Berezin symbol and invertibility of operators on the functional Hilbert spaces

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Abstract

We give in terms of reproducing kernel and Berezin symbol the sufficient conditions ensuring the invertibility of some linear bounded operators on some functional Hilbert spaces.

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1. Introduction

Let \( T \) be the unit circle \( T = \{ \zeta \in \mathbb{C}: |\zeta| = 1 \} \), \( \varphi \in L^{\infty} = L^{\infty}(T) \), and let \( T_{\varphi} \) be the Toeplitz operator acting in the Hardy space \( H^{2}(\mathbb{D}) \) on the unit disc \( \mathbb{D} = \{ z \in \mathbb{C}: |z| < 1 \} \) by the formula \( T_{\varphi}f = P_{+}\varphi f \), where \( P_{+} \) is the Riesz projector. Let \( \tilde{\varphi} \) denote the harmonic extension of the function \( \varphi \) to \( \mathbb{D} \). In [5] Douglas posed the following problem: if \( \varphi \) is a function in \( L^{\infty} \) for which \( |\tilde{\varphi}(z)| \geq \delta > 0, z \in \mathbb{D} \), then is \( T_{\varphi} \) invertible?

In [18] Tolokonnikov firstly gave a positive answer to this question under the condition that \( \delta \) is near enough to 1, namely, he proved that if

\[
1 \geq |\tilde{\varphi}(z)| \geq \delta > \frac{45}{46}, \quad z \in \mathbb{D},
\]

then \( T_{\varphi} \) is invertible and

\[
\| T_{\varphi}^{-1} \| \leq (1 - 46(1 - \delta))^{-1}.
\]

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This assertion was also proved by Wolff [19]. Nikolskii [15] has somewhat improved the result of Tolokonnikov proving invertibility of $T_\varphi$ and the estimate
\[ \| T_\varphi^{-1} \| \leq (24\delta - 23)^{-1/2} \]
under condition $\delta > 23/24$. Finally, Wolff [19] has constructed a function $\varphi \in L^\infty$ such that $\inf_D |\tilde{\varphi}(z)| > 0$ but the corresponding operator $T_\varphi$ is not invertible, and thus showed that the answer to the question of Douglas is negative in general. Since $\tilde{\varphi}$ coincides with the Berezin symbol $\tilde{T}_\varphi$ of the operator $T_\varphi$ (see Lemma 2.1), in this context the following natural problem arises.

**Problem 1.** Let $A$ be a linear bounded operator acting in the functional Hilbert space $\mathcal{H}(\Omega)$ of complex-valued functions over the some (non-empty) set $\Omega$, such that $|\tilde{A}(z)| \geq \delta$ for all $z \in \Omega$ and for some $\delta > 0$. To find the number $\delta_0$, which can be (more or less) easily computed from the data of $A$, and due to which the inequality
\[ |\tilde{A}(z)| \geq \delta > \delta_0, \quad z \in D, \]
ensures the invertibility of $A$, where $\tilde{A}$ denotes the Berezin symbol of the operator $A$.

In particular, the following problem is also interesting, which is closely related with the finite section method of Böttcher and Silbermann [3].

**Problem 2.** Let $E \subset \mathcal{H}(\Omega)$ be a closed subspace of the functional Hilbert space $\mathcal{H}(\Omega)$, and let $A$ be a linear bounded operator acting in $\mathcal{H}(\Omega)$ such that
\[ |\tilde{A}(z)| \geq \delta \]
for all $z \in \Omega$ and for some $\delta > 0$. To find a number $\delta_0$, such that $\delta > \delta_0$ ensures the invertibility of operator $P_E A | E$ (the compression of the operator $A$ to the subspace $E$), where $P_E$ is an orthogonal projection from $\mathcal{H}(\Omega)$ onto $E$.

In this article we solve these problems in some special cases. Our argument uses the concept of reproducing kernel and Berezin symbol.

