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Berezin symbol and invertibility of operators on the functional Hilbert spaces

Mubariz T. Karaev

Suleyman Demirel University, Isparta (MYO) Vocation School, 32260 Isparta, Turkey

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Abstract

We give in terms of reproducing kernel and Berezin symbol the sufficient conditions ensuring the invertibility of some linear bounded operators on some functional Hilbert spaces.

Keywords: Hardy space; Bergman space; Toeplitz operator; Berezin symbol; Reproducing kernel

1. Introduction

Let $\mathbf{T}$ be the unit circle $\mathbf{T} = \{\zeta \in \mathbf{C}\colon |\zeta| = 1\}$, $\varphi \in L^\infty = L^\infty(\mathbf{T})$, and let T_φ be the Toeplitz operator acting in the Hardy space $H^2(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbf{C}\colon |z| < 1\}$ by the formula $T_\varphi f = P_+\varphi f$, where P_+ is the Riesz projector. Let $\tilde{\varphi}$ denote the harmonic extension of the function φ to $\mathbb{D}$. In [5] Douglas posed the following problem: if φ is a function in L^∞ for which $|\tilde{\varphi}(z)| \geqslant \delta > 0$, $z \in \mathbb{D}$, then is T_φ invertible?

In [18] Tolokonnikov firstly gave a positive answer to this question under the condition that δ is near enough to 1, namely, he proved that if

$$1 \geqslant \left|\tilde{\varphi}(z)\right| \geqslant \delta > \frac{45}{46}, \quad z \in \mathbb{D},$$

then T_φ is invertible and

$$\left\|T_\varphi^{-1}\right\| \leqslant \left(1 - 46(1-\delta)\right)^{-1}.$$

E-mail address: garayev@fef.sdu.edu.tr.

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This assertion was also proved by Wolff [19]. Nikolskii [15] has somewhat improved the result of Tolokonnikov proving invertibility of T_φ and the estimate

$$\left\| T_\varphi^{-1} \right\| \leqslant (24\delta - 23)^{-1/2}$$

under condition $\delta > 23/24$. Finally, Wolff [19] has constructed a function $\varphi \in L^\infty$ such that $\inf_{\mathbf{D}} |\tilde{\varphi}(z)| > 0$ but the corresponding operator T_φ is not invertible, and thus showed that the answer to the question of Douglas is negative in general. Since $\tilde{\varphi}$ coincides with the Berezin symbol $\widetilde{T}_\varphi$ of the operator T_φ (see Lemma 2.1), in this context the following natural problem arises.

Problem 1. *Let A be a linear bounded operator acting in the functional Hilbert space $\mathcal{H}(\Omega)$ of complex-valued functions over the some (non-empty) set Ω, such that $|\widetilde{A}(z)| \geqslant \delta$ for all $z \in \Omega$ and for some $\delta > 0$. To find the number δ_0, which can be (more or less) easily computed from the data of A, and due to which the inequality*

$$\left|\widetilde{A}(z)\right| \geqslant \delta > \delta_0, \quad z \in \mathbf{D},$$

ensures the invertibility of A, where $\widetilde{A}$ denotes the Berezin symbol of the operator A.

In particular, the following problem is also interesting, which is closely related with the finite section method of Böttcher and Silbermann [3].

Problem 2. *Let $E \subset \mathcal{H}(\Omega)$ be a closed subspace of the functional Hilbert space $\mathcal{H}(\Omega)$, and let A be a linear bounded operator acting in $\mathcal{H}(\Omega)$ such that*

$$\left|\widetilde{A}(z)\right| \geqslant \delta$$

for all $z \in \Omega$ and for some $\delta > 0$. To find a number δ_0, such that $\delta > \delta_0$ ensures the invertibility of operator $P_E A \mid E$ (the compression of the operator A to the subspace E), where P_E is an orthogonal projection from $\mathcal{H}(\Omega)$ onto E.

In this article we solve these problems in some special cases. Our argument uses the concept of reproducing kernel and Berezin symbol.

