Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 325 (2007) 612–623
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Vector Lyapunov functions for practical stability of nonlinear impulsive functional differential equations

Ivanka M. Stamova

Department of Mathematics, Bourgas Free University, 8000 Bourgas, Bulgaria

Received 29 September 2005
Available online 9 March 2006
Submitted by William F. Ames

Abstract

This paper studies the practical stability of the solutions of nonlinear impulsive functional differential equations. The obtained results are based on the method of vector Lyapunov functions and on differential inequalities for piecewise continuous functions. Examples are given to illustrate our results.

Keywords: Practical stability; Impulsive functional differential equations; Vector Lyapunov functions

1. Introduction

Impulsive differential equations arise naturally from a wide variety of applications such as aircraft control, inspection process in operations research, drug administration, and threshold theory in biology. There has been a significant development in the theory of impulsive differential equations in the past 10 years (see monographs [3,4,13,20]). Now there also exists a well-developed qualitative theory of functional differential equations [2,9–12]. However, not so much has been developed in the direction of impulsive functional differential equations. In the few publications dedicated to this subject, earlier works were done by Anokhin [1] and Gopalsamy and Zhang [8]. Recently, some qualitative properties (oscillation, asymptotic behavior and stability) are investigated by several authors (see [5–7,18,21,23,24]).

The efficient applications of impulsive functional differential equations to mathematical simulation requires the finding of criteria for stability of their solutions.

E-mail address: stamova@bfu.bg.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.02.019

In the study of Lyapunov stability, an interesting set of problems deal with bringing sets close to a certain state, rather than the state $x = 0$. The desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Many problems fall into this category including the travel of a space vehicle between two points, an aircraft or a missile which may oscillate around a mathematically unstable course yet its performance may be acceptable, the problem in a chemical process of keeping the temperature within certain bounds, etc. Such considerations led to the notion of practical stability which is neither weaker nor stronger than Lyapunov stability. The main results in this prospect are due to Martynyuk [14,16,17].

It is well known that employing several Lyapunov functions in the investigation of the qualitative behavior of the solutions of differential equations is more useful than employing a single one, since each function can satisfy less rigid requirements. Hence, the corresponding theory, known as the method of vector Lyapunov functions, offers a very flexible mechanism [15].

In this paper, we use piecewise continuous vector Lyapunov functions to study practical stability of the solutions of nonlinear impulsive functional differential equations. The main results are obtained by means of the comparison principle coupled with the Razumikhin technique [14,19]. Examples are given to illustrate our results.

2. Statement of the problem. Preliminary notes and definitions

Let R^n be the n-dimensional Euclidean space with norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$, Ω be a bounded domain in R^n containing the origin and $R_+ = [0, \infty)$.

Let $t_0 \in R$, $\tau > 0$.

Consider the system of impulsive functional differential equations

$$\begin{cases} \dot{x}(t) = f(t, x(t), x_t), & t > t_0,\ t \neq t_k, \\ \Delta x(t_k) = x(t_k + 0) - x(t_k) = I_k(x(t_k)), & t_k > t_0,\ k = 1, 2, \ldots, \end{cases} \tag{1}$$

where $f : (t_0, \infty) \times \Omega \times D \to R^n$; $D = \{\phi : [-\tau, 0] \to \Omega,\ \phi(t)$ is continuous everywhere except at finite number of points $\tilde{t}$ at which $\phi(\tilde{t} - 0)$ and $\phi(\tilde{t} + 0)$ exist and $\phi(\tilde{t} - 0) = \phi(\tilde{t})\}$; $I_k : \Omega \to \Omega$, $k = 1, 2, \ldots$; $t_0 < t_1 < t_2 < \cdots$; $\lim_{k\to\infty} t_k = \infty$ and for $t > t_0$, $x_t \in D$ is defined by $x_t = x(t + s)$, $-\tau \leqslant s \leqslant 0$.

Let $\varphi_0 \in D$. Denote by $x(t; t_0, \varphi_0)$ the solution of system (1) satisfying the initial conditions:

$$\begin{cases} x(t; t_0, \varphi_0) = \varphi_0(t - t_0), & t_0 - \tau \leqslant t \leqslant t_0, \\ x(t_0 + 0; t_0, \varphi_0) = \varphi_0(0). \end{cases} \tag{2}$$

The solution $x(t) = x(t; t_0, \varphi_0)$ of the initial value problem (1), (2) is characterized by the following:

(a) For $t_0 - \tau \leqslant t \leqslant t_0$ the solution $x(t)$ satisfied the initial conditions (2).
(b) For $t_0 < t \leqslant t_1$, $x(t)$ coincides with the solution of the problem

$$\dot{x}(t) = f\big(t, x(t), x_t\big), \quad t > t_0,$$
$$x_{t_0} = \varphi_0(s), \quad -\tau \leqslant s \leqslant 0.$$

At the moment $t = t_1$ the mapping point $(t, x(t; t_0, \varphi_0))$ of the extended phase space jumps momentarily from the position $(t_1, x(t_1; t_0, \varphi_0))$ to the position $(t_1, x(t_1; t_0, \varphi_0) + I_1(x(t_1; t_0, \varphi_0)))$.

