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Interaction between the Geometry of the Boundary and Positive Solutions of a Semilinear Neumann Problem with Critical Nonlinearity*

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We consider the problem: $-\Delta u+\lambda u=u^{(n+2)/(n-2)}$, $u>0$ in Ω, $\partial u/\partial \nu=0$ on $\partial\Omega$, where Ω is a bounded smooth domain in $\mathbb{R}^n$ $(n\geqslant 3)$. We show that, for λ large, least-energy solutions of the above problem have a unique maximum point P_λ on $\partial\Omega$ and the limit points of P_λ, as $\lambda\to\infty$ are contained in the set of the points of maximum mean curvature. We also prove that, if $\partial\Omega$ has k peaks then the equation has at least k solutions for λ large.

1. INTRODUCTION

In the last few years there has been considerable interest in nonlinear elliptic problems of the form

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)}+\alpha u && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ is a bounded domain with smooth boundary and α is a smooth function on $\bar{\Omega}$. As it is well known, the main difficulty in studying (1.1) is that the corresponding variational problem lacks of compactness, i.e., the functionals related to (1.1) do not satisfy the Palais–Smale condition.

The first result, concerning (1.1), was obtained by Pohozaev [26], who proved that there are no solutions in the case $\alpha = 0$, whenever Ω is star-shaped. After several years Brezis and Nirenberg [10], proved an existence theorem for (1.1), supposing α not identically zero. Since then many other interesting results have been obtained either exploiting the function α or using the topology of Ω (see [9, 11, 17, 27, 31]).

More recently some interest has also grown up for the corresponding Neumann problem

$$\begin{aligned} -\Delta u + \lambda u &= u^p && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $1 < p < \infty$ and λ is a positive constant. The study of (1.2) is also motivated by the fact that problems like this arise in pattern formation in various models in mathematical biology. Indeed, (1.2) is equivalent to the steady state problem for a chemotactic aggregation model by Keller and Segal [18, 29]. It may also be viewed as the "shadow systems" of some activator–inhibitor systems [19, 20].

In the subcritical case $(p < (n+2)/(n-2))$, (1.2) has been extensively studied in the papers of Lin and Ni [19] and Lin *et al.* [20]. In particular they have proved that (1.2) admits a nonconstant solution for λ sufficiently large and does not have any nonconstant solution for λ small. Furthermore Ni and Takagi in [24, 25] have analyzed the least-energy solutions of (1.2), i.e., the functions which minimize the functional

$$J_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2}{(\int_\Omega |u|^{p+1})^{2/p+1}} \tag{1.3}$$

in the space $H^1(\Omega)$, and they have proved that, if λ is sufficiently large, the maximum of u_λ is attained at a unique point on the boundary of Ω.

In the critical case $(p = (n+2)/(n-2))$ the situation is quite different. In fact, when $n = 4, 5, 6$ and Ω is a ball, Adimurthi and Yadava [4] and Budd *et al.* [13], showed that even for λ small, (1.2) has a nonconstant solution. The existence result for general domains and λ large has been obtained by Adimurthi and Mancini in [1] and Wang in [32] (see also [3]). More precisely, let us set

$$S_\lambda = \inf\{Q_\lambda(u) : u \in H^1(\Omega) \setminus \{0\}\}, \tag{1.4}$$

where

$$Q_\lambda(u)=\frac{\int_\Omega |\nabla u|^2+\lambda \int_\Omega u^2}{(\int_\Omega |u|^q)^{2/q}}, \qquad q=\frac{2n}{n-2}, \tag{1.5}$$

$$H_0=\max\{H(x) : x\in\partial\Omega\}, \tag{1.6}$$

where $H(x)$ denotes the mean curvature at $x\in\partial\Omega$ with respect to the unit outward normal. Then the result of Adimurthi and Mancini is the following

THEOREM 1.1. *There exists $\lambda_0>0$ such that for all $\lambda>\lambda_0$, the problem*

$$\begin{aligned} -\Delta u+\lambda u&=u^{(n+2)/(n-2)} && \text{in } \Omega\\ u&>0 && \text{in } \Omega\\ \frac{\partial u}{\partial \nu}&=0 && \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

admits a solution u_λ which minimizes Q_λ (i.e., $Q_\lambda(u_\lambda)=S_\lambda$). Moreover there exists a positive constant A_n, depending only on n, such that as $\varepsilon\to 0$

$$S_\lambda<S/2^{2/n}-A_nH_0\begin{cases}\varepsilon\log\dfrac{1}{\varepsilon}+o(\varepsilon) & \text{if } n=3\\ \varepsilon+o(\varepsilon) & \text{if } n\geqslant 4,\end{cases} \tag{1.8}$$

where S is the best constant for the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^{2n/(n-2)}(\Omega)$ and $o(\varepsilon)$ means $o(\varepsilon)/\varepsilon\to 0$ as $\varepsilon\to 0$.

In view of the results obtained by Ni and Takagi (see [25]) in the subcritical case (i.e., the maximum of the least-energy solutions is attained at only one point on $\partial\Omega$) it is natural to ask whether the same happens in the critical case. Moreover, in order to understand better the shape of these solutions, it would be interesting to localize precisely the maximum point on $\partial\Omega$. In view of (1.8) it is reasonable to expect that least energy solutions should attain the maximum near the points on $\partial\Omega$, of maximum mean curvature.

In this paper, using a blow-up technique (see, for instance, [6, 12, 16, 21, 30, 31]) we answer both questions affirmatively.

THEOREM 1.2. *Let $u_\lambda\in H^1(\Omega)$ be a solution of (1.7) such that $Q_\lambda(u_\lambda)<S/2^{2/n}$. Then there exists $\lambda_0>0$ such that, for all $\lambda>\lambda_0$*

(a) *u_λ attains its maximum at only one point $P_\lambda\in\partial\Omega$*

(b) *further if* $n \geqslant 7$ *and* u_λ *minimizes* Q_λ *(i.e.,* $Q_\lambda(u_\lambda) = S_\lambda$*) then the limit points of* $\{P_\lambda\}$*, as* $\lambda \to \infty$*, are contained in the set of the points of maximum mean curvature.*

This theorem emphasizes the role of the mean curvature of $\partial\Omega$ in understanding the qualitative behaviour of the least-energy solutions of (1.7). The next theorem will show how the mean curvature plays also a role in obtaining multiplicity results.

THEOREM 1.3. *Let* $n \geqslant 7$ *and* $P_0 \in \partial\Omega$ *be a strict local maximum point of* $H(x)$*, such that* $H(P_0) > 0$*. Then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$*, problem* (1.7) *has a solution* u_λ *with* $Q_\lambda(u_\lambda) < S/2^{2/n}$*. Moreover the functions* u_λ *concentrate at* P_0 *as* $\lambda \to \infty$*.*

The restriction on the dimension ($n \geqslant 7$) is essential in our proof. We do not know whether the same result holds in lower dimension.

Of course from this theorem we deduce the existence of k positive solutions of (1.7) in any domain Ω with k "peaks," i.e., with k points on the boundary of strict local maximum for the mean curvature $H(x)$.

Let us remark that, as it will be clear from the proof, both theorems give results about the concentration of solutions of (1.7) as $\lambda \to \infty$. In the Dirichlet case, similar concentration phenomena have been analyzed by Rey [27] and Han [17] (see also [6]).

In a forthcoming paper L^∞-estimates (similar to those of Han [17] and Rey [28]) and the blow-up points of solutions u_λ of (1.7) with $Q_\lambda(u_\lambda) < S/2^{2/n}$ as the critical points of the mean curvature have also been obtained.

The paper is organized as follows. In Section 2 we will prove Theorem 1.2 and 1.3 postponing the proofs of some technical lemmata to Section 3. In Section 4 we generalize these results to some mixed boundary value problems and make some final remarks.

After this manuscript was completed, we learned of a recent paper of Ni *et al.* [23], where they have obtained results similar to (a) of Theorem 1.2.

2. PROOFS OF THE THEOREMS

In order to prove the theorems, we need a few lemmata. Let $p = (n+2)/(n-2)$, $1 < q < \infty$, $u \in H^1(\Omega)$, $\varepsilon > 0$, $y \in \mathbb{R}^n$ and define

$$|u|_q^q = \int_\Omega |u|^q; \qquad \|u\|^2 = \int_\Omega |\nabla u|_2^2 + |u|_2^2$$

$$U(x) = \left[\frac{n(n-2)}{n(n-2) + |x|^2}\right]^{(n-2)/2}; \qquad U_{\varepsilon, y}(x) = \frac{1}{\varepsilon^{(n-2)/2}}\, U\left(\frac{x-y}{\varepsilon}\right).$$

We have

$$-\Delta U_{\varepsilon, y} = U^p_{\varepsilon, y} \qquad \text{in } \mathbb{R}^n.$$

Define $\beta_i \colon \mathbb{R}_+ \to \mathbb{R}$ for $i = 1, 2, 3$ by

$$\beta_1(t) = \begin{cases} t \log 1/t & \text{if } n = 3 \\ t & \text{if } n \geqslant 4 \end{cases}$$

$$\beta_2(t) = \begin{cases} t & \text{if } n = 3 \\ t^2 \log 1/t & \text{if } n = 4 \\ t^2 & \text{if } n \geqslant 5 \end{cases}$$

$$\beta_3(t) = \begin{cases} t^{1/2} & \text{if } n = 3 \\ t(\log 1/t)^{2/3} & \text{if } n = 4 \\ t & \text{if } n \geqslant 5. \end{cases}$$

Then

LEMMA 2.1. *Let $\alpha \in C^1(\bar{\Omega})$. Then there exist positive constants A_n and a_n such that for all $\varepsilon > 0$, $y \in \partial\Omega$ and $\lambda \geqslant 1$*

$$Q_{\lambda\alpha}(U_{\varepsilon, y}) = \frac{S}{2^{2/n}} - A_n H(y)\, \beta_1(\varepsilon) + a_n \alpha(y)\, \lambda\beta_2(\varepsilon) + O(\beta_2(\varepsilon)) + o(\lambda\beta_2(\varepsilon)) \tag{2.1}$$

where $O(s)$ and $o(s)$ mean, as $s \to 0$, that $O(s)/s$ is bounded and $o(s)/s$ tends to zero.

Proof. Since α is in $C^1(\bar{\Omega})$, there exists an a_n depending only on n such that

$$\frac{\lambda \int_\Omega \alpha(x)\, U^2_{\varepsilon, y}(x)\, dx}{|U_{\varepsilon, y}|^2_{p+1}} = \frac{\lambda\alpha(y)\, |U_{\varepsilon, y}|^2_2}{|U_{\varepsilon, y}|^2_{p+1}} + \frac{O(\lambda \int_\Omega |x-y|\, U^2_{\varepsilon, y}\, dx)}{|U_{\varepsilon, y}|^2_{p+1}}$$
$$= a_n \lambda\beta_2(\varepsilon) + o(\lambda\beta_2(\varepsilon)).$$

Hence Lemma 2.1 follows from (2.10) of Lemma 2.2 in [1]. ∎

LEMMA 2.2. *Let u_λ be a solution of* (1.7) *with $Q_\lambda(u_\lambda) < S/2^{2/n}$ and let $P_\lambda \in \bar{\Omega}$ be such that*

$$M_\lambda = u_\lambda(P_\lambda) = \max_{x \in \bar{\Omega}} u_\lambda(x). \tag{2.2}$$

Then there exists a $\lambda_0 > 0$ *such that for* $\lambda > \lambda_0$,

(i) $P_\lambda \in \partial\Omega$ *and is unique*

(ii) $\lim_{\lambda\to\infty} |\nabla(u_\lambda - U_{\varepsilon_\lambda, P_\lambda})|_2 = 0$ *where* $\varepsilon_\lambda^{(n-2)/2} = 1/M_\lambda$.