2. Notations and preliminaries

2.1. Recall that a functional Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on a (non-empty) set Ω, which has the property that point evaluations are continuous (i.e., for each $\lambda \in \Omega$, the map $f \to f(\lambda)$ is a continuous linear functional on $\mathcal{H}$). Then the Riesz representation theorem ensures that for each $\lambda \in \Omega$ there is a unique element k_λ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\lambda\colon \lambda \in \Omega\}$ is called the reproducing kernel of $\mathcal{H}$. It is well known (see, for instance, [8, Problem 37] that if $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by

$$k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z).$$

For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. For a bounded linear operator A on $\mathcal{H}$, the function $\widetilde{A}$ defined on Ω by

$$\widetilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$$

is the Berezin symbol of A, which firstly have been introduced by Berezin [1,2]. It is clear that the Berezin symbol $\widetilde{A}$ is the bounded function on Ω whose values lies in the numerical range of the operator A, and hence

$$\begin{aligned} \sup_{z\in\mathbb{D}} \left|\widetilde{A}(z)\right| &\overset{\text{def}}{=} ber(A) \quad (\text{“Berezin number”}) \\ &\leqslant w(A) \quad (\text{numerical radius}). \end{aligned}$$

More typical examples of functional Hilbert spaces are the Hardy and Bergman spaces.

2.2. Let dm_2 denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space $L_a^2 = L_a^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dm_2)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $k_\lambda \in L_a^2$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for every $f \in L_a^2$. It is well known that $k_\lambda(z) = \frac{1}{(1-\bar{\lambda}z)^2}$. The normalized Bergman reproducing kernel $\hat{k}_\lambda$ is the function $\frac{k_\lambda}{\|k_\lambda\|_2} = \frac{1-|\lambda|^2}{(1-\bar{\lambda}z)^2}$.

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space of analytic functions $f(z) = \sum_{n\geqslant 0} a_n z^n$ defined in the unit disc $\mathbb{D} = \{z \in \mathbf{C}\colon\ |z| < 1\}$, such that $\sum_{n\geqslant 0} |a_n|^2 < \infty$. Alternately, it can be identified with a closed subspace of the Lebesgue space $L^2 = L^2(\mathbf{T})$ on the unit circle, by associating to each analytic function its radial limit. The algebra of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞. Any $\varphi \in H^\infty$ acts as a multiplication operator on H^2, that we will denote by T_φ.

Norm and inner product in L^2 or H^2 will be denoted by $\|.\|$ and $\langle\cdot,\cdot\rangle$, respectively. Evaluations at points $\lambda \in \mathbb{D}$ are bounded functionals on H^2 and the corresponding reproducing kernel is $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$; thus, $f(\lambda) = \langle f, k_\lambda \rangle$. If $\varphi \in H^\infty$, then k_λ is an eigenvector for T_φ^* and $T_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$. By normalizing k_λ we obtain

$$\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|} = \sqrt{1-|\lambda|^2}\, k_\lambda.$$

2.3. The Berezin symbol have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis (see, for instance, [9–14,16,17,20]). In particular, it is known (see [11,20]) the following result which we will use in what follows.

Lemma 2.1. *The Berezin symbol $\widetilde{T_\varphi}$ of the Toeplitz operator T_φ, $\varphi \in L^\infty$, on the Hardy space H^2 coincides with the harmonic extension $\tilde{\varphi}$ of the function φ into the unit disc $\mathbb{D}$, that is $\widetilde{T_\varphi}(\lambda) = \tilde{\varphi}(\lambda)$ for all $\lambda \in \mathbb{D}$.*

Suppose now θ is an inner function. We define the corresponding model space by the formula $K_\theta = H^2 \ominus \theta H^2$.

2.4. We recall some basic definitions concerning geometric properties of sequences in a Hilbert space. For most of the definitions and facts below, one can use [7,15] as a main references (see also [4,6]).

Let H be a complex Hilbert space. If $\{x_n\}_{n\geqslant 1} \subset H$, we denote by $span\{x_n\colon n \geqslant 1\}$ the closure of the linear hull generated by $\{x_n\}_{n\geqslant 1}$. The sequence $X \stackrel{\text{def}}{=} \{x_n\}_{n\geqslant 1}$ is called:

- complete if $span\{x_n\colon n \geqslant 1\} = H$;
- minimal if for all $n \geqslant 1$, $x_n \notin span\{x_m\colon m \neq n\}$;
- uniformly minimal if $\inf_{n\geqslant 1} dist(\frac{x_n}{\|x_n\|}, span(x_m\colon m \neq n)) > 0$;
- a Riesz basis if there exists an isomorphism U mapping X onto an orthonormal family $\{Ux_n\colon n \geqslant 1\}$;
- the operator U will be called the orthogonalizer of X.