(c) For $t_1 < t \leqslant t_2$ the solution $x(t)$ coincides with the solution of

$$\begin{cases} \dot{y}(t) = f(t, y(t), y_t), & t > t_1, \\ y_{t_1} = \varphi_1, & \varphi_1 \in D, \end{cases}$$

where

$$\varphi_1(t - t_1) = \begin{cases} \varphi_0(t - t_1), & t \in [t_0 - \tau, t_0] \cap [t_1 - \tau, t_1], \\ x(t; t_0, \varphi_0), & t \in (t_0, t_1) \cap [t_1 - \tau, t_1], \\ x(t; t_0, \varphi_0) + I_1(t; t_0, \varphi_0), & t = t_1. \end{cases}$$

At the moment $t = t_2$ the mapping point $(t, x(t))$ jumps momentarily, etc.

The solution $x(t; t_0, \varphi_0)$ of problem (1), (2) is a piecewise continuous function for $t > t_0$ with points of discontinuity of the first kind $t = t_k, k = 1, 2, \ldots,$ at which it is continuous from the left.

Introduce the following notations:

$$I = [t_0 - \tau, \infty); \qquad I_0 = [t_0, \infty);$$
$$G_k = \{(t, x) \in I_0 \times \Omega\colon\ t_{k-1} < t < t_k\}, \quad k = 1, 2, \ldots;$$
$$G = \bigcup_{k=1}^{\infty} G_k; \qquad \|\phi\| = \sup_{s \in [-\tau, 0]} |\phi(s)| \quad \text{is the norm of the function } \phi \in D.$$

Together with system (1) we shall consider the system

$$\begin{cases} \dot{u} = g(t, u), & t > t_0,\ t \neq t_k, \\ \Delta u(t_k) = B_k(u(t_k)), & t_k > t_0,\ k = 1, 2, \ldots, \end{cases} \tag{3}$$

where $g : (t_0, \infty) \times R_+^m \to R_+^m$, $B_k : R_+^m \to R_+^m$, $k = 1, 2, \ldots$.

Denote by $u^+(t; t_0, u_0)$ the maximal solution of system (3) satisfying the initial condition $u^+(t_0 + 0; t_0, u_0) = u_0 \in R_+^m$.

Definition 1. System (1) is said to be:

(PS$_1$) *practically stable with respect to* (λ, A) if given (λ, A) with $0 < \lambda < A$, we have $\|\varphi_0\| < \lambda$ implies $|x(t; t_0, \varphi_0)| < A$, $t \geqslant t_0$ for some $t_0 \in R$;
(PS$_2$) *uniformly practically stable with respect to* (λ, A) if (PS$_1$) holds for every $t_0 \in R$;
(PS$_3$) *practically asymptotically stable with respect to* (λ, A) if (PS$_1$) holds and $\lim_{t\to\infty} |x(t; t_0, \varphi_0)| = 0$.

Other practical stability notions can be defined based on this definition. See [14] for details.

Introduce in R^m a partial ordering defined in the following natural way: For $u, v \in R^m$ we will write $u \geqslant v$ $(u > v)$ if and only if $u_j \geqslant v_j$ $(u_j > v_j)$ for any $j = 1, 2, \ldots, m$.

Definition 2. The function $\psi : R_+^m \to R_+^m$ is said to be *monotone nondecreasing* in R_+^m if $\psi(u) > \psi(v)$ for $u > v$ and $\psi(u) \geqslant \psi(v)$ for $u \geqslant v$ $(u, v \in R_+^m)$.

Definition 3. The function $g : (t_0, \infty) \times R_+^m \to R_+^m$ is said to be *quasi monotone nondecreasing* in $(t_0, \infty) \times R_+^m$ if for each pair of points (t, u) and (t, v) from $(t_0, \infty) \times R_+^m$ and for $j \in \{1, 2, \ldots, m\}$ the inequality $g_j(t, u) \geqslant g_j(t, v)$ holds whenever $u_j = v_j$ and $u_j \geqslant v_j$ for

$j = 1, 2, \ldots, m$, $i \neq j$, i.e., for any fixed $t \in (t_0, \infty)$ and any $j \in \{1, 2, \ldots, m\}$ the function $g_j(t, u)$ is nondecreasing with respect to $(u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_m)$.

Let $J \subset R$ be an interval. Define the following classes of functions:

$$PC\big[J, R^n\big] = \big\{\sigma : J \to R^n\text{: } \sigma(t) \text{ is continuous everywhere except some points } t_k \text{ at which } \sigma(t_k - 0) \text{ and } \sigma(t_k + 0) \text{ exist and } \sigma(t_k - 0) = \sigma(t_k),\ k = 1, 2, \ldots\big\};$$

$$PC^1\big[J, R^n\big] = \big\{\sigma \in PC\big[J, R^n\big]\text{: } \sigma(t) \text{ is continuously differentiable everywhere except some points } t_k \text{ at which } \dot\sigma(t_k - 0) \text{ and } \dot\sigma(t_k + 0) \text{ exist and } \dot\sigma(t_k - 0) = \dot\sigma(t_k),\ k = 1, 2, \ldots\big\};$$

$$K = \big\{a \in C[R_+, R_+]\text{: } a(u) \text{ is strictly increasing and such that } a(0) = 0\big\};$$

$$CK = \big\{a \in C\big[[t_0, \infty) \times R_+, R_+\big]\text{: } a(t, u) \in K \text{ for each } t \in [t_0, \infty)\big\};$$

$$S(\alpha) = \big\{x \in R^n\text{: } |x| < \alpha\big\}.$$

In the further considerations we shall use the class V_0 of piecewise continuous auxiliary functions $V : [t_0, \infty) \times \Omega \to R_+^m$ which are analogues of Lyapunov's functions [22].

