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Spectral Method of Decoupling the Vorticity and Stream Function for the Incompressible Fluid Flows

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Abstract—A spectral method is proposed for the vorticity-stream function equations of the incompressible fluid flows. It is effective to overcome the lack of vorticity boundary condition. This method decouples the vorticity and stream function. At each time step, first, the vorticity is explicitly solved and the stream function is evaluated by a Poisson-like equation; then the vorticity is determined by a Poisson-like equation again. The numerical experiments show that this method is of efficiency and high accuracy.

Keywords—Spectral method, Vorticity-stream function formulation.

1. INTRODUCTION

For the incompressible fluid flows, there are many numerical studies in the literature based on the vorticity-stream function formulation (see [1,2]). One of the main difficulties for solving such a problem is due to the lack of vorticity boundary condition. The vorticity and stream function are coupled together through the boundary conditions and the nonlinear term. Physically, the vorticity on the boundary is unknown and the boundary conditions for the stream function are given. Many efforts have been devoted in developing discrete approximations to the vorticity on the boundary. A usual method is to approximate the vorticity by a finite difference using the boundary conditions of the stream function. Dennis *et al.* [3–5] first presented some special integral conditions for the vorticity. Numerical results show that the methods with such integral boundary conditions are more efficient and stable for solving the incompressible flow problems. However, the structure of the discrete system is much more complicated. The biharmonic formulation of the stream function may be taken as another alternative to overcome the difficulty of the boundary condition for vorticity. But it is very difficult to solve the biharmonic equation, and there is scarcely any standard fast algorithm.

In this paper, a spectral method is developed for remedying the lack of vorticity boundary condition. To decouple the vorticity from the stream function at each time step, first, the vorticity transport equation is explicitly discretized, and a tau method is adopted to solve the

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system. An important feature for this tau method is that the expansion coefficients of the vorticity are decoupled and only part of them appears in the discrete vorticity transport equation. The remaining coefficients and the stream function will be determined by the Poisson equation with the boundary conditions of the stream function. However, at this step it is not necessary to evaluate the remaining coefficients of the vorticity. Second, this Poisson equation for the stream function is solved by Galerkin method where the equations related to the remaining unknown coefficients of the vorticity are replaced by the Neumann boundary condition of the stream function. Finally, we solve the Poisson-like equation of the vorticity with the boundary conditions determined by the stream function. The advantages of this method are simple, robust, and efficient. It can be implemented easily and is of high accuracy. Numerical experiments show these features.

2. DESCRIPTION OF THE METHOD

The vorticity-stream function formulation of the viscous incompressible flow is as follows:

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + (\mathbf{u} \cdot \nabla)\zeta - \nu \nabla^2 \zeta &= f, \qquad & x,t \in \Omega \times (0,T), \\ \nabla^2 \psi &= -\zeta, \qquad & x,t \in \Omega \times (0,T), \\ \psi|_{\partial\Omega} &= a, \\ \left.\frac{\partial \psi}{\partial n}\right|_{\partial\Omega} &= b, \end{aligned} \tag{2.1}$$

where ψ and ζ are the dimensionless stream function and vorticity, respectively, $\mathbf{u} = (u,v) = (\partial_y \psi, -\partial_x \psi)$ is the velocity, and ν is the kinetic viscosity. The initial velocity field $\mathbf{u}_0 = (u_0, v_0)$ provides the initial condition for the vorticity

$$\zeta_0 = -\partial_y u_0 + \partial_x v_0.$$

In this formulation ζ and ψ are coupled through the nonlinear term and the boundary conditions. In the temporal discretization, the nonlinear term can be explicitly treated. Then the discrete system is coupled only by the boundary conditions.

Let X and Y denote the finite dimensional spaces of polynomials. The dimension of X depends upon the dimension of Y and $\frac{\partial\psi}{\partial n}|_{\partial\Omega}$. Let $(\cdot,\cdot)$ be the L^2 scalar product, and $H^1(\Omega)$ and $H_0^1(\Omega)$ denote the usual Sobolev spaces.

The spectral method for solving (2.1) is as follows.