2. Notations and preliminaries

2.1. Recall that a functional Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on a (non-empty) set $\Omega$, which has the property that point evaluations are continuous (i.e., for each $\lambda \in \Omega$, the map $f \to f(\lambda)$ is a continuous linear functional on $\mathcal{H}$). Then the Riesz representation theorem ensures that for each $\lambda \in \Omega$ there is a unique element $k_\lambda$ of $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The collection $\{ k_\lambda : \lambda \in \Omega \}$ is called the reproducing kernel of $\mathcal{H}$. It is well known (see, for instance, [8, Problem 37]) that if $\{ e_n \}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by
\[ k_\lambda(z) = \sum_n \overline{e_n(\lambda)} e_n(z). \]
For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. For a bounded linear operator $A$ on $\mathcal{H}$, the function $\tilde{A}$ defined on $\Omega$ by

$$\tilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$$

is the Berezin symbol of $A$, which firstly have been introduced by Berezin [1,2]. It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on $\Omega$ whose values lies in the numerical range of the operator $A$, and hence

$$\sup_{z \in \mathbb{D}} |\tilde{A}(z)| \overset{\text{def}}{=} ber(A) \quad \text{("Berezin number")}$$

$$\leq w(A) \quad \text{(numerical radius)}.$$ 

More typical examples of functional Hilbert spaces are the Hardy and Bergman spaces.

2.2. Let $dm_2$ denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space $L^2_a = L^2_a(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dm_2)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $k_\lambda \in L^2_a$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for every $f \in L^2_a$. It is well known that $k_\lambda(z) = \frac{1}{(1-\lambda z)^2}$. The normalized Bergman reproducing kernel $\hat{k}_\lambda$ is the function $\frac{k_\lambda}{\|k_\lambda\|} = \frac{1-|\lambda|^2}{(1-\lambda z)^2}$.

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space of analytic functions $f(z) = \sum_{n \geq 0} a_n z^n$ defined in the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$, such that $\sum_{n \geq 0} |a_n|^2 < \infty$. Alternately, it can be identified with a closed subspace of the Lebesgue space $L^2 = L^2(\mathbb{T})$ on the unit circle, by associating to each analytic function its radial limit. The algebra of bounded analytic functions on $\mathbb{D}$ is denoted by $H^\infty$. Any $\varphi \in H^\infty$ acts as a multiplication operator on $H^2$, that we will denote by $T_{\varphi}$.

Norm and inner product in $L^2$ or $H^2$ will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. Evaluations at points $\lambda \in \mathbb{D}$ are bounded functionals on $H^2$ and the corresponding reproducing kernel is $k_\lambda(z) = \frac{1}{1-\lambda z}$; thus, $f(\lambda) = \langle f, k_\lambda \rangle$. If $\varphi \in H^\infty$, then $k_\lambda$ is an eigenvector for $T_{\varphi}^*$ and $T_{\varphi}^* k_\lambda = \varphi(\lambda) k_\lambda$. By normalizing $k_\lambda$ we obtain

$$\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|} = \sqrt{1-|\lambda|^2} k_\lambda.$$

2.3. The Berezin symbol have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis (see, for instance, [9–14,16,17,20]). In particular, it is known (see [11,20]) the following result which we will use in what follows.

**Lemma 2.1.** The Berezin symbol $\tilde{T}_{\varphi}$ of the Toeplitz operator $T_{\varphi}$, $\varphi \in L^\infty$, on the Hardy space $H^2$ coincides with the harmonic extension $\tilde{\varphi}$ of the function $\varphi$ into the unit disc $\mathbb{D}$, that is $\tilde{T}_{\varphi}(\lambda) = \tilde{\varphi}(\lambda)$ for all $\lambda \in \mathbb{D}$.

Suppose now $\theta$ is an inner function. We define the corresponding model space by the formula $K_\theta = H^2 \ominus \theta H^2.$
2.4. We recall some basic definitions concerning geometric properties of sequences in a Hilbert space. For most of the definitions and facts below, one can use [7,15] as a main references (see also [4,6]).

Let \( H \) be a complex Hilbert space. If \( \{x_n\}_{n \geq 1} \subset H \), we denote by \( \text{span}\{x_n: n \geq 1\} \) the closure of the linear hull generated by \( \{x_n\}_{n \geq 1} \).

The sequence \( X = \{x_n\}_{n \geq 1} \) is called:

- complete if \( \text{span}\{x_n: n \geq 1\} = H \);
- minimal if for all \( n \geq 1, x_n \notin \text{span}\{x_m: m \neq n\} \);
- uniformly minimal if \( \inf_{n \geq 1} \text{dist}(x_n, \text{span}\{x_m: m \neq n\}) > 0 \);
- a Riesz basis if there exists an isomorphism \( U \) mapping \( X \) onto an orthonormal family \( \{Ux_n: n \geq 1\} \);
- the operator \( U \) will be called the orthogonalizer of \( X \).