Proof. In this proof we follow the blow up technique used in Gidas and Spruck [15]. The method of the proof is similar to the one used by Ni and Takagi [24, 25] in the subcritical case. Integrating the equation (1.7), we obtain that there exists an $x \in \bar{\Omega}$ such that $u_\lambda(x) > \lambda^{1/p-1}$ and hence $M_\lambda > \lambda^{1/p-1}$. This implies that

$$\lambda\varepsilon_\lambda^2 = \frac{\lambda}{M_\lambda^{p-1}} < 1. \tag{2.3}$$

Thus we can assume that for a subsequence of λ, as $\lambda \to \infty$, $P_\lambda \to P_0$, $\lambda\varepsilon_\lambda^2 \to a$ and $\Omega_\lambda = \Omega - \{P_\lambda\}/\varepsilon_\lambda \to \Omega_\infty$. For $x \in \Omega_\lambda$, define $\tilde{u}_\lambda(x) = \varepsilon_\lambda^{(n-2)/2} u_\lambda(\varepsilon_\lambda x + P_\lambda)$. Then clearly $\tilde{u}_\lambda$ satisfies

$$\begin{aligned} -\Delta\tilde{u}_\lambda + \lambda\varepsilon_\lambda^2\tilde{u}_\lambda = \tilde{u}_\lambda^p \qquad & \text{in } \Omega_\lambda \\ \frac{\partial\tilde{u}_\lambda}{\partial\nu} = 0 \qquad & \text{on } \partial\Omega_\lambda \\ 0 < \tilde{u}_\lambda \leqslant 1, \qquad & \tilde{u}_\lambda(0) = 1. \end{aligned} \tag{2.4}$$

From the elliptic regularity theory it follows that

$$\tilde{u}_\lambda \to \omega \text{ in } C^2_{\text{loc}}(\Omega_\infty) \tag{2.5}$$

$$0 \leqslant \omega \leqslant 1, \qquad \omega(0) = 1 \tag{2.6}$$

$$\int_{\Omega_\infty} |\nabla\omega|^2 \leqslant \varliminf_{\lambda\to\infty} \int_{\Omega_\lambda} |\nabla\tilde{u}_\lambda|^2 = \varliminf_{\lambda\to\infty} \int_\Omega |\nabla u_\lambda|^2 < \infty \tag{2.7}$$

$$\int_{\Omega_\infty} |\omega|^{p+1} \leqslant \varliminf_{\lambda\to\infty} \int_{\Omega_\lambda} |\tilde{u}_\lambda|^{p+1} = \varliminf_{\lambda\to\infty} \int_\Omega |u_\lambda|^{p+1} < \infty \tag{2.8}$$

$$\begin{aligned} -\Delta\omega + a\omega = \omega^p \qquad & \text{in } \Omega_\infty \\ \frac{\partial\omega}{\partial\nu} = 0 \qquad & \text{on } \partial\Omega_\infty. \end{aligned} \tag{2.9}$$

Suppose that $\lim_{\lambda\to\infty} d(P_\lambda, \partial\Omega)/\varepsilon_\lambda = \infty$, then $\Omega_\infty = \mathbb{R}^n$. Hence from (2.5)–(2.9) and Pohozaev's identity [26] we get $a = 0$ and $\omega = U$. Moreover from (3.59) and (1.7) it follows that $\lim_{\lambda\to\infty} |\nabla u_\lambda|^2 = S^{n/2}/2$. But from (2.7) we have

$$S^{n/2} = \int_{\mathbb{R}_n} |\nabla U|^2 \leqslant \lim_{\lambda\to\infty} \int_\Omega |\nabla u_\lambda|^2 = \frac{S^{n/2}}{2},$$

which is a contradiction. Hence

$$\lim_{\lambda \to \infty} \frac{d(P_\lambda, \partial\Omega)}{\varepsilon_\lambda} = \alpha < \infty. \tag{2.10}$$

This implies that $P_\lambda \to P_0 \in \partial\Omega$. Without loss of generality we can assume that $P_0 = 0$ and in a neighborhood $B(R)$ of 0, Ω and $\partial\Omega$ are described as in (3.1). Let ψ be the map given by (3.4). Then, if $v_\lambda(y) = u_\lambda(\psi^{-1}(y))$, from (3.6), we obtain that v_λ satisfies

$$\begin{aligned} -\sum a_{jk}(y) \frac{\partial^2 v_\lambda}{\partial y_j\, \partial y_k} + \sum b_j(y) \frac{\partial v_\lambda}{\partial y_j} + \lambda v_\lambda = v_\lambda^p \qquad &\text{in} \quad B(R_0)^+ \\ \frac{\partial v_\lambda}{\partial y_n} = 0 \qquad &\text{on} \quad y_n = 0 \end{aligned} \tag{2.11}$$

$$v_\lambda(q_\lambda) = M_\lambda, \tag{2.12}$$

where a_{jk}, b_j are given by (3.7), $\psi^{-1}(q_\lambda) = p_\lambda$ and $B(R_0)^+ = B(R_0) \cap \{y_n > 0\}$. Let $q_\lambda = (q_{1\lambda}, ..., q_{n\lambda})$, $B_\lambda = (B(R_0)^+ - \{q_\lambda\})/\varepsilon_\lambda$, $\tilde{v}_\lambda(x) = \varepsilon_\lambda^{(n-2)/2} v_\lambda(\varepsilon_\lambda x + q_\lambda)$ for $x \in B_\lambda$. Then from (2.10) and (2.11) we have

$$\lim_{\lambda \to \infty} \frac{q_{n\lambda}}{\varepsilon_\lambda} = \alpha \tag{2.13}$$

$$\begin{aligned} -\sum (\delta_{ij} + O(\varepsilon_\lambda x + q_\lambda)) \frac{\partial^2 \tilde{v}_\lambda}{\partial x_i\, \partial x_j} + \varepsilon_\lambda \sum b_i(\varepsilon_\lambda x + q_\lambda) \frac{\partial \tilde{v}_\lambda}{\partial x_i} \\ + \lambda \varepsilon_\lambda^2 \tilde{v}_\lambda = \tilde{v}_\lambda^p \qquad \text{in } B_\lambda \\ \frac{\partial \tilde{v}_\lambda}{\partial x_n} = 0 \qquad \text{on} \quad B_\lambda \cap \left\{ x_n = -\frac{q_{n\lambda}}{\varepsilon_\lambda} \right\} \end{aligned} \tag{2.14}$$

$$\tilde{v}_\lambda(0) = 1, \qquad 0 \leqslant \tilde{v}_\lambda \leqslant 1. \tag{2.15}$$

Hence by the elliptic regularity theory, we obtain that $\tilde{v}_\lambda \to \omega_1$ in $C^2_{\text{loc}}(\bar{B}_\infty)$ where $B_\infty = \{x; x_n > -\alpha\}$ and $\omega_1 \in H^1(B_\infty)$ is such that

$$\begin{aligned} -\Delta \omega_1 + a\omega_1 = \omega_1^p \qquad &\text{in } B_\infty \\ \frac{\partial \omega_1}{\partial x_n} = 0 \qquad &\text{on } \partial B_\infty \end{aligned} \tag{2.16}$$

$$0 \leqslant \omega_1 \leqslant 1, \qquad \omega_1(0) = 1. \tag{2.17}$$

Hence by Pohozaev's identity $a = 0$ and thus $\omega_1(x) = 1/\beta^{(n-2)/2} U((x - x_0)/\beta)$ for some $\beta > 0$, $x_0 \in \mathbb{R}^n$. Since ω_1 attains its maximum at 0 and $\partial\omega_1/\partial x_n = 0$ on ∂B_∞, we deduce that $x_0 = 0$, $\alpha = 0$ and $\beta = 1$. Therefore $\omega_1 = U$.

Now suppose $P_\lambda \notin \partial\Omega$, then $q_\lambda \notin \{y_n = 0\}$. Since $\partial v_\lambda/\partial y_n = 0$ on $y_n = 0$, by taking the reflection of v_λ about $y_n = 0$ and using Rolle's theorem, we obtain a point $R_\lambda = (q_{1\lambda} \cdots q_{n-1,\lambda}, \bar{q}_{n\lambda})$ with $0 \leqslant \bar{q}_{n\lambda} < q_{n\lambda}$ and

$$\frac{\partial^2 v_\lambda}{\partial y_n^2}(R_\lambda) \geqslant 0. \tag{2.18}$$

Let $x_\lambda = (R_\lambda - q_\lambda)/\varepsilon_\lambda$. Since $\alpha = 0$, we obtain from (2.13) that $|x_\lambda| \to 0$ as $\lambda \to \infty$. Thus from (2.18)

$$0 > \frac{\partial^2 U}{\partial x_n^2}(0) = \lim_{\lambda \to \infty} \frac{\partial^2 \tilde{v}_\lambda}{\partial x_n^2}(x_\lambda) \geqslant 0, \tag{2.19}$$

which is a contradiction. Hence $P_\lambda \in \partial\Omega$ and $\Omega_\infty = \mathbb{R}^n_+$.

From (2.4)–(2.9) and Pohozaev's identity we get $a = 0$ and $\tilde{u}_\lambda \to U$ in $C^2_{\text{loc}}(\mathbb{R}^n_+)$. Moreover from Lemma 3.6,

$$\frac{S^{n/2}}{2} = \int_{\mathbb{R}^n_+} |\nabla U|^2 \leqslant \varliminf_{\lambda \to \infty} \int_{\Omega_\lambda} |\nabla \tilde{u}_\lambda|^2 = \frac{S^{n/2}}{2}$$

and hence

$$\lim_{\lambda \to \infty} \int_\Omega |\nabla(u_\lambda - U_{\varepsilon_\lambda, P_\lambda})|^2 = 0. \tag{2.20}$$

This proves (ii).

The last thing to show is that P_λ is unique for λ large.

Suppose that this is not true then there exist P^i_λ, $i = 1, 2$, such that $M_\lambda = u_\lambda(P^i_\lambda)$ for $i = 1, 2$. Consequently $\{P^i_\lambda\}$ satisfy, either

$$\lim_{\lambda \to \infty} \frac{|P^1_\lambda - P^2_\lambda|}{\varepsilon_\lambda} = \infty \tag{2.21}$$

or

$$\lim_{\lambda \to \infty} \frac{|P^1_\lambda - P^2_\lambda|}{\varepsilon_\lambda} < \infty. \tag{2.22}$$

Suppose that (2.21) holds. Then from (2.20) we obtain

$$\lim_{\lambda \to \infty} \int_\Omega |\nabla(U_{\varepsilon_\lambda, P^1_\lambda} - U_{\varepsilon_\lambda, P^2_\lambda})|^2 = 0.$$

That is for $\Omega_\lambda = (\Omega - \{P^1_\lambda\})/\varepsilon_\lambda$ and $z_\lambda = (P^1_\lambda - P^2_\lambda)/\varepsilon_\lambda$,

$$0 = S^{n/2} - 2 \lim_{\lambda \to \infty} \int_{\Omega_\lambda} \nabla U \, \nabla U_{1, z_\lambda}. \tag{2.23}$$

Since $|z_\lambda| \to \infty$, we obtain $\lim_{\lambda \to \infty} \int_{\Omega_\lambda} \nabla U \nabla U_{1, z_\lambda} = 0$. This contradicts (2.23) and hence (2.21) does not hold.

Suppose that (2.22) holds. Let $q_\lambda^i = \psi(P_\lambda^i)$ for $i = 1, 2$ and $B_\lambda = (B(R_0)^+ - \{q_\lambda^1\})/\varepsilon_\lambda$, $\tilde{v}_\lambda(x) = \varepsilon_\lambda^{(n-2)/2} v_\lambda(\varepsilon_\lambda x + q_\lambda^1)$ where v_λ is defined as in (2.11). Since $q_\lambda^i \in \{y_n = 0\}$, by an orthogonal rotation in $(n-1)$ variables, we can assume that q_λ^i lie on the x_1-axis. Let $x_\lambda = (q_\lambda^1 - q_\lambda^2)/\varepsilon_\lambda$, then from (2.22) we can assume that for a subsequence $x_\lambda \to x_0$ as $\lambda \to \infty$. Again from Rolle's theorem, we can find $0 < t_\lambda < 1$, such that if $z_\lambda = t_\lambda x_\lambda$, then

$$\frac{\partial \tilde{v}_\lambda}{\partial x_1}(z_\lambda) = 0, \qquad \frac{\partial^2 \tilde{v}_\lambda}{\partial x_1^2}(z_\lambda) \geqslant 0. \tag{2.24}$$

Let $z_\lambda \to z_0$. By arguing in the same way as in (2.14), we obtain $\tilde{v}_\lambda \to U$ in $C^2_{\mathrm{loc}}(\bar{\mathbb{R}}^n_+)$ and hence from (2.24) we have $(\partial U/\partial x_1)(z_0) = 0$. This implies that $z_0 = 0$. Since z_0 lies on the x_1-axis, again from (2.24), $(\partial^2 U/\partial x_1^2)(0) \geqslant 0$, which is a contradiction. This proves the uniqueness of P_λ and hence the lemma. ▌

LEMMA 2.3. *Let $\delta_k > 0$, $\lambda_k > 0$, $P_k \in \partial\Omega$ and $0 \leqslant u_k \in H^1(\Omega)$ be such that as $k \to \infty$, $\lambda_k \to \infty$, $P_k \to P_0$, $\delta_k \to 0$, $u_k \to 0$ weakly in $H^1(\Omega)$ and*

$$\lim_{k \to \infty} |\nabla(u_k - U_{\delta_k, P_k})|_2 = 0. \tag{2.25}$$

Then there exist $\varepsilon_k > 0$, $C_k \in \mathbb{R}$, $y_k \in \partial\Omega$, $\omega_k \in H^1(\Omega)$ such that for a subsequence, as $k \to \infty$, $\varepsilon_k/\delta_k \to 1$, $C_k \to 1$, $y_k \to P_0$ and

$$u_k = C_k U_{\varepsilon_k, y_k} + \omega_k. \tag{2.26}$$

Moreover, if $n \geqslant 7$, then

$$Q_{\lambda_k}(u_k) = \frac{S}{2^{2/n}} - A_n H(y_k)\,\beta_1(\varepsilon_k) + a_n \lambda_k \beta_2(\varepsilon_k) + R_k \lambda_k\, |\omega_k|_2^2$$
$$+ O(\beta_2(\varepsilon_k)) + O(\lambda_k \beta_2(\varepsilon_k)) + O(\|\omega_k\|^2 + \beta_3(\varepsilon_k)\, \|\omega_k\|) \tag{2.27}$$

where $R_k > 0$ and $\lim_{k \to \infty} R_k = R_0 > 0$. Furthermore if $Q_{\lambda_k}(u_k) < S/2^{2/n}$, then there exists an $r > 2$ such that

$$Q_{\lambda_k}(u_k) \geqslant \frac{S}{2^{2/n}} - A_n H(y_k)\,\beta_1(\varepsilon_k) + a_n \lambda_k \beta_2(\varepsilon_k) + o(\lambda_k \beta_2(\varepsilon_k))$$
$$+ O(\beta_2(\varepsilon_k) + \beta_3(\varepsilon_k)\,\beta_1(\varepsilon_k)^{1/2} + \beta_1(\varepsilon_k)^{r/2}) \tag{2.28}$$

$$\lambda_k \beta_2(\varepsilon_k) = O(\beta_1(\varepsilon_k)). \tag{2.29}$$

Proof. Since $\{u_k\}$ satisfies the hypotheses of Lemma 3.1, there exist C_k, ε_k, $y_k \in \partial\Omega$, $\omega_k \in H^1(\Omega)$ such that (2.26) holds. Because of (2.25) we have that $d(u_k, M) \to 0$, and hence $\|\omega_k\| \to 0$. Therefore from (ii) of Lemma 3.1, $C_k \to C_0 \neq 0$. Let

$$\alpha = \lim_{k \to \infty} |\nabla U_{\varepsilon_k, y_k}|_2^2 = \lim_{k \to \infty} |\nabla U_{\delta_k, P_k}|_2^2$$

$$\beta = \lim_{k \to \infty} \int_\Omega \nabla U_{\varepsilon_k, y_k} \nabla U_{\delta_k, P_k}.$$