$$V_0 = \Big\{V : I_0 \times \Omega \to R_+^m\text{: } V \in C\big[G, R_+^m\big],\ V(t, 0) = 0 \text{ for } t \in [t_0, \infty),\ V \text{ is locally Lipschitzian in } x \in \Omega \text{ on each of the sets } G_k,\ V(t_k - 0, x) = V(t_k, x) \text{ and } V(t_k + 0, x) = \lim_{\substack{t \to t_k \\ t > t_k}} V(t, x) \text{ exists}\Big\}.$$

We also introduce the following class of functions:

$$\Omega_1 = \big\{x \in PC[I_0, \Omega]\text{: } V\big(s, x(s)\big) \leqslant V\big(t, x(t)\big),\ t - \tau \leqslant s \leqslant t,\ t \in I_0,\ V \in V_0\big\}.$$

Let $V \in V_0$. For $x \in PC[I_0, \Omega]$ and $t \in I_0$, $t \neq t_k$, $k = 1, 2, \ldots$, we define the function

$$D_- V\big(t, x(t)\big) = \liminf_{h \to 0^-} h^{-1}\big[V\big(t + h, x(t) + hf\big(t, x(t), x_t\big)\big) - V\big(t, x(t)\big)\big].$$

Introduce the following conditions:

(H1) $f \in C[(t_0, \infty) \times \Omega \times D, R^n]$.
(H2) The function f is Lipschitz continuous with respect to its second and third arguments in $(t_0, \infty) \times \Omega \times D$ uniformly on $t \in (t_0, \infty)$.
(H3) $f(t, 0, 0) = 0$, for $t \in (t_0, \infty)$.
(H4) $I_k \in C[\Omega, \Omega]$, $k = 1, 2, \ldots$.
(H5) $I_k(0) = 0$, $k = 1, 2, \ldots$.
(H6) The functions $(I + I_k) : \Omega \to \Omega$, $k = 1, 2, \ldots$, where I is the identity in Ω.
(H7) $t_0 < t_1 < t_2 < \cdots$.
(H8) $\lim_{k \to \infty} t_k = \infty$.

In the proof of the main results we shall use the following lemma:

Lemma 1. [5,7] *Let the following conditions hold*:

1. *Conditions* (H1)*,* (H2)*,* (H4)*,* (H6)–(H8) *are met.*
2. *The function g is quasimonotone nondecreasing, continuous in the sets $(t_k, t_{k+1}] \times R^m_+$, $k \in N \cup \{0\}$ and for each $k \in N \cup \{0\}$ and $v \in R^m_+$ there exists the finite limit*

$$\lim_{\substack{(t,u)\to(t,v)\\ t>t_k}} g(t,u).$$

3. *The functions $\psi_k : R^m_+ \to R^m_+$, $\psi_k(u) = u + B_k(u)$, $k = 1, 2, \ldots,$ are monotone nondecreasing in R^m_+.*
4. *The maximal solution $u^+(t; t_0, u_0)$ of system* (3) *is defined in the interval I_0.*
5. *The solution $x = x(t; t_0, \varphi_0)$ of problem* (1)*,* (2) *is such that $x \in PC[I, \Omega] \cap PC^1[I_0, \Omega]$.*
6. *The function $V \in V_0$ is such that*

$$V\big(t_0, \varphi_0(t_0)\big) \leqslant u_0$$

and the inequalities

$$D_- V\big(t, x(t)\big) \leqslant g\big(t, V\big(t, x(t)\big)\big), \quad t \neq t_k,\ k = 1, 2, \ldots,$$
$$V\big(t_k + 0, x(t_k) + I_k\big(x(t_k)\big)\big) \leqslant \psi_k\big(V\big(t_k, x(t_k)\big)\big), \quad k = 1, 2, \ldots,$$

are valid for $t \in I_0$ and $x \in \Omega_1$.
Then

$$V\big(t, x(t; t_0, \varphi_0)\big) \leqslant u^+(t; t_0, u_0) \quad \text{for } t \in I_0.$$

3. Main results

Theorem 1. *Assume that*:

1. *The conditions of Lemma* 1 *are satisfied.*
2. *$0 < \lambda < A$ is given and $S(A) \subset \Omega$.*
3. *$g(t, 0) = 0$ for $t \in I_0$.*
4. *$B_k(0) = 0$, $k = 1, 2, \ldots$.*
5. *There exist functions $a, b \in K$ such that*

$$a\big(|x|\big) \leqslant L_0(t, x) \leqslant b\big(|x|\big) \quad (t, x) \in I_0 \times S(A),$$

where $L_0(t, x) = \sum_{i=1}^m V_i(t, x)$.