The expression “a Riesz basis in \( H \)” means a Riesz basis \( X \) with the completeness property \( \text{span}(X) = H \). It is well known that \( X \) is a Riesz basis in its closed linear span if there are positive constants \( C_1, C_2 \) such that

\[
C_1 \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \leq \left\| \sum_{n \geq 1} a_n x_n \right\| \leq C_2 \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2}
\]

for all finite complex sequences \( \{a_n\}_{n \geq 1} \). Note that if \( U \) is an orthogonalizer of the family \( X \) then the product \( r(X) \) characterizes the deviation of the basis \( X \) from an orthonormal one and \( U \) will be referred to as the Riesz constant of the family \( X \).

3. Results

In this section we partially solve Problems 1 and 2.
**Theorem 3.1.** Let \( \Lambda = \{ \lambda_n \}_{n \geq 1} \) be a Carleson sequence of distinct points in \( \mathbb{D} \), \( B \) the corresponding Blaschke product, and

\[
X \overset{\text{def}}{=} \{ \hat{k}_{\lambda_n}; n \geq 1 \}
\]

be a corresponding Riesz basis in the model space \( K_B = H^2 \ominus BH^2 \) (see assertion (i3) above), and denote by \( r(X) = \| U \| U^{-1} \) the corresponding Riesz constant of the family \( X \). Let \( A \) be a linear bounded operator on the Hardy space \( H^2 \) such that \( A^* K_B \subset K_B \), and denote \( M_A \overset{\text{def}}{=} P_B A|K_B \), where \( P_B \) is an orthogonal projection from \( H^2 \) onto \( K_B \). Suppose that:

(1) \[
\left( \sum_{n=1}^{\infty} \| A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \|^2 \right)^{1/2} \overset{\text{def}}{=} \tau_A < +\infty \quad \text{and}
\]

(2) \[
\left( \sum_{n=1}^{\infty} \| A^* \hat{k}_{\lambda_n} - \tilde{A}^*(\lambda_n) \hat{k}_{\lambda_n} \|^2 \right)^{1/2} \overset{\text{def}}{=} \tau_{A^*} < +\infty,
\]

where \( \tilde{A} \) denotes the Berezin symbol of the operator \( A \). If

\[
\inf_{z \in \mathbb{D}} |\tilde{A}(z)| \overset{\text{def}}{=} \delta > \delta_0 \overset{\text{def}}{=} r(X) \| U \| \max \{ \tau_A, \tau_{A^*} \},
\]

then the operator \( M_A \) is invertible in \( K_B \) and

\[
\| M_A^{-1} \| \leq \left( \frac{\delta}{r(X)} - \| U \| \tau_A \right)^{-1}.
\]

**Proof.** Since \( A \in (C) \), the sequence \( X = \{ \hat{k}_{\lambda_n}; n \geq 1 \} \) is a Riesz basis in \( K_B \) (see assertion (i3) above). If \( U \) is an orthogonalizer of \( X \), then \( \| U \|^{-1} \) and \( \| U^{-1} \| \) are corresponding best constants appearing in (1), that is

\[
\| U \|^{-1} \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \leq \sum_{n \geq 1} a_n \hat{k}_{\lambda_n} \leq \| U^{-1} \| \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2}
\]

for all finite complex sequences \( \{a_n\}_{n \geq 1} \). Now it is clear from (2) and the condition \( |\tilde{A}(z)| \geq \delta > 0, z \in \mathbb{D} \), of the theorem that

\[
\| \sum_{n=1}^{N} a_n \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \| \geq \| U \|^{-1} \left( \sum_{n=1}^{N} |a_n \tilde{A}(\lambda_n)|^2 \right)^{1/2} \geq \delta \| U \|^{-1} \left( \sum_{n=1}^{N} |a_n|^2 \right)^{1/2} \geq \frac{\delta}{\| U \| \| U^{-1} \|} \left( \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right) \| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \|
\]

and hence

\[
\| \sum_{n=1}^{N} a_n \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \| \geq \frac{\delta}{r(X)} \left( \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right), \quad (3)
\]
where $r(X)$ is the Riesz constant of the family $X = \{ \hat{k}_{\lambda_n} : n \geq 1 \}$.

