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Journal of Algebra 320 (2008) 1851–1877
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Third homology of general linear groups

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Received 17 October 2006
Available online 9 July 2008
Communicated by Michel Van den Bergh

Abstract

The third homology group of $\mathrm{GL}_n(R)$ is studied, where R is a 'ring with many units' with center $Z(R)$. The main theorem states that if $K_1(Z(R))\otimes\mathbb{Q}\simeq K_1(R)\otimes\mathbb{Q}$ (e.g. R a commutative ring or a central simple algebra), then $H_3(\mathrm{GL}_2(R),\mathbb{Q})\to H_3(\mathrm{GL}_3(R),\mathbb{Q})$ is injective. If R is commutative, $\mathbb{Q}$ can be replaced by a field k such that $1/2\in k$. For an infinite field R (resp. an infinite field R such that $R^*=R^{*2}$), we get the better result that $H_3(\mathrm{GL}_2(R),\mathbb{Z}[\frac{1}{2}])\to H_3(\mathrm{GL}_3(R),\mathbb{Z}[\frac{1}{2}])$ (resp. $H_3(\mathrm{GL}_2(R),\mathbb{Z})\to H_3(\mathrm{GL}_3(R),\mathbb{Z})$) is injective. As an application we study the third homology group of $\mathrm{SL}_2(R)$ and the indecomposable part of $K_3(R)$.

Keywords: K-theory; Homology of groups

1. Introduction

The Hurewicz theorem relates homotopy groups to homology groups, which are much easier to calculate. This in turn provides a homomorphism from the Quillen K_n-group of a ring R to the nth integral homology of stable linear group $\mathrm{GL}(R)$, $h_n: K_n(R)\to H_n(\mathrm{GL}(R),\mathbb{Z})$. One can also define Milnor K-groups, $K_n^M(R)$, and when R is commutative there is a canonical map $K_n^M(R)\to K_n(R)$ [8].

One of the approaches to investigate K-groups is by means of the homology stability. Suslin's stability theorem states that for an infinite field F, the natural map

$$H_i\big(\mathrm{GL}_n(F),\mathbb{Z}\big)\to H_i\big(\mathrm{GL}(F),\mathbb{Z}\big)$$

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doi:10.1016/j.jalgebra.2008.04.012

is bijective if $n \geqslant i$ [18]. Using this result, Suslin constructed a map from $H_n(\mathrm{GL}_n(F), \mathbb{Z})$ to $K_n^M(F)$ such that the sequence

$$H_n\big(\mathrm{GL}_{n-1}(F), \mathbb{Z}\big) \xrightarrow{H_n(\mathrm{inc})} H_n\big(\mathrm{GL}_n(F), \mathbb{Z}\big) \to K_n^M(F) \to 0$$

is exact. Combining these two results he constructed a map from $K_n(F)$ to $K_n^M(F)$ such that the composite homomorphism

$$K_n^M(F) \to K_n(F) \to K_n^M(F)$$

coincides with the multiplication by $(-1)^{n-1}(n-1)!$ [18, Section 4].

These results have been generalized by Nesterenko and Suslin [14] to commutative local rings with infinite residue fields, and by Sah [16] and Guin [8] to a wider class of rings which is now called 'rings with many units.'

Except for $n = 1, 2$, there is no precise information about the kernel of $H_n(\mathrm{inc})$. In this direction Suslin posed a problem, which is now referred to as 'a conjecture by Suslin' (see [3, 7.7], [17, 4.13]).

Injectivity Conjecture. *For any infinite field F the natural homomorphism*

$$H_n\big(\mathrm{GL}_{n-1}(F), \mathbb{Q}\big) \to H_n\big(\mathrm{GL}_n(F), \mathbb{Q}\big)$$

is injective.

This conjecture is easy if $n = 1, 2$. For $n = 3$ the conjecture was proved positively by Sah [17] and Elbaz-Vincent [7]. The case $n = 4$ is proved by the author in [13]. The conjecture is proved in full for number fields by Borel and Yang [3].

When $n = 3$, in [7], Elbaz-Vincent proves the conjecture for a wider class of commutative rings (called H1-ring in [7]). In fact he proves that for any commutative ring with many units $H_3(\mathrm{GL}_2(R), \mathbb{Q}) \to H_3(\mathrm{GL}_3(R), \mathbb{Q})$ is injective. We will generalize this further, to include some class of non-commutative rings.

The above conjecture says that the kernel of $H_n(\mathrm{inc})$ is in fact torsion. Our main goal, in this paper, is to study the map $H_3(\mathrm{inc})$ in such a way that we lose less information on its kernel. Here is our main result.

Theorem 5.4. *Let R be a ring with many units with center $Z(R)$. Let k be a field such that $1/2 \in k$.*

(i) *If $K_1(Z(R)) \otimes \mathbb{Q} \simeq K_1(R) \otimes \mathbb{Q}$, then $H_3(\mathrm{GL}_2(R), \mathbb{Q}) \to H_3(\mathrm{GL}_3(R), \mathbb{Q})$ is injective. If R is commutative, then $\mathbb{Q}$ can be replaced by k.*
(ii) *If R is an infinite field or a quaternion algebra over an infinite field, then $H_3(\mathrm{GL}_2(R), \mathbb{Z}[\frac{1}{2}]) \to H_3(\mathrm{GL}_3(R), \mathbb{Z}[\frac{1}{2}])$ is injective.*
(iii) *Let R be either $\mathbb{R}$ or an infinite field such that $R^* = R^{*2}$. Then $H_3(\mathrm{GL}_2(R), \mathbb{Z}) \to H_3(\mathrm{GL}_3(R), \mathbb{Z})$ is injective.*
(iv) *The map $H_3(\mathrm{GL}_2(\mathbb{H}), \mathbb{Z}) \to H_3(\mathrm{GL}_3(\mathbb{H}), \mathbb{Z})$ is bijective, where $\mathbb{H}$ is the ring of quaternion.*

Examples of non-commutative rings with many units which satisfy the condition $K_1(Z(R)) \otimes \mathbb{Q} \simeq K_1(R) \otimes \mathbb{Q}$ of (i) in the above theorem are Azumaya algebras over commutative local rings with infinite residue fields.

As an application we generalize and give an easier proof of the main theorem of Sah in [17, Theorem 3.0]. Our proof of the next theorem avoids the case by case analysis done in [17].

Theorem 6.1. *Let R be a commutative ring with many units. Let k be a field such that $1/2 \in k$.*

- (i) *The map $H_0(R^*, H_3(\mathrm{SL}_2(R), k)) \to H_3(\mathrm{SL}(R), k)$ is injective.*
- (ii) *For an infinite field R, $H_0(R^*, H_3(\mathrm{SL}_2(R), \mathbb{Z}[\frac{1}{2}])) \to H_3(\mathrm{SL}(R), \mathbb{Z}[\frac{1}{2}])$ is injective.*
- (iii) *If R is either $\mathbb{R}$ or an infinite field such that $R^* = R^{*2}$, then $H_3(\mathrm{SL}_2(R), \mathbb{Z}) \to H_3(\mathrm{SL}(R), \mathbb{Z})$ is injective.*
- (iv) *The map $H_3(\mathrm{SL}_2(\mathbb{H}), \mathbb{Z}) \to H_3(\mathrm{SL}_3(\mathbb{H}), \mathbb{Z})$ is bijective.*

We use these results to study the third K-group of a field. Let $K_3(R)^{\mathrm{ind}} = \mathrm{coker}(K_3^M(R) \to K_3(R))$ be the indecomposable part of $K_3(R)$. In this article we prove that if R is an infinite field,

$$K_3(R)^{\mathrm{ind}} \otimes \mathbb{Z}\left[\frac{1}{2}\right] \simeq H_0\left(R^*, H_3\left(\mathrm{SL}_2(R), \mathbb{Z}\left[\frac{1}{2}\right]\right)\right).$$

Furthermore if $R^* = R^{*2}$ or $R = \mathbb{R}$, then

$$K_3(R)^{\mathrm{ind}} \simeq H_3\big(\mathrm{SL}_2(R), \mathbb{Z}\big).$$

To prove these claims, our general strategy will be the same as in [17] and [7]. We will introduce some spectral sequences similar to ones in [7], smaller but still big enough to do some computations. The main theorem will come out of the analysis of these spectral sequences.

Here we establish some notations. In this paper, by $H_i(G)$ we mean the ith integral homology of the group G. We use the bar resolution to define the homology of a group [4, Chap. I, Section 5]. Define $\mathbf{c}(g_1, g_2, \ldots, g_n) = \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma)[g_{\sigma(1)}|g_{\sigma(2)}|\ldots|g_{\sigma(n)}] \in H_n(G)$, where $g_i \in G$ pairwise commute and Σ_n is the symmetric group of degree n. By GL_n we mean the general linear group $\mathrm{GL}_n(R)$, where R is a *ring with many units*. By $Z(R)$ we mean the center of R.

