where, \(t_1 = t_1\) for the left crack tip and \(t_2 = t_2 + r_{eff}^{(i)}\) for the right crack tip, respectively. Here, plastic zone correction is only considered at the right crack tip, because the crack tip opening exists at the sharp tip only. By solving Eqs. (20) and (21), the effective stress intensity factor is obtained in the expression:

\[
K_{eff}^{(i)}(t_m) = \lim_{\xi \to t_2} \frac{2\mu_3 \sqrt{2\pi}}{K_3 + 1} \sqrt{t_2 - \xi} B_y(x) = \frac{2\mu_3}{K_3 + 1} \frac{b_y'}{\sqrt{(t_2 - t_1)/2}} F_y(1). \tag{22}
\]

Here, \(F_y(1)\) is the value from solving the singular integral equations of (20) and (21).

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**Fig. 4.** Effect of shear modulus ratio \(\mu_3/\mu_1\) (inclusion/matrix), with \(t_1 = 1.05b, t_2 = 1.35b, \mu_3/\mu_1 = 2.33, v_2 = 0.28\) and \(v_1 = 0.3\): (a) the normalized SIF of left crack tip; (b) the normalized SIF of right crack tip; (c) the normalized PZS of the left crack tip; (d) the normalized PZS of the right crack tip.
4. Numerical examples and discussion

In this section, numerical examples are given to show the results of stress intensity factors (SIFs), plastic zone sizes and CTOD (at the sharp tip) for a Zener–Stroh crack near a coated circular inclusion solved by the current method. Only the displacement loading \( b_0 \) is considered.

The stress intensity factors, the plastic zone sizes and the CTOD are normalized separately by:

\[
K_0 = \frac{2\mu_0 b_0^2}{(1 + \nu_0)\sqrt{\pi(t_2 - t_1)/2}}, \quad r_{p0} = \frac{(K_0')^2}{2\pi\sigma_{ys}}, \quad \delta_{p0} = \frac{4\sigma_0 K_0''}{\pi E \sigma_{ys}}.
\]  

(23)

Here, \( K_0 \), \( r_{p0} \) and \( \delta_{p0} \) are the mode I stress intensity factor, the plastic zone size and the CTOD separately for the same Zener–Stroh crack in a homogeneous material without the inclusion. Here, \( K_0' = (1 - 2\nu_2)K_0 \) in plane strain when the mode II stress intensity factor is zero.

4.1. Effects of the shear modulus of the inclusion

As the mode II stress intensity factor is vanished \( \left(K_0'' = 0\right) \), Eqs. (5), (6) and (10) are re-written as:

\[
r_y^{(\text{II})} = \frac{1}{2\pi} \frac{(1 - 2\nu_2)\left(K_0''\right)^2}{\sigma_{ys}}.
\]

(24)

\[
\delta_y^{(\text{II})} = \frac{4}{\pi} \left(1 - 2\nu_3\right)K_0''^2.
\]

(25)

Considering Eq. (23), the normalized plastic zone size \( r_y^{(\text{II})}/r_{p0} \) and the normalized CTOD \( \delta_y^{(\text{II})}/\delta_{p0} \) are written as:

\[
\frac{r_y^{(\text{II})}}{r_{p0}} = \frac{\delta_y^{(\text{II})}}{\delta_{p0}} = \left(\frac{K_0''}{K_0'}\right)^2.
\]

Hence, only the normalized plastic zone size \( r_y^{(\text{II})}/r_{p0} \) is emphasized in the following sections to reduce the number of figures in the work. However, the real values of the plastic zone size and CTOD are not the same and the plastic zone size is much larger. When the mode II stress intensity factor does not equal to zero for mixed mode problems, the normalized values of PZS and CTOD will different.