Now using condition (1) of the theorem and inequalities (2) and (3), for every finite $N > 0$ and for arbitrary numbers $a_n \in \mathbb{C}$ ($n = 1, 2, \ldots, N$) we have:

$$
\left\| MA \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \\
= \left\| (PB | KB) \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| = \left\| PB A \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \\
= \left\| \sum_{n=1}^{N} a_n PB (A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n}) + \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\| \\
\geq \left\| \sum_{n=1}^{N} a_n PB \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\| - \left\| \sum_{n=1}^{N} a_n PB (A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n}) \right\| \\
\geq \left\| \sum_{n=1}^{N} a_n \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\| - \left\| \sum_{n=1}^{N} |a_n| \left\| A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\| \right\| \\
\geq \frac{\delta}{r(X)} \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| - \left( \sum_{n=1}^{N} |a_n|^2 \right)^{1/2} \left( \left\| \sum_{n=1}^{N} A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\|^2 \right)^{1/2} \\
\geq \frac{\delta}{r(X)} \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| - \| U \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \left( \sum_{n=1}^{N} \| A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \|^2 \right)^{1/2} \\
\geq \frac{\delta}{r(X)} \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| - \| U \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \\
= \left( \frac{\delta}{r(X)} - \| U \left\| \right. \right) \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\|.
$$

Thus

$$
\left\| MA \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \geq \left( \frac{\delta}{r(X)} - \| U \left\| \right. \right) \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \\
\|MAf\| \geq \left( \frac{\delta}{r(X)} - \| U \left\| \right. \right) \|f\| \\
$$

for all finite $N > 0$ and complex numbers $a_n$, $n = 1, 2, \ldots, N$. Since the Carleson condition implies the Blaschke condition, we have from (4) that

$$
\|MAf\| \geq \left( \frac{\delta}{r(X)} - \| U \left\| \right. \right) \|f\| \\
$$

for all $f \in KB$ (see assertion (i2) above).
Since $A^* K_B \subset K_B$, it is easy to see that

$$M_A^* = (P_B A \mid K_B)^* = A^* \mid K_B.$$  

Analogously, using condition (2) of the theorem, the equality $|\widetilde{A}^*(z)| = |\widetilde{A}(z)|$ ($z \in \mathbb{D}$) and the inequalities (2) it can be proved that (we omit it)

$$\left\| M_A^* \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\| \geq \left( \frac{\delta}{r(X)} - \|U\| \tau^A \right) \left\| \sum_{n=1}^{N} a_n \hat{k}_{\lambda_n} \right\|$$

for all finite $N > 0$ and complex numbers $a_n, n = 1, 2, \ldots, N$, which yields that

$$\left\| M_A^* f \right\| \geq \left( \frac{\delta}{r(X)} - \|U\| \tau^A \right) \|f\|$$

for all $f \in K_B$.

Now by considering that

$$\delta > r(X)\|U\| \max \{\tau^A, \tau^A\} = \delta_0,$$

we deduce from the estimates (5) and (6) that the operator $M_A$ is invertible in $K_B$ and

$$\left\| M_A^{-1} \right\| \leq \left( \frac{\delta}{r(X)} - \|U\| \tau^A \right)^{-1},$$

which completes the proof. □

It is necessary to note that when $A$ is an analytic Toeplitz operator (i.e., $A = T_\varphi, \varphi \in H^\infty$) in Theorem 3.1, the invertibility of the operator $M_A$ follows only from the condition

$$\inf_{z \in \mathbb{D}} |\tilde{A}(z)| > 0,$$

because in this case $A$ is invertible.

Our next result concerns to the Bergman space operators. We recall that

$$\hat{k}_{\lambda}(z) = \frac{1 - |\lambda|^2}{(1 - \bar{\lambda}z)^2}$$

are the normalized reproducing kernels of the Bergman space $L^2_a$. These normalized reproducing kernels are the right building blocks for $L^2_a$. In some sense, they play the role of an orthonormal basis for $L^2_a$, although they are clearly not mutually orthogonal. (This and other properties of Bergman kernel can be found in Zhu [20].)

The following key lemma (see [20, Theorem 4.4.6]) gives the so-called atomic decomposition for functions in the Bergman space $L^2_a$.