The expression "a Riesz basis in H" means a Riesz basis X with the completeness property $span(X) = H$. It is well known that X is a Riesz basis in its closed linear span if there are positive constants C_1, C_2 such that

$$C_1\Big(\sum_{n\geqslant 1}|a_n|^2\Big)^{1/2} \leqslant \Big\|\sum_{n\geqslant 1} a_n x_n\Big\| \leqslant C_2\Big(\sum_{n\geqslant 1}|a_n|^2\Big)^{1/2} \tag{1}$$

for all finite complex sequences $\{a_n\}_{n\geqslant 1}$. Note that if U is an orthogonalizer of the family X then the product $r(X) \stackrel{\text{def}}{=} \|U\|\|U^{-1}\|$ characterizes the deviation of the basis X from an orthonormal one and $\|U\|^{-1}$ and $\|U^{-1}\|$ are the best constants in the inequality (1); $r(X)$ will be referred to as the Riesz constant of the family X. Obviously, $r(X) \geqslant 1$.

We now recall some well-known facts (see [15]) concerning reproducing kernels in H^2. Let $\Lambda = \{\lambda_n\}_{n\geqslant 1}$ be a sequence of distinct points in $\mathbb{D}$. Then we have:

(i$_1$) $\{k_{\lambda_n}\}_{n\geqslant 1}$ is minimal if and only if $\{\lambda_n\}_{n\geqslant 1}$ is Blaschke sequence (which means that $\sum_{n\geqslant 1}(1 - |\lambda_n|) < \infty$). As usual, we denote by

$$B = B_\Lambda = \prod_{n\geqslant 1} b_{\lambda_n}, \quad \text{where } b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n}\frac{\lambda_n - z}{1 - \overline{\lambda_n} z}.$$

(i$_2$) If $\{\lambda_n\}_{n\geqslant 1}$ is a Blaschke sequence, then $\{k_{\lambda_n}\}_{n\geqslant 1}$ is complete in K_B.

(i$_3$) $\{\hat{k}_{\lambda_n}\}_{n\geqslant 1}$ is a Riesz basis of K_B if and only if it is uniformly minimal which is equivalent to $\{\lambda_n\}_{n\geqslant 1}$ satisfies the Carleson condition

$$\inf_{n\geqslant 1}\big|B_n(\lambda_n)\big| > 0,$$

where $B_n = B/b_{\lambda_n}$; we will write in this case $\{\lambda_n\}_{n\geqslant 1} \in (\mathrm{C})$.

3. Results

In this section we partially solve Problems 1 and 2.

Theorem 3.1. *Let $\Lambda = \{\lambda_n\}_{n\geqslant 1}$ be a Carleson sequence of distinct points in $\mathbb{D}$, B the corresponding Blaschke product, and*

$$X \stackrel{\text{def}}{=} \{\hat{k}_{\lambda_n}\colon n \geqslant 1\}$$

be a corresponding Riesz basis in the model space $K_B = H^2 \ominus BH^2$ (see assertion (i$_3$) *above), and denote by $r(X) = \|U\|\|U^{-1}\|$ the corresponding Riesz constant of the family X. Let A be a linear bounded operator on the Hardy space H^2 such that $A^\star K_B \subset K_B$, and denote $M_A \stackrel{\text{def}}{=} P_B A|K_B$, where P_B is an orthogonal projection from H^2 onto K_B. Suppose that*:

$$\left(\sum_{n=1}^{\infty}\left\|A\hat{k}_{\lambda_n} - \widetilde{A}(\lambda_n)\hat{k}_{\lambda_n}\right\|^2\right)^{1/2} \stackrel{\text{def}}{=} \tau_A^\Lambda < +\infty \quad \textit{and} \tag{1}$$

$$\left(\sum_{n=1}^{\infty}\left\|A^\star\hat{k}_{\lambda_n} - \widetilde{A^\star}(\lambda_n)\hat{k}_{\lambda_n}\right\|^2\right)^{1/2} \stackrel{\text{def}}{=} \tau_{A^\star}^\Lambda < +\infty, \tag{2}$$

where $\widetilde{A}$ denotes the Berezin symbol of the operator A. If

$$\inf_{z\in\mathbb{D}}\left|\widetilde{A}(z)\right| \stackrel{\text{def}}{=} \delta > \delta_0 \stackrel{\text{def}}{=} r(X)\|U\|\max\left\{\tau_A^\Lambda, \tau_{A^\star}^\Lambda\right\},$$

then the operator M_A is invertible in K_B and

$$\left\|M_A^{-1}\right\| \leqslant \left(\frac{\delta}{r(X)} - \|U\|\tau_A^\Lambda\right)^{-1}.$$