The curves of the normalized SIF and the normalized PZS with the shear modulus ratio \( \mu_2/\mu_3 \) are depicted in Fig. 4(a–d). A series of coating layer thicknesses are taken: \( b/a = 1.05, 1.2, 1.5 \). Other parameters are: \( t_1 = 1.05b, \quad t_2 = 1.35b, \quad \mu_2/\mu_3 = 2.33, \quad \nu_1 = 0.3, \quad \nu_2 = 0.28 \) and \( \nu_3 = 0.3 \). From Fig. 4(b), it is observed that the values of the normalized SIF at the right crack tip are always greater than 1, which indicates that a harder inclusion makes a Zener–Stroh easier to propagate. Similar phenomena also observed in the curves of the normalized plastic zone size, which signifies that a harder inclusion causes a larger plastic zone size and CTOD, as shown in

Fig. 5. Effect of shear modulus ratio \( \mu_2/\mu_3 \) (coating phase/matrix), with \( \nu_1 = \nu_2 = \nu_3 = 0.3, b/a = 1.1, t_1 = 1.05b, \) and \( t_2 = 1.35b \): (a) the normalized SIF of left crack tip; (b) the normalized SIF of right crack tip; (c) the normalized PZS of the left crack tip; (d) the normalized PZS of the right crack tip.
Fig. 4(d). For the left (blunt) crack tip, the normalized SIF is always below zero as shown in Fig. 4(a) and the normalized plastic zone size below 1 as shown in Fig. 4(c), which agrees with the physical phenomena that the propagation of a Zener–Stroh crack always occur at the sharp tip.

4.2. Effects of the shear modulus of coating phase

In this section, the influence of the shear modulus of the coating phase is investigated. The parameters are taken as: $v_1 = v_2 = v_3 = 0.3$, $b/a = 1.1$, $t_1 = 1.05b$ and $t_2 = 1.35b$, the shear modulus ratio: $\mu_1/\mu_2 = 0.1$, 0.5, 1, 2, 5, 10, including the conditions that the inclusion is softer and harder than the matrix. The coating phase is assumed to be “harder” than the matrix ($\mu_2/\mu_3 > 1$).

The normalized SIF and the normalized plastic zone size at the left and right crack tips are shown in Fig. 5(a–d). With the increasing coating phase shear modulus ($\mu_2/\mu_3$), the normalized SIFs at both crack tips increase. The normalized plastic zone sizes at the right crack tip increases as well, while the normalized plastic zone size at the left tip decreases. Also we observed that the normalized SIFs and the normalized plastic zone sizes are insensitive to the change of the inclusion phase shear modulus ($\mu_1/\mu_2$) when $\mu_2/\mu_3$ is increasing. It is concluded that a harder coating phase can reduce/shield the influence of the inclusion on the stress and displacement fields near the crack tips.

4.3. Effects of the coating phase thickness

In this section, the influence of the coating phase thickness on the elastic–plastic fracture behavior of the Zener–Stroh crack is discussed. The involved parameters are set as: $v_1 = v_2 = v_3 = 0.3$, $\mu_1/\mu_2 = 0.1$, 0.5, 1, 1.86, 5, 10, $t_1 = 1.05b$, $t_2 = 1.35b$, and $\mu_2/\mu_3 = 1.86$. When $\mu_1/\mu_3 = 1.86$, the coating phase and the inclusion have the same material and the current problem is simplified to the two-phase problem. The variable $b/a$ changes from 1 to 2.

Fig. 6(a–d) depict the curves of the normalized SIF and the normalized plastic zone size at the both crack tips. In these figures, it is shown that when the coating thickness increases, the influence of the shear modulus ratio $\mu_1/\mu_3$ decreases. In other words, a thick coating will reduce the influence of the inclusion on the crack. The figures also illustrate that, when $\mu_1 > \mu_3$, the normalized SIF at both crack tips and the normalized plastic zone size at the right tip decrease with the increasing $b/a$. When $\mu_1 < \mu_2$, the three normalized quantities all increase with the increasing $b/a$. On the other hand, for the normalized plastic zone size at the left crack tip (blunt tip), it increases with the increasing $b/a$ when $\mu_1 > \mu_2$, and decreases with increasing $b/a$ when $\mu_1 < \mu_2$. From the above
observations, it can be concluded that when the inclusion is harder than the coating phase \((\mu_1 > \mu_2)\), a thicker coating phase helps avoid fracture failures. Conversely, when the inclusion is softer than the coating phase \((\mu_1 < \mu_2)\), a thinner coating phase is better for avoiding fracture failures.