**Lemma 3.2.** There exists a sequence $\Lambda = \{\lambda_n\}_{n \geq 1}$ in $\mathbb{D}$ and a constant $C > 0$ with the following properties:

...
(a) For any \( \{a_n\} \) in \( l^2 \), the function
\[
f(z) = \sum_{n=1}^{\infty} a_n \frac{1 - |\lambda_n|^2}{(1 - \lambda_n z)^2}
\]
is in \( L^2_a \) with \( \| f \| \leq C \| \{a_n\} \|_2 \);
(b) If \( f \in L^2_a \), then there is \( \{a_n\} \) in \( l^2 \), such that
\[
f(z) = \sum_{n=1}^{\infty} a_n \frac{1 - |\lambda_n|^2}{(1 - \lambda_n z)^2}
\]
and \( \| \{a_n\} \|_2 \leq C \| f \| \).

**Theorem 3.3.** Let \( A = \{\lambda_n\}_{n \geq 1} \) and \( C > 0 \) are the same as in Lemma 3.2. Let \( A \) be a linear bounded operator on the Bergman space \( L^2_a \) satisfying

1. \( \sum_{n=1}^{\infty} \| A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \|^2 \triangleq \tau_A^A < +\infty \)
2. \( \sum_{n=1}^{\infty} \| A^* \hat{k}_{\lambda_n} - \tilde{A}^*(\lambda_n) \hat{k}_{\lambda_n} \|^2 \triangleq \tau_A^{A^*} < +\infty \),

where \( \tilde{A} \) denotes the Berezin symbol of operator \( A \). If \( |\tilde{A}(z)| \geq \delta > \max\{C^3 \tau_A^A, C^3 \tau_A^{A^*}\} \), \( z \in \mathbb{D} \), then \( A \) is invertible and
\[
\| A^{-1} \| \leq \frac{C^2}{\delta - C^3 \tau_A^{A^*}}.
\]

**Proof.** If \( f \in L^2_a \) is an arbitrary function then by Lemma 3.2, there exists \( \{a_n\} \) in \( l^2 \) such that
\[
f(z) = \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n}(z) \quad \text{and} \quad \| \{a_n\} \|_2 \leq C \| f \|.
\]

Since \( \sup_{z \in \mathbb{D}} |\tilde{A}(z)| = ber(A) \leq \| A \| \), we have that \( \{a_n \tilde{A}(\lambda_n)\}_{n \geq 1} \in l^2 \), and therefore it follows from the claim (a) of Lemma 3.2 that the function \( \sum_{n=1}^{\infty} a_n \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \) is in \( L^2_a \) with
\[
\left\| \sum_{n=1}^{\infty} a_n \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\| \leq C \left\{ \{a_n \tilde{A}(\lambda_n)\} \right\}_2.
\]

Since
\[
\| \{a_n \tilde{A}(\lambda_n)\} \|_2 \leq ber(A) \| \{a_n\} \|_2 \leq C ber(A) \| f \| = C ber(A) \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n},
\]
the inequality (7) means that the diagonal operator \( D_{\{\tilde{A}(\lambda_n)\}} \) defined in \( L^2_a \) by the formula
\[
D_{\{\tilde{A}(\lambda_n)\}} \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n} = \sum_{n=1}^{\infty} a_n \tilde{A}(\lambda_n) \hat{k}_{\lambda_n},
\]

\( D_{\{\tilde{A}(\lambda_n)\}} \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n} = \sum_{n=1}^{\infty} a_n \tilde{A}(\lambda_n) \hat{k}_{\lambda_n}, \)
is bounded operator with
\[ \| D(\tilde{A}(\lambda_n)) \| \leq C^2 \text{ber}(A). \]

On the other hand, it follows from the condition of theorem that \( |\tilde{A}(\lambda_n)| \geq \delta \) for all \( n \geq 1 \), and therefore \( D(\tilde{A}(\lambda_n)) \) is an invertible in \( L^2_\alpha \), \( D^{-1}(\tilde{A}(\lambda_n)) = D\left(\frac{1}{\tilde{A}(\lambda_n)}\right) \), and
\[
\| D\left(\frac{1}{\tilde{A}(\lambda_n)}\right) \| \leq \frac{C^2}{\delta}. \tag{8}
\]