(1) The initial values are computed by

$$\begin{aligned} \mathbf{u}^0 &= (u^0, v^0) \in X^2, \\ (\mathbf{u}^0, \phi) &= (\mathbf{u}_0, \phi), \qquad & \forall \phi \in X \cap H_0^1(\Omega), \\ \mathbf{u}^0|_{\partial\Omega} &= \mathbf{u}_0|_{\partial\Omega}, \end{aligned}$$

$$\begin{aligned} \zeta^0 &\in X, \\ (\zeta^0, \phi) &= (\zeta_0, \phi), \qquad & \forall \phi \in X \cap H_0^1(\Omega), \\ \zeta^0|_{\partial\Omega} &= \zeta_0|_{\partial\Omega}. \end{aligned}$$

(2) For the vorticity transport equation, an explicit temporal discretization is employed with the step size τ. The tau method is used in the spatial discretization to find $\zeta^* \in X$ such that

$$\left(\frac{\zeta^* - \zeta^n}{\tau}, \phi\right) = \left(f^n + \nu \nabla^2 \zeta^n - \mathbf{u}^n \cdot \nabla \zeta^n, \phi\right), \qquad \forall \phi \in Y.$$

The expansion coefficients of ζ^* are partly defined by the above equation. The remaining coefficients will be determined by the coupled boundary condition. But in practical computation, it is not necessary to evaluate them. Assume that $\{P_k\}_{k=0}^{M}$ are the orthogonal bases for X. Then the expansion of ζ^* is

$$\zeta^* = \sum_{k=0}^{M} a_k^* P_k.$$

Denote by m the number of the discrete Neumann conditions of the stream function. We set $Y := \{P_k\}_{k=0}^{M-m}$. The expansion coefficients a_k^*, $k = 0, \dots, M-m$, are obtained by the tau method, while the expansion coefficients a_k^*, $k = M-m+1, \dots, M$, are unknown. They will be determined by the boundary condition $\frac{\partial \psi}{\partial n}|_{\partial\Omega}$.

(3) For the Poisson equation of the stream function, the Dirichlet problem is considered. Its weak form is to find $\psi \in H^1(\Omega)$,

$$(\nabla\psi, \nabla\phi) = (\zeta, \phi), \qquad \forall\, \phi \in H_0^1(\Omega),$$
$$\psi|_{\partial\Omega} = a.$$

The Galerkin solution $\psi \in X$ satisfies

$$(\nabla\psi^{n+1}, \nabla\phi) = (\zeta^*, \phi), \qquad \forall\, \phi \in X \cap H_0^1(\Omega),$$
$$\psi^{n+1}|_{\partial\Omega} = a.$$

The equations associated with the undetermined coefficients of ζ^* are replaced by the boundary condition $\frac{\partial \psi}{\partial n}|_{\partial\Omega}$. Then the stream function ψ^{n+1} is uniquely determined.

We can choose the suitable basis functions for $X \cap H_0^1(\Omega)$. For example, take P_k as a Legendre polynomial in the one-dimensional case. Then

$$\phi_k = \begin{cases} P_k - P_0, & k \text{ even}, \\ P_k - P_1, & k \text{ odd}, \end{cases}$$

is a basis function for $X \cap H_0^1(\Omega)$. In the two-dimensional case, the basis function is introduced by tensor product. By the orthogonal property of the Legendre polynomials, we obtain a total of m equations related with a_k^*, $k = M-m+1, \dots, M$. These equations are replaced by the discrete Neumann conditions of the stream function.

(4) Solve the Poisson-like equation of the vorticity. The vorticity on the boundary is obtained by the stream function ψ^{n+1}:

$$\left(\frac{\zeta^{n+1} - \zeta^n}{\tau} - \nu\nabla^2\zeta^{n+1}, \phi\right) = (f^n - \mathbf{u}^n \cdot \nabla\zeta^n, \phi), \qquad \forall\, \phi \in X \cap H_0^1(\Omega),$$
$$\zeta^{n+1}\big|_{\partial\Omega} = -\nabla^2\psi^{n+1}\big|_{\partial\Omega}.$$

(5) Update the velocity. We have

$$u^{n+1} = \partial_y \psi^{n+1},$$
$$v^{n+1} = -\partial_x \psi^{n+1}.$$

This method is simple and can be implemented easily. The standard fast algorithms can be used to solve these two Poisson-like equations. The above approximation to the vorticity on the boundary is more reasonable and natural and is expected to be more accurate since the approximate solution ψ^n exactly satisfies the two physical boundary conditions. This is one of the reasons the method achieves the high accuracy.