When the coating phase thickness is fixed, the influence of inclusion size is also studied as shown in Fig. 7(a–d), for the normalized SIFs and normalized plastic zone size respectively. The parameters are set as: \(v_1 = v_2 = v_3 = 0.3\), \(t_1 = 2.05\), \(t_2 = 2.35\), \(b - a = 0.1\) and \(\mu_2/\mu_3 = 1.86\). It is found from the figures that: increase the ratio \(2a/(t_2 - t_3)\), the normalized SIFs and normalized plastic zone size at both crack tips will change a lot. Physically, increase \(2a/(t_2 - t_1)\), the inclusion is moving towards the fixed crack and the influence of the inclusion can be better felt by the crack. When the ratio is quite small, the distance between the crack and the inclusion becomes very larger, the problem can reduce to a homogeneous case and the normalized values all approach to 1.

4.4. Effects of the distance between the crack and the inclusion

In this section, the influence of the distance between the crack and the inclusion is studied. The normalized SIF and the normalized plastic zone size at both crack tips are plotted in Fig. 8(a–d). The length of the Zener–Stroh crack is fixed at \(t_2 - t_1 = 0.5b\). Other parameters are set as: \(\mu_1/\mu_3 = 5.43\), \(v_1 = v_2 = v_3 = 0.25\) and \(b/a = 1.1\). The coating phase shear modulus is set to: \(\mu_2/\mu_3 = 1, 2, 5, 10\).

From Fig. 8(a–d), we observe that, with the increasing distance between the crack and the inclusion, the normalized SIF at both crack tips, and the normalized plastic zone size at the right crack tip decrease, while the normalized plastic zone size at the left crack tip increases. When the shear modulus of the coating phase is larger, its influence on the normalized SIF and the normalized plastic zone size will be greater.

4.5. Effective stress intensity factors

The curves of the effective stress intensity factor at the right crack tip (sharp tip) are shown in Fig. 9. The parameters are taken as: \(t_1 = 1.05b\), \(t_2 = 1.35b\), \(\mu_2/\mu_3 = 2.33\), \(v_1 = 0.3\), \(v_2 = 0.28\), \(v_3 = 0.3\). Three different coating thicknesses, \(b/a = 1.02, 1.2, 1.5\), are studied. The shear modulus of the inclusion is varied.

The effective stress intensity factor is normalized by the SIF value \((K_1^{(t)}/K_r)\) of the same crack without plastic zone correction.

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*Fig. 7. Effect of inclusion size \(2a/(t_2 - t_3)\), with \(v_1 = v_2 = v_3 = 0.3\), \(t_1 = 2.05\), \(t_2 = 2.35\), \(b - a = 0.1\) and \(\mu_2/\mu_3 = 1.86\): (a) the normalized SIF of left crack tip; (b) the normalized SIF of right crack tip; (c) the normalized PZS of the left crack tip; (d) the normalized PZS of the right crack tip.*
When the plastic zone correction is considered, Fig. 9 illustrates the normalized effective SIF which is slightly smaller than 1. This result means when the displacement loading $b_T$ is fixed, a longer crack length will induce to smaller stress intensity factors, which is the major difference between a Zener–Stroh and a Griffith crack.

4.6. Comparison with the Dugdale model

In this section, the current results are compared with those obtained by Hoh et al. (2011) where the plastic zone size and CTOD were solved by Dugdale mode. By taking the same materials constants, a comparison has been made between the result of Hoh and our current work as shown in Table 1, where $\mu_1 = 10\mu_3$, $\mu_2 = \mu_3$, $t_1 = 1.6a$, $t_2 = 2.2a$, and 0.45. We can see that when the current problem is reduced to the two-phase problem, the results from the two models agree well, which proves that the current Irwin model can be a great improvement for the Dugdale model since the relationship among the SIF, PZS and CTOD are obtained.