Note that GL_0 is the trivial group and $\mathrm{GL}_1 = R^*$. By R^{*m} we mean $R^* \times \cdots \times R^*$ (m-times) or, when R is commutative and $m \geqslant 2$, the subgroup $\{a^m \mid a \in R^*\}$ of R^*, depending on the context. This will not cause any confusion. The ith factor of $R^{*m} = R^* \times \cdots \times R^*$ (m-times), is denoted by R_i^*.

2. Rings with many units

The study of rings with many units is originated by W. van der Kallen in [19],[1] where he shows that K_2 of such commutative rings behave very much like K_2 of fields. According to [19], in order to have a nice description of $K_2(R)$ in terms of generators and relations or in order

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[1] This notion is introduced by W. van der Kallen.

to have a nice stability property for $K_2(R)$, the ring should have 'enough invertible elements,' and 'more invertible elements' the ring has, a better description of $K_2(R)$ one gets. In this direction, see Proposition 2.6 for a homological proof of a theorem of Van der Kallen [19], due to Nesterenko and Suslin [14, Corollary 4.3].

In [14], another definition of rings with many units is given, where the authors prove very nice homology stability results for the homology of general linear groups over these rings. They further prove that when the ring is a local ring with infinite residue field, the homology stability bound can be very sharp.

In [8], Guin shows that if a ring satisfies both the definition of Van der Kallen and of Suslin, then most of the main results of Suslin in [18] are still true. Following [19] and [14], we call such rings, *rings with many units*.

Definition 2.1. We say that R is a *ring with many units* if it has the following properties:

(H1) Hypothesis 1. For any finite number of surjective linear forms $f_i : R^n \to R$, there exist $v \in R^n$ such that $f_i(v) \in R^*$.
(H2) Hypothesis 2. For any $n \geqslant 1$, there exist n elements of the center of R such that the sum of each nonempty subfamily belongs to R^*.

Remark 2.2.

(i) (H1) implies that the stable range of R is one, $\mathrm{sr}(R) = 1$ [8, Proposition 1.4].
(ii) (H1) implies (H2) if R is commutative [8, Proposition 1.3].
(iii) Property (H1) is considered by Van der Kallen [19, Section 1] and property (H2) is studied by Nesterenko and Suslin [14, §1].

Example 2.3.

(i) Let R satisfy property (H2). Then a semilocal ring R is a ring with many units if and only if $R/\mathrm{Jac}(R)$ is a ring with many units, where $\mathrm{Jac}(R)$ denotes the Jacobson radical of R.
(ii) Product of rings with many units is a ring with many units.
(iii) Let D be a finite-dimensional F-division algebra, F an infinite field. Then $M_n(D)$, $n \geqslant 1$, is a ring with many units.
(iv) Let F be an infinite field. Then any finite-dimensional F-algebra is a semilocal ring [10, §20]. Therefore, it is a ring with many units.
(v) Let R be a commutative semilocal ring with many units. Then any Azumaya R-algebra is a ring with many units (see [10, §20]).

Here we give two known results which are used in the construction of spectral sequences in the coming section. They show the need for properties (H1) and (H2).

Lemma 2.4. *Let R satisfy the property* (H1). *Let $n \geqslant 2$ and assume T_i, $1 \leqslant i \leqslant l$, are finite subsets of R^n such that each T_i is a basis of a free summand of R^n with k elements, where $k \leqslant n-1$. Then there is a vector $v \in R^n$, such that $T_i \cup \{v\}$, $1 \leqslant i \leqslant l$, is a basis of a free summand of R^n.*

Proof. This is well-known and easy to prove. We leave the proof to the reader. □

The next result is due to Suslin.

Proposition 2.5. *Let R satisfy the property* (H2). *Let G_i be subgroups of* GL_{n_i}, $i = 1, 2$, *and assume that at least one of them contains the subgroup of diagonal matrices. Let M be a submodule of $M_{n_1,n_2}(R)$ such that $G_1M = M = MG_2$. Then the inclusion*

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \to \begin{pmatrix} G_1 & M \\ 0 & G_2 \end{pmatrix}$$

induces isomorphism on the homology with coefficients in $\mathbb{Z}$.

Proof. See [18, Theorem 1.9]. □

The next proposition is rather well-known. We refer the reader to [8, 3.2] for the definition of the Milnor K-groups $K_n^M(R)$ of a ring R.

Proposition 2.6. *Let R be a commutative ring with many units. Then*

(i) $\mathrm{SK}_1(R) = 0$.
(ii) (*Van der Kallen* [19])

$$K_2(R) \simeq K_2^M(R) = R^* \otimes_{\mathbb{Z}} R^* / \big\langle a \otimes (1-a)\colon\ a, 1-a \in R^* \big\rangle.$$

Proof. (i) By the homology stability theorem [8, Theorem 1]

$$K_1(R) = H_1\big(\mathrm{GL}(R)\big) \simeq H_1\big(\mathrm{GL}_1(R)\big) \simeq R^*,$$

but we also have $K_1(R) \simeq R^* \times \mathrm{SK}_1(R)$. Thus $\mathrm{SK}_1(R) = 0$.

(ii) (Nesterenko–Suslin) By easy analysis of the Lyndon–Hochschild–Serre spectral sequence associated to

$$1 \to \mathrm{SL} \to \mathrm{GL} \to R^* \to 1,$$

using part (i) and the homology stability theorem, one sees that $K_2(R) \simeq H_2(\mathrm{GL}_2)/H_2(\mathrm{GL}_1)$ (see [14, Lemma 4.2]). By [8, Theorem 2], we have $K_2^M(R) \simeq H_2(\mathrm{GL}_2)/H_2(\mathrm{GL}_1)$. Therefore, $K_2^M(R) \simeq K_2(R)$. For the rest, see [8, Proposition 3.2.3]. □

In this paper we always assume that R is a ring with many units.

3. The spectral sequences

Let $C_l(R^n)$ and $D_l(R^n)$ be the free abelian groups with a basis consisting of $(\langle v_0\rangle, \dots, \langle v_l\rangle)$ and $(\langle w_0\rangle, \dots, \langle w_l\rangle)$ respectively, where every $\min\{l+1, n\}$ of $v_i \in R^n$ and every $\min\{l+1, 2\}$ of $w_i \in R^n$ is a basis of a free direct summand of R^n. By $\langle v_i\rangle$ and $\langle w_i\rangle$ we mean the submodules of R^n generated by v_i and w_i respectively. Let $\partial_0 : C_0(R^n) \to C_{-1}(R^n) := \mathbb{Z}$, $\sum_i n_i(\langle v_i\rangle) \mapsto \sum_i n_i$ and $\partial_l = \sum_{i=0}^{l} (-1)^i d_i : C_l(R^n) \to C_{l-1}(R^n)$, $l \geqslant 1$, where

$$d_i\big((\langle v_0\rangle, \dots, \langle v_l\rangle)\big) = \big(\langle v_0\rangle, \dots, \widehat{\langle v_i\rangle}, \dots, \langle v_l\rangle\big).$$

Define the differential $\tilde{\partial}_l = \sum_{i=0}^{l}(-1)^i \tilde{d}_i : D_l(R^n) \to D_{l-1}(R^n)$ similar to ∂_l. By Lemma 2.4 it is easy to see that the complexes

$$\begin{aligned} C_*\colon &\quad 0 \leftarrow C_{-1}\big(R^n\big) \leftarrow C_0\big(R^n\big) \leftarrow \cdots \leftarrow C_{l-1}\big(R^n\big) \leftarrow \cdots, \\ D_*\colon &\quad 0 \leftarrow D_{-1}\big(R^n\big) \leftarrow D_0\big(R^n\big) \leftarrow \cdots \leftarrow D_{l-1}\big(R^n\big) \leftarrow \cdots \end{aligned}$$

are exact. Consider $C_i(R^n)$ and $D_i(R^n)$ as a left GL_n-module in a natural way and convert this action to the right action by the definition $m.g := g^{-1}m$.

Take a free left GL_n-resolution $P_* \to \mathbb{Z}$ of $\mathbb{Z}$ with trivial GL_n-action. From the double complexes $C_* \otimes_{\mathrm{GL}_n} P_*$ and $D_* \otimes_{\mathrm{GL}_n} P_*$, using Proposition 2.5, we obtain two first quadrant spectral sequences converging to zero with

$$E^1_{p,q}(n) = \begin{cases} H_q(R^{*p} \times \mathrm{GL}_{n-p}) & \text{if } 0 \leqslant p \leqslant n, \\ H_q(\mathrm{GL}_n, C_{p-1}(R^n)) & \text{if } p \geqslant n+1, \end{cases}$$

$$\tilde{E}^1_{p,q}(n) = \begin{cases} H_q(R^{*p} \times \mathrm{GL}_{n-p}) & \text{if } 0 \leqslant p \leqslant 2, \\ H_q(\mathrm{GL}_n, D_{p-1}(R^n)) & \text{if } p \geqslant 3. \end{cases}$$

For $1 \leqslant p \leqslant n$ and $q \geqslant 0$, $d^1_{p,q}(n) = \sum_{i=1}^{p}(-1)^{i+1} H_q(\alpha_{i,p})$, where

$$\begin{aligned} &\alpha_{i,p} : R^{*p} \times \mathrm{GL}_{n-p} \to R^{*p-1} \times \mathrm{GL}_{n-p+1}, \\ &(a_1, \ldots, a_p, A) \mapsto \left(a_1, \ldots, \widehat{a_i}, \ldots, a_p, \begin{pmatrix} a_i & 0 \\ 0 & A \end{pmatrix}\right). \end{aligned}$$

In particular, for $0 \leqslant p \leqslant n$,

$$d^1_{p,0}(n) = \begin{cases} \mathrm{id}_{\mathbb{Z}} & \text{if } p \text{ is odd}, \\ 0 & \text{if } p \text{ is even}. \end{cases}$$

So $E^2_{p,0}(n) = 0$ for $p \leqslant n-1$. It is also easy to see that $E^2_{n,0}(n) = E^2_{n+1,0}(n) = 0$. See the proof of [12, Theorem 3.5] for more details.