Proof. Since $\Lambda \in (\mathrm{C})$, the sequence $X = \{\hat{k}_{\lambda_n}\colon n \geqslant 1\}$ is a Riesz basis in K_B (see assertion (i$_3$) above). If U is an orthogonalizer of X , then $\|U\|^{-1}$ and $\|U^{-1}\|$ are corresponding best constants appearing in (1), that is

$$\|U\|^{-1}\left(\sum_{n\geqslant 1}|a_n|^2\right)^{1/2} \leqslant \left\|\sum_{n\geqslant 1}a_n\hat{k}_{\lambda_n}\right\| \leqslant \|U^{-1}\|\left(\sum_{n\geqslant 1}|a_n|^2\right)^{1/2} \tag{2}$$

for all finite complex sequences $\{a_n\}_{n\geqslant 1}$. Now it is clear from (2) and the condition $|\widetilde{A}(z)| \geqslant \delta > 0$, $z \in \mathbb{D}$, of the theorem that

$$\begin{aligned}\left\|\sum_{n=1}^{N}a_n\widetilde{A}(\lambda_n)\hat{k}_{\lambda_n}\right\| &\geqslant \|U\|^{-1}\left(\sum_{n=1}^{N}\left|a_n\widetilde{A}(\lambda_n)\right|^2\right)^{1/2} \geqslant \delta\|U\|^{-1}\left(\sum_{n=1}^{N}|a_n|^2\right)^{1/2} \\ &\geqslant \frac{\delta}{\|U\|\|U^{-1}\|}\left\|\sum_{n=1}^{N}a_n\hat{k}_{\lambda_n}\right\| = \frac{\delta}{r(X)}\left\|\sum_{n=1}^{N}a_n\hat{k}_{\lambda_n}\right\|\end{aligned}$$

and hence

$$\left\|\sum_{n=1}^{N}a_n\widetilde{A}(\lambda_n)\hat{k}_{\lambda_n}\right\| \geqslant \frac{\delta}{r(X)}\left\|\sum_{n=1}^{N}a_n\hat{k}_{\lambda_n}\right\|, \tag{3}$$

where $r(X)$ is the Riesz constant of the family $X = \{\hat{k}_{\lambda_n}\colon n \geqslant 1\}$.

Now using condition (1) of the theorem and inequalities (2) and (3), for every finite $N > 0$ and for arbitrary numbers $a_n \in \mathbb{C}$ $(n = 1, 2, \ldots, N)$ we have:

$$\begin{aligned}
&\left\| M_A \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \\
&= \left\| (P_B A \mid K_B) \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| = \left\| P_B A \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| = \left\| \sum_{n=1}^{N} a_n P_B A \hat{k}_{\lambda_n} \right\| \\
&= \left\| \sum_{n=1}^{N} a_n P_B \big(A \hat{k}_{\lambda_n} - \widetilde{A}(\lambda_n) \hat{k}_{\lambda_n} + \widetilde{A}(\lambda_n) \hat{k}_{\lambda_n} \big) \right\| \\
&\geqslant \left\| \sum_{n=1}^{N} a_n P_B \widetilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\| - \left\| \sum_{n=1}^{N} a_n P_B \big(A \hat{k}_{\lambda_n} - \widetilde{A}(\lambda_n) \hat{k}_{\lambda_n} \big) \right\| \\
&\geqslant \left\| \sum_{n=1}^{N} a_n \widetilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\| - \sum_{n=1}^{N} |a_n| \big\| A \hat{k}_{\lambda_n} - \widetilde{A}(\lambda_n) \hat{k}_{\lambda_n} \big\| \\
&\geqslant \frac{\delta}{r(X)} \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| - \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2} \left(\sum_{n=1}^{N} \big\| A \hat{k}_{\lambda_n} - \widetilde{A}(\lambda_n) \hat{k}_{\lambda_n} \big\|^2 \right)^{1/2} \\
&\geqslant \frac{\delta}{r(X)} \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| - \|U\| \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \left(\sum_{n=1}^{\infty} \big\| A \hat{k}_{\lambda_n} - \widetilde{A}(\lambda_n) \hat{k}_{\lambda_n} \big\|^2 \right)^{1/2} \\
&\geqslant \frac{\delta}{r(X)} \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| - \|U\| \tau_A^{\Lambda} \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \\
&= \left(\frac{\delta}{r(X)} - \|U\| \tau_A^{\Lambda} \right) \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\|.
\end{aligned}$$

Thus

$$\left\| M_A \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \geqslant \left(\frac{\delta}{r(X)} - \|U\| \tau_A^{\Lambda} \right) \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \tag{4}$$

for all finite $N > 0$ and complex numbers a_n, $n = 1, 2, \ldots, N$. Since the Carleson condition implies the Blaschke condition, we have from (4) that

$$\|M_A f\| \geqslant \left(\frac{\delta}{r(X)} - \|U\| \tau_A^{\Lambda} \right) \|f\| \tag{5}$$

for all $f \in K_B$ (see assertion (i_2) above).