Additionally, the blunt tip CTOD was not considered in this model as the crack tip is under compression and not able to be opened.

The numerical results of normalized SIF, normalized PZS, normalized CTOD at the right crack tip (sharp tip) when realistic material combinations being considered are shown in Table 2. The material properties including Young's modulus, Poisson's ratio and yield stress can be found in the work of Hahn (1993). From the table, it is observed that changing any one of the three materials (fiber, coating and matrix), the normalized values of SIF, PZS...
Table 1
Comparison of the normalized CTOD from Hoh et al. (2011) and the current work, with \(\mu_1 = 10\mu_2, \mu_2 = \mu, t_1 = 1.6a, t_2 = 2.2a,\) and 0.45.

<table>
<thead>
<tr>
<th>(v_1 = v_2 = v_3)</th>
<th>(\delta^{(1)}/\delta_0)</th>
<th>(\delta^{(1)}/\delta_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.2136</td>
<td>1.0293</td>
</tr>
<tr>
<td>1/3</td>
<td>1.1837</td>
<td>1.0261</td>
</tr>
<tr>
<td>0.45</td>
<td>1.1599</td>
<td>1.0242</td>
</tr>
</tbody>
</table>

Table 2
Numerical results of normalized stress intensity factor, normalized plastic zone size and normalized CTOD for realistic material combination, with \(t_1 = 2.05, t_2 = 2.35, b = 2, a = 1.9.\)

<table>
<thead>
<tr>
<th>Fiber (E, v)</th>
<th>Coating (E, v)</th>
<th>Matrix (E, v)</th>
<th>(K_0^{(1)}/K_0)</th>
<th>(\Delta\delta^{(1)}/\Delta\delta_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SiC W Al</td>
<td>Si T Al</td>
<td>1.20447</td>
<td>1.45075</td>
<td>1.45075</td>
</tr>
<tr>
<td>Si W T Al</td>
<td>1.07135</td>
<td>1.14829</td>
<td>1.14829</td>
<td></td>
</tr>
<tr>
<td>SiN4 W T Al</td>
<td>1.08379</td>
<td>1.17462</td>
<td>1.17462</td>
<td></td>
</tr>
<tr>
<td>SiC Mo Al</td>
<td>1.19367</td>
<td>1.24324</td>
<td>1.24324</td>
<td></td>
</tr>
<tr>
<td>Si Mo T Al</td>
<td>1.04999</td>
<td>1.10248</td>
<td>1.10248</td>
<td></td>
</tr>
<tr>
<td>SiN4 Mo Al</td>
<td>1.17346</td>
<td>1.29088</td>
<td>1.29088</td>
<td></td>
</tr>
</tbody>
</table>

and CTOD will change accordingly. The performance of metal-matrix composites can be improved by choosing proper coating phase material.

5. Conclusions

In this paper, the elastic–plastic stress investigation for a Zener–Stroh crack interacting with a near-by coated inclusion has been carried out. The generalized Irwin model is developed for the first time to evaluate the plastic zone size. We focus on the influence of the coating phase properties on the stress intensity factor (SIF), effective stress intensity factor, plastic zone size and CTOD. The following conclusions are drawn from our study:

1. When the shear modulus ratio \(\mu_1/\mu_2 > 1,\) or the inclusion is “harder” than the matrix, a displacement loaded Zener–Stroh crack is easier to propagate than the same crack in homogeneous material.
2. Increasing the shear modulus of the coating phase (\(\mu_2/\mu_3\)), the influence of the inclusion on the normalized SIF and normalized PZS will be reduced. Increasing the coating phase thickness, the influence of the inclusion on the quantities will be reduced as well.
3. When the inclusion is “harder” than the coating layer, the normalized SIF at both crack tips, and the normalized PZS at the right tip decrease with increasing coating layer thickness \(b/a.\) While when the inclusion is “softer” than the coating phase, these quantities increase with the increasing \(b/a.\) The curvilinear trend of normalized plastic zone size at the left crack tip is convex.
4. For the influence of the coating phase, when the thickness is fixed (\(b/a = 1.1\) for instance), a lower shear modulus will induce to smaller normalized SIF (for both tips) and smaller normalized PZS at the right (the sharp) crack tip, but larger normalized PZS at the left (blunt) tip. When the shear modulus ratio is fixed (\(\mu_1/\mu_2 = 1.86,\)) a smaller coating phase thickness causes smaller normalized SIF (for both tips) and smaller normalized PZS at the right tip, but larger normalized PZS at the left tip, under the condition that the inclusion is “softer” than coating. When the inclusion is “harder” than coating, the situation becomes inverse.

5. Comparison between the current method and the Dugdale model (Hoh et al., 2011) shows that the current is accurate and much more convenient for matrix cracking problem.

Appendix A

To solve the singular integral equations given in Eqs. (12) and (15), the numerical method developed by Erdogan and Gupta (1972) is used. The integral interval is shifted from \((t_1, t_2)\) to \((-1, 1)\) by:

\[
x = \frac{t_2 - t_1}{2} t + \frac{t_2 + t_1}{2}, \quad \xi = \frac{t_2 - t_1}{2} + s + \frac{t_2 + t_1}{2}.
\]

Then, Eqs. (12) and (15) are rewritten in terms of \(s, t:\)

\[
\frac{1}{\pi} \int_{-1}^{1} B_1(s) ds + \frac{1}{\pi} \int_{-1}^{1} k_{11}(t, s) B_1(s) ds = 0, \quad -1 < s, \quad t < 1,
\]

\[
\int_{-1}^{1} B_2(s) ds = \frac{2b_1}{t_2 - t_1}.
\]

Here,

\[
k_{11}(t, s) = \frac{t_2 - t_1}{2} \left( \frac{t_2 - t_1}{2} t + s + t_2 - t_1 \right).
\]

The expression of \(k_1(s, \xi)\) was given in Eq. (13). \(B_2(s)\) is the dislocation density in \(y\)-direction and \(b_1\) is the total sum of Burgers vectors of the net dislocation inside the Zener–Stroh crack.

The discretized forms of Eqs. (A.2) and (A.3) are:

\[
\sum_{k=1}^{n} \frac{1}{n} F_1(S_k) \left[ \begin{array}{c} k_1(U_k, S_k) \\ k_1(U_k, S_k) \end{array} \right] = 0,
\]

\[
\sum_{k=1}^{n} \frac{1}{n} F_2(S_k) = \frac{b_1}{\pi} \frac{2}{t_2 - t_1}.
\]

in which,

\[
s_k = \cos \frac{\pi}{2n}(2k - 1), \quad u_r = \cos \frac{\pi r}{n}, \quad k = 1, \ldots, n,
\]

\[
r = 1, \ldots, n - 1,
\]

Eqs. (A.5) and (A.6) provide a system of \(n\) linear algebraic equations to determine the values of \(F_1(S_k), \ldots, F_2(S_k)\).

With the numerical solutions of the dislocation density functions, the stress intensity factor of the left (blunt) and right (sharp) crack tips can be obtained:

\[
K_1^{(1)} = \lim_{\xi \to -1} \frac{-2\mu_2 \sqrt{2\pi}}{k_3 + 1} \sqrt{\xi - t_1} B_1(x) = \frac{-2\mu_2}{k_3 + 1} \sqrt{\pi(t_2 - t_1)} F_1(-1),
\]

\[
K_2^{(1)} = \lim_{\xi \to -1} \frac{-2\mu_2 \sqrt{2\pi}}{k_4 + 1} \sqrt{t_2 - \xi} B_1(x) = \frac{-2\mu_2}{k_4 + 1} \sqrt{\pi(t_2 - t_1)} F_1(1).
\]

Here, \((t_1)\) and \((t_2)\) represent the left and right crack tips, respectively which have been mentioned before.

References