We will use $\tilde{E}^i_{p,q}(n)$ and $E^i_{p,q}(n)$ only for $n = 3$, so from now on by $\tilde{E}^i_{p,q}$ and $E^i_{p,q}$ we mean $\tilde{E}^i_{p,q}(3)$ and $E^i_{p,q}(3)$ respectively. We describe $\tilde{E}^1_{p,q}$ for $p = 3, 4$. Let

$$w_1 = \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle\big), \qquad w_2 = \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + e_2\rangle\big) \in D_2\big(R^3\big)$$

and $u_1, \ldots, u_5, u_{6,a} \in D_3(R^3)$, $a, a-1 \in R^*$, where

$$\begin{aligned} u_1 &= \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2+e_3\rangle\big), & u_2 &= \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_2\rangle\big), \\ u_3 &= \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_2+e_3\rangle\big), & u_4 &= \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1+e_3\rangle\big), \\ u_5 &= \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_3\rangle\big), & u_{6,a} &= \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+ae_2\rangle\big) \end{aligned}$$

(see [8, Lemma 3.3.3]). By the Shapiro lemma

$$\tilde{E}^1_{3,q}=H_q\big(\mathrm{Stab}_{\mathrm{GL}_3}(w_1)\big)\oplus H_q\big(\mathrm{Stab}_{\mathrm{GL}_3}(w_2)\big),$$
$$\tilde{E}^1_{4,q}=\bigoplus_{j=1}^{5}H_q\big(\mathrm{Stab}_{\mathrm{GL}_3}(u_j)\big)\oplus\Bigg[\bigoplus_{a,a-1\in R^*}H_q\big(\mathrm{Stab}_{\mathrm{GL}_3}(u_{6,a})\big)\Bigg].$$

So by Proposition 2.5 we get

$$\begin{aligned}\tilde{E}^1_{3,q}&=H_q\big(R^{*3}\big)\oplus H_q\big(R^*I_2\times R^*\big),\\ \tilde{E}^1_{4,q}&=H_q\big(R^*I_3\big)\oplus H_q\big(R^*I_2\times R^*\big)\oplus H_q\big(R^*\times R^*I_2\big)\oplus H_q(T)\\ &\quad\oplus H_q\big(R^*I_2\times R^*\big)\oplus\Bigg[\bigoplus_{a,a-1\in R^*}H_q\big(R^*I_2\times R^*\big)\Bigg],\end{aligned}$$

where $T=\{(a,b,a)\in R^3\colon\ a,b\in R^*\}$. Note that $\tilde{d}^1_{p,q}=d^1_{p,q}$ for $p=1,2$, $\tilde{d}^1_{3,q}|_{H_q(R^{*3})}=d^1_{3,q}$ and $\tilde{d}^1_{3,q}|_{H_q(R^*I_2\times R^*)}=H_q(\mathrm{inc})$, where $\mathrm{inc}: R^*I_2\times R^*\to R^{*3}$.

Lemma 3.1. *The group $\tilde{E}^2_{p,0}$ is trivial for $0\leqslant p\leqslant 5$.*

Proof. Triviality of $\tilde{E}^2_{p,0}$ is easy for $0\leqslant p\leqslant 2$. To prove the triviality of $\tilde{E}^2_{3,0}$, note that $\tilde{E}^1_{2,0}=\mathbb{Z}$, $\tilde{E}^1_{3,0}=\mathbb{Z}\oplus\mathbb{Z}$ and $\tilde{d}^1_{3,0}((n_1,n_2))=n_1+n_2$, so if $(n_1,n_2)\in\ker(\tilde{d}^1_{3,0})$, then $n_2=-n_1$. It is easy to see that this is contained in $\mathrm{im}(\tilde{d}^1_{4,0})$. We prove the triviality of $\tilde{E}^2_{5,0}$. Triviality of $\tilde{E}^2_{4,0}$ is similar but much easier. This proof is taken from [7, Section 1.3.3].

Triviality of $\tilde{E}^2_{5,0}$. The proof will be in four steps.

Step 1. The sequence $0\to C_*(R^3)\otimes_{\mathrm{GL}_3}\mathbb{Z}\to D_*(R^3)\otimes_{\mathrm{GL}_3}\mathbb{Z}\to Q_*(R^3)\otimes_{\mathrm{GL}_3}\mathbb{Z}\to 0$ is exact, where $Q_*(R^3):=D_*(R^3)/C_*(R^3)$.

Step 2. The group $H_4(Q_*(R^3)\otimes_{\mathrm{GL}_3}\mathbb{Z})$ is trivial.

Step 3. The map induced in homology by $C_*(R^3)\otimes_{\mathrm{GL}_3}\mathbb{Z}\to D_*(R^3)\otimes_{\mathrm{GL}_3}\mathbb{Z}$ is zero in degree 4.

Step 4. The group $\tilde{E}^2_{5,0}$ is trivial.

Proof of Step 1. For $i\geqslant -1$, $D_i(R^3)\simeq C_i(R^3)\oplus Q_i(R^3)$. This decomposition is compatible with the action of GL_3, so we get an exact sequence of GL_3-modules

$$0\to C_i\big(R^3\big)\to D_i\big(R^3\big)\to Q_i\big(R^3\big)\to 0$$

which splits as a sequence of GL_3-modules. One can easily deduce the desired exact sequence from this. Note that this exact sequence does not split as complexes.

Proof of Step 2. The complex $Q_*(R^3)$ induces a spectral sequence

$$\hat{E}^1_{p,q} = \begin{cases} 0 & \text{if } 0 \leqslant p \leqslant 2, \\ H_q(\mathrm{GL}_3, Q_{p-1}(R^3)) & \text{if } p \geqslant 3, \end{cases}$$

which converges to zero. To prove the claim it is sufficient to prove that $\hat{E}^2_{5,0} = 0$, and this follows from $\hat{E}^2_{3,1} = 0$ which we now show. One can see that $\hat{E}^1_{3,1} = H_1(R^* I_2 \times R^*)$. If $w = (\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1 + e_2\rangle) \in Q_3(R^3)$, then $H_1(\mathrm{Stab}_{\mathrm{GL}_3}(w)) \simeq H_1(R^* I_2 \times R^*)$ is a summand of $\hat{E}^1_{4,1}$ and $\hat{d}^1_{4,1} : H_1(\mathrm{Stab}_{\mathrm{GL}_3}(w)) \to \hat{E}^1_{3,1}$ is an isomorphism. So $\hat{d}^1_{4,1}$ is surjective and therefore $\hat{E}^2_{3,1} = 0$.

Proof of Step 3. Consider the following commutative diagram

$$\begin{tikzcd}
C_5(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z} \arrow[r] \arrow[d] & C_4(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z} \arrow[r] \arrow[d] & C_3(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z} \arrow[d] \\
D_5(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z} \arrow[r] & D_4(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z} \arrow[r] & D_3(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z}.
\end{tikzcd}$$

The generators of $C_4(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z}$ are of the form $x_{a,b} \otimes 1$, where $x_{a,b} = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + ae_2 + be_3\rangle, \langle e_3\rangle, \langle e_1 + e_2 + e_3\rangle)$, $a, a-1, b, b-1, a-b \in R^*$ (see [8, Lemma 3.3.3]). Since $C_3(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z} = \mathbb{Z}$, the elements $(x_{a,b} - x_{c,d}) \otimes 1$ generate $\ker(\partial_4 \otimes 1)$. Hence to prove this step it is sufficient to prove that $(x_{a,b} - x_{c,d}) \otimes 1 \in \operatorname{im}(\tilde{\partial}_5 \otimes 1)$.