Since $A^\star K_B \subset K_B$, it is easy to see that

$$M_A^\star = (P_B A \mid K_B)^\star = A^\star \mid K_B.$$

Analogously, using condition (2) of the theorem, the equality $|\widetilde{A^\star}(z)| = |\widetilde{A}(z)|$ $(z \in \mathbb{D})$ and the inequalities (2) it can be proved that (we omit it)

$$\left\| M_A^\star \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \geqslant \left(\frac{\delta}{r(X)} - \|U\| \tau_{A^\star}^{\Lambda} \right) \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\|$$

for all finite $N > 0$ and complex numbers a_n, $n = 1, 2, \ldots, N$, which yields that

$$\|M_A^\star f\| \geqslant \left(\frac{\delta}{r(X)} - \|U\| \tau_{A^\star}^{\Lambda} \right) \|f\| \tag{6}$$

for all $f \in K_B$.

Now by considering that

$$\delta > r(X)\|U\| \max\{\tau_A^{\Lambda}, \tau_{A^\star}^{\Lambda}\} = \delta_0,$$

we deduce from the estimates (5) and (6) that the operator M_A is invertible in K_B and

$$\|M_A^{-1}\| \leqslant \left(\frac{\delta}{r(X)} - \|U\| \tau_A^{\Lambda} \right)^{-1},$$

which completes the proof. □

It is necessary to note that when A is an analytic Toeplitz operator (i.e., $A = T_\varphi$, $\varphi \in H^\infty$) in Theorem 3.1, the invertibility of the operator M_A follows only from the condition

$$\inf_{z \in \mathbb{D}} |\widetilde{A}(z)| > 0,$$

because in this case A is invertible.

Our next result concerns to the Bergman space operators. We recall that

$$\hat{k}_\lambda(z) = \frac{1 - |\lambda|^2}{(1 - \bar{\lambda} z)^2}$$

are the normalized reproducing kernels of the Bergman space L_a^2. These normalized reproducing kernels are the right building blocks for L_a^2. In some sense, they play the role of an orthonormal basis for L_a^2, although they are clearly not mutually orthogonal. (This and other properties of Bergman kernel can be found in Zhu [20].)

The following key lemma (see [20, Theorem 4.4.6]) gives the so-called atomic decomposition for functions in the Bergman space L_a^2.

Lemma 3.2. *There exists a sequence $\Lambda = \{\lambda_n\}_{n \geqslant 1}$ in $\mathbb{D}$ and a constant $C > 0$ with the following properties*:

(a) *For any* $\{a_n\}$ *in* l^2, *the function*

$$f(z)=\sum_{n=1}^{\infty}a_n\frac{1-|\lambda_n|^2}{(1-\overline{\lambda_n}z)^2}$$

is in L_a^2 *with* $\|f\|\leqslant C\|\{a_n\}\|_{l^2}$;

(b) *If* $f\in L_a^2$, *then there is* $\{a_n\}$ *in* l^2, *such that*

$$f(z)=\sum_{n=1}^{\infty}a_n\frac{1-|\lambda_n|^2}{(1-\overline{\lambda_n}z)^2}\quad \textit{and}\quad \big\|\{a_n\}\big\|_{l^2}\leqslant C\|f\|.$$

Theorem 3.3. *Let* $\Lambda=\{\lambda_n\}_{n\geqslant 1}$ *and* $C>0$ *are the same as in Lemma* 3.2. *Let* A *be a linear bounded operator on the Bergman space* L_a^2 *satisfying*

(1) $\sum_{n=1}^{\infty}\|A\hat{k}_{\lambda_n}-\widetilde{A}(\lambda_n)\hat{k}_{\lambda_n}\|^2\stackrel{\text{def}}{=}\tau_A^{\Lambda}<+\infty$ *and*