The above method is different from the influence matrix method for the spectral solution of the vorticity-stream function equations. In the influence matrix method, the vorticity and stream function are decomposed into three components. Each of them is the solution of a second-order equation supplemented by the Dirichlet boundary conditions. The linear combination of the solutions to the second-order equations satisfies the Neumann boundary condition of the stream function and gives the vorticity boundary condition. Both spectral methods have the spectral convergence.

3. NUMERICAL RESULTS

In this section, two numerical experiments are implemented with the method described in the above section. One is the Plane Poiseuille flow and the other is an artificial problem with the exact solution. In the plane channel flow, the height (y-direction) between the walls is 2. The boundary conditions are periodic in the x-direction and nonperiodic in the y-direction. For the artificial problem, we take all the functions be 2π-periodic for the variable x. The schemes are constructed by using the Fourier spectral method in the periodic direction and the Chebyshev spectral method in the nonperiodic direction. So the standard FFT can be applied in both directions.

Let N_x and N_y be positive integers. P_{N_x} is the space of all the trigonometric polynomials of degree $\leq N_x$, and P_{N_y} the space of all the algebraic polynomials of degree $\leq N_y$. Since the problems are periodic in the x-direction, the boundary conditions for the stream function in the y-direction are

$$\psi(x,-1,t) = a_1, \qquad \psi(x,1,t) = a_2,$$
$$\frac{\partial \psi}{\partial y}(x,-1,t) = b_1, \qquad \frac{\partial \psi}{\partial y}(x,1,t) = b_2.$$

Let

$$X = P_{N_x} \times P_{N_y}, \qquad Y = P_{N_x} \times P_{N_y-2}.$$

The Chebyshev polynomials are mutually orthogonal on $(-1,1)$ with respect to the weight function $w = (1-y^2)^{-1/2}$. The associated inner product with the weight function w is introduced as in [6].

3.1. Plane Poiseuille Flow

The velocities of the equilibrium state are

$$\bar{u}(x,y) = 1 - y^2, \qquad \bar{v}(x,y) = 0.$$

The corresponding vorticity and stream function are

$$\bar{\zeta} = 2y, \qquad \bar{\psi} = y - \frac{1}{3}y^3.$$

Small perturbations are added to this state, i.e.,

$$u(x,y,t) = \bar{u} + \epsilon \,\mathrm{Real}\left\{\hat{u}(y)e^{i(\alpha x - \omega t)}\right\},$$
$$v(x,y,t) = \bar{v} + \epsilon \,\mathrm{Real}\left\{\hat{v}(y)e^{i(\alpha x - \omega t)}\right\},$$

where Real$\{z\}$ represents the real part of z and $\hat{u}(y)$, $\hat{v}(y)$, and ω are the solutions of the Orr-Sommerfeld eigenvalue problem (see [6, (6.4)]). When Re $= 7500$ ($\nu = 1/\mathrm{Re}$) and $\alpha = 1$, the eigenvalue of the only growing mode is $\omega = \omega_r + i\omega_i = 0.24989154 + i \times 0.00223497$. When $\alpha = -1$, the eigenvalue of the conjugate growing mode is $\omega = -\omega_r + i\omega_i$. The initial values are

$$u(x,y,0) = \partial_y \psi = \bar{u} + \epsilon \,\mathrm{Real}\left\{\hat{u}(y)e^{ix}\right\} = 1 - y^2 + \epsilon(\hat{u}_r \cos x - \hat{u}_i \sin x),$$
$$v(x,y,0) = -\partial_x \psi = \epsilon \,\mathrm{Real}\left\{\hat{v}(y)e^{ix}\right\} = \epsilon(\hat{v}_r \cos x - \hat{v}_i \sin x).$$

Here we take $\epsilon = 0.0001$.