Set $w'_a = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + ae_2 + e_3\rangle, \langle e_3\rangle, \langle e_1 + e_2\rangle, \langle e_1 + ae_2\rangle) \in D_5(R^3)$, where $a, a-1 \in R^*$. Let g, g', and g'' be the matrices

$$\begin{pmatrix} 0 & a^{-1} & 0 \\ -1 & 1+a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively, then

$$g\big(\tilde{d}_1(w'_a)\big) = \tilde{d}_0(w'_a), \qquad g'\big(\tilde{d}_3(w'_a)\big) = \tilde{d}_2(w'_a), \qquad g''\big(\tilde{d}_4(w'_a)\big) = v'_1$$

and so $(\tilde{\partial}_5 \otimes 1)(w'_a \otimes 1) = (v'_1 - v'_a) \otimes 1$, where

$$v'_a = \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + ae_2 + e_3\rangle, \langle e_3\rangle, \langle e_1 + e_2\rangle\big).$$

Note that the elements of the form $(gw - w) \otimes 1$ are zero in $D_* \otimes_{\mathrm{GL}_3} \mathbb{Z}$. If

$$u'_a = \big(\langle e_3\rangle, \langle e_1 + ae_2 + e_3\rangle, \langle e_1\rangle, \langle e_1 + e_2\rangle, \langle e_1 + ae_2\rangle\big),$$
$$u''_a = \big(\langle e_1 + ae_2 + e_3\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_1 + e_2\rangle, \langle e_1 + ae_2\rangle\big),$$

where $a, a-1 \in R^*$, then

$$gu'_a = \big(\langle e_3\rangle, \langle e_1 + ae_2 + e_3\rangle, \langle e_2\rangle, \langle e_1 + e_2\rangle, \langle e_1 + ae_2\rangle\big),$$
$$g'u''_a = \big(\langle e_3\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_1 + e_2\rangle, \langle e_1 + ae_2\rangle\big).$$

So if $a, a-1, c, c-1 \in R^*$, then

$$(\tilde{\partial}_5 \otimes 1)\big((z_a - z_c) \otimes 1\big) = (t_c - t_a) \otimes 1,$$

where

$$\begin{aligned} z_a &= \big(\langle e_3\rangle, \langle e_1 + ae_2 + e_3\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_1 + e_2\rangle, \langle e_1 + ae_2\rangle\big), \\ t_a &= \big(\langle e_3\rangle, \langle e_1 + ae_2 + e_3\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_1 + e_2\rangle\big). \end{aligned}$$

If g_1, g_2, g_3 and g_4 are the matrices

$$\begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ \frac{b-1}{1-a} & \frac{1-b}{1-a} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ \frac{b-a}{1-a} & \frac{a-b}{1-a} & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1-b}{b} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{b} \end{pmatrix},$$

respectively, then

$$\begin{aligned} g_1\big(\tilde{d}_0(y_{a,b})\big) &= t_{\frac{1}{1-b}}, & g_2\big(\tilde{d}_1(y_{a,b})\big) &= t_{\frac{-a}{b-a}}, \\ g_3\big(\tilde{d}_3(y_{a,b})\big) &= v'_{\frac{a-b}{1-b}}, & g_4\big(\tilde{d}_3(y_{a,b})\big) &= v'_a, \end{aligned}$$

where

$$y_{a,b} = \big(\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + ae_2 + be_3\rangle, \langle e_3\rangle, \langle e_1 + e_2 + e_3\rangle, \langle e_1 + e_2\rangle\big).$$

(Here by $\frac{r}{s} \in R^*$ we mean $s^{-1}r$.) By an easy computation

$$(\tilde{\partial}_5 \otimes 1)(y_{a,b} \otimes 1) = t_{\frac{1}{1-b}} \otimes 1 - t_{\frac{-a}{b-a}} \otimes 1 + v'_1 \otimes 1 - v'_{\frac{a-b}{1-b}} \otimes 1 + v'_a \otimes 1 - x_{a,b} \otimes 1.$$

Now it is easy to see that $(x_{a,b} - x_{c,d}) \otimes 1 \in (\tilde{\partial}_5 \otimes 1)(D_5(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z})$. This completes the proof of Step 3.

Proof of Step 4. From the homology long exact sequence of the short exact sequence obtained in the first step, we get the exact sequence

$$H_4\big(C_*\big(R^3\big) \otimes_{\mathrm{GL}_3} \mathbb{Z}\big) \to H_4\big(D_*\big(R^3\big) \otimes_{\mathrm{GL}_3} \mathbb{Z}\big) \to H_4\big(Q_*\big(R^3\big) \otimes_{\mathrm{GL}_3} \mathbb{Z}\big).$$

By Steps 2 and 3, $H_4(D_*(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z}) = 0$, but $\tilde{E}^2_{5,0} = H_4(D_*(R^3) \otimes_{\mathrm{GL}_3} \mathbb{Z})$. This completes the proof of the triviality of $\tilde{E}^2_{5,0}$. □

Lemma 3.2. *The group* $\tilde{E}^2_{p,1}$ *is trivial for* $0 \leqslant p \leqslant 4$.

Proof. Triviality of $\tilde{E}^2_{p,1}$, $p=0,1$, is a result of Lemma 3.1 and the fact that the spectral sequence converges to zero (one can also prove this directly). If $(a_0,b_0,c_0)\in\ker(\tilde{d}^1_{2,1})$, $a_0,b_0,c_0\in H_1(R^*)$, then $a_0=b_0$. It is easy to see that this element is contained in $\mathrm{im}(\tilde{d}^1_{3,1})$. Let $x=(x_1,\dots,x_5,(x_{6,a}))\in\tilde{E}^1_{4,1}$, where $x_2=(a_2,a_2,b_2)$, $x_3=(a_3,b_3,b_3)$, $x_4=(a_4,b_4,a_4)$, $x_5=(a_5,a_5,b_5)$, $a_i,b_i\in H_1(R^*)$. By a direct calculation $\tilde{d}_{4,1}(x)=(p_1,p_2)$, where

$$\begin{aligned} p_1&=-(a_2,a_2,b_2)-(a_3,b_3,b_3)+(b_4,a_4,a_4)+(a_5,a_5,b_5),\\ p_2&=(a_2,a_2,b_2)+(b_3,b_3,a_3)-(a_4,a_4,b_4)-(a_5,a_5,b_5). \end{aligned}$$

If $y=((a_0,b_0,c_0),(d_0,d_0,e_0))\in\ker(\tilde{d}^1_{3,1})$, $a_0,b_0,c_0,d_0,e_0\in H_1(R^*)$, then $b_0+d_0=a_0-b_0+c_0+e_0=0$. Let $x'_2=(-b_0,-b_0,-c_0)$, $x'_3=(-a_0+b_0,0,0)$ and set $x'=(0,x'_2,x'_3,0,0,0)\in\tilde{E}^1_{4,1}$, then $y=\tilde{d}_{4,1}(x')$.

To prove the triviality of $\tilde{E}^2_{4,1}$; let $x\in\ker(\tilde{d}_{4,1})$ and set

$$\begin{aligned} w_1&=\big(\langle e_1\rangle,\langle e_2\rangle,\langle e_1+e_2\rangle,\langle e_3\rangle,\langle e_1+ae_2\rangle\big),\\ w_2&=\big(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle,\langle e_1+e_3\rangle,\langle e_1+be_3\rangle\big),\\ w_3&=\big(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle,\langle e_1+e_2+e_3\rangle,\langle e_2+e_3\rangle\big),\\ w_{4,a}&=\big(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle,\langle e_1+e_2\rangle,\langle e_1+ae_2\rangle\big),\\ w_5&=\big(\langle e_1\rangle,\langle e_2\rangle,\langle e_3\rangle,\langle e_1+e_2+e_3\rangle,\langle e_1+ae_2+be_3\rangle\big), \end{aligned}$$

where $a,a-1,b,b-1,a-b\in R^*$, b fixed. The groups $T_i=H_1(\mathrm{Stab}_{\mathrm{GL}_3}(w_i))$, $i=1,2,3,5$ and $T_4=\bigoplus_{a,a-1\in R^*}H_1(\mathrm{Stab}_{\mathrm{GL}_3}(w_{4,a}))$ are summands of $\tilde{E}^1_{5,1}$. Note that $T_1=H_1(R^*I_2\times R^*)$, $T_2=H_1(T)$, $T_3=T_5=H_1(R^*I_3)$ and $T_4=\bigoplus_{a,a-1\in R^*}H_1(R^*I_2\times R^*)$. The restriction of $\tilde{d}^1_{5,1}$ on these summands is as follows,

$$\begin{aligned} \tilde{d}^1_{5,1}|_{T_1}\big((c_1,c_1,d_1)\big)&=\big(0,(c_1,c_1,d_1),0,0,(c_1,c_1,d_1),-(c_1,c_1,d_1)\big),\\ \tilde{d}^1_{5,1}|_{T_2}\big((c_2,d_2,c_2)\big)&=\big(0,0,(d_2,c_2,c_2),(c_2,d_2,c_2),0,-(c_2,c_2,d_2)\big),\\ \tilde{d}^1_{5,1}|_{T_3}\big((c_3,c_3,c_3)\big)&=\big((c_3,c_3,c_3),(c_3,c_3,c_3),-(c_3,c_3,c_3),0,0,0\big),\\ \tilde{d}^1_{5,1}|_{T_{4,a}}\big((c_4,c_4,d_4)\big)&=\big(0,0,0,0,0,(c_4,c_4,d_4)\big),\\ \tilde{d}^1_{5,1}|_{T_5}&=\mathrm{id}_{H_1(R^*I_3)}. \end{aligned}$$