Now using condition (1) of the theorem and inequality (8), we have
\[
\| Af \|_2 = \left\| \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n} \right\|
= \left\| \sum_{n=1}^{\infty} a_n A \hat{k}_{\lambda_n} \right\|
\geq \left\| \sum_{n=1}^{\infty} a_n \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \right\| - \left\| \sum_{n=1}^{\infty} a_n (A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n}) \right\|
\geq \| D\left(\frac{1}{\tilde{A}(\lambda_n)}\right) \|^{-1} \left\| \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n} \right\| - \left( \sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \| A \hat{k}_{\lambda_n} - \tilde{A}(\lambda_n) \hat{k}_{\lambda_n} \|^2 \right)^{1/2}
\geq \left( \frac{\delta}{C^2} - C \tau_A A \right) \left\| \sum_{n=1}^{\infty} a_n \hat{k}_{\lambda_n} \right\|
= \left( \frac{\delta}{C^2} - C \tau_A A \right) \| f \|.
\]
Thus
\[
\| Af \|_2 \geq \left( \frac{\delta}{C^2} - C \tau_A A \right) \| f \| \tag{9}
\]
for all \( f \in L^2_\alpha \).

Since \( |\tilde{A}^*(z)| = |\tilde{A}(z)|, z \in \mathbb{D} \), by similar arguments we can prove that
\[
\| A^* f \| \geq \left( \frac{\delta}{C^2} - C \tau_A^* A^* \right) \| f \| \tag{10}
\]
for all \( f \in L^2_\alpha \).

Also, since \( \delta/C^2 > C \max\{\tau_A A, \tau_A^* A^*\} \) (see the condition of theorem), inequalities (9) and (10) mean that \( A \) is invertible in \( L^2_\alpha \) and
\[
\| A^{-1} \| \leq \left( \frac{\delta}{C^2} - C \tau_A A \right)^{-1},
\]
which proves the theorem. \( \square \)
Our more general result is the following theorem.

**Theorem 3.4.** Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a functional Hilbert space of complex-valued functions on a (non-empty) set $\Omega$ with the orthonormal basis $\{e_n(z)\}_{n \geq 0}$, and let $A$ be a linear bounded operator on $\mathcal{H}$ such that:

(a) $|\tilde{A}(z)| \geq \delta > 0$, $z \in \Omega$;
(b) there exists a sequence $\Lambda = \{\lambda_n\}_{n \geq 0} \subset \Omega$ such that:

1. $\left( \sum_{n=0}^{\infty} \|A e_n(z) - \tilde{A}(\lambda_n) e_n(z)\|^2 \right)^{1/2} \overset{\text{def}}{=} \delta^A < +\infty$;
2. $\left( \sum_{n=0}^{\infty} \|A^* e_n(z) - \tilde{A}^*(\lambda_n) e_n(z)\|^2 \right)^{1/2} \overset{\text{def}}{=} \delta^A^* < +\infty$.

If $\delta > \max\{\delta^A, \delta^A^*\}$, then $A$ is invertible and

$$\|A^{-1}\| \leq (\delta - \delta^A)^{-1}.$$

**Proof.** Let us consider the diagonal operator $D_{\{\tilde{A}(\lambda_n)\}}$ with respect to the orthonormal basis $\{e_n(z)\}_{n \geq 0}$ of the space $\mathcal{H}$, that is $D_{\{\tilde{A}(\lambda_n)\}} e_n(z) = \tilde{A}(\lambda_n) e_n(z)$, $n \geq 0$. Since $\delta \leq |\tilde{A}(\lambda_n)| \leq \|A\|$ for all $n \geq 0$, we have that

$$\|D_{\{\tilde{A}(\lambda_n)\}}\| = \sup_{n \geq 0} |\tilde{A}(\lambda_n)| \leq \|A\|,$$

$D_{\{\tilde{A}(\lambda_n)\}}^{-1}$ exists and

$$D_{\{\tilde{A}(\lambda_n)\}}^{-1} = D_{\{\frac{1}{\tilde{A}(\lambda_n)}\}} \quad \text{and} \quad \|D_{\{\tilde{A}(\lambda_n)\}}^{-1}\| \leq \frac{1}{\delta}.$$