Then clearly $|\beta| \leqslant \alpha$. From (2.25) we have

$$0 = \lim_{k \to \infty} |\nabla(U_{\delta_k, P_k} - C_k U_{\varepsilon_k, y_k})|_2^2 = \alpha C_0^2 - 2C_0 \beta + \alpha. \tag{2.30}$$

Since $|\beta| \leqslant \alpha$, $C_0 = \pm 1$. Suppose $C_0 = -1$, then again from (2.26)

$$\lim_{k \to \infty} |\nabla(U_{\delta_k, P_k} + |C_k|\, U_{\varepsilon_k, y_k})|_2 = 0.$$

Hence from Lemma 3.2, $|U_{\delta_k, P_k} + |C_k|\, U_{\varepsilon_k, y_k}|_{p+1} \to 0$, which is a contradiction. This proves that $C_0 = 1$. Therefore from (2.30), we obtain $y_k \to P_0$ and $\varepsilon_k/\delta_k \to 1$. Taking $q = p+1$ and $L = 1$ in Lemma 3.5, we obtain from (3.57) and (3.15)

$$\begin{aligned} |u_k|_{p+1}^{p+1} &= C_k^{p+1}\, |U_{\varepsilon_k, y_k}|_{p+1}^{p+1} + (p+1)\, C_k^p \int_\Omega U_{\varepsilon_k, y_k}^p \omega_k + O(\|\omega_k\|^2) \\ &= C_k^{p+1}\, |U_{\varepsilon_k, y_k}|_{p+1}^{p+1} + O(\beta_3(\varepsilon_k)\, \|\omega_k\| + \|\omega_k\|^2). \end{aligned} \tag{2.31}$$

Since $n \geqslant 7$, we have

$$\left| \int_\Omega U_{\varepsilon_k, y_k} \omega_k \right| \leqslant \left(\int_\Omega U_{\varepsilon_k, y_k}^{2n/n+2} \right)^{(n+2)/2n} |\omega_k|_{p+1} = O(\varepsilon_k^2\, \|\omega_k\|). \tag{2.32}$$

Hence

$$\begin{aligned} &Q_{\lambda_k}(u_k) \\ &\quad = \frac{C_k^2(|\nabla U_{\varepsilon_k, y_k}|_2^2 + \lambda_k\, |U_{\varepsilon_k, y_k}|_2^2) + |\nabla \omega_k|_2^2 + \lambda_k\, |\omega_k|_2^2 + O(\lambda_k \varepsilon_k^2\, \|\omega_k\|)}{C_k^2\, |U_{\varepsilon_k, y_k}|_{p+1}^2\, (1 + O(\beta_3(\varepsilon_k)\, \|\omega_k\| + \|\omega_k\|^2))^{2/p+1}} \\ &\quad = Q_{\lambda_k}(U_{\varepsilon_k, y_k}) + \lambda_k R_k\, |\omega_k|_2^2 + O(\beta_3(\varepsilon_k)\, \|\omega_k\| + \|\omega_k\|^2 + \lambda_k \varepsilon_k^2\, \|\omega_k\|), \end{aligned} \tag{2.33}$$

where $R_k > 0$, and $\lim_{k \to \infty} R_k = R_0 > 0$. Thus (2.27) follows from (2.33) and (2.1).

Let $Q_{\lambda_k}(u_k) < S/2^{2/n}$. Since $\|\omega_k\| \to 0$, from (2.27) we obtain that $\lambda_k \beta_2(\varepsilon_k) \to 0$ and hence

$$\lim_{k \to \infty} \lambda_k \varepsilon_k^2 = 0. \tag{2.34}$$

Taking $q = p+1$ and $L = 2$ in Lemma 3.5 we have for some $r > 2$,

$$|u_k|_{p+1}^{p+1} = C_k^{p+1} \, |U_{\varepsilon_k, y_k}|_{p+1}^{p+1} + C_k^p (p+1) \int_\Omega U_{\varepsilon_k, y_k}^p \omega_k + \frac{p(p+1)\, C_k^{p-1}}{2} \int_\Omega U_{\varepsilon_k, y_k}^{p-1} \omega_k^2 + O(\|\omega_k\|^r).$$

Hence from (3.15)

$$\begin{aligned} |u_k|_{p+1}^{-2} &= C_k^{-2} \, |U_{\varepsilon_k, y_k}|_{p+1}^{-2} \\ &\quad \times \left\{ 1 + \frac{p(p+1)}{2C_k^2 \, |U_{\varepsilon_k, y_k}|_{p+1}^{p+1}} \right. \\ &\quad \left. \times \int_\Omega U_{\varepsilon_k, y_k}^{p-1} \omega_k^2 + O(\beta_3(\varepsilon_k) \, \|\omega_k\| + \|\omega_k\|^r) \right\}^{-2/p+1} \\ &= C_k^{-2} \, |U_{\varepsilon_k, y_k}|_{p+1}^{-2} \\ &\quad \times \left\{ 1 - \frac{p \int_\Omega U_{\varepsilon_k, y_k}^{p-1} \omega_k^2}{C_k^2 \, |U_{\varepsilon_k, y_k}|_{p+1}^{p+1}} + O(\beta_3(\varepsilon_k) \, \|\omega_k\| + \|\omega_k\|^r) \right\}. \end{aligned}$$

Thus from (2.1) we have

$$\begin{aligned} Q_{\lambda_k}(u_k) &= \left\{ Q_{\lambda_k}(U_{\varepsilon_k, y_k}) + \frac{|\nabla \omega_k|_2^2 + \lambda_k \, |\omega_k|_2^2 + O(\lambda_k \varepsilon_k^2 \, \|\omega_k\|)}{C_k^2 \, |U_{\varepsilon_k, y_k}|_{p+1}^2} \right\} \\ &\quad \times \left\{ 1 - \frac{p \int_\Omega U_{\varepsilon_k, y_k}^{p-1} \omega_k^2}{C_k^2 \, |U_{\varepsilon_k, y_k}|_{p+1}^{p+1}} + O(\beta_3(\varepsilon_k) \, \|\omega_k\| + \|\omega_k\|^r) \right\} \\ &= \frac{S}{2^{2/n}} - A_n H(y_k) \, \beta_1(\varepsilon_k) + a_n \lambda_k \beta_2(\varepsilon_k) \\ &\quad + O(\beta_2(\varepsilon_k)) + o(\lambda_k \beta_2(\varepsilon_k)) \\ &\quad + \frac{|\nabla \omega_k|_2^2 + \lambda_k \, |\omega_k|_2^2}{C_k^2 \, |U_{\varepsilon_k, y_k}|_{p+1}^2} - \frac{pS \int_\Omega U_{\varepsilon_k, y_k}^{p-1} \omega_k^2}{2^{2/n} C_k^2 \, |U_{\varepsilon_k, y_k}|_{p+1}^{p+1}} \\ &\quad + O(\beta_3(\varepsilon_k) \, \|\omega_k\| + \|\omega_k\|^r + \lambda_k \, |\omega_k|_2^2 \, \beta_3(\varepsilon_k) \, \|\omega_k\| \\ &\quad + \lambda_k \, |\omega_k|_2^2 \, \|\omega_k\|^2). \end{aligned} \tag{2.35}$$

Now as $k \to \infty$, $|U_{\varepsilon_k, y_k}|_{p+1}^{p+1} \to S/2^{2/n}$, hence from (2.34) and Lemma 3.4, there exists a $\delta > 0$ such that

$$|\nabla \omega_k|_2^2 + \lambda_k \, |\omega_k|_2^2 \geqslant (p + \delta) \int_\Omega U_{\varepsilon_k, y_k}^{p-1} \omega_k^2 + O(\beta_3(\varepsilon_k)^2 \, \|\omega_k\|^2). \tag{2.36}$$

Since $Q_{\lambda_k}(u_k) < S/2^{2/n}$, from (2.35), (2.36) and using

$$2\beta_3(\varepsilon_k)\,\|\omega_k\| \leqslant \gamma\,\|\omega_k\|^2 + \frac{\beta_3(\varepsilon_k)^2}{\gamma}$$

for any $\gamma > 0$, we obtain

$$|\nabla\omega_k|_2^2 + \lambda_k\,|\omega_k|_2^2 = O(\beta_1(\varepsilon_k) + \beta_3(\varepsilon_k)^2) = O(\beta_1(\varepsilon_k)). \tag{2.37}$$

Hence $\|\omega_k\| = O(\beta_1(\varepsilon_k)^{1/2})$. This implies

$$\|\omega_k\|^r + \beta_3(\varepsilon_k)\,\|\omega_k\| = O(\beta_3(\varepsilon_k)\,\beta_1(\varepsilon_k)^{1/2}) + O(\beta_1(\varepsilon_k)^{r/2}). \tag{2.38}$$

From (2.35) and (2.36)–(2.38) we obtain

$$Q_{\lambda_k}(u_k) \geqslant \frac{S}{2^{2/n}} - A_n H(y_k)\,\beta_1(\varepsilon_k) + a_n\lambda_k\beta_2(\varepsilon_k) + O(\beta_2(\varepsilon_k) + \beta_3(\varepsilon_k)\,\beta_1(\varepsilon_k)^{1/2})$$
$$+ o(\lambda_k\beta_2(\varepsilon_k)) + O(\beta_1(\varepsilon_k)^{r/2}).$$

This proves (2.28). Since $Q_{\lambda_k}(u_k) < S/2^{2/n}$, from (2.28) we get $\lambda_k\beta_2(\varepsilon_k) = O(\beta_1(\varepsilon_k))$ and hence the lemma is completely proved. ∎

Here we would like to remark that if u_k is a solution of (1.7), then we can improve the estimate (2.28) as follows.

COROLLARY 2.1. *Let $n \geqslant 7$ and assume that u_k solves (1.7) and $Q_{\lambda_k}(u_k) < S/2^{2/n}$. Then*

$$Q_{\lambda_k}(u_k) = \frac{S}{2^{2/n}} - A_n H(y_k)\,\beta_1(\varepsilon_k) + a_n\lambda_k\beta_2(\varepsilon_k) + o(\lambda_k\beta_2(\varepsilon_k))$$
$$+ O(\beta_2(\varepsilon_k) + \beta_3(\varepsilon_k)^2).$$

Proof. Since u_k solves (1.7), from Lemma 2.2, u_k satisfies the hypotheses of Lemma 2.3. Now from Lemma 3.5 we have

$$\int_\Omega \nabla u_k\cdot\nabla\omega_k + \lambda_k\int_\Omega u_k\omega_k = \int_\Omega u_k^p\omega_k$$

$$\int_\Omega |\nabla\omega_k|^2 + \lambda_k\int_\Omega \omega_k^2 + \lambda_k\int_\Omega U_{\varepsilon_k,y_k}\omega_k$$
$$= \int_\Omega (C_k U_{\varepsilon_k,y_k} + \omega_k)^p\,\omega_k \tag{2.40}$$
$$= C_k^p\int_\Omega U^p_{\varepsilon_k,y_k}\omega_k + pC_k^{p-1}\int_\Omega U^{p-1}_{\varepsilon_k,y_k}\omega_k^2 + O(\|\omega_k\|^r)$$

for some $r>2$. From (2.32), (2.36), (3.15) and using $C_k \to 1$ as $k \to \infty$, we obtain from (2.40)

$$|\nabla \omega_k|_2^2 + \lambda_k\, |\omega_k|_2^2 = O(\beta_3(\varepsilon_k)\, \|\omega_k\| + \lambda_k \varepsilon_k^2\, \|\omega_k\|). \tag{2.41}$$

Hence

$$|\nabla \omega_k|_2^2 + \lambda_k\, |\omega_k|_2^2 = O(\beta_3(\varepsilon_k)^2 + \lambda_k^2 \varepsilon_k^4). \tag{2.42}$$

Now (2.39) follows from (2.42), (2.35) and (2.36) and this proves the corollary. ∎