Let $z_1=(a_5,a_5,b_5)\in T_1$ and $z_2=(a_4,b_4,a_4)\in T_2$. Then $x-\tilde{d}^1_{5,1}(z_1+z_2)=(x'_1,x'_2,x'_3,0,0,(x'_{6,a}))$, so we can assume that $x_4=x_5=0$. An easy calculation shows that $a_2=b_2=-a_3=-b_3$. If $z_3=(a_2,a_2,a_2)\in T_3$, then $x-\tilde{d}_{5,1}(z_3)=(x'_1,0,0,0,0,(x'_{6,a}))$. Again we can assume that $x_2=x_3=0$. If $z_4=(x_{6,a})\in T_4$, then $x-\tilde{d}^1_{5,1}(z_4)=(x'_1,0,0,0,0,0)$. Once more we can assume that $x_{6,a}=0$. These reduce x to an element of the form $(x_1,0,0,0,0,0)$. If $x_1\in T_5$, then $\tilde{d}^1_{5,1}(x_1)=(x_1,0,0,0,0,0)$. This completes the triviality of $\tilde{E}^2_{4,1}$. □

Lemma 3.3. *The group $\tilde{E}^2_{p,2}$ is trivial for $0\leqslant p\leqslant 3$.*

Proof. Triviality of $\tilde{E}^2_{0,2}$ and $\tilde{E}^2_{1,2}$ is a result of Lemmas 3.1 and 3.2 and the fact that the spectral sequence converges to zero. Let

$$\begin{aligned}
\tilde{E}^1_{1,2} &= H_2\big(R^* \times \mathrm{GL}_2\big) = H_2\big(R^*\big) \oplus H_2(\mathrm{GL}_2) \oplus H_1\big(R^*\big) \otimes H_1(\mathrm{GL}_2),\\
\tilde{E}^1_{2,2} &= H_2\big(R^{*3}\big) = \bigoplus_{i=1}^{6} T_i,\\
\tilde{E}^1_{3,2} &= H_2\big(R^{*3}\big) \oplus H_2\big(R^* I_2 \times R^*\big) = \bigoplus_{i=1}^{9} T_i,
\end{aligned}$$

where

$$\begin{aligned}
&T_i = H_2\big(R^*_i\big) \quad \text{for } i = 1, 2, 3, && T_4 = H_1\big(R^*_1\big) \otimes H_1\big(R^*_2\big),\\
&T_5 = H_1\big(R^*_1\big) \otimes H_1\big(R^*_3\big), && T_6 = H_1\big(R^*_2\big) \otimes H_1\big(R^*_3\big),\\
&T_7 = H_2\big(R^* I_2\big), && T_8 = H_2\big(I_2 \times R^*\big),\\
&T_9 = H_1\big(R^* I_2\big) \otimes H_1\big(I_2 \times R^*\big).
\end{aligned}$$

If $y = (y_1, y_2, y_3, \sum r \otimes s, \sum t \otimes u, \sum v \otimes w) \in \tilde{E}^1_{2,2}$ and

$$x = \Big(x_1, x_2, x_3, \sum a \otimes b, \sum c \otimes d, \sum e \otimes f, x_7, x_8, \sum g \otimes h\Big) \in \tilde{E}^1_{3,2},$$

$a, b, \dots, h, r, \dots, w \in H_1(R^*)$, then $\tilde{d}^1_{2,2}(y) = (h_1, h_2, h_3)$, where

$$h_1 = -y_1 + y_2,$$
$$h_3 = -\sum s \otimes \mathrm{diag}(1, r) - \sum r \otimes \mathrm{diag}(1, s) - \sum t \otimes \mathrm{diag}(1, u) + \sum v \otimes \mathrm{diag}(1, w)$$

and $\tilde{d}^1_{3,2}(x) = (z_i)_{1 \leqslant i \leqslant 6}$, where

$$\begin{aligned}
&z_1 = z_2 = x_2 + x_7, \qquad z_3 = x_1 + x_3 - x_2 + x_8,\\
&z_4 = \sum a \otimes b - \sum c \otimes d + \sum e \otimes f,\\
&z_5 = -\sum b \otimes a - \sum a \otimes b + \sum c \otimes d + \sum g \otimes h,\\
&z_6 = -\sum d \otimes c + \sum f \otimes e + \sum e \otimes f + \sum g \otimes h.
\end{aligned}$$

If $y \in \ker(\tilde{d}^1_{2,2})$, then $y_1 = y_2$ and $h_3 = 0$. By the isomorphism $H_1(R^*) \otimes H_1(\mathrm{GL}_1) \simeq H_1(R^*) \otimes H_1(\mathrm{GL}_2)$ and the triviality of h_3, we have

$$-\sum s \otimes r - \sum r \otimes s - \sum t \otimes u + \sum v \otimes w = 0.$$

If

$$z = \Big(y_1, y_1, y_3, 0, \sum t \otimes u, \sum r \otimes s + \sum t \otimes u, 0, 0, 0\Big) \in \tilde{E}^1_{3,2},$$

then $y=\tilde{d}^1_{3,2}(z)$ and therefore $\tilde{E}^2_{2,2}=0$.

Let $\tilde{d}^1_{3,2}(x)=0$. Consider the summands $S_2=H_2(\mathrm{Stab}_{\mathrm{GL}_3}(u_2))=H_2(R^*I_2\times R^*)$ and $S_3=H_2(\mathrm{Stab}_{\mathrm{GL}_3}(u_3))=H_2(R^*\times R^*I_2)$ of $\tilde{E}^1_{4,2}$. Then $S_i\simeq H_2(R^*)\oplus H_2(R^*)\oplus H_1(R^*)\otimes H_1(R^*)$ and by a direct calculation

$$\tilde{d}^1_{4,2}|_{S_2}\big((y_1,y_2,s\otimes t)\big)=(-y_1,-y_1,-y_2,0,-s\otimes t,-s\otimes t,y_1,y_2,s\otimes t),$$
$$\tilde{d}^1_{4,2}|_{S_3}\big((q_1,q_2,p\otimes q)\big)=(-q_1,-q_2,-q_2,-p\otimes q,-p\otimes q,0,q_2,q_1,-q\otimes p).$$

Choose $z'_2=(-x_2,-x_3,-\sum e\otimes f)\in S_2$ and $z'_3=(x_3+x_8,0,-\sum a\otimes b)\in S_3$. Then $x=\tilde{d}^1_{4,2}(z'_2+z'_3)$ and therefore $\tilde{E}^2_{3,2}=0$. □

Lemma 3.4. *The groups $\tilde{E}^2_{0,3}$, $\tilde{E}^2_{1,3}$ and $\tilde{E}^3_{0,4}$ are trivial.*

Proof. This follows from Lemmas 3.1, 3.2, and 3.3 and the fact that the spectral sequence converges to zero. □

Corollary 3.5.

(i) *The complex*

$$H_2\big(R^{*3}\times\mathrm{GL}_0\big)\xrightarrow{d^1_{3,2}}H_2\big(R^{*2}\times\mathrm{GL}_1\big)\xrightarrow{d^1_{2,2}}H_2\big(R^{*}\times\mathrm{GL}_2\big)\xrightarrow{d^1_{1,2}}H_2(\mathrm{GL}_3)\to 0$$

is exact, where $d^1_{3,2}=H_2(\alpha_{1,3})-H_2(\alpha_{2,3})+H_2(\alpha_{3,3})$, $d^1_{2,2}=H_2(\alpha_{1,2})-H_2(\alpha_{2,2})$ *and* $d^1_{1,2}=H_2(\mathrm{inc})$.

(ii) *The complex*

$$H_3\big(R^{*2}\times\mathrm{GL}_1\big)\xrightarrow{d^1_{2,3}}H_3\big(R^{*}\times\mathrm{GL}_2\big)\xrightarrow{d^1_{1,3}}H_3(\mathrm{GL}_3)\to 0$$

is exact, where $d^1_{2,3}=H_3(\alpha_{1,2})-H_3(\alpha_{2,2})$ *and* $d^1_{1,3}=H_3(\mathrm{inc})$.