Then by considering these and the conditions of theorem, for all $f(z) = \sum_{n=1}^{\infty} a_n e_n(z) \in \mathcal{H}$ we have

$$\|Af\|_{\mathcal{H}} = \left\| A \sum_{n=0}^{\infty} a_n e_n(z) \right\|_{\mathcal{H}} = \left\| \sum_{n=0}^{\infty} a_n A e_n(z) \right\|_{\mathcal{H}}$$

$$\geq \left\| \sum_{n=0}^{\infty} a_n \tilde{A}(\lambda_n) e_n(z) \right\|_{\mathcal{H}} - \left\| \sum_{n=0}^{\infty} a_n (A e_n(z) - \tilde{A}(\lambda_n) e_n(z)) \right\|_{\mathcal{H}}$$

$$\geq \frac{1}{\|D_{\{\tilde{A}(\lambda_n)\}}^{-1}\|} \left\| \sum_{n=0}^{\infty} a_n e_n(z) \right\|_{\mathcal{H}} - \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} \|A e_n(z) - \tilde{A}(\lambda_n) e_n(z)\|^2 \right)^{1/2}$$

$$= \frac{1}{\sup_{n \geq 0} \left| \frac{1}{\tilde{A}(\lambda_n)} \right|} \||f|_{\mathcal{H}} - \delta^A |f|_{\mathcal{H}}\| = \inf_{n \geq 0} |\tilde{A}(\lambda_n)| \|f|_{\mathcal{H}} - \delta^A |f|_{\mathcal{H}}$$

$$\geq \delta \|f|_{\mathcal{H}} - \delta^A |f|_{\mathcal{H}} = (\delta - \delta^A) \|f|_{\mathcal{H}}.$$
Analogously, we can show that
\[ \| A^* f \|_\mathcal{H} \geq (\delta - \delta_A^A) \| f \|_\mathcal{H} \quad \text{for all } f \in \mathcal{H}, \]
and hence \( A \) is invertible and
\[ \| A^{-1} \| \leq (\delta - \delta_A^A)^{-1}. \]
The theorem is proved. \( \Box \)

**Corollary 3.5.** Let \( \varphi \in L^\infty(\mathbb{T}) \), and denote as before, by \( \tilde{\varphi} \) its harmonic extension (by the Poisson formula) into \( \mathbb{D} \). Let \( T\varphi \) be a corresponding Toeplitz operator on the Hardy space \( H^2 \). Suppose that \( \delta \overset{\text{def.}}{=} \inf_{z \in \mathbb{D}} |\tilde{\varphi}(z)| > 0 \) and there exists a sequence \( \Lambda = \{\lambda_n\}_{n \geq 0} \subset \mathbb{D} \) such that
\[ \sum_{n=0}^{\infty} \left( |\tilde{\varphi}|^2(0) - 2 \Re \tilde{\varphi}(\lambda_n) \tilde{\varphi}(0) + |\tilde{\varphi}(\lambda_n)|^2 \right) \overset{\text{def.}}{=} \nu_\varphi^A < +\infty. \] (11)
If \( \delta > \nu_\varphi^A \), then \( T\varphi \) is invertible and
\[ \| T\varphi^{-1} \| \leq (\delta - \delta_{T\varphi}^A)^{-1}. \]

**Proof.** An easy computation shows that
\[ \| T\varphi z^n - \tilde{\varphi}(\lambda_n) z^n \|_2^2 = \| T\varphi z^n \|_2^2 - 2 \Re \tilde{\varphi}(\lambda_n) \langle T\varphi z^n, z^n \rangle + |\tilde{\varphi}(\lambda_n)|^2 \]
\[ \leq \| \varphi z^n \|_{L^2(\mathbb{T})}^2 - 2 \Re \tilde{\varphi}(\lambda_n) \tilde{\varphi}(0) + |\tilde{\varphi}(\lambda_n)|^2 \]
\[ = |\tilde{\varphi}|^2(0) - 2 \Re \tilde{\varphi}(\lambda_n) \tilde{\varphi}(0) + |\tilde{\varphi}(\lambda_n)|^2, \]
from which by replacing \( \varphi \) with \( \tilde{\varphi} \) and by considering that \( T\varphi^* = T\tilde{\varphi} \) and \( |T\varphi(z)| = |T\tilde{\varphi}(z)| \), we also have
\[ \| T\varphi^* z^n - \tilde{\varphi}(\lambda_n) z^n \|_2^2 \leq |\tilde{\varphi}|^2(0) - 2 \Re \tilde{\varphi}(\lambda_n) \tilde{\varphi}(0) + |\tilde{\varphi}(\lambda_n)|^2. \]

Now, by considering the conditions \( \delta > \nu_\varphi^A \) and (11), and by applying Lemma 2.1 and Theorem 3.4, we have the desired result. \( \Box \)

**References**