Proof of Theorem 1.2. (a) follows from Lemma 2.2. Let u_λ be a minimal energy solution of (1.7). Suppose that for a sequence $\lambda_k \to \infty$, $P_{\lambda_k} \to P_0$. Then from Lemma 2.2, $u_k = u_{\lambda_k}$, $P_k = P_{\lambda_k}$, $\delta_k = (u_k(P_k))^{-2/(n-2)}$ satisfy the hypotheses of Lemma 2.3. Therefore by Lemma 2.3, there exist ε_k, y_k satisfying (2.39) and (2.29). Suppose P_0 is not a point of maximum mean curvature and let y_0 be a point of maximum mean curvature. Then since $Q_{\lambda_k}(u_k) \leqslant Q_{\lambda_k}(U_{\varepsilon_k, y_0})$, it follows from (2.39), (2.29) and (2.1) that

$$A_n(H(y_0) - H(y_k))\, \beta_1(\varepsilon_k) + o(\beta_1(\varepsilon_k)) \leqslant 0,$$

which is a contradiction. This proves the theorem. ∎

Proof of Theorem 1.3. Let $x_0 \in \partial\Omega$ and $R_0 > 0$ such that for all $x \in \partial\Omega \cap \overline{B(x_0, R_0))} \backslash \{x_0\}$

$$0 \leqslant H(x) < H(x_0). \tag{2.43}$$

For $u \in H^1(\Omega)$, define the concentration function

$$\rho(u) = \frac{\int_\Omega x\, |\nabla u|^2}{\int_\Omega |\nabla u|^2} \tag{2.44}$$

and for $\lambda > 0$, let

$$A = \{u \in H^1(\Omega); \qquad \rho(u) \in \overline{B(x_0, R_0)}\} \tag{2.45}$$

$$S(\lambda, x_0) = \inf\{Q_\lambda(u) : u \in A\}. \tag{2.46}$$

A is not empty because $U_{\varepsilon, x_0} \in A$ for ε sufficiently small and from (2.1), $S(\lambda, x_0) < S/2^{2/n}$. Thus $S(\lambda, x_0)$ is achieved (see for instance Lemma 3.2 in [3]) by u_λ. Since $u \in A$ then also $|u| \in A$ and hence we can assume that $u_\lambda \geqslant 0$ and $|\nabla u_\lambda|_2^2 + \lambda\, |u_\lambda|_2^2 = |u_\lambda|_{p+1}^{p+1}$. We claim that there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, $\rho(u_\lambda) \in B(x_0, R_0)$.

Suppose that this is not true. Then there exist sequences $\lambda_k \to \infty$, $u_k = u_{\lambda_k}$ such that $\rho(u_k) \in \partial B(x_0, R_0)$. From Lemma 3.6 we get

$$\lim_{k \to \infty} |\nabla u_k|_2^2 = \lim_{k \to \infty} |u_k|_{p+1}^{p+1} = \frac{S^{n/2}}{2}.$$

Hence from Lemma 3.7 there exist $\delta_k > 0$, $x_k \in \partial\Omega$ such that, as $k \to \infty$,

$$|\nabla(u_k - U_{\delta_k, x_k})|_2^2 \to 0.$$

Consequently u_k satisfies the hypotheses of Lemma 2.3 with $x_k \to x_1 \in \partial\Omega \cap \partial B(x_0, R_0)$. From Lemma 2.3 there exist a sequence $\varepsilon_k \to 0$, $y_k \in \partial\Omega$ with $y_k \to x_1$ such that $Q_{\lambda_k}(u_k)$ satisfies (2.28). From (2.28), (2.29), and $Q_{\lambda_k}(u_k) \leqslant Q_{\lambda_k}(U_{\varepsilon_k, x_0})$ we get

$$A_n(H(x_0) - H(y_k))\, \beta_1(\varepsilon_k) + o(\beta_1(\varepsilon_k)) \leqslant 0. \tag{2.47}$$

From (2.43), $\lim_{k \to \infty}(H(x_0) - H(y_k)) = H(x_0) - H(x_1) > 0$ and this contradicts (2.47) and hence proves that $\rho(u_\lambda) \in B(x_0, R_0)$ for λ sufficiently large.

From this we deduce that for any $\phi \in H^1(\Omega)$ there exists $\varepsilon_0 > 0$ such that $\rho(u_\lambda + \varepsilon\phi) \in B(x_0, R_0)$ for all $|\varepsilon| < \varepsilon_0$. This implies that

$$\frac{d}{d\varepsilon}\, Q_\lambda(u_\lambda + \varepsilon\phi)_{\varepsilon = 0} = 0. \tag{2.48}$$

Since $|\nabla u_\lambda|_2^2 + \lambda\, |u_\lambda|_2^2 = |u_\lambda|_{p+1}^{p+1}$, from (2.48) we obtain that u_λ is a solution of (1.7). From Theorem 1.2 we get that P_λ, the maximum of u_λ tends to x_0. This proves Theorem 1.3. ∎

3. Technical Lemmata

Let $P_0 \in \partial\Omega$. By a translation and rotation, we can assume that $P_0 = 0$ and there exist $R > 0$ and a smooth function $g : B(2R) \cap \{x_n = 0\} \to \mathbb{R}$, $B(2R) = \{x; |x| < 2R\}$ such that locally Ω is of the form

$$\begin{aligned} \Omega \cap B(R) &= \{(x', x_n) \in B(R) : x_n > g(x')\} \\ \partial\Omega \cap B(R) &= \{(x', x_n) \in B(R) : x_n = g(x')\} \\ g(0) &= 0, \qquad \nabla g(0) = 0. \end{aligned} \tag{3.1}$$

Then $\nu(x)$ the unit outward normal vector and $T_x(\partial\Omega)$ the tangent space at x to $\partial\Omega$ are given by

$$\nu(x) = \frac{1}{(1 + |\nabla g(x')|^2)^{1/2}}\, (\nabla g(x'), -1)$$

$$T_x(\partial\Omega) = \operatorname{span}\{\tau_1(x), \ldots, \tau_{n-1}(x)\}$$

where, for $1 \leqslant i \leqslant n - 1$,

$$\tau_i(x) = \left(0 \cdots 0, 1, 0 \cdots 0, \frac{\partial g}{\partial x_i}(x')\right).$$

The normal and the $(n-1)$ tangential derivatives are given by

$$\frac{\partial}{\partial \nu}=\frac{1}{(1+|\nabla g(x')|^2)^{1/2}}\left\{\sum_{i=1}^{n-1}\frac{\partial g}{\partial x_i}\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_n}\right\}_{x_n=g(x')} \tag{3.2}$$

and for $1\leqslant i\leqslant n-1$

$$\frac{\partial}{\partial \tau_i}=\left\{\frac{\partial}{\partial x_i}+\frac{\partial g}{\partial x_i}(x')\frac{\partial}{\partial x_n}\right\}_{x_n=g(x')}. \tag{3.3}$$

Define the map $\psi=(\psi_1, ..., \psi_n): B(R)\to\mathbb{R}^n$ by

$$\begin{aligned}\psi_j(x)&=x_j-\frac{(g(x')-x_n)}{(1+|\nabla g(x')|^2)}\frac{\partial g}{\partial x_j}(x') \qquad 1\leqslant j\leqslant n-1\\ \psi_n(x)&=x_n-g(x').\end{aligned} \tag{3.4}$$

Then clearly the determinant of the Jacobian of ψ at 0 is one and hence we can choose $R_0<R$ and $\tilde{B}\subset B(R)$ an open neighbourhood of zero such that

$$\begin{aligned}&\psi: \tilde{B}\to B(R_0) \text{ is an diffeomorphism}\\ &\psi: \Omega\cap\tilde{B}\to B(R_0)^+=\{(y', y_n)\in B(R_0),\ y_n>0\}\\ &\psi: \partial\Omega\cap\tilde{B}\to\{(y', y_n)\in B(R_0),\ y_n=0\}.\end{aligned} \tag{3.5}$$

Let u be a smooth function on $\tilde{B}$ and let us define v on $B(R_0)$ by $v(y)=u(\psi^{-1}(y))$. Then by a direct calculation, it follows that

$$\begin{aligned}(\Delta u)(\psi^{-1}(y))&=\sum_{j,k}a_{j,k}(y)\frac{\partial^2 v}{\partial y_j\,\partial y_k}+\sum_j b_j(y)\frac{\partial v}{\partial y_j}\\ \frac{\partial u}{\partial \nu}(\psi^{-1}(y))&=\alpha(y)\frac{\partial v}{\partial y_n}(y) \qquad \text{on} \quad y_n=0,\end{aligned} \tag{3.6}$$

where a_{jk}, b_j and $\alpha>0$ are smooth functions satisfying

$$a_{jk}(y)=\delta_{jk}+O(|y|). \tag{3.7}$$

For $u\in H^1(\Omega)$, $y\in\mathbb{R}^n$, $p=(n+2)/(n-2)$ and $\varepsilon>0$ define

$$U(x)=\left[\frac{n(n-2)}{n(n-2)+|x|^2}\right]^{(n-2)/2} \tag{3.8}$$

$$U_{\varepsilon,y}(x)=\frac{1}{\varepsilon^{(n-2)/2}}U\left(\frac{x-y}{\varepsilon}\right) \tag{3.9}$$

$$M=\{CU_{\varepsilon,y}: C\in\mathbb{R},\ \varepsilon>0,\ y\in\partial\Omega\} \tag{3.10}$$

$$d(u, M)=\inf\{|\nabla(u-\psi)|_2^2;\ \psi\in M\}. \tag{3.11}$$

Then we have the following

LEMMA 3.1. *Let $\delta > 0$ and $\{\phi_l\} \subset H^1(\Omega)$ be such that*

$$\phi_l \to 0 \text{ weakly in } H^1(\Omega) \tag{3.12}$$

$$d(\phi_l, M)^2 \leqslant |\nabla\phi_l|_2^2 - 2\delta. \tag{3.13}$$

Then there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $d(\phi_l, M)$ is achieved by some $C_l U_{\varepsilon_l, y_l}$. Furthermore if ω_l is defined by

$$\phi_l = C_l U_{\varepsilon_l, y_l} + \omega_l. \tag{3.14}$$

then for a subsequence

(i) $\lim_{l \to \infty} \varepsilon_l = 0$

(ii) *If $d(\phi_l, M) \to 0$ as $l \to \infty$, then $\lim_{l \to \infty} C_l = C_0 \neq 0$*

(iii) *For any bounded tangential derivative D at y_l, we have*

$$\int_\Omega \omega_l U^p_{\varepsilon_l, y_l} = O \begin{cases} \varepsilon_l^{1/2} \|\omega_l\| & \text{if } n = 3 \\ \varepsilon_l (\log 1/\varepsilon_l)^{2/3} \|\omega_l\| & \text{if } n = 4 \\ \varepsilon_l \|\omega_l\| & \text{if } n \geqslant 5 \end{cases} \tag{3.15}$$

$$\int_\Omega \omega_l U^{p-1}_{\varepsilon_l, y_l} D U_{\varepsilon_l, y_l} = O(\|\omega_l\|). \tag{3.16}$$

Proof. Let $\{C_{kl} U_{\varepsilon_{kl}, y_{kl}}\}$ be a minimizing sequence for $d(\phi_l, M)$. Then there exists $k_0 > 0$ such that for all $k \geqslant k_0$ we have from (3.13)

$$\begin{aligned} &-2(\operatorname{sign} C_{kl}) \frac{|C_{kl}|}{\varepsilon_{kl}^{(n+2)/2}} \int_\Omega \frac{\nabla\phi_l \cdot (x - y_{kl})}{(n(n-2) + |(x - y_{kl})/\varepsilon_{kl}|^2)^{n/2}} \\ &+ (n(n-2))^{n/2} \left(\frac{C_{kl}}{\varepsilon_{kl}^{(n+2)/2}}\right)^2 \\ &\times \int_\Omega \frac{|x - y_{kl}|^2}{(n(n-2) + |(x - y_{kl})/\varepsilon_{kl}|^2)^n} \leqslant -\delta_1 \end{aligned} \tag{3.17}$$

where $\delta_1 = \delta/(n(n-2))^{n/2} (n-2)$.

Let us set for a subsequence of k and l

$$\begin{aligned} &\lim_{k \to \infty} y_{kl} = y_l, \qquad \lim_{k \to \infty} \frac{|C_{kl}|}{\varepsilon_{kl}^{(n+2)/2}} = \mu_l, \qquad \lim_{k \to \infty} \frac{C_{kl}}{\varepsilon_{kl}} = \nu_l \\ &\lim_{k \to \infty} \varepsilon_{kl} = \varepsilon_l, \qquad \lim_{k \to \infty} C_{kl} = C_l \\ &\lim_{l \to \infty} y_l = y_0, \qquad \lim_{l \to \infty} \mu_l = \mu_0 \\ &\lim_{l \to \infty} \varepsilon_l = \varepsilon_0, \qquad \lim_{l \to \infty} C_l = C_0. \end{aligned} \tag{3.18}$$

We claim that there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon_l \in (0, \infty)$ and $C_l \in \mathbb{R} \backslash \{0\}$.

Suppose that for a subsequence of l, $\varepsilon_l = \infty$, then from (3.17), we have

$$\pm 2\mu_l \int_\Omega \nabla\phi_l \cdot (x - y_l) + \mu_l^2 \int_\Omega |x - y_l|^2 \leqslant -\delta/(n-2). \tag{3.19}$$

This implies that $\{\mu_l\}$ is bounded and $\mu_0 < \infty$. Since $\phi_l \to 0$ weakly in $H^1(\Omega)$, letting $l \to \infty$ in (3.19) we get

$$\mu_0^2 \int_\Omega |x - y_0|^2 \leqslant -\delta/(n-2)$$

which is a contradiction. Hence $\varepsilon_l < \infty$.