6. *$b(\lambda) < a(A)$.*

Then, the practical stability properties of system (3) *with respect to $(b(\lambda), a(A))$, imply the corresponding practical stability properties of system* (1) *with respect to (λ, A).*

Proof. 1. We shall first prove practical stability of (1). Suppose that (3) is practically stable with respect to $(b(\lambda), a(A))$. Then we have

$$\sum_{i=1}^m u_{i0} < b(\lambda) \quad \text{implies} \quad \sum_{i=1}^m u_i(t; t_0, u_0) < a(A), \quad t \geqslant t_0, \tag{4}$$

for some given $t_0 \in R$, where $u_0 = (u_{01}, \dots, u_{0m})^T$ and $u(t; t_0, u_0)$ is any solution of (3) defined in the interval I_0.

Setting $u_0 = V(t_0, \varphi_0(0))$, we get by Lemma 1,

$$V\big(t, x(t; t_0, \varphi_0)\big) \leqslant u^+\big(t; t_0, V\big(t_0, \varphi_0(0)\big)\big) \quad \text{for } t \in I_0. \tag{5}$$

Let

$$\|\varphi_0\| < \lambda. \tag{6}$$

Then, because of condition 5 of Theorem 1 and (6), it follows

$$L_0\big(t_0, \varphi_0(0)\big) \leqslant b\big(\big|\varphi_0(0)\big|\big) \leqslant b\big(\|\varphi_0\|\big) \leqslant b(\lambda)$$

which due to (4) implies

$$\sum_{i=1}^m u_i^+\big(t; t_0, V\big(t_0, \varphi_0(0)\big)\big) < a(A), \quad t \geqslant t_0. \tag{7}$$

Consequently, from condition 5 of Theorem 1, (5) and (7) we obtain

$$a\big(\big|x(t; t_0, \varphi_0)\big|\big) \leqslant L_0\big(t, x(t; t_0, \varphi_0)\big) \leqslant \sum_{i=1}^m u_i^+\big(t; t_0, V\big(t_0, \varphi_0(0)\big)\big) < a(A), \quad t \geqslant t_0.$$

Hence $|x(t; t_0, \varphi_0)| < A$, $t \geqslant t_0$ for the given $t_0 \in R$ which proves the practical stability of (1).

2. Suppose that (3) is uniformly practically stable with respect $(b(\lambda), a(A))$. Therefore, we have that

$$\sum_{i=1}^m u_{i0} < b(\lambda) \quad \text{implies} \quad \sum_{i=1}^m u_i(t; t_0, u_0) < a(A), \quad t \geqslant t_0, \tag{8}$$

for every $t_0 \in R$.

We claim that $\|\varphi_0\| < \lambda$ implies $|x(t; t_0, \varphi_0)| < A$, $t \geqslant t_0$ for every $t_0 \in R$. If the claim is not true, there exists $t_0 \in R$, a corresponding solution $x(t; t_0, \varphi_0)$ of (1) with $\|\varphi_0\| < \lambda$, and $t^* > t_0$ such that,

$$\big|x(t^*; t_0, \varphi_0)\big| \geqslant A, \quad \big|x(t; t_0, \varphi_0)\big| < A, \quad t_0 \leqslant t < t_k, \tag{9}$$

where $t^* \in (t_k, t_{k+1}]$ for some k.

Then, due to (H6) and condition 6 of Lemma 1, we can find $t^0 \in (t_k, t^*)$ such that

$$\big|x\big(t^0; t_0, \varphi_0\big)\big| \geqslant A \quad \text{and} \quad x\big(t^0; t_0, \varphi_0\big) \in \Omega. \tag{10}$$

Hence, setting $u_0 = V(t_0, \varphi_0(t^0 - t_k))$, since all the conditions of Lemma 1 are satisfied, we get

$$V\big(t, x(t; t_0, \varphi_0)\big) \leqslant u^+\big(t; t_0, V\big(t_0, \varphi_0\big(t^0 - t_k\big)\big)\big) \quad \text{for } t_0 \leqslant t \leqslant t^0. \tag{11}$$

From (10), condition 5 of Theorem 1, (11) and (8), it follows that

$$\begin{aligned} a(A) &\leqslant a\big(\big|x\big(t^0; t_0, \varphi_0\big)\big|\big) \leqslant L_0\big(t^0, x(t; t_0, \varphi_0)\big) \\ &\leqslant \sum_{i=1}^m u_i^+\big(t^0; t_0, V\big(t_0, \varphi_0\big(t^0 - t_k\big)\big)\big) < a(A). \end{aligned}$$

The contradiction obtained proves that (1) is uniformly practically stable. The proof is complete. □

Note that in Theorem 1, we have used the function $L_0(t,x) = \sum_{i=1}^m V_i(t,x)$ as a measure and consequently, we need to modify the definition of practical stability of (3) as follows: For example, (3) is practically stable with respect to $(b(\lambda), a(A))$ if (4) is satisfied for some given $t_0 \in R$. We could use other convenient measures such as

$$L_0(t,x) = \max_{1 \leqslant i \leqslant m} V_i(t,x),$$

$$L_0(t,x) = \sum_{i=1}^m d_i V_i(t,x),$$

where $d \in R_+^m$, or

$$L_0(t,x) = Q\big(V(t,x)\big),$$

where $Q : R_+^m \to R_+$ and $Q(u)$ is nondecreasing in u, and appropriate modifications of practical stability definitions are employed for system (3).