Proof. The case (i) follows from the proof of Lemma 3.3 and (ii) follows from Lemma 3.4. □

Lemma 3.6. *The groups $E^3_{0,4}$, $E^3_{5,0}$ are trivial.*

Proof. Using 3.5, one sees that $E^2_{p,q}$-terms are of the form

$$\begin{array}{ccccccc}
E^2_{0,4} & * & & & & & \\
0 & 0 & E^2_{2,3} & * & * & * & \\
0 & 0 & 0 & * & * & * & \\
0 & 0 & 0 & E^2_{3,1} & * & * & * \\
0 & 0 & 0 & 0 & 0 & E^2_{5,0} & *.
\end{array}$$

From this description we get $E^3_{3,1}\simeq E^\infty_{3,1}=0$. So we obtain the exact sequence

$$0 \to E^3_{5,0} \to E^2_{5,0} \xrightarrow{d^2_{5,0}} E^2_{3,1} \to 0.$$

The map of spectral sequences $E_{p,q} \to \tilde{E}_{p,q}$ induces the following commutative diagram

$$\begin{tikzcd}
E^1_{3,3} \arrow[r, "d^1_{3,3}"] \arrow[d] & E^1_{2,3} \arrow[r, "d^1_{2,3}"] \arrow[d] & E^1_{1,3} \arrow[d] \\
\tilde{E}^1_{3,3} \arrow[r, "\tilde{d}^1_{3,3}"] & \tilde{E}^1_{2,3} \arrow[r, "\tilde{d}^1_{2,3}"] & \tilde{E}^1_{1,3}.
\end{tikzcd}$$

Since $E^1_{p,q} = \tilde{E}^1_{p,q}$ for $p = 0, 1, 2$, the diagram induces the surjective map $E^2_{2,3} \twoheadrightarrow \tilde{E}^2_{2,3}$. Now look at the commutative diagram

$$\begin{tikzcd}
E^2_{2,3} \arrow[r, "d^2_{2,3}"] \arrow[d] & E^2_{0,4} \arrow[d] \\
\tilde{E}^2_{2,3} \arrow[r, "\tilde{d}^2_{2,3}"] & \tilde{E}^2_{0,4}.
\end{tikzcd}$$

From the definitions of the spectral sequences

$$E^2_{0,4} = \tilde{E}^2_{0,4} = H_4(\mathrm{GL}_3)/\operatorname{im} H_4\big(R^* \times \mathrm{GL}_2\big).$$

By Lemma 3.4, $\tilde{d}^2_{2,3}$ is surjective, so the surjectivity of $d^2_{2,3}$ follows from the commutativity of the diagram and the surjectivity of the left-hand column map. Therefore $E^3_{0,4} = 0$.

Using this it is easy to see that $E^3_{5,0} \simeq E^\infty_{5,0}$. Since the spectral sequence converges to zero, we have $E^3_{5,0} = 0$. □

Following [20, Section 3] we define

Definition 3.7. Let F be an infinite field. We call

$$\wp^n(F)_{\mathrm{cl}} := H\big(C_{n+2}(F^n)_{\mathrm{GL}_n} \to C_{n+1}(F^n)_{\mathrm{GL}_n} \to C_n(F^n)_{\mathrm{GL}_n}\big)$$

the nth classical Bloch group.

Proposition 3.8. *Let F be an infinite field. We have an isomorphism $\wp^3(F)_{\mathrm{cl}} \simeq F^*$. In particular if F is algebraically closed, then $\wp^3(F)_{\mathrm{cl}}$ is divisible.*

Proof. In the proof of Lemma 3.6, we obtained the exact sequence

$$0 \to E^3_{5,0} \to E^2_{5,0} \xrightarrow{d^2_{5,0}} E^2_{3,1} \to 0.$$

By Lemma 3.6, $E^3_{5,0}=0$. By the above definition $E^2_{5,0}=\wp^3(F)_{\mathrm{cl}}$. It is also easy to see that $E^2_{3,1}=H_1(F^*)$. This proves the first part of the proposition. The second part follows from the fact that for an algebraically closed field F, F^* is divisible. □

Remark 3.9. From Proposition 3.8 and the existence of a surjective map $\wp^3(F)_{\mathrm{cl}}\to\wp^3(F)$ [20, Proposition 3.11] we deduce that $\wp^3(F)$ is divisible. See [20, 2.7] for the definition of $\wp^3(F)$. This gives a positive answer to Conjecture 0.2 in [20] for $n=3$.

4. Künneth theorem for $H_3(F^*\times F^*)$

Let F be an infinite field. The Künneth theorem for $H_3(\mu_F\times\mu_F)$ provides the following form

$$0\to H_3(\mu_F)\oplus H_3(\mu_F)\to H_3(\mu_F\times\mu_F)\to \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)\to 0.$$

Clearly $H_3(\mu_F)\oplus H_3(\mu_F)\to H_3(\mu_F\times\mu_F)$ is the map $\alpha:=H_3(i_1)+H_3(i_2)$, where $i_l:\mu_F\to\mu_F\times\mu_F$ is the usual injection, $l=1,2$. Let

$$\beta: H_3(p_1)\oplus H_3(p_2): H_3(\mu_F\times\mu_F)\to H_3(\mu_F)\oplus H_3(\mu_F),$$

where $p_l:\mu_F\times\mu_F\to\mu_F$ is the usual projection, $l=1,2$. Since $\beta\circ\alpha=\mathrm{id}$, the above exact sequence splits canonically. Thus we have the canonical decomposition

$$H_3(\mu_F\times\mu_F)=H_3(\mu_F)\oplus H_3(\mu_F)\oplus\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F).$$

We construct a splitting map $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)\to H_3(\mu_F\times\mu_F)$. The elements of the group $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)=\mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mu_F),H_1(\mu_F))$ are of the form $\langle\xi,n,\xi\rangle=\langle[\xi],n,[\xi]\rangle$, where ξ is an element of order n in F^* [11, Chap. V, Section 6]. It is easy to see that $\partial_2(\sum_{i=1}^n[\xi|\xi^i])=n[\xi]$ in $(B_1)_{\mu_F}$. For the definition of ∂_2 and B_* see [11, Chap. IV, Section 5]. By [11, Chap. V, Proposition 10.6] a map $\phi:\mathrm{Tor}_1^{\mathbb{Z}}(H_1(\mu_F),H_1(\mu_F))\to H_3((B_*)_{\mu_F}\otimes(B_*)_{\mu_F})$ can be defined as

$$a:=\langle[\xi],n,[\xi]\rangle\mapsto[\xi]\otimes\sum_{i=1}^n[\xi|\xi^i]+\sum_{i=1}^n[\xi|\xi^i]\otimes[\xi].$$

Considering the isomorphism $(B_*)_{\mu_F}\otimes(B_*)_{\mu_F}\simeq(B_*)_{\mu_F\times\mu_F}$ we have $\phi(a)=\chi(\xi)\in H_3(\mu_F\times\mu_F)$, where

$$\begin{aligned}\chi(\xi):=\sum_{i=1}^n\Big(&\big[(\xi,1)\big|(1,\xi)\big|(1,\xi^i)\big]-\big[(1,\xi)\big|(\xi,1)\big|(1,\xi^i)\big]+\big[(1,\xi)\big|(1,\xi^i)\big|(\xi,1)\big]\\&+\big[(\xi,1)\big|(\xi^i,1)\big|(1,\xi)\big]-\big[(\xi,1)\big|(1,\xi)\big|(\xi^i,1)\big]+\big[(1,\xi)\big|(\xi,1)\big|(\xi^i,1)\big]\Big).\end{aligned}$$

Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_3(\mu_F)\oplus H_3(\mu_F) \arrow[r] \arrow[d] & H_3(\mu_F\times\mu_F) \arrow[r] \arrow[d] & \mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i+j=3} H_i(F^*)\otimes H_j(F^*) \arrow[r] & H_3(F^*\times F^*) \arrow[r] & \mathrm{Tor}_1^{\mathbb{Z}}(F^*,F^*) \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)\simeq\mathrm{Tor}_1^{\mathbb{Z}}(F^*,F^*)$, we see that the second horizontal exact sequence in the above diagram splits canonically. So we proved the following proposition.

Proposition 4.1. *Let F be an infinite field. Then we have the canonical decomposition*

$$H_3\bigl(F^*\times F^*\bigr)=\bigoplus_{i+j=3} H_i\bigl(F^*\bigr)\otimes H_j\bigl(F^*\bigr)\oplus\mathrm{Tor}_1^{\mathbb{Z}}\bigl(F^*,F^*\bigr),$$

where a splitting map $\mathrm{Tor}_1^{\mathbb{Z}}(F^*,F^*)=\mathrm{Tor}_1^{\mathbb{Z}}(\mu_F,\mu_F)\to H_3(F^*\times F^*)$ is defined by $\langle[\xi],n,[\xi]\rangle\mapsto\chi(\xi)$.

5. The injectivity theorem

Lemma 5.1. *Let $K_1(Z(R))\otimes\mathbb{Z}[\frac{1}{n}]\overset{\theta}{\simeq}K_1(R)\otimes\mathbb{Z}[\frac{1}{n}]$ be induced by the usual inclusion $Z(R)\to R$. Then for all $i\geqslant 1$,*

$$H_i\biggl(Z(R)^*,\mathbb{Z}\biggl[\frac{1}{n}\biggr]\biggr)\simeq H_i\biggl(K_1(R),\mathbb{Z}\biggl[\frac{1}{n}\biggr]\biggr).$$

Proof. Since the map θ is an isomorphism in the localized category of $\mathbb{Z}[\frac{1}{n}]$-modules, it induces an isomorphism on the group homology in this category. □

Example 5.2.

(i) If R is commutative, then $K_1(Z(R))=K_1(R)$.
(ii) Let R be a (finite-dimensional) division F-algebra of rank $[R:F]=n^2$. Note that $F=Z(R)$. Then $K_1(F)\otimes\mathbb{Z}[\frac{1}{n}]\simeq K_1(R)\otimes\mathbb{Z}[\frac{1}{n}]$. This is also true if R is an Azumaya S-algebra, where S is a commutative local ring [9, Corollary 2.3].

These are the examples one should keep in mind in the rest of this section.