Suppose that for a subsequence $\varepsilon_l = 0$, then from (3.17) we obtain

$$(n(n-2))^{n/2} v_l^2 \int_{\mathbb{R}_+^n} |\nabla U_{0, y_l}|^2 \leqslant -\delta_1$$

which is a contradiction. This proves that $\varepsilon_l \in (0, \infty)$ and from (3.17) we have that $C_l \neq 0$ and $C_l < \infty$.

Therefore $d(\phi_l, M)$ is achieved by $C_l U_{\varepsilon_l, y_l}$ and from (3.17) we obtain

$$\begin{aligned} &-2(\operatorname{sign} C_l) \frac{|C_{kl}|}{\varepsilon_l^{(n+2)/2}} \int_\Omega \frac{\nabla\phi_l \cdot (x - y_l)}{[n(n-2) + |(x - y_l)/\varepsilon_l|^2]^{n/2}} \\ &\quad + (n(n-2))^{n/2} \left(\frac{C_l}{\varepsilon^{(n+2)/2}}\right)^2 \\ &\quad \times \int_\Omega \frac{|x - y_l|^2}{[n(n-2) + |(x - y_l)/\varepsilon_l|^2]^n} \leqslant -\delta_1. \end{aligned} \tag{3.20}$$

Since $\phi_l \to 0$ weakly in $H^1(\Omega)$, letting $l \to \infty$ in (3.20) we obtain that $\varepsilon_l \to \varepsilon_0 = 0$.

If $d(\phi_l, M) \to 0$, then from (3.13), it follows that $|\nabla\phi_l|_2 \nrightarrow 0$. Hence from (3.14)

$$0 \neq \lim_{l \to \infty} |\nabla\phi_l|_2^2 = C_0^2 \lim_{l \to \infty} |\nabla U_{\varepsilon_l, y_l}|^2$$

and this implies that $C_0 \neq 0$.

Since (3.12) and (3.13) are invariant under translations and rotations, without loss of generality we can assume that $y_0 = 0$. Let $R > 0$ and g be given by (3.1). Choose l_0 large such that for $l \geqslant l_0$, $y_l \in B(R)$.

Observe that $M \backslash \{0\}$ is a manifold of dimension $(n+1)$ and the tangent space $T_{C_l, U_{\varepsilon_l, y_l}}(M)$ at $(C_l, \varepsilon_l, y_l)$ is given by

$$T_{C_l, U_{\varepsilon_l, y_l}}(M) = \operatorname{Span} \left\{ U_{\varepsilon, y}, C \frac{\partial}{\partial\varepsilon} U_{\varepsilon, y}, C \frac{\partial}{\partial\tau_i} U_{\varepsilon, y}, 1 \leqslant i \leqslant n-1 \right\}_{(C_l, \varepsilon_l, y_l)} \tag{3.21}$$

where $\partial/\partial\tau_i$ is the tangential derivative at y_l given by (3.3). From (3.14), we obtain

$$\int_\Omega \nabla\omega_l \nabla U_{\varepsilon_l, y_l} = 0 \tag{3.22}$$

$$\int_\Omega \nabla\omega_l \nabla \frac{\partial}{\partial\tau_i} U_{\varepsilon_l, y_l} = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant n-1. \tag{3.23}$$

Integrating by parts, we have from (3.22)

$$\begin{aligned}
-\int_\Omega \omega_l U^p_{\varepsilon_l, y_l} &= \int_{\partial\Omega} \omega_l \frac{\partial U_{\varepsilon_l, y_l}}{\partial\nu} \\
&= \int_{B(R)\cap\partial\Omega} \omega_l \frac{\partial U_{\varepsilon_l, y_l}}{\partial\nu} + \int_{\partial\Omega\setminus B(R)} \omega_l \frac{\partial U_{\varepsilon_l, y_l}}{\partial\nu} \qquad (3.24)\\
&= \int_{B(R)\cap\partial\Omega} \omega_l \frac{\partial U_{\varepsilon_l, y_l}}{\partial\nu} + O\left(\varepsilon_l^{(n-2)/2} \int_{\partial\Omega} |\omega_l|\right).
\end{aligned}$$

Using Taylor's formula we have for $x \in B(R) \cap \partial\Omega$,

$$\begin{aligned}
\frac{\partial}{\partial\nu} |x - y_l|^2 &= \frac{2}{(1 + |\nabla g|^2)^{1/2}} \left\{ \sum_{i=1}^{n-1} (x_i - y_{il}) \frac{\partial g}{\partial x_i} (y_l') - (g(x') - g(y_l')) \right\} \\
&= O(|x' - y_l'|^2) \qquad (3.25)\\
\frac{\partial}{\partial\nu} U_{\varepsilon_l, y_l} &= \frac{1}{\varepsilon_l^{(n+2)/2}} \frac{-[n(n-2)]^{(n-2)/2} (n-2)(\partial/\partial\nu) |x - y_l|^2}{[n(n-2) + |(x - y_l)/\varepsilon_l|^2]^{n/2}}.
\end{aligned}$$

Hence for $q = (2(n-1))/n$, we have from (3.25)

$$\begin{aligned}
&\left(\int_{\partial\Omega\cap B(R)} \left|\frac{\partial}{\partial\nu} U_{\varepsilon_l, y_l}\right|^q\right)^{1/q} \\
&= O\left(\varepsilon_l^{(n-1)-((n-2)/2)q} \int_{|\xi| \leqslant 2R/\varepsilon_l} \frac{|\xi|^{2q}}{(1 + |\xi|^2)^{nq/2}}\right)^{1/q} \qquad (3.26)\\
&= O \begin{cases} \varepsilon_l^{1/2} & \text{if} \quad n = 3 \\ \varepsilon_l (\log 1/\varepsilon_l)^{2/3} & \text{if} \quad n = 4 \\ \varepsilon_l & \text{if} \quad n \geqslant 5. \end{cases}
\end{aligned}$$

Thus by trace embedding theorems and by Hoelder's inequality we get from (3.24)–(3.26)

$$\int_\Omega \omega_l U^p_{\varepsilon_l, y_l} = O \begin{cases} \varepsilon_l^{1/2} \|\omega_l\| & \text{if} \quad n = 3 \\ \varepsilon_l (\log 1/\varepsilon_l)^{2/3} \|\omega_l\| & \text{if} \quad n = 4 \\ \varepsilon_l \|\omega_l\| & \text{if} \quad n \geqslant 5. \end{cases}$$

This proves (3.15).

Now for $1 \leqslant i \leqslant n-1$

$$-\Delta \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l} = p U_{\varepsilon_l, y_l}^{p-1} \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l}.$$

Hence from (3.23), we have

$$0 = \int_\Omega \nabla \omega_l \nabla \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l} = \int_\Omega \omega_l (-\Delta) \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l} + \int_{\partial\Omega} \omega_l \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l}$$

that is

$$\begin{aligned} p \int_\Omega \omega_l U_{\varepsilon_l, y_l}^{p-1} \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l} &= -\int_{B(R) \cap \partial\Omega} \omega_l \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l} - \int_{\partial\Omega \setminus B(R)} \omega_l \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l} \\ &= -\int_{B(R) \cap \partial\Omega} \omega_l \frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l} + O(\varepsilon_l^{(n-2)/2} \|\omega_l\|). \end{aligned} \tag{3.27}$$

In $B(R) \cap \partial\Omega$,

$$\frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l}(x) = \frac{C_n}{\varepsilon_l^{(n+2)/2}} \frac{(x_i - y_{il}) + (x_n - g(y_l'))(\partial g/\partial y_i)(y'l)}{[n(n-2) + (|x' - y_l'|^2 - g(y_l')|^2)/\varepsilon_l^2]^{n/2}}$$

where $C_n = [n(n-2)]^{(n-2)/2}(n-2)$.

Hence

$$\begin{aligned} &\frac{\partial}{\partial \nu} \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l}(x)\Big|_{x_n = g(x')} \\ &= \frac{C_n}{\varepsilon_l^{(n+2)/2}} \frac{(\partial x_i/\partial \nu) + (\partial x_n/\partial \nu)(\partial g/\partial y_i)(y_l')}{[n(n-2) + (|x' - y_l'|^2 + |g(x') - g(y_l')|^2)/\varepsilon_l^2]^{n/2}} \\ &\quad - \frac{nC_n}{\varepsilon_l^{(n+6)/2}} \frac{[(x_i - y_{il}) + (g(x') - g(y_l'))(\partial g/\partial y_i)(y_l')][(\partial/\partial \nu)|x - y_l|^2]}{[n(n-2) + (|x' - y_l'|^2 + (g(x') - g(y_l')|^2)/\varepsilon_l^2]^{(n+2)/2}}. \end{aligned}$$

Now from (3.25)

$$\begin{aligned} \frac{\partial x_i}{\partial \nu} + \frac{\partial x_n}{\partial \nu} \frac{\partial g}{\partial y_i}(y_l') &= \frac{1}{(1 + |\nabla g(x')|^2)^{1/2}} \left\{ \frac{\partial g}{\partial x_i}(x') - \frac{\partial g}{\partial y_i}(y_l') \right\} \\ &= O(|x' - y_l'|), \end{aligned}$$

$$\left[(x_i - y_{il}) + (g(x') - g(y_l')) \frac{\partial g}{\partial y_i}(y_l') \right] \frac{\partial}{\partial \nu} |x - y_l|^2 = O(|x' - y_l'|^3).$$

Hence for $q = 2(n-1)/n$

$$\frac{\partial}{\partial \nu}\frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l}(x) = O\left(\frac{|x' - y_l'|}{\varepsilon_l^{(n+2)/2}\left[1 + \frac{|x' - y'|^2}{\varepsilon_l^2}\right]^{n/2}}\right)$$

and

$$\int_{\partial\Omega \cap B(R)} \left|\frac{\partial}{\partial \nu}\frac{\partial}{\partial_{\tau_i}} U_{\varepsilon_l, y_l}\right|^q$$

$$= O\left(\frac{1}{\varepsilon_l^{nq/2 - (n-1)}} \int_{|\xi| \leqslant 2R/\varepsilon_l} \frac{|\xi|^q}{(1 + |\xi|^2)^{nq/2}}\right) = O(1). \tag{3.28}$$

Hence by trace embedding theorems, (3.27) and (3.28) we get for $1 \leqslant i \leqslant n-1$,

$$\int_\Omega \omega_l U_{\varepsilon_l, y_l}^{p-1} \frac{\partial}{\partial \tau_i} U_{\varepsilon_l, y_l} = \left(\int_{\partial\Omega} |\omega_l|^{2(n-1)/(n-2)}\right)^{(n-2)/2(n-1)} O(1)$$

$$+ O(\varepsilon_l^{(n-2)/2} \|\omega_l\|) = O(\|\omega_l\|). \tag{3.29}$$

Now if D is any bounded tangential derivatives, then D is a linear combination of $\partial/\partial\tau_i$, $1 \leqslant i \leqslant n-1$ and hence from (3.29), (3.16) follows. This proves the lemma. ▮

LEMMA 3.2. *For every $\delta > 0$, there exists $C(\delta) > 0$ such that for $u \in H^1(\Omega)$*

$$|u|_{p+1}^2 \leqslant \left(\frac{2^{2/n}}{S} + \delta\right) |\nabla u|_2^2 + C(\delta)\, |u|_2^2. \tag{3.30}$$

This lemma is due to Cherrier. For the proof we refer to Aubin [7, p. 51].