The boundary conditions of the stream function are as follows:

$$\psi(x,\pm 1,t) = \pm\frac{2}{3},$$
$$\frac{\partial \psi}{\partial y}(x,\pm 1,t) = 0,$$

and the initial value of the vorticity is

$$\zeta(x,y,0) = -\partial_y u(x,y,0) + \partial_x v(x,y,0).$$

The perturbation kinetic energy is

$$E(t) = \frac{1}{2}\int_{-1}^{1}\int_{0}^{2\pi}\left[(u-\bar{u})^2 + v^2\right]\,dx\,dy = E(0)e^{2\omega_i t}.$$

The calculations are carried out with $N_x = 8$, $N_y = 32$, $\tau = 10^{-3}$, and $T = 50$. In Figure 1, $\ln(E(t)/E(0))$ is given. The exact straight line $2\omega_i t$ and the numerical results of our method are shown in Figure 1. The numerical results are much better than the results obtained by the component-consistent pressure correction (CCPC) projection method on a 129×128 grid in [7]. This shows that our method has high accuracy.

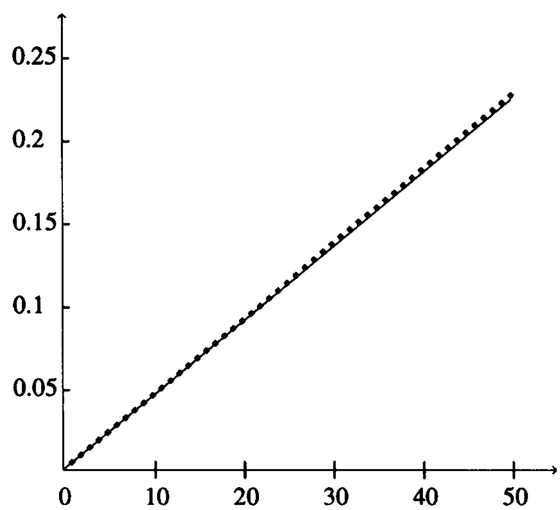

Figure 1. $\ln(E(t)/E(0))$ of plane Poiseuille flow.

3.2. A Problem Having Exact Solutions

To show the accuracy of the method described in Section 2, we choose f, a, and b suitably such that (2.1) admits the following solution:

$$\zeta(x,y,t) = 4ce^{bt}\left(y^2-1\right)\left(y^2-8\right)\sin 2x - \frac{12c}{100}e^{bt}\left(y^2-1\right),$$
$$\psi(x,y,t) = ce^{bt}\left(y^2-1\right)\left(y^2-5\right)\left(\sin 2x + \frac{1}{100}\right).$$

The relative errors $e(\zeta(t))$ and $e(\psi(t))$ are defined by

$$e(\zeta(t)) := \frac{\|\zeta - \zeta^n\|_{L^2(\Omega)}}{\|\zeta\|_{L^2(\Omega)}},$$
$$e(\psi(t)) := \frac{\|\psi - \psi^n\|_{L^2(\Omega)}}{\|\psi\|_{L^2(\Omega)}}.$$

We take $b = c = 0.1$, $N_x = N_y = 8$, and $\tau = 0.005$. In Tables 1 and 2, the numerical results are reported for our method. Obviously, it is very accurate. These display the advantages of our methods.

Table 1. $\nu = 1/5000$.

	$e(\zeta(t))$	$e(\psi(t))$
$t = 0.5$	0.173509E−3	0.128661E−4
$t = 1.0$	0.294191E−3	0.249387E−4
$t = 1.5$	0.376736E−3	0.363026E−4
$t = 2.0$	0.432002E−3	0.470319E−4
$t = 2.5$	0.477841E−3	0.575869E−4

Table 2. $\nu = 1/10000$.

	$e(\zeta(t))$	$e(\psi(t))$
$t = 0.5$	0.184288E−3	0.127560E−4
$t = 1.0$	0.334653E−3	0.247946E−4
$t = 1.5$	0.457028E−3	0.361716E−4
$t = 2.0$	0.560271E−3	0.471344E−4
$t = 2.5$	0.848337E−3	0.681665E−4

4. THE CONCLUSION

We proposed a new spectral method to overcome the lack of the boundary condition of the vorticity. The method only solves two Poisson-like equations. It can be easily implemented and has high accuracy. At each time level, the approximate solution of the stream function satisfies the boundary conditions $\psi|_{\partial\Omega}$ and $\frac{\partial\psi}{\partial n}|_{\partial\Omega}$ exactly. It avoids the errors of the approximation of the stream function on the boundary.

Though we only use the first-order temporal discretization in this paper, classical Runge-Kutta methods can be used to improve the accuracy of the time integration. In addition, high order multilevel schemes can be constructed in a similar way.

The numerical results show that the method is very efficient and accurate.

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