The following example will demonstrate Theorem 1.

Example 1. Consider the system

$$\begin{cases} \dot{x}(t) = n(t)y(t) + m(t)x(t)[x^2(t-h) + y^2(t-h)], & t \neq t_k,\ t > 0, \\ \dot{y}(t) = -n(t)x(t) + m(t)y(t)[x^2(t-h) + y^2(t-h)], & t \neq t_k,\ t > 0, \\ \Delta x(t_k) = c_k x(t_k), \quad \Delta y(t_k) = d_k y(t_k), & k = 1, 2, \ldots, \end{cases} \tag{12}$$

where $x, y \in R$, $h > 0$, the functions $n(t)$ and $m(t)$ are continuous in $(0, \infty)$, $-1 \leqslant c_k < 0$, $-1 \leqslant d_k < 0$, $0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$.

Let

$$\begin{cases} x(s) = \varphi_1(s), & s \in [-h, 0], \\ y(s) = \varphi_2(s), & s \in [-h, 0], \end{cases}$$

where the functions φ_1 and φ_2 are continuous in $[-h, 0]$.

Choose

$$V(t,x,y) = x^2 + y^2 = r^2(s).$$

Then

$$\Omega_1 = \big\{\mathrm{col}\big(x(t), y(t)\big) \in PC\big[R_+, R^2\big]:\ r^2(s) \leqslant r^2(t),\ t-h \leqslant s \leqslant t,\ t \geqslant 0\big\} \tag{13}$$

and for $t > 0$, $t \neq t_k$, $(x, y) \in \Omega_1$ we have

$$\begin{aligned} D_- V\big(t, x(t), y(t)\big) &= 2m(t)x^2(t)r^2(t-h) + 2m(t)y^2(t)r^2(t-h) \\ &\leqslant 2m(t)V^2\big(t, x(t), y(t)\big). \end{aligned}$$

Also

$$\begin{aligned} &V\big(t_k + 0, x(t_k) + c_k x(t_k), y(t_k) + d_k y(t_k)\big) \\ &\quad = (1 + c_k)^2 x^2(t_k) + (1 + d_k)^2 y^2(t_k) \leqslant V\big(t_k, x(t_k), y(t_k)\big). \end{aligned}$$

Consider the comparison system

$$\begin{cases} \dot{u}(t) = 2m(t)u^2(t), & t \neq t_k,\ t > 0, \\ u(0) = u_0, \\ u(t_k + 0) = u(t_k), & k = 1, 2, \dots, \end{cases} \tag{14}$$

where $u \in R_+$ and $u_0 = \varphi_1^2(0) + \varphi_2^2(0) = r^2(0)$.

The general solution of system (14) is given by

$$u(t) = \left[u_0^{-1} - 2\int_0^t m(t)\,dt \right]^{-1}. \tag{15}$$

It is clear that the trivial solution of (14) is stable if $m(t) \leqslant 0$, $t \geqslant 0$. If $m(t) > 0$, $t \geqslant 0$, then the trivial solution of (14) is stable when the integral

$$\int_0^t m(t)\,dt \tag{16}$$

is bounded and unstable when (16) is unbounded.

Let $A = 2\lambda$. We can take $a(u) = b(u) = u^2$. Suppose that $\int_0^t m(t)\,dt = \beta > 0$. It therefore follows from (15) that system (14) is practically stable if $\beta \leqslant \frac{3}{8\lambda^2}$.

Hence we get, by Theorem 1 that system (12) is practically stable if $\beta \leqslant \frac{3}{8\lambda^2}$.

In Example 1, we have used the single Lyapunov function $V(t,x)$. In this case the function $L_0(t,x) = V(t,x)$.

To demonstrate the advantage of employing several Lyapunov functions, let us consider the following example.

Example 2. Consider the system

$$\begin{cases} \dot{x}(t) = e^{-t}x(t-h) + y(t-h)\sin t - (x^3 + xy^2)\sin^2 t, & t \neq t_k, \\ \dot{y}(t) = x(t-h)\sin t + e^{-t}y(t-h) - (x^2y + y^3)\sin^2 t, & t \neq t_k, \\ \Delta x(t) = a_k x(t) + b_k y(t), & t = t_k,\ k = 1, 2, \dots, \\ \Delta y(t) = b_k x(t) + a_k y(t), & t = t_k,\ k = 1, 2, \dots, \end{cases} \tag{17}$$

where $t > 0$, $h > 0$,

$$\begin{aligned} &a_k = \frac{1}{2}\big(\sqrt{1 + c_k} + \sqrt{1 + d_k} - 2\big), \quad b_k = \frac{1}{2}\big(\sqrt{1 + c_k} - \sqrt{1 + d_k}\big), \\ &\quad -1 < c_k \leqslant 0,\ -1 < d_k \leqslant 0,\ k = 1, 2, \dots, \\ &0 < t_1 < t_2 < \cdots \quad \text{and} \quad \lim_{k\to\infty} t_k = \infty. \end{aligned}$$

Suppose that we choose a single Lyapunov function $V(t,x,y) = x^2 + y^2$. Then the set Ω_1 is given by (13). Hence, using the inequality $2|ab| \leqslant a^2 + b^2$ and observing that $(x^2 + y^2)^2\sin^2 t \geqslant 0$, we get

$$D_-V\big(t, x(t), y(t)\big) = 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t) \leqslant 4\big[\big|e^{-t}\big| + |\sin t|\big]V\big(t, x(t), y(t)\big),$$

for $t \geqslant 0$, $t \neq t_k$ and $(x,y) \in \Omega_1$.