In the following lemma, we study some eigenvalue problems in Ω which are related to an eigenvalue problem in $\mathbb{R}^n_+$. It was shown in Bianchi and Egnell [8] and Rey [27] that the following problem

$$-\Delta\phi = \nu U^{p-1}\phi \text{ in } \mathbb{R}^n$$

$$\int_{\mathbb{R}^n} U^{p-1} |\phi|^2 < \infty$$

admits a discrete set of eigenvalues $\nu_1 < \nu_2 \leqslant \nu_3 \cdots$ such that $\nu_1 = 1$, $\nu_i = p$ for $2 \leqslant i \leqslant n+1$ and $\nu_{n+2} > p$. The eigenspaces V_1 and V_p corresponding to 1 and p are given by

$$V_1 = \operatorname{span}\{U\}$$
$$V_p = \operatorname{span}\left\{ \left.\frac{\partial U_{1,y}}{\partial y_i}\right|_{y=0} \quad 1 \leqslant i \leqslant n \right\}.$$

As an immediate consequence of this we have

COROLLARY 3.1. *The following eigenvalue problem*

$$-\Delta\phi = \mu U^{p-1}\phi \qquad \text{in} \quad \mathbb{R}^n_+$$
$$\frac{\partial \phi}{\partial \nu} = 0 \qquad \text{on } \partial\mathbb{R}^n_+$$
$$\int_{R^n_+} U^{p-1} |\phi|^2 < \infty$$

admits a discrete spectrum $\mu_1 < \mu_2 \leqslant \mu_3 \cdots$ *such that* $\mu_1 = 1$, $\mu_2 = \mu_3 \cdots = \mu_n = p$, $\mu_{n+1} > p$. *The eigenspaces* V_1 *and* V_p *corresponding to* 1 *and* p *are given by*

$$V_1 = \operatorname{span}\{U\}$$
$$V_p = \operatorname{span}\left\{ \left.\frac{\partial U_{1,y}}{\partial y_i}\right|_{y=0} \quad 1 \leqslant i \leqslant n-1 \right\}.$$

Let $\varepsilon > 0$, $\nu(\varepsilon) > 0$ and $P_\varepsilon \in \partial\Omega$ with $\lim_{\varepsilon \to 0} P_\varepsilon = P_0$. Let $\{\phi_{i\varepsilon}\}_{i=1}^{\infty}$ be the complete set of orthonormal eigenfunctions with eigenvalues $0 < \mu_{1\varepsilon} < \mu_{2\varepsilon} \leqslant \mu_{3\varepsilon} \cdots$ for the weighted eigenvalue problem

$$\begin{aligned} -\Delta u + \nu(\varepsilon) u &= \mu U_{\varepsilon, P_\varepsilon}^{p-1} u \qquad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.31}$$

We can assume, without loss of generality, that $\phi_{1\varepsilon} > 0$ and

$$\int_\Omega \phi_{i\varepsilon}\phi_{j\varepsilon} U_{\varepsilon, P_\varepsilon}^{p-1} = \delta_{ij}. \tag{3.32}$$

Let $\Omega_\varepsilon = (\Omega - \{P_\varepsilon\})/\varepsilon$. For any function v on Ω let us define $\tilde{v}$ on Ω_ε by

$$\tilde{v}(x) = \varepsilon^{(n-2)/2} v(\varepsilon x + P_\varepsilon). \tag{3.33}$$

Then we have

LEMMA 3.3. *Assume that* $\nu(\varepsilon) \to \infty$, $\varepsilon^2\nu(\varepsilon) \to 0$ *as* $\varepsilon \to 0$. *Then up to a subsequence* $\Omega_\varepsilon \to \mathbb{R}^n_+$

$$\lim_{\varepsilon \to 0} \mu_{i\varepsilon} = \mu_i \tag{3.34}$$

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} U^{p-1}(\tilde{\phi}_{i\varepsilon} - \tilde{\phi}_i)^2 = 0 \tag{3.35}$$

where μ_i *and* $\tilde{\phi}_i$ *satisfy*

$$\begin{aligned} -\Delta\tilde{\phi}_i &= \mu_i U^{p-1}\tilde{\phi}_i \qquad && \text{in } \mathbb{R}^n_+ \\ \frac{\partial \tilde{\phi}_i}{\partial x_n} &= 0 && \text{on } \partial\mathbb{R}^n_+ \end{aligned} \tag{3.36}$$

$$\int_{\mathbb{R}^n_+} |\nabla\tilde{\phi}_i|^2 + \int_{\mathbb{R}^n_+} U^{p-1}\tilde{\phi}_i^2 < \infty.$$

In particular from Corollary 3.1, $\mu_1 = 1$, $\tilde{\phi}_1 = CU$ *for some* $C > 0$ *and there exists* $2 \leqslant k_0 \leqslant n$ *such that* $\mu_i = p$ *for* $2 \leqslant i \leqslant k_0$ *and* $\mu_{k_0+1} > p$. *Furthermore* $\{\tilde{\phi}_i\}$ *for* $2 \leqslant i \leqslant k_0$ *is in the span of* $\{(\partial U_{1,y}/\partial y_i)|_{y=0},\ 1 \leqslant i \leqslant n-1\}$.

Proof. First we prove the lemma for $i = 1$. By the Rayleigh quotient, $\mu_{1\varepsilon}$ is given by

$$\begin{aligned} \mu_{1\varepsilon} &= \inf\left\{\int_\Omega |\nabla u|^2 + \nu(\varepsilon)\int_\Omega u^2;\ \int_\Omega U^{p-1}_{\varepsilon,p_\varepsilon} u^2 = 1\right\} \\ &= \inf\left\{\int_{\Omega_\varepsilon} |\nabla v|^2 + \varepsilon^2\nu(\varepsilon)\int_{\Omega_\varepsilon} v^2;\ \int_{\Omega_\varepsilon} U^{p-1}v^2 = 1\right\}. \end{aligned} \tag{3.37}$$

Since $\tilde{\Omega}_\varepsilon \to \mathbb{R}^n_+$ as $\varepsilon \to 0$, we can choose $x_0 \in \mathbb{R}^n$, $\varepsilon_0 > 0$ and $R > 0$ such that

$$B(x_0, R) \subset \Omega_\varepsilon \qquad \text{for all} \quad 0 < \varepsilon < \varepsilon_0. \tag{3.38}$$

Let $\chi \in C_0^\infty(B(x_0, R))$ with $\int_{R^n} \chi^2 U^{p-1} = 1$. Then from (3.37), (3.38) and the fact that $\{\varepsilon^2\nu(\varepsilon)\}$ is bounded, we have for all $\varepsilon < \varepsilon_0$

$$\mu_{1\varepsilon} \leqslant \int_{\mathbb{R}^n} |\nabla\chi|^2 + \varepsilon^2\nu(\varepsilon)\int_{\mathbb{R}^n} \chi^2$$

and this proves that $\{\mu_{1\varepsilon}\}$ is bounded. Since

$$\int_\Omega |\nabla\phi_{1\varepsilon}|^2 + \nu(\varepsilon)\int_\Omega \phi_{1\varepsilon}^2 = \mu_{1\varepsilon}, \tag{3.39}$$

we deduce that $\{\|\phi_{1\varepsilon}\|\}$ is bounded and $|\phi_{1\varepsilon}|_2^2 \leqslant \mu_{1\varepsilon}/\nu(\varepsilon) \to 0$. Therefore $\phi_{1\varepsilon} \to 0$ weakly in $H^1(\Omega)$.

Up to a subsequence we have

$$\lim_{\varepsilon \to 0} \mu_{1\varepsilon} = \mu_1. \tag{3.40}$$

Suppose $\mu_1 = 0$. Then from (3.39), $\|\phi_{1\varepsilon}\| \to 0$. Hence from (3.30), $|\phi_{1\varepsilon}|_{p+1} \to 0$. Now by Hoelder's inequality, we get from (3.32)

$$1 = \int_\Omega \phi_{1\varepsilon}^2 U_{\varepsilon, p_\varepsilon}^{p-1} \leqslant \left(\int_\Omega U_{\varepsilon, p_\varepsilon}^{p+1}\right)^{(p-1)/(p+1)} \left(\int_\Omega \phi_{1\varepsilon}^{p+1}\right)^{2/p+1} \to 0$$

which is a contradiction. This proves that $\mu_1 \neq 0$ and $\|\phi_{1\varepsilon}\| \not\to 0$.

In Ω_ε, $\tilde{\phi}_{1\varepsilon}$ satisfies

$$\left.\begin{aligned} -\Delta\tilde{\phi}_{1\varepsilon} + v(\varepsilon)\,\varepsilon^2\tilde{\phi}_{1\varepsilon} &= \mu_{1\varepsilon} U^{p-1}\tilde{\phi}_{1\varepsilon} && \text{in } \Omega_\varepsilon \\ \tilde{\phi}_{1\varepsilon} &> 0 && \text{in } \Omega_\varepsilon \\ \frac{\partial\tilde{\phi}_{1\varepsilon}}{\partial v} &= 0 && \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \tilde{\phi}_{1\varepsilon}^2 U^{p-1} &= 1. \end{aligned}\right\} \tag{3.41}$$

By the elliptic regularity theory, $\tilde{\phi}_{1\varepsilon}$ converges to $\tilde{\phi}_1$ in $C^2_{\text{loc}}(\mathbb{R}^n_+)$. Since $\varepsilon^2 v(\varepsilon) \to 0$, we obtain that $\tilde{\phi}_1$ satisfies (3.36). Now for $\delta > 0$, choose $\varepsilon(\delta) > 0$ and $K_\delta \Subset \Omega_\varepsilon$ for $\varepsilon \leqslant \varepsilon(\delta)$ such that

$$\int_{\mathbb{R}_n \setminus K_\delta} U^{p+1} < \delta. \tag{3.42}$$

Then for $\varepsilon < \varepsilon(\delta)$

$$\begin{aligned} &\overline{\lim_{\varepsilon \to 0}} \int_{\Omega_\varepsilon} U^{p-1}(\tilde{\phi}_{1\varepsilon} - \tilde{\phi}_1)^2 \\ &\quad = \overline{\lim_{\varepsilon \to 0}} \left\{\int_{K_\delta} U^{p-1}(\tilde{\phi}_{1\varepsilon} - \tilde{\phi}_1)^2 + \int_{\Omega_\varepsilon \setminus K_\delta} U^{p-1}(\tilde{\phi}_{1\varepsilon} - \tilde{\phi}_1)^2\right\} \\ &\quad \leqslant \overline{\lim_{\varepsilon \to 0}} \left\{\int_{K_\delta} U^{p-1}(\tilde{\phi}_{1\varepsilon} - \tilde{\phi}_1)^2 \right. \\ &\qquad \left. + \left(\int_{\mathbb{R}_n \setminus K_\delta} U^{p+1}\right)^{(p-1)/(p+1)} \left(\int_{\Omega_\varepsilon} |\tilde{\phi}_{1\varepsilon} - \tilde{\phi}_1|^{p+1}\right)^{2/p+1}\right\} \\ &\quad \leqslant O(\delta^{(p-1)/(p+1)}). \end{aligned} \tag{3.43}$$

This proves (3.35), since δ is arbitrary and $1=\int_{\mathbb{R}^n_+} U^{p-1}\tilde{\phi}_1^2$. Now the proof of the lemma follows by induction. Assume that for $1 \leqslant i \leqslant L-1$, $\{\mu_{i\varepsilon}\}$ are bounded and

$$\lim_{\varepsilon \to 0} \mu_{i\varepsilon} = \mu_i$$

$$\lim_{\varepsilon \to 0} \tilde{\phi}_{1\varepsilon} = \tilde{\phi}_i \qquad \text{in } C^2_{\mathrm{loc}} \tag{3.44}$$

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} U^{p-1}(\tilde{\phi}_{1\varepsilon} - \tilde{\phi}_i)^2 = 0,$$

where $\tilde{\mu}_i$ is an eigenvalue with eigenfunction $\tilde{\phi}_i$ of (3.36). Now by the Rayleigh quotient,

$$\mu_{L,\varepsilon} = \inf\left\{\int_{\Omega_\varepsilon} |\nabla v|^2 + \varepsilon^2 \nu(\varepsilon) \int_{\Omega_\varepsilon} v^2; \int_{\Omega_\varepsilon} U^{p-1} v^2 = 1, \right.$$
$$\left. \int_{\Omega_\varepsilon} U^{p-1} \tilde{\phi}_{i\varepsilon} v = 0 \text{ for } 1 \leqslant i \leqslant L-1 \right\}. \tag{3.45}$$

From (3.38) and (3.44), we can choose $\chi_\varepsilon \in C_0^\infty(B(x_0, R))$ such that

$$\int_{\Omega_\varepsilon} U^{p-1}\chi_\varepsilon^2 = 1, \qquad \int_{\Omega_\varepsilon} U^{p-1}\tilde{\phi}_{i\varepsilon}\chi_\varepsilon = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant L-1$$
$$\sup_\varepsilon \left\{\int |\nabla \chi_\varepsilon|^2 + \int \chi_\varepsilon^2\right\} = M < \infty. \tag{3.46}$$

Hence from (3.46) and (3.45), $\{\mu_{L,\varepsilon}\}$ is uniformly bounded. Now repeating the same procedure as for $i=1$ we conclude that for a subsequence $\mu_{L,\varepsilon} \to \mu_L$, $\tilde{\phi}_{L\varepsilon} \to \tilde{\phi}_L$ in the sense of (3.35) and μ_L is an eigenvalue of (3.36) having $\tilde{\phi}_L$ as corresponding eigenfunction. This proves the lemma. ∎

LEMMA 3.4. *Assume that $P_0 = 0$ and in a neighbourhood of P_0, Ω is described as in* (3.1). *Further assume that ε and $\nu(\varepsilon)$ satisfy the hypotheses of Lemma* 3.3. *Then there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,*

$$|\nabla \omega|_2^2 + \nu(\varepsilon)\, |\omega|_2^2 \geqslant (p+\delta) \int_\Omega U_{\varepsilon, P_\varepsilon}^{p-1} \omega^2 + O(\beta_3(\varepsilon)^2\, \|\omega\|^2) \tag{3.47}$$

where ω is orthogonal to $T_{1, U_\varepsilon, P_\varepsilon}(M)$ (see 3.21).

Proof. For $A, B \in H^1(\Omega_\varepsilon)$, define

$$\langle A, B\rangle = \int_{\Omega_\varepsilon} U^{p-1} AB \tag{3.48}$$

$$\|A\|_\varepsilon^2 = \int_{\Omega_\varepsilon} |\nabla A|^2 + \varepsilon^2 \nu(\varepsilon) \int_{\Omega_\varepsilon} |A|^2. \tag{3.49}$$

By Lemma 3.3, there exist constants $C>0$, $a_{ij}\in\mathbb{R}$ such that $\tilde{\phi}_1 = CU$ and for $2 \leqslant i \leqslant k_0$

$$\tilde{\phi}_i = \sum_{j=1}^{n-1} a_{ij} U^j \tag{3.50}$$

where $U^j(x) = \frac{\partial}{\partial y_j} U_{1,y}\,|_{y=0}$ for $1 \leqslant j \leqslant n$.