(2) $\sum_{n=1}^{\infty}\|A^{\star}\hat{k}_{\lambda_n}-\widetilde{A^{\star}}(\lambda_n)\hat{k}_{\lambda_n}\|^2\stackrel{\text{def}}{=}\tau_{A^{\star}}^{\Lambda}<+\infty$,

where $\widetilde{A}$ *denotes the Berezin symbol of operator* A. *If* $|\widetilde{A}(z)|\geqslant\delta>\max\{C^3\tau_A^{\Lambda},C^3\tau_{A^{\star}}^{\Lambda}\}$, $z\in\mathbb{D}$, *then* A *is invertible and*

$$\|A^{-1}\|\leqslant\frac{C^2}{\delta-C^3\tau_A^{\Lambda}}.$$

Proof. If $f\in L_a^2$ is an arbitrary function then by Lemma 3.2, there exists $\{a_n\}$ in l^2 such that

$$f(z)=\sum_{n=1}^{\infty}a_n\hat{k}_{\lambda_n}(z)\quad\text{and}\quad\big\|\{a_n\}\big\|_{l^2}\leqslant C\|f\|.$$

Since $\sup_{z\in\mathbb{D}}|\widetilde{A}(z)|=ber(A)\leqslant\|A\|$, we have that $\{a_n\widetilde{A}(\lambda_n)\}_{n\geqslant 1}\in l^2$, and therefore it follows from the claim (a) of Lemma 3.2 that the function $\sum_{n=1}^{\infty}a_n\widetilde{A}(\lambda_n)\hat{k}_{\lambda_n}$ is in L_a^2 with

$$\left\|\sum_{n=1}^{\infty}a_n\widetilde{A}(\lambda_n)\hat{k}_{\lambda_n}\right\|\leqslant C\big\|\{a_n\widetilde{A}(\lambda_n)\}\big\|_{l^2}. \tag{7}$$

Since

$$\big\|\{a_n\widetilde{A}(\lambda_n)\}\big\|_{l^2}\leqslant ber(A)\big\|\{a_n\}\big\|_{l^2}\leqslant C\,ber(A)\|f\|=C\,ber(A)\left\|\sum_{n=1}^{\infty}a_n\hat{k}_{\lambda_n}\right\|,$$

the inequality (7) means that the diagonal operator $\mathcal{D}_{\{\widetilde{A}(\lambda_n)\}}$ defined in L_a^2 by the formula

$$\mathcal{D}_{\{\widetilde{A}(\lambda_n)\}}\sum_{n=1}^{\infty}a_n\hat{k}_{\lambda_n}=\sum_{n=1}^{\infty}a_n\widetilde{A}(\lambda_n)\hat{k}_{\lambda_n},$$

is bounded operator with

$$\|\mathcal{D}_{\{\widetilde{A}(\lambda_n)\}}\| \leqslant C^2\, ber(A).$$

On the other hand, it follows from the condition of theorem that $|\widetilde{A}(\lambda_n)| \geqslant \delta$ for all $n \geqslant 1$, and therefore $\mathcal{D}_{\{\widetilde{A}(\lambda_n)\}}$ is an invertible in L^2_a, $\mathcal{D}^{-1}_{\{\widetilde{A}(\lambda_n)\}} = \mathcal{D}_{\{\frac{1}{\widetilde{A}(\lambda_n)}\}}$, and

$$\left\|\mathcal{D}_{\{\frac{1}{\widetilde{A}(\lambda_n)}\}}\right\| \leqslant \frac{C^2}{\delta}. \tag{8}$$

Now using condition (1) of the theorem and inequality (8), we have

$$\begin{aligned}
\|Af\|_2 &= \left\| A\sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n} \right\| = \left\| \sum_{n=1}^{\infty} a_n A\hat{k}_{\lambda_n} \right\| \\
&\geqslant \left\| \sum_{n=1}^{\infty} a_n \widetilde{A}(\lambda_n)\hat{k}_{\lambda_n} \right\| - \left\| \sum_{n=1}^{\infty} a_n \big(A\hat{k}_{\lambda_n} - \widetilde{A}(\lambda_n)\hat{k}_{\lambda_n}\big) \right\| \\
&\geqslant \left\|\mathcal{D}_{\{\frac{1}{\widetilde{A}(\lambda_n)}\}}\right\|^{-1} \left\| \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n} \right\| - \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \left\|\big(A\hat{k}_{\lambda_n} - \widetilde{A}(\lambda_n)\hat{k}_{\lambda_n}\big)\right\|^2\right)^{1/2} \\
&\geqslant \left(\frac{\delta}{C^2} - C\tau_A^{\Lambda}\right) \left\| \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n} \right\| \\
&= \left(\frac{\delta}{C^2} - C\tau_A^{\Lambda}\right) \|f\|.
\end{aligned}$$

Thus

$$\|Af\|_2 \geqslant \left(\frac{\delta}{C^2} - C\tau_A^{\Lambda}\right) \|f\| \tag{9}$$

for all $f \in L^2_a$.