Also

$$\begin{aligned}
&V\big(t_k+0, x(t_k)+a_k x(t_k)+b_k y(t_k), y(t_k)+b_k x(t_k)+a_k y(t_k)\big)\\
&\quad=\big[(1+a_k)x(t_k)+b_k y(t_k)\big]^2+\big[(1+a_k)y(t_k)+b_k x(t_k)\big]^2\\
&\quad\leqslant V\big(t_k, x(t_k), y(t_k)\big)+2|c_k-d_k|V\big(t_k, x(t_k), y(t_k)\big), \quad k=1,2,\ldots.
\end{aligned}$$

It is clear that

$$\begin{cases}\dot u(t)=4[|e^{-t}|+|\sin t|]u(t), & t\neq t_k,\ t>0,\\ \Delta u(t_k)=2|c_k-d_k|u(t_k), & k=1,2,\ldots,\end{cases}$$

where $u\in R_+$, is not practically stable and consequently, we cannot deduce any information about the practical stability of system (17) from Theorem 1, even though system (17) is practically stable.

Now, let us take the function $V=(V_1,V_2)$, where the functions V_1 and V_2 are defined by $V_1(t,x,y)=\frac{1}{2}(x+y)^2$, $V_2(t,x,y)=\frac{1}{2}(x-y)^2$ so that $L_0(t,x,y)=x^2+y^2$. This means that we can take $a(u)=b(u)=u^2$. Then

$$\Omega_1=\big\{(x,y)\in PC\big[R_+,R_+^2\big]\colon\ V\big(s,x(s),y(s)\big)\leqslant V\big(t,x(t),y(t)\big),\ t-h\leqslant s\leqslant t,\ t\geqslant 0\big\}.$$

Moreover, for $t\geqslant 0$ and $(x,y)\in\Omega_1$ the following vectorial inequalities:

$$\begin{aligned}
&D_-V\big(t,x(t),y(t)\big)\leqslant g\big(t,V\big(t,x(t),y(t)\big)\big), \quad t\neq t_k,\ k=1,2,\ldots,\\
&V\big(t_k+0,x(t_k)+\Delta x(t_k),y(t_k)+\Delta y(t_k)\big)\leqslant\psi_k\big(V\big(t_k,x(t_k),y(t_k)\big)\big), \quad k=1,2,\ldots,
\end{aligned}$$

are satisfied with $g=(g_1,g_2)$, where

$$\begin{aligned}
&g_1(t,u_1,u_2)=2\big(e^{-t}+\sin t\big)u_1,\\
&g_2(t,u_1,u_2)=2\big(e^{-t}-\sin t\big)u_2
\end{aligned}$$

and $\psi_k(u)=u+C_k u$, $k=1,2,\ldots$, where $C_k=\left(\begin{smallmatrix}c_k & 0\\ 0 & d_k\end{smallmatrix}\right)$.

It is obvious that the functions g and ψ_k satisfy conditions 2 and 3 of Lemma 1 and the comparison system

$$\begin{cases}\dot u_1(t)=2(e^{-t}+\sin t)u_1(t), & t\neq t_k,\\ \dot u_2(t)=2(e^{-t}-\sin t)u_2(t), & t\neq t_k,\\ \Delta u_1(t_k)=c_k u_1(t_k), \quad \Delta u_2(t_k)=d_k u_2(t_k), & k=1,2,\ldots,\end{cases}$$

is practically stable for any $0<\lambda<A$, which satisfy, for example, $\exp(e^{-t_1}+2)<(\frac{A}{\lambda})^2$. Hence Theorem 1 implies that system (17) is also practically stable.

We have assumed in Theorem 1 stronger requirements on L_0 only to unify all the practical results in one theorem. This puts burden on the comparison system (3). However, to obtain only nonuniform practical stability criteria, we could weaken certain assumptions of Theorem 1 as in the next result.

Theorem 2. *Assume that the conditions of Theorem* 1 *hold with the following changes in conditions* 5 *and* 6:

5*. *There exist functions $a\in K$ and $b\in CK$, such that*

$$a\big(|x|\big)\leqslant L_0(t,x)\leqslant b\big(t,|x|\big) \quad (t,x)\in I_0\times\Omega.$$

6*. $b(t_0,\lambda)<a(A)$ *for some* $t_0\in R$.

Then, the uniform or nouniform practical stability properties of system (3) *with respect to* $(b(t_0,\lambda), a(A))$, *imply the corresponding nouniform practical stability properties of system* (1) *with respect to* (λ, A).

We shall next consider a result which gives practical asymptotic stability of (1). We will use two Lyapunov like functions.