For $1 \leqslant i \leqslant n-1$, define ψ_i on Ω such that $\tilde{\psi}_i$ on Ω_ε is given by

$$\tilde{\psi}_i = \sum_{j=1}^{n-1} a_{ij} \left(U^j + \frac{\partial g}{\partial y_j}(P'_\varepsilon)\, U^n \right) \tag{3.51}$$

and hence from (3.3)

$$\psi_i = \sum_{j=1}^{n-1} a_{ij} \frac{\partial}{\partial \tau_j} U_{\varepsilon, P_\varepsilon}. \tag{3.52}$$

Let

$$\delta_\varepsilon = \sup_{i,j} |a_{ij}|\, |\nabla g(P'_\varepsilon)| + \int_{\Omega_\varepsilon} U^{p-1} (\tilde{\phi}_{i\varepsilon} - \tilde{\phi}_i)^2. \tag{3.53}$$

Then from (3.26), we have by Hoelder's inequality

$$\begin{aligned} |\langle \tilde{\omega}, \tilde{\phi}_{1\varepsilon} \rangle| &\leqslant |\langle \tilde{\omega}, \tilde{\phi}_{1\varepsilon} - \tilde{\phi}_1 \rangle| + |\langle \tilde{\omega}, \tilde{\phi}_1 \rangle| \\ &\leqslant \left(\int_{\Omega_\varepsilon} U^{p-1} \tilde{\omega}^2 \right)^{1/2} \left(\int_{\Omega_\varepsilon} U^{p-1} (\tilde{\phi}_{1\varepsilon} - \tilde{\phi}_1)^2 \right)^{1/2} + C \left| \int_\Omega U^p_{\varepsilon, P_\varepsilon} \omega \right| \\ &\leqslant \left(\delta_\varepsilon \int_{\Omega_\varepsilon} U^{p-1} \tilde{\omega}^2 \right)^{1/2} + O(\beta_3(\varepsilon)\, \|\omega\|). \end{aligned} \tag{3.54}$$

For $2 \leqslant i \leqslant k_0$, we have from (3.50)–(3.52) and (3.29)

$$\begin{aligned} |\langle \tilde{\omega}, \tilde{\phi}_{i\varepsilon} \rangle| &\leqslant |\langle \tilde{\omega}, \tilde{\phi}_{i\varepsilon} - \tilde{\psi}_i \rangle| + |\langle \tilde{\omega}, \tilde{\psi}_i \rangle| \\ &\leqslant |\langle \tilde{\omega}, \tilde{\phi}_{i\varepsilon} - \tilde{\phi}_i \rangle| + \sum_{j=1}^{n-1} |a_{ij}| \left| \frac{\partial g}{\partial y_j}(P'_\varepsilon) \right| |\langle |\omega|, |U^n| \rangle| + |\langle \tilde{\omega}, \tilde{\psi}_i \rangle| \\ &\leqslant \left(\int_{\Omega_\varepsilon} U^{p-1} \tilde{\omega}^2 \right)^{1/2} \left(\int_{\Omega_\varepsilon} U^{p-1} (\tilde{\phi}_{i\varepsilon} - \tilde{\Phi}_i)^2 \right)^{1/2} \\ &\quad + \delta_\varepsilon \left(\int_{\Omega_\varepsilon} U^{p-1} \tilde{\omega}^2 \right)^{1/2} \left(\int_{\Omega_\varepsilon} U^{p-1} |U^n|^2 \right)^{1/2} \\ &\quad + \varepsilon \sum_{j=1}^{n-1} |a_{ij}| \left| \int_\Omega U^{p-1}_{\varepsilon, p_\varepsilon} \omega \frac{\partial}{\partial \tau_j} U_{\varepsilon, p_\varepsilon} \right| \\ &= O\left(\delta_\varepsilon^{1/2} \left(\int_{\Omega_\varepsilon} U^{p-1} \tilde{\omega}^2 \right)^{1/2} \right) + O(\varepsilon\, \|\omega\|). \end{aligned} \tag{3.55}$$

Let

$$z = \tilde{\omega} - \sum_{i=1}^{k_0} \langle \tilde{\omega}, \tilde{\phi}_{i\varepsilon} \rangle \tilde{\phi}_{i\varepsilon}.$$

Then from (3.54) and (3.55)

$$\begin{aligned}\|z\|_\varepsilon^2 &\geqslant \mu_{k_0+1,\varepsilon} \int_{\Omega_\varepsilon} U^{p-1} z^2 \\ &= \mu_{k_0+1,\varepsilon} \int_{\Omega_\varepsilon} U^{p-1} \tilde{\omega}^2 + O\left(\sum_{i=1}^{k_0} \langle \tilde{\omega}, \tilde{\phi}_{i\varepsilon} \rangle^2 \right) \\ &= (\mu_{k_0+1,\varepsilon} + O(\delta_\varepsilon)) \int_{\Omega_\varepsilon} U^{p-1} \tilde{\omega}^2 + O(\beta_3(\varepsilon)^2 \|\omega\|^2). \qquad (3.56)\end{aligned}$$

Hence from Lemma 3.3 we can choose $\varepsilon_0 > 0$, $\delta > 0$ such that for $o < \varepsilon < \varepsilon_0$,

$$\begin{aligned}\|\tilde{\omega}\|_\varepsilon^2 &= \sum_{i=1}^{k_0} \mu_{i\varepsilon} |\langle \tilde{\omega}, \tilde{\phi}_{i\varepsilon} \rangle|^2 + \|z\|_\varepsilon^2 \\ &\geqslant (\mu_{k_0+1,\varepsilon} + O(\delta_\varepsilon)) \int_{\Omega_\varepsilon} U^{p-1} \tilde{\omega}^2 + O(\beta_3(\varepsilon)^2 \|\omega\|^2) \\ &\geqslant (p + 2\delta) \int_\Omega U_{\varepsilon, p_\varepsilon}^{p-1} \omega^2 + O(\beta_3(\varepsilon)^2 \|\omega\|^2).\end{aligned}$$

This proves (3.47) and hence the lemma. ∎

LEMMA 3.5. *Let $q > 1$ and L be a non negative integer with $L \leqslant q$. Let V and ω be measurable functions on Ω with $V \geqslant 0$, $V + \omega \geqslant 0$. Then*

$$\int_\Omega (V+\omega)^q = \sum_{i=0}^{L} \frac{q(q-1)\cdots(q-i+1)}{i!} \int_\Omega V^{q-i} \omega^i + O\left(\int_\Omega V^{q-r} |\omega|^r + |\omega|^q \right) \qquad (3.57)$$

$$\int_\Omega (V+\omega)^q \omega = \sum_{i=0}^{L} \frac{q(q-1)\cdots(q-i+1)}{i!} \int_\Omega V^{q-i} \omega^{i+1} + O\left(\int_\Omega V^{q-r} |\omega|^{r+1} + |\omega|^{q+1} \right), \qquad (3.58)$$

where $r = \min(L+1, q)$.

Proof. Since $L \leqslant q$, it follows that for any $x \geqslant -1$,

$$(1+x)^q = \sum_{i=0}^{L} \frac{q(q-1)\cdots(q-i+1)}{i!} x^i + O(|x|^r + |x|^q).$$

Hence

$$\int_\Omega (V+\omega)^q = \int_\Omega V^q \left(1+\frac{\omega}{V}\right)^q$$
$$= \sum_{i=0}^{L} \frac{q(q-1)\cdots(q-i+1)}{i!}$$
$$\times \int_\Omega V^{q-i}\omega^i + O\left(\int_\Omega V^{q-r}|\omega|^r + |\omega|^q\right).$$

This proves (3.57). Similarly (3.58) follows and hence the lemma. ∎

LEMMA 3.6. *With the same notations as in Section* 1, *we have*

$$\lim_{\lambda\to\infty} S_\lambda = S/2^{2/n} \tag{3.59}$$

$$\lim_{\lambda\to\infty} \lambda\,|v_\lambda|_2^2 = 0, \tag{3.60}$$

where $v_\lambda \in H^1(\Omega)$ *is such that* $Q_\lambda(v_\lambda) \leqslant S/2^{2/n}$ *and*

$$0 < \underline{\lim}_{\lambda\to\infty} |v_\lambda|_{p+1} \leqslant \overline{\lim}_{\lambda\to\infty} |v_\lambda|_{p+1} < \infty.$$

Proof. Clearly $\lambda \to S_\lambda$ is a non decreasing function. From Theorem 1.1, $S_\lambda < S/2^{2/n}$ and S_λ is achieved by u_λ satisfying (1.7). Let us set $\lim_{\lambda\to\infty} S_\lambda = S_\infty$ and suppose that $S_\infty < S/2^{2/n}$. Let us choose $\delta > 0$, and $0 < a < 1$ such that

$$S_\infty \left(\frac{2^{2/n}}{S} + \delta\right) = a < 1. \tag{3.61}$$

From (3.30) and (3.61) we have

$$|\nabla u_\lambda|_2^2 + \lambda\,|u_\lambda|_2^2 = S_\lambda\,|u_\lambda|_{p+1}^2$$
$$\leqslant S_\lambda \left(\frac{2^{2/n}}{S} + \delta\right) |\nabla u_\lambda|_2^2 + C(\delta)\,|u_\lambda|_2^2$$
$$\leqslant a\,|\nabla u_\lambda|_2^2 + C(\delta)\,|u_\lambda|_2^2.$$

Hence

$$(1-a)\,|\nabla u_\lambda|_2^2 + (\lambda - C(\delta))\,|u_\lambda|_2^2 \leqslant 0,$$

which is a contradiction. This proves (3.59).

From the hypothesis, it follows that $\{\lambda |v_\lambda|_2^2\}$ is bounded and hence $v_\lambda \to 0$ weakly in $H^1(\Omega)$. From (3.30) we have for a $\delta > 0$

$$1 \leqslant \left(\frac{2^{2/n}}{S} + \delta\right) \frac{|\nabla v_\lambda|_2^2}{|v_\lambda|_{p+1}^2} + C(\delta) \frac{|v_\lambda|_2^2}{|v_\lambda|_{p+1}^2}$$

and hence

$$\frac{S}{2^{2/n}} \leqslant \lim_{\lambda \to \infty} \frac{|\nabla v_\lambda|_2^2}{|v_\lambda|_{p+1}^2} \leqslant \lim_{\lambda \to \infty} Q_\lambda(v_\lambda) \leqslant S/2^{2/n}.$$

This implies that $\lim_{\lambda \to \infty} |\nabla v_\lambda|_2^2/|v_\lambda|_{p+1}^2 = S/2^{2/n}$ and $\lim_{\lambda \to \infty} \lambda |v_\lambda|_2^2 = 0$. This proves (3.60) and hence the lemma. ∎

Lemma 3.7. *Let $\{u_k\}$ be a sequence in $H^1(\Omega)$ such that $u_k \to 0$ weakly and $\lim_{k \to \infty} |\nabla u_k|^2 = \lim_{k \to \infty} |u_k|_{p+1}^{p+1} = S^{n/2}/2$. Then there exist $\delta_k > 0$, $y_k \in \partial\Omega$ with $\delta_k \to 0$ such that*

$$\lim_{k \to 0} |\nabla(u_k - U_{\delta_k, y_k})|_2^2 = 0. \tag{3.62}$$

Proof. The proof of this lemma is well known (see Struwe [30, 31]; Grossi and Pacella [16]). For sake of completeness we will give the proof here.

Since $\lim_{k \to \infty} |\nabla u_k|_2^2/|u_k|_{p+1}^{2/p+1} = S/2^{2/n}$, for a subsequence there exist $\alpha \geqslant 0$, $\delta_k > 0$, $x_k \in \bar{\Omega}$ such that $\delta_k \to 0$,

$$\lim_{k \to \infty} |\nabla(u_k - U_{\delta_k, x_k})|_2 = 0 \tag{3.63}$$

$$\lim_{k \to \infty} \frac{d(x_k, \partial\Omega)}{\delta_k} = \alpha. \tag{3.64}$$

Let $x_k \to x_0 \in \partial\Omega$. By rotation and translation, we can assume that $x_0 = 0$ and if $x_k = (x'_k, x_{kn})$, then

$$\lim_{k \to \infty} \frac{d(x_k, \partial\Omega)}{\delta_k} = \lim_{k \to \infty} \frac{x_{kn}}{\delta_k} = \alpha. \tag{3.65}$$

Let $\Omega_k = (\Omega - \{(x'_k, 0)\})/\delta_k$. Then $\Omega_k \to \mathbb{R}^n_+$ and hence from (3.63), we have with $e_n = (0 \cdots 0, 1)$

$$\begin{aligned}
\frac{S^{n/2}}{2} &= \lim_{k \to \infty} |\nabla u_k|_2^2 = \lim_{k \to \infty} |\nabla U_{\delta_k, x_k}|_2^2 \\
&= [n(n-2)]^{n-2} (n-2)^2 \lim_{k \to \infty} \int_{\Omega_k} \frac{|x - (x_{kn}/\delta_k)\, e_n|^2}{[n(n-2) + |x - (x_{kn}/\delta_k)\, e_n|^2]^n} \\
&= [n(n-2)]^{n-2} (n-2)^2 \int_{\mathbb{R}^n_+} \frac{|x - \alpha e_n|^2}{[n(n-2) + |x - \alpha e_n|^2]^n}.
\end{aligned}$$

This implies that $\alpha=0$. Let $y_k \in \partial\Omega$ such that $|x_k - y_k| = d(x_k, \partial\Omega)$ and hence $|x_k - y_k|/\delta_k \to 0$. It follows easily that

$$\lim_{k \to \infty} |\nabla(U_{\delta_k, x_k} - U_{\delta_k, y_k})|_2^2 = 0.$$

This together with (3.63) proves the lemma. ∎

4. Final Remarks

Remark 4.1. The usual proof of the Cherrier embedding (Lemma 3.2) gives $C(\delta) = O(1/\delta^2)$, whereas using Corollary 2.1, we can improve this order to $O(1/\delta)$ as follows.