Since $|\widetilde{A^\star}(z)| = |\widetilde{A}(z)|$, $z \in \mathbb{D}$, by similar arguments we can prove that

$$\|A^\star f\| \geqslant \left(\frac{\delta}{C^2} - C\tau_{A^\star}^{\Lambda}\right) \|f\| \tag{10}$$

for all $f \in L^2_a$.

Also, since $\delta/C^2 > C\max\{\tau_A^{\Lambda}, \tau_{A^\star}^{\Lambda}\}$ (see the condition of theorem), inequalities (9) and (10) mean that A is invertible in L^2_a and

$$\left\|A^{-1}\right\| \leqslant \left(\frac{\delta}{C^2} - C\tau_A^{\Lambda}\right)^{-1},$$

which proves the theorem. □

Our more general result is the following theorem.

Theorem 3.4. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a functional Hilbert space of complex-valued functions on a (non-empty) set Ω with the orthonormal basis $\{e_n(z)\}_{n\geqslant 0}$, and let A be a linear bounded operator on $\mathcal{H}$ such that*:

(a) $|\widetilde{A}(z)| \geqslant \delta > 0$, $z \in \Omega$;
(b) *there exists a sequence* $\Lambda = \{\lambda_n\}_{n\geqslant 0} \subset \Omega$ *such that*:
  (1) $(\sum_{n=0}^{\infty} \|Ae_n(z) - \widetilde{A}(\lambda_n)e_n(z)\|^2)^{1/2} \stackrel{\text{def}}{=} \delta_A^{\Lambda} < +\infty$;
  (2) $(\sum_{n=0}^{\infty} \|A^{\star}e_n(z) - \widetilde{A^{\star}}(\lambda_n)e_n(z)\|^2)^{1/2} \stackrel{\text{def}}{=} \delta_{A^{\star}}^{\Lambda} < +\infty$.

If $\delta > \max\{\delta_A^{\Lambda}, \delta_{A^{\star}}^{\Lambda}\}$, *then A is invertible and*

$$\|A^{-1}\| \leqslant \left(\delta - \delta_A^{\Lambda}\right)^{-1}.$$

Proof. Let us consider the diagonal operator $\mathcal{D}_{\{\widetilde{A}(\lambda_n)\}}$ with respect to the orthonormal basis $\{e_n(z)\}_{n\geqslant 0}$ of the space $\mathcal{H}$, that is $\mathcal{D}_{\{\widetilde{A}(\lambda_n)\}}e_n(z) = \widetilde{A}(\lambda_n)e_n(z)$, $n \geqslant 0$. Since $\delta \leqslant |\widetilde{A}(\lambda_n)| \leqslant \|A\|$ for all $n \geqslant 0$, we have that

$$\|\mathcal{D}_{\{\widetilde{A}(\lambda_n)\}}\| = \sup_{n\geqslant 0}\left|\widetilde{A}(\lambda_n)\right| \leqslant \|A\|,$$

$\mathcal{D}^{-1}_{\{\widetilde{A}(\lambda_n)\}}$ exists and

$$\mathcal{D}^{-1}_{\{\widetilde{A}(\lambda_n)\}} = \mathcal{D}_{\{\frac{1}{\widetilde{A}(\lambda_n)}\}} \quad \text{and} \quad \left\|\mathcal{D}^{-1}_{\{\widetilde{A}(\lambda_n)\}}\right\| \leqslant \frac{1}{\delta}.$$

Then by considering these and the conditions of theorem, for all $f(z) = \sum_{n=1}^{\infty} a_n e_n(z) \in \mathcal{H}$ we have