Let A be a commutative ring with trivial GL_3-action. Let $P_*\to A$ be a free left $A[\mathrm{GL}_3]$-resolution of A. Consider the complex

$$D'_*:0\leftarrow D'_0\bigl(R^3\bigr)\leftarrow D'_1\bigl(R^3\bigr)\leftarrow\cdots\leftarrow D'_l\bigl(R^3\bigr)\leftarrow\cdots,$$

where $D'_i(R^3):=D_i(R^3)\otimes A$. The double complex $D'_*\otimes_{\mathrm{GL}_3}P_*$ induces a first quadrant spectral sequence $\mathcal{E}^1_{p,q}\Rightarrow H_{p+q}(\mathrm{GL}_3,A)$, where $\mathcal{E}^1_{p,q}=\tilde{E}^1_{p+1,q}(3)\otimes A$ and $\mathfrak{d}^1_{p,q}=\tilde{d}^1_{p+1,q}\otimes\mathrm{id}_A$.

Lemma 5.3. *The groups* $\mathcal{E}^2_{3,0}$, $\mathcal{E}^2_{4,0}$, $\mathcal{E}^2_{2,1}$, $\mathcal{E}^2_{3,1}$, $\mathcal{E}^2_{1,2}$ *and* $\mathcal{E}^2_{2,2}$ *are trivial.*

Proof. This follows from the above spectral sequence and Lemmas 3.1, 3.2, and 3.3. □

Theorem 5.4. *Let* $Z(R)$ *be the center of* R. *Let* k *be a field such that* $1/2 \in k$.

- (i) *If* $K_1(Z(R)) \otimes \mathbb{Q} \simeq K_1(R) \otimes \mathbb{Q}$, *then* $H_3(\mathrm{GL}_2, \mathbb{Q}) \to H_3(\mathrm{GL}_3, \mathbb{Q})$ *is injective. If* R *is commutative, then* $\mathbb{Q}$ *can be replaced by* k.
- (ii) *If* R *is an infinite field or a quaternion algebra over an infinite field, then* $H_3(\mathrm{GL}_2, \mathbb{Z}[\frac{1}{2}]) \to H_3(\mathrm{GL}_3, \mathbb{Z}[\frac{1}{2}])$ *is injective.*
- (iii) *Let* R *be either* $\mathbb{R}$ *or an infinite field such that* $R^* = R^{*2}$. *Then* $H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)$ *is injective.*
- (iv) *The map* $H_3(\mathrm{GL}_2(\mathbb{H})) \to H_3(\mathrm{GL}_3(\mathbb{H}))$ *is bijective.*

Proof. Let $A = \mathbb{Z}$, $\mathbb{Z}[\frac{1}{2}]$, $\mathbb{Q}$ or k (depending on parts (i), …, (iv)). By Lemma 5.3, $\mathcal{E}^2_{0,3} \simeq \mathcal{E}^\infty_{0,3} \simeq H_3(\mathrm{GL}_3, A)$, so to prove the theorem it is sufficient to prove that $H_3(\mathrm{GL}_2, A)$ is a summand of $\mathcal{E}^2_{0,3}$. To prove this it is sufficient to define a map $\varphi : H_3(R^* \times \mathrm{GL}_2, A) \to H_3(\mathrm{GL}_2, A)$ such that $\varphi|_{H_3(\mathrm{GL}_2, A)}$ is the identity map and $\partial^1_{1,3}(H_3(R^{*2} \times \mathrm{GL}_1, A)) \subseteq \ker(\varphi)$.

We have the canonical decomposition $H_3(R^* \times \mathrm{GL}_2, A) = \bigoplus_{i=0}^4 S_i$, where

$$S_i = H_i\big(R^*, A\big) \otimes H_{3-i}(\mathrm{GL}_2, A), \quad 0 \leqslant i \leqslant 3,$$
$$S_4 = \mathrm{Tor}_1^A\big(H_1\big(R^*, A\big), H_1(\mathrm{GL}_2, A)\big).$$

In case of (i) this follows from the Künneth theorem and the fact that $S_4 = 0$. In other cases it follows again from the Künneth theorem and an argument in the line of the previous section. Note that for parts (ii), (iii) and (iv), the splitting map is

$$S_4 \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu_{Z(R)}, \mu_{Z(R)}) \otimes A \xrightarrow{\phi} H_3\big(R^* \times R^*, A\big) \xrightarrow{q_*} H_3\big(R^* \times \mathrm{GL}_2, A\big),$$

where ϕ can be defined as in the previous section, and

$$q : R^* \times R^* \to R^* \times \mathrm{GL}_2, \quad (a, b) \mapsto \big(a, \mathrm{diag}(b, 1)\big).$$

Define $\varphi|_{S_0} : S_0 \to H_3(\mathrm{GL}_2, A)$ the identity map,

$$\varphi|_{S_2} : S_2 \simeq H_2\big(R^*, A\big) \otimes H_1(\mathrm{GL}_1, A) \to H_3\big(R^* \times \mathrm{GL}_1, A\big) \to H_3(\mathrm{GL}_2, A)$$

the shuffle product, $\varphi|_{S_3} : S_3 \to H_3(\mathrm{GL}_2, A)$ the map induced by $R^* \to \mathrm{GL}_2$, $a \mapsto \mathrm{diag}(a, 1)$, and $\varphi|_{S_4} : S_4 \to H_3(\mathrm{GL}_2, A)$ the composition

$$S_4 \xrightarrow{\phi} H_3\big(R^* \times R^*, A\big) \xrightarrow{\mathrm{inc}_*} H_3(\mathrm{GL}_2, A).$$

By the homology stability theorem [8, Theorem 1] and a theorem of Dennis [5, Corollary 8] (see also [1, Theorem 1]) we have the decomposition

$$H_2(\mathrm{GL}_2) = H_2\big(K_1(R)\big) \oplus K_2(R).$$

So using Lemma 5.1 we have $S_1 = S_1' \oplus S_1''$, where

$$S_1' = H_1\big(R^*, A\big) \otimes H_2\big(Z(R)^*, A\big),$$
$$S_1'' = H_1\big(R^*, A\big) \otimes K_2(R) \otimes A.$$

Define $\varphi|_{S_1'} : S_1' \to H_3(\mathrm{GL}_2, A)$ to be the shuffle product and define the map $\varphi|_{S_1''} : S_1'' \to H_3(\mathrm{GL}_2, A)$ as the composition

$$\begin{aligned} &H_1\big(Z(R)^*, A\big) \otimes K_2(R) \otimes A \xrightarrow{f} H_1\big(Z(R)^*, A\big) \otimes H_2(\mathrm{GL}_2, A) \\ &\xrightarrow{g} H_3\big(Z(R)^* \times \mathrm{GL}_2, A\big) \xrightarrow{h} H_3(\mathrm{GL}_2, A), \end{aligned}$$

where $f = \frac{1}{2}\lambda$, λ being the natural map

$$\lambda : K_2(R) \otimes A = H_2\big(E(R), A\big) \to H_2\big(\mathrm{GL}(R), A\big) \simeq H_2(\mathrm{GL}_2, A),$$

and g is the shuffle product. Here h is induced by the map

$$Z(R)^* \times \mathrm{GL}_2 \to \mathrm{GL}_2, \quad (a, B) \mapsto aB.$$

By Proposition 4.1 we have $H_3(R^{*2} \times \mathrm{GL}_1, A) = \bigoplus_{i=0}^{8} T_i$, where

$$\begin{aligned} T_0 &= H_3(\mathrm{GL}_1, A), \\ T_1 &= \bigoplus_{i=1}^{3} H_i\big(R_1^*, A\big) \otimes H_{3-i}(\mathrm{GL}_1, A), \\ T_2 &= \bigoplus_{i=1}^{3} H_i\big(R_2^*, A\big) \otimes H_{3-i}(\mathrm{GL}_1, A), \\ T_3 &= H_1\big(R_1^*, A\big) \otimes H_1\big(R_2^*, A\big) \otimes H_1(\mathrm{GL}_1, A), \\ T_4 &= \mathrm{Tor}_1^A\big(H_1\big(R_1^*, A\big), H_1\big(R_2^*, A\big)\big), \\ T_5 &= \mathrm{Tor}_1^A\big(H_1\big(R_1^*, A\big), H_1(\mathrm{GL}_1, A)\big), \\ T_6 &= \mathrm{Tor}_1^A\big(H_1\big(R_2^*, A\big), H_1(\mathrm{GL}_1, A)\big), \\ T_7 &= H_1\big(R_1^*, A\big) \otimes H_2\big(R_2^*, A\big), \\ T_8 &= H_2\big(R_1^*, A\big) \otimes H_1\big(R_2^*, A\big). \end{aligned}$$