PROPOSITION 4.1. *Let $n \geqslant 7$. For every $\delta > 0$, there exist a $C(\delta) > 0$ such that*

$$\left(\frac{S}{2^{2/n}} - \delta\right) |u|_{p+1}^2 \leqslant |\nabla u|_2^2 + C(\delta)\, |u|_2^2 \tag{4.1}$$

where $C(\delta) = O(1/\delta)$ as $\delta \to 0$.

Proof. From Corollary 2.1, we have that there exists a constant C depending only on $\partial\Omega$ and Ω such that

$$S_\lambda \geqslant \frac{S}{2^{2/n}} - C\beta_1(\varepsilon_\lambda) \qquad \text{and} \qquad \lambda\beta_2(\varepsilon_\lambda) = O(\beta_1(\varepsilon_\lambda)).$$

Hence by the definition of S_λ we have, for $u \in H^1(\Omega)$

$$\left(\frac{S}{2^{2/n}} - C\beta_1(\varepsilon_\lambda)\right) |u|_{p+1}^2 \leqslant |\nabla u|_2^2 + \lambda\, |u|_2^2.$$

Let $\delta = C\beta_1(\varepsilon_\lambda)$, using that $\lambda\beta_2(\varepsilon_\lambda) = O(\beta_1(\varepsilon_\lambda))$ we obtain (4.1). ∎

Remark 4.2. When Ω is a ball, Theorem 1.2 generalizes a result of Adimurthi and Yadava [5] who proved, by a different method, that for λ large least energy solutions of (1.7) are nonradial.

Remark 4.3. A similar analysis holds also for the mixed boundary value problem. Let $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where Γ_0 and Γ_1 are disjoint components of $\partial\Omega$.

Let $\alpha \in C^1(\bar{\Omega})$, $\lambda > 0$ and consider the following problem

$$\begin{aligned} -\Delta u + \lambda \alpha u &= u^{(n+2)/(n-2)} && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \Gamma_1. \end{aligned} \tag{4.2}$$

Assume that for every $\lambda > 0$,

(i) $-\Delta + \lambda \alpha > 0$ on $H^1(\Gamma_0)$ where

$$H^1(\Gamma_0) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$$

(ii) Γ_1 has points where the mean curvature is positive

(iii) $\alpha|_{\Gamma_1}$ is a positive function such that

$$\min_{\Gamma_1} \alpha = \min\{\alpha(x) : x \in \Gamma_1 \text{ and } H(x) \text{ is maximum}\}.$$

THEOREM 4.1. *There exists $\lambda_0 > 0$ such that if u_λ is a solution of* (4.2) *with $Q_{\lambda\alpha}(u_\lambda) < S/2^{2/n}$, then for $\lambda > \lambda_0$*

(a) *u_λ attains its unique maximum at $P_\lambda \in \Gamma_1$.*

(b) *if $n \geqslant 7$ and u_λ is a least energy solution, then the limit points of $\{P_\lambda\}$ as $\lambda \to \infty$ are contained in the set of the points on Γ_1 of maximum mean curvature. Furthermore these limit points are contained in the set of the points of Γ_1 where α achieves its minimum.*

It should be noted that because of [1], the least energy solution of (4.2) always exists, under the assumption (ii) while it may not exist if the mean curvature is negative in any point of Γ_1 (see [14, Proposition 3.1]).

THEOREM 4.2. *Let $n \geqslant 7$ and $P_0 \in \Gamma_1$ be a point which is a strict local maximum for the mean curvature H with $H(P_0) > 0$. Suppose that P_0 is a local minimum for α. Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$,* (4.2) *admits a solution u_λ with $Q_{\lambda\alpha}(u_\lambda) < S/2^{2/n}$ and u_λ concentrate at P_0 as $\lambda \to \infty$.*

The proof of Theorem 4.1 and 4.2 is similar to that of Theorem 1.2 and 1.3 therefore we will only briefly sketch it.

Part (a) of Theorem 4.1 follows exactly in the same way as for Theorem 1.2. In order to prove (b), we first observe that (2.1) holds for any $y \in \Gamma_1$ with $U_{\varepsilon,y}$ replaced by $\phi U_{\varepsilon,y}$ where $\phi \in H^1(\Gamma_0)$ and $\phi \equiv 1$ in a neighbourhood of Γ_1. Now for a sequence $\lambda_k \to \infty$, let u_k be a solution of (4.2) and $P_k \in \Gamma_1$ be the maximum point of u_k. Then we can find $\varepsilon_k \to 0$, $y_k \in \Gamma_1$ with $\lim_{k\to\infty} y_k = \lim_{k\to\infty} P_k = P_0$ such that we have (see Corollary 2.1)

$$Q_{\lambda_k \alpha}(u_k) = \frac{S}{2^{2/n}} - A_n H(y_k)\,\beta_1(\varepsilon_k) + a_n \alpha(y_k)\,\lambda_k \beta_2(\varepsilon_k) + O(\beta_3(\varepsilon_k)^2) + O(\beta_2(\varepsilon_k)) + o(\lambda_k \beta_2(\varepsilon_k)). \tag{4.3}$$

Let $Q \in \Gamma_1$ be such that $H(Q)$ is the maximum of H and $\alpha(Q)$ is the minimum of α. Then from Lemma 2.1, (4.3) and using that $Q_{\lambda_k\alpha}(u_k) \leqslant Q_{\lambda_k\alpha}(\phi U_{\varepsilon_k, Q})$, we obtain

$$A_n(H(Q) - H(y_k))\,\beta_1(\varepsilon_k) + a_n(\alpha(y_k) - \alpha(Q))\,\lambda_k \beta_2(\varepsilon_k) + O(\beta_3(\varepsilon_k)^2) + O(\beta_2(\varepsilon_k)) + o(\lambda_k \beta_2(\varepsilon_k)) \leqslant 0. \tag{4.4}$$

Since $\alpha(y_k) - \alpha(Q) \geqslant 0$ and $\lambda_k \beta_2(\varepsilon_k) = O(u_1(\varepsilon_k))$ we obtain from (4.4)

$$A_n(H(Q) - H(y_k))\,\beta_1(\varepsilon_k) + o(\beta_1(\varepsilon_k)) \leqslant 0.$$

This implies that $H(P_0) = \lim_{k\to\infty} H(y_k) = H(Q)$. Furthermore from (4.4) we have

$$a_n(\alpha(y_k) - \alpha(Q)) + O\left(\frac{1}{\lambda_k}\right) + o(1) \leqslant 0.$$

This implies that $\alpha(P_0) = \lim_{k\to\infty} \alpha(y_k) = \alpha(Q)$. This proves Theorem 4.1. In the same way the proof of Theorem 4.2 follows.

Theorem 4.2 shows that α controls the concentration points. In fact as an immediate consequence of this we have the following

COROLLARY 4.1. *In addition to the assumption of Theorem* 4.1, *let us assume that at a point $P_0 \in \Gamma_1$, $H(P_0)$ is maximum, $\alpha(P_0) < \alpha(x)$ for any $x \in \Gamma_1 \setminus \{P_0\}$. Then the least energy solutions concentrate at P_0 as $\lambda \to \infty$.*

Some other results concerning problem (4.2) are contained in [2].

Remark 4.4. Suppose that $\partial\Omega$ has k points of strict local maximum for the mean curvature H where H is positive. Then from Theorem 1.3 we deduce that for λ large, (1.7) admits at least k distinct solutions which concentrate at these points.

Of course Theorem 1.3 does not hold in the case when Ω is a ball. On the other hand, given a positive integer k we know that there exists $\lambda(k)$ such that for $\lambda > \lambda(k)$, (1.7) admits at least k number of radial solutions (see [22, 4]). Moreover in [5] it is shown that, for λ large, (1.7) admits infinitely many rotationally equivalent solutions of least energy. Therefore the interesting question is whether it is possible to obtain more nonradial solutions which are not rotationally equivalent.

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REFERENCES

1. ADIMURTHI AND G. MANCINI, The Neumann problem for elliptic equations with critical non-linearity, A tribute in honour of G. Prodi, *Scuola Norm. Sup. Pisa*, (1991), 9–25.
2. ADIMURTHI AND G. MANCINI, Effect of geometry and topology of the boundary in the critical Neumann problem, preprint.
3. ADIMURTHI AND S. L. YADAVA, Critical Sobolev exponent problem in $\mathbb{R}^n (n \geqslant 4)$ with Neumann boundary condition, *Proc. Indian Acad. Sci.* **100** (1990), 275–284.
4. ADIMURTHI AND S. L. YADAVA, Existence and nonexistence of positive radial solutions of Neumann problems with critical Sobolev exponents, *Arch. Rat. Mech. Anal.* **115** (1991), 275–296.
5. ADIMURTHI AND S. L. YADAVA, Existence of nonradial positive solutions for critical exponent problem with Neumann boundary condition, *J. Differential Equations*, to appear.
6. A. BAHRI, "Critical Point at Infinity in Some Variational Problems," Pitman Research Notes in Math. Series, Vol. 182, Longman, Green, New York, 1989.
7. T. AUBIN, "Nonlinear Analysis on Manifold: Monge–Ampere Equations," Springer-Verlag, New York, 1982.
8. G. BIANCHI AND H. EGNELL, A note on the Sobolev inequality, *J. Funct. Anal.* **100** (1991), 18–24.
9. A. BAHRI AND J. M. CORON, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, *Comm. Pure Appl. Math.* **41** (1988), 253–290.
10. H. BREZIS AND L. NIRENBERG, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.*, **XXXVI** (1983), 437–477.
11. H. BREZIS, Elliptic equations with limiting Sobolev exponent. The impact of topology, *Comm. Pure Appl. Math.* **39** (1986), S17–S39.
12. H. BREZIS AND J. M. CORON, Convergence of solution of *H*-system or how to blow bubbles, *Arch. Rat. Mech. Anal.* **89** (1985), 21–56.
13. C. BUDD, M. C. KNAAP, AND L. A. PELETIER, Asymptotic behaviour of solutions of elliptic equations with critical exponent and Neumann boundary conditions, *Proc. Roy. Soc. Edinburgh* **117A** (1991), 225–250.
14. H. EGNELL, F. PACELLA, AND M. TRICARICO, Some remarks on Sobolev inequalities, *Nonlinear Anal. T.M.A.* (1989), 763–773.
15. B. GIDAS AND J. SPRUCK, Apriori bounds for positive solutions of nonlinear elliptic equations, *Comm. Partial Differential Equations* **6** (1981), 883–901.

16. M. GROSSI AND F. PACELLA, Positive solutions of nonlinear elliptic equations with critical Sobolev exponent and mixed boundary conditions, *Proc. Roy. Soc. Edinburgh* **116A** (1990), 23–43.
17. Z. C. HAN, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré* **8** (1991), 159–174.
18. E. KELLER AND L. SEGAL, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.* **26** (1970), 399–415.
19. C. S. LIN AND W. M. NI, "On the Diffusion Coefficient of a Semilinear Neumann Problem," Springer Lecture Notes, Vol. 1340, Springer, New York/Berlin, 1986.
20. C. S. LIN, W. M. NI, AND I. TAKAGI, Large amplitude stationary solutions to a chemotaxis system, *J. Differential Equations* **72** (1988), 1–27.
21. P. L. LIONS, The concentration compactness principle in the calculus of variations. The limit case (Part 1 and Part 2), *Rev. Mat. Iberoamerocana* **1** (1985) 145–201, 45–121.
22. W. M. NI, On the positive radial solutions of some semilinear elliptic equations on $\mathbb{R}^n$, *Appl. Math. Optim.* **9** (1983), 373–380.
23. W. M. NI, X. B. PAN, AND I. TAKAGI, Singular behaviour of least energy solutions of a semilinear Neumann problem involving critical Sobolev exponents, preprint.
24. W. M. NI AND I. TAKAGI, On the existence and the shape of solutions to a semilinear Neumann problem, "Proceedings of the Conference on Nonlinear Diffusion Equations and Their Equilibrium States, Gregynog, Wales, August 1989," to appear.
25. W. M. NI AND I. TAKAGI, On the shape of least-energy solutions to a semilinear Neumann problem, *Comm. Pure Appl. Math.* **XLIV** (1991), 819 851.
26. S. POHOZAEV, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet Math. Dokl.* **6** (1965), 1408–1411.
27. O. REY, The role of the Green's function in a non-linear elliptic equation involving the critical Sobolev exponent, *J. Funct. Anal.* **89** (1990), 1–52.
28. O. REY, Proof of two conjectures of H. Brezis and L. A. Peletier, *Manuscripta Math.* **65** (1989), 19–37.
29. R. SCHAAF, Stationary solutions of chemotaxis systems, *Trans. Amer. Math. Soc.* **292** (1985), 531–566.
30. M. STRUWE, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.* **187** (1984), 511–517.
31. M. STRUWE, "Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems," Springer-Verlag, Berlin/New York, 1990.
32. X. J. WANG, Neumann problem of semilinear elliptic equations involving critical Sobolev exponents, *J. Differential Equations* **93** (1991), 283–310.