$$\begin{aligned}
\|Af\|_{\mathcal{H}} &= \left\|A\sum_{n=0}^{\infty} a_n e_n(z)\right\|_{\mathcal{H}} = \left\|\sum_{n=0}^{\infty} a_n Ae_n(z)\right\|_{\mathcal{H}} \\
&\geqslant \left\|\sum_{n=0}^{\infty} a_n \widetilde{A}(\lambda_n)e_n(z)\right\|_{\mathcal{H}} - \left\|\sum_{n=0}^{\infty} a_n\left(Ae_n(z) - \widetilde{A}(\lambda_n)e_n(z)\right)\right\|_{\mathcal{H}} \\
&\geqslant \frac{1}{\|\mathcal{D}^{-1}_{\{\widetilde{A}(\lambda_n)\}}\|}\left\|\sum_{n=0}^{\infty} a_n e_n(z)\right\|_{\mathcal{H}} - \left(\sum_{n=0}^{\infty}|a_n|^2\right)^{1/2}\left(\sum_{n=0}^{\infty}\left\|\left(Ae_n(z) - \widetilde{A}(\lambda_n)e_n(z)\right)\right\|^2\right)^{1/2} \\
&= \frac{1}{\sup_{n\geqslant 0}\left|\frac{1}{\widetilde{A}(\lambda_n)}\right|}\|f\|_{\mathcal{H}} - \delta_A^{\Lambda}\|f\|_{\mathcal{H}} = \inf_{n\geqslant 0}\left|\widetilde{A}(\lambda_n)\right|\|f\|_{\mathcal{H}} - \delta_A^{\Lambda}\|f\|_{\mathcal{H}} \\
&\geqslant \delta\|f\|_{\mathcal{H}} - \delta_A^{\Lambda}\|f\|_{\mathcal{H}} = \left(\delta - \delta_A^{\Lambda}\right)\|f\|_{\mathcal{H}}.
\end{aligned}$$

Analogously, we can show that

$$\|A^\star f\|_{\mathcal{H}} \geqslant \left(\delta - \delta_{A^\star}^{\Lambda}\right)\|f\|_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H},$$

and hence A is invertible and

$$\left\|A^{-1}\right\| \leqslant \left(\delta - \delta_A^{\Lambda}\right)^{-1}.$$

The theorem is proved. □

Corollary 3.5. *Let $\varphi \in L^\infty(\mathbb{T})$, and denote as before, by $\tilde{\varphi}$ its harmonic extension (by the Poisson formula) into $\mathbb{D}$. Let T_φ be a corresponding Toeplitz operator on the Hardy space H^2. Suppose that $\delta \stackrel{\text{def}}{=} \inf_{z\in\mathbb{D}} |\tilde{\varphi}(z)| > 0$ and there exists a sequence $\Lambda = \{\lambda_n\}_{n\geqslant 0} \subset \mathbb{D}$ such that*

$$\sum_{n=0}^{\infty}\left(\widehat{|\varphi|^2}(0) - 2\operatorname{Re}\overline{\tilde{\varphi}(\lambda_n)}\hat{\varphi}(0) + \left|\tilde{\varphi}(\lambda_n)\right|^2\right) \stackrel{\text{def}}{=} \nu_\varphi^\Lambda < +\infty. \tag{11}$$

If $\delta > \nu_\varphi^\Lambda$, then T_φ is invertible and

$$\left\|T_\varphi^{-1}\right\| \leqslant \left(\delta - \delta_{T_\varphi}^{\Lambda}\right)^{-1}.$$

Proof. An easy computation shows that

$$\begin{aligned}\left\|T_\varphi z^n - \tilde{\varphi}(\lambda_n) z^n\right\|^2 &= \left\|T_\varphi z^n\right\|^2 - 2\operatorname{Re}\overline{\tilde{\varphi}(\lambda_n)}\left\langle T_\varphi z^n, z^n\right\rangle + \left|\tilde{\varphi}(\lambda_n)\right|^2 \\ &\leqslant \left\|\varphi z^n\right\|_{L^2(\mathbb{T})}^2 - 2\operatorname{Re}\overline{\tilde{\varphi}(\lambda_n)}\hat{\varphi}(0) + \left|\tilde{\varphi}(\lambda_n)\right|^2 \\ &= \widehat{|\varphi|^2}(0) - 2\operatorname{Re}\overline{\tilde{\varphi}(\lambda_n)}\hat{\varphi}(0) + \left|\tilde{\varphi}(\lambda_n)\right|^2,\end{aligned}$$

from which by replacing φ with $\bar{\varphi}$ and by considering that $T_\varphi^\star = T_{\bar{\varphi}}$ and $|\widetilde{T}_\varphi(z)| = |\widetilde{T}_\varphi^\star(z)|$, we also have

$$\left\|T_\varphi^\star z^n - \tilde{\bar{\varphi}}(\lambda_n) z^n\right\|^2 \leqslant \widehat{|\varphi|^2}(0) - 2\operatorname{Re}\tilde{\varphi}(\lambda_n)\bar{\hat{\varphi}}(0) + \left|\tilde{\varphi}(\lambda_n)\right|^2.$$

Now, by considering the conditions $\delta > \nu_\varphi^\Lambda$ and (11), and by applying Lemma 2.1 and Theorem 3.4, we have the desired result. □

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