Theorem 3. *Assume that*:

1. $0<\lambda<A$ *is given and* $S(A)\subset\Omega$.
2. *The functions* $V, W \in V_0$ *and* $a, c \in K$, $b \in CK$ *are such that*

$$a(|x|) \leqslant L_0(t,x) \leqslant b(t,|x|) \quad (t,x)\in I_0\times S(A), \tag{18}$$

$$c(|x|)e \leqslant W(t,x) \quad (t,x)\in I_0\times S(A), \tag{19}$$

   where $e\in R_+^m$, $e=(1,1,\dots,1)$.
3. *The inequalities*

$$D_-V(t,x(t)) \leqslant -d(L_1(t,x(t)))e, \quad t\neq t_k,\ k=1,2,\dots, \tag{20}$$

   where $L_1(t,x)=\sum_{i=1}^m W_i(t,x)$,

$$V(t_k+0, x(t_k)+I_k(x(t_k))) \leqslant V(t_k, x(t_k)), \quad k=1,2,\dots, \tag{21}$$

$$W(t_k+0, x(t_k)+I_k(x(t_k))) \leqslant W(t_k, x(t_k)), \quad k=1,2,\dots, \tag{22}$$

   are valid for $d\in K$, $t\in I_0$ *and* $x\in\Omega_1$.
4. *The function* $D_-W(t,x(t))$ *is bounded in* G.
5. $b(t_0,\lambda)<a(A)$ *for some* $t_0\in R$.

Then, system (1) *is practically asymptotically stable with respect to* (λ, A).

Proof. By Theorem 1 with $g(t,u)\equiv -d(u)e$ and $\psi_k(u)\equiv u$, $t\in I_0$, $k=1,2,\dots,$ it follows because of conditions for the function $W\in V_0$ that system (1) is practically stable. Hence, it is enough to prove that every solution $x(t)=x(t;t_0,\varphi_0)$ with $\|\varphi_0\|<\lambda$ satisfies $\lim_{t\to\infty}|x(t;t_0,\varphi_0)|=0$.

Suppose that this is not true. Then there exist $\varphi_0\in D$: $\|\varphi_0\|<\lambda$, $\beta>0$, $r>0$, and a sequence $\{\xi_k\}_{k=1}^\infty\in I_0$ such that for $k=1,2,\dots,$ the following inequalities are valid:

$$\xi_k-\xi_{k-1}\geqslant\beta, \qquad |x(\xi_k;t_0,\varphi_0)|\geqslant r. \tag{23}$$

From the last inequality and (19) we get

$$W(\xi_k, x(\xi_k;t_0,\varphi_0)) \geqslant c(r)e, \quad k=1,2,\dots. \tag{24}$$

From condition 4 of Theorem 3 it follows that there exists a constant $M\in R_+$ such that

$$\sup\{D_-W(t,x(t)):\ t\in G\} \leqslant Me. \tag{25}$$

By (22), (24) and (25) we obtain

$$\begin{aligned}
W\big(t, x(t; t_0, \varphi_0)\big) &\geqslant W\big(\xi_k, x(\xi_k; t_0, \varphi_0)\big) + \int_{\xi_k}^{t} D_-W\big(s, x(s; t_0, \varphi_0)\big)\,ds \\
&= W\big(\xi_k, x(\xi_k; t_0, \varphi_0)\big) - \int_{t}^{\xi_k} D_-W\big(s, x(s; t_0, \varphi_0)\big)\,ds \\
&\geqslant c(r)e - Me(\xi_k - t) \geqslant c(r)e - Me\varepsilon > \frac{c(r)e}{2}
\end{aligned}$$

for $t \in [\xi_k - \varepsilon, \xi_k]$, where $0 < \varepsilon < \min\{\beta, \frac{c(r)}{2M}\}$.

From the estimate obtained, making use of (20) and (21), we conclude that for $\xi_R \in \{\xi_k\}_{k=1}^{\infty}$, we have

$$\begin{aligned}
0 &\leqslant V\big(\xi_R, x(\xi_R; t_0, \varphi_0)\big) \\
&\leqslant V\big(t_0, \varphi_0(0)\big) + \int_{t_0}^{\xi_R} D_-V\big(s, x(s; t_0, \varphi_0)\big)\,ds \\
&\leqslant V\big(t_0, \varphi_0(0)\big) + \sum_{k=1}^{R} \int_{\xi_k-\varepsilon}^{\xi_k} D_-V\big(s, x(s; t_0, \varphi_0)\big)\,ds \\
&\leqslant V\big(t_0, \varphi_0(0)\big) - \sum_{k=1}^{R} \int_{\xi_k-\varepsilon}^{\xi_k} d\big(L_1\big(s, x(s; t_0, \varphi_0)\big)\big)\,ds \\
&\leqslant V\big(t_0, \varphi_0(0)\big) - Rd\left(\frac{mc(r)}{2}\right)\varepsilon e
\end{aligned}$$

which contradicts (18) for large R.

Thus $\lim_{t\to\infty} |x(t; t_0, \varphi_0)| = 0$. The proof is therefore complete. □

Corollary 1. *In Theorem* 3*, the following choices of* $W(t, x)$ *are admissible to yield the same conclusion*:

(i) $W(t, x) = |x|e$ *provided that* f *is bounded on* $(t_0, \infty) \times S(A) \times D$;
(ii) $W(t, x) = V(t, x)$.

Example 3. Consider

$$\begin{cases}
\dot{x}(t) = [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2]x(t-h), \quad t \neq t_k,\ t > 0, \\
x(s) = \varphi(s), \quad s \in [-h, 0], \\
\Delta x(t_k) = \beta_k x(t_k), \quad k = 1, 2, \ldots,
\end{cases} \tag{26}$$

where $x \in R$; $h > 0$; $-1 \leqslant \beta_k < 0$; the function $\varphi(t)$ is continuous in $[-h, 0]$; $0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$.

Let

$$V(x) = |x|^2.$$

Then

$$\Omega_1 = \{x(t) \in PC[R_+, R]: \ |x|^2(s) \leqslant |x|^2(t), \ t-h \leqslant s \leqslant t, \ t \geqslant 0\}.$$

For $t > 0$ and $x \in \Omega_1$ we have

$$D_- V(x(t)) \leqslant \lambda'(t) V(x(t)) = -W(t, x(t)), \quad t \neq t_k,$$
$$V(x(t_k) + \beta_k x(t_k)) \leqslant (1+\beta_k)^2 V(x(t_k)), \quad k = 1, 2, \ldots,$$
$$W(t_k + 0, x(t_k) + \beta_k x(t_k)) \leqslant (1+\beta_k)^2 W(t_k, x(t_k)), \quad k = 1, 2, \ldots,$$

where $\lambda(t) = \exp[-2(t+1)(2 - \sin \ln(t+1))]$.

Hence all conditions of Theorem 3 are satisfied and therefore (26) is practically asymptotically stable.

References

[1] A.V. Anokhin, On linear impulsive systems for functional differential equations, Soviet Math. Dokl. 33 (1986) 220–223.

[2] N.A. Azbelev, V.P. Maksimov, L.F. Rakhmatullina, Introduction to the Theory of Functional Differential Equations, Nauka, Moscow, 1991 (in Russian).

[3] D.D. Bainov, P.S. Simeonov, Systems with Impulse Effect: Stability Theory and Applications, Ellis Horwood, Chichester, 1989.

[4] D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations: Periodic Solutions and Applications, Longman, Harlow, 1993.

[5] D.D. Bainov, I.M. Stamova, Global stability of the solutions of impulsive differential-difference equations with variable impulsive perturbations, COMPEL 1 (1997) 3–16.

[6] D.D. Bainov, I.M. Stamova, Strong stability of impulsive differential-difference equations, Panamer. Math. J. 9 (1999) 87–95.

[7] D.D. Bainov, I.M. Stamova, Stability of the solutions of impulsive functional differential equations by Lyapunov's direct method, ANZIAM J. 2 (2001) 269–278.

[8] K. Gopalsamy, B.G. Zhang, On delay differential equation with impulses, J. Math. Anal. Appl. 139 (1989) 110–122.

[9] J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, Berlin, 1977.

[10] J.K. Hale, V. Lunel, Introduction to Functional Differential Equations, Springer-Verlag, Berlin, 1993.

[11] J. Kato, Stability problems in functional differential equation with infinite delays, Funkcial. Ekvac. 21 (1978) 63–80.

[12] V.B. Kolmanovskii, V.R. Nosov, Stability of Functional Differential Equations, Academic Press, New York, 1986.

[13] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[14] V. Lakshmikantham, S. Leela, A.A. Martynyuk, Practical Stability Analysis of Nonlinear Systems, World Scientific, Singapore, 1990.

[15] V. Lakshmikantham, V.M. Matrosov, S. Sivasundaram, Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems, Kluwer, The Netherlands, 1991.

[16] A.A. Martynyuk, Practical Stability of Motion, Naukova Dumka, Kiev, 1983 (in Russian).

[17] A.A. Martynyuk, Practical stability conditions for hybrid systems, 12th World Congress of IMACS, Paris, 1988, pp. 344–347 (in Russian).

[18] M. Rama Mohana Rao, S.K. Srivastava, Stability of Volterra integro-differential equations with impulsive effect, J. Math. Anal. Appl. 163 (1992) 47–59.

[19] B.S. Razumikhin, Stability of Systems with Retardation, Nauka, Moscow, 1988 (in Russian).

[20] A.M. Samoilenko, N.A. Perestyuk, Differential Equations with Impulse Effect, Visca Skola, Kiev, 1987 (in Russian).

[21] J. Shen, J. Yan, Razumikhin type stability theorems for impulsive functional differential equations, Nonlinear Anal. 33 (1998) 519–531.

[22] P.S. Simeonov, D.D. Bainov, Stability with respect to part of the variables in systems with impulse effect, J. Math. Anal. Appl. 117 (1986) 247–263.

[23] I.M. Stamova, G.T. Stamov, Lyapunov–Razumikhin method for impulsive functional differential equations and applications to the population dynamics, J. Comput. Appl. Math. 130 (2001) 163–171.

[24] J. Yan, J. Shen, Impulsive stabilization of impulsive functional differential equations by Lyapunov–Razumikhin functions, Nonlinear Anal. 37 (1999) 245–255.