Note that here $R_i^* = R^*$, $i = 1, 2$, is the ith summand of $R^{*2} = R^* \times R^*$. We know that $\mathfrak{d}_{1,3}^1 = \sigma_1 - \sigma_2$, where $\sigma_i = H_3(\alpha_{i,2})$. It is not difficult to see that $\mathfrak{d}_{1,3}^1(T_0 \oplus T_1 \oplus T_2 \oplus T_7 \oplus T_8) \subseteq \ker(\varphi)$. Here one should use the isomorphism $H_1(\mathrm{GL}_1, A) \simeq H_1(\mathrm{GL}_2, A)$. Now $(\sigma_1 - \sigma_2)(T_4) \subseteq S_4$, $\sigma_1(T_5) \subseteq S_0$ and $\sigma_2(T_5) \subseteq S_4$, $\sigma_1(T_6) \subseteq S_4$ and $\sigma_2(T_6) \subseteq S_0$. With this description one can see that $\mathfrak{d}_{1,3}^1(T_4 \oplus T_5 \oplus T_6) \subseteq \ker(\varphi)$. To finish the proof of the claim we have to prove that $\mathfrak{d}_{1,3}^1(T_3) \subseteq \ker(\varphi)$. Let $x = a \otimes b \otimes c \in T_3$. By Lemma 5.1, we may assume that $a, b, c \in Z(R)^*$. Then

$$\begin{aligned}\mathfrak{d}^1_{1,3}(x) &= -b\otimes \mathbf{c}\big(\mathrm{diag}(a,1),\mathrm{diag}(1,c)\big) - a\otimes \mathbf{c}\big(\mathrm{diag}(b,1),\mathrm{diag}(1,c)\big)\in S_1\\ &= \big(-b\otimes \mathbf{c}(a,c) - a\otimes \mathbf{c}(b,c), b\otimes [a,c] + a\otimes [b,c]\big)\in S_1'\oplus S_1'',\end{aligned}$$

where

$$\begin{aligned}[][a,c] &:= \mathbf{c}\big(\mathrm{diag}\big(a,1,a^{-1}\big),\mathrm{diag}\big(b,b^{-1},1\big)\big)\in H_2\big(E(R),A\big)\\ &= \mathbf{c}\big(\mathrm{diag}(a,1),\mathrm{diag}\big(b,b^{-1}\big)\big)\in H_2(\mathrm{GL}_2,A).\end{aligned}$$

Thus,

$$\begin{aligned}\varphi\big(\mathfrak{d}^1_{1,3}(x)\big) &= -\mathbf{c}\big(\mathrm{diag}(b,1),\mathrm{diag}(1,a),\mathrm{diag}(1,c)\big) - \mathbf{c}\big(\mathrm{diag}(a,1),\mathrm{diag}(1,b),\mathrm{diag}(1,c)\big)\\ &\quad + \frac{1}{2}\mathbf{c}\big(\mathrm{diag}(b,b),\mathrm{diag}(a,1),\mathrm{diag}\big(c,c^{-1}\big)\big)\\ &\quad + \frac{1}{2}\mathbf{c}\big(\mathrm{diag}(a,a),\mathrm{diag}(b,1),\mathrm{diag}\big(c,c^{-1}\big)\big).\end{aligned}$$

Set $p := \mathrm{diag}(p,1)$, $\bar{q} := \mathrm{diag}(1,q)$, $p\bar{q}\bar{r} := \mathbf{c}(\mathrm{diag}(p,1),\mathrm{diag}(1,q),\mathrm{diag}(1,r))$, etc. Conjugation by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ induces the equality $p\bar{q}\bar{r} = \bar{p}qr$ and it is easy to see that $pqr = -qpr$ and $\overline{p^{-1}}qr = -\bar{p}qr$. With these notations and the above relations we have

$$\begin{aligned}\varphi\big(\mathfrak{d}^1_{1,3}(x)\big) &= -b\overline{ac} - a\overline{bc} + \frac{1}{2}\big(bac + ba\overline{c^{-1}} + \bar{b}ac + \bar{b}a\overline{c^{-1}}\big)\\ &\quad + \frac{1}{2}\big(abc + ab\overline{c^{-1}} + \bar{a}bc + \bar{a}b\overline{c^{-1}}\big) = 0.\end{aligned}$$

This proves that $H_3(\mathrm{GL}_2,A)$ is a summand of $\mathcal{E}^2_{0,3}$. This proves (i) and (ii).

The proof of (iii) is almost the same as the proof of (i), only we need to modify the definition of the map f. If $R^* = R^{*2}$, f should be induced by the map

$$K_2(R) = K_2^M(R)\to H_2(\mathrm{GL}_2),\quad \{a,b\}\mapsto \mathbf{c}\big(\mathrm{diag}(\sqrt{a},1),\mathrm{diag}\big(b,b^{-1}\big)\big).$$

Note that if R is commutative and $R^* = R^{*2}$, then $K_2^M(R)$ is uniquely 2-divisible [2, Proposition 1.2], so in this case f is well-defined.

Now let $R = \mathbb{R}$. It is well-known that $K_2^M(\mathbb{R}) = \langle\{-1,-1\}\rangle\oplus K_2^M(\mathbb{R})^\circ$, where $\langle\{-1,-1\}\rangle$ is a group of order 2 generated by $\{-1,-1\}$ and $K_2^M(\mathbb{R})^\circ$ is a uniquely divisible group. In fact every element of $K_2^M(\mathbb{R})$ can be uniquely written as $m\{-1,-1\} + \sum\{a_i,b_i\}$, $a_i, b_i > 0$ and $m = 0$ or 1. Now we define the map $K_2^M(\mathbb{R})\to H_2(\mathrm{GL}_2(\mathbb{R}))$ by $\{-1,-1\}\mapsto 0$ and $\{a,b\}\mapsto \mathbf{c}(\mathrm{diag}(\sqrt{a},1),\mathrm{diag}(b,b^{-1}))$ for $a,b>0$.

For the proof of (iv) we should mention that $\mathbb{R}^{>0} = K_1^M(\mathbb{R})^\circ\simeq K_1(\mathbb{H})$ and $K_2^M(\mathbb{R})^\circ\simeq K_2(\mathbb{H})$ [17, p. 188]. Since $K_2(\mathbb{H})$ and $H_2(\mathbb{R}^{>0})$ are uniquely divisible, the proof of injectivity is similar to the above approach. Surjectivity follows from [8, Theorem 2] and the fact that $K_n^M(\mathbb{H})$ are trivial for $n\geqslant 2$ [16, Remark B.15]. □

Corollary 5.5. *Let $Z(R)$ be the center of R. Let k be a field such that $1/2 \in k$.*

(i) *If $K_1(Z(R)) \otimes \mathbb{Q} \simeq K_1(R) \otimes \mathbb{Q}$, then we have the exact sequence*

$$0 \to H_3(\mathrm{GL}_2, \mathbb{Q}) \to H_3(\mathrm{GL}_3, \mathbb{Q}) \to K_3^M(R) \otimes \mathbb{Q} \to 0.$$

If R is commutative, then $\mathbb{Q}$ can be replaced by k.

(ii) *If R is an infinite field or a quaternion algebra with an infinite center, then we have the split exact sequence*

$$0 \to H_3\left(\mathrm{GL}_2, \mathbb{Z}\left[\frac{1}{2}\right]\right) \to H_3\left(\mathrm{GL}_3, \mathbb{Z}\left[\frac{1}{2}\right]\right) \to K_3^M(R) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to 0.$$

(iii) *Let R be an infinite field such that $R^* = R^{*2}$. Then we have the split exact sequence*

$$0 \to H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3) \to K_3^M(R) \to 0.$$

(iv) *We have the (non-split) exact sequence*

$$0 \to H_3\big(\mathrm{GL}_2(\mathbb{R})\big) \to H_3\big(\mathrm{GL}_3(\mathbb{R})\big) \to K_3^M(\mathbb{R}) \to 0.$$

Proof. The exactness in all cases follows from Theorem 5.4 and the following exact sequence [8, Theorem 2]

$$H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3) \to K_3^M(R) \to 0.$$

If R is commutative, we have a natural map $K_3^M(R) \to K_3(R)$ such that the composition

$$K_3^M(R) \to K_3(R) \to H_3(\mathrm{GL}_3) \to K_3^M(R)$$

coincides with the multiplication by 2 [8, Proposition 4.1.1]. Now splitting maps can be constructed easily. □

Remark 5.6.

(i) Let $R = M_m(D)$, where D is a finite-dimensional division F-algebra. Then $\mathrm{GL}_n(R) \simeq \mathrm{GL}_{mn}(D)$. So by the stability theorem and [8, Theorem 2], $K_i^M(R) = 0$ for $m \geqslant 2$ and $i \geqslant 2$.

(ii) It seems that it is not known whether for a finite-dimensional division F-algebra D, $H_2(\mathrm{GL}_1(D), \mathbb{Q}) \to H_2(\mathrm{GL}_2(D), \mathbb{Q})$ is injective. The only case that is known to us is when $D = \mathbb{H}$. This follows from applying the Künneth theorem to $\mathrm{GL}_n(\mathbb{H}) = \mathrm{SL}_n(\mathbb{H}) \times \mathbb{R}^{>0}$ for $n = 1, 2$ and the isomorphism $K_2(\mathbb{H}) \simeq H_2(\mathrm{SL}_1(\mathbb{H}))$ from [17, p. 287].

6. Third homology of SL_2 and the indecomposable K_3

In this section we assume that R is a commutative ring with many units, unless it is mentioned otherwise. When a group G acts on a module M, we use the standard definition M_G

for $H_0(G, M)$. Consider the action of R^* on SL_n defined by

$$a.B := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} B \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where $a \in R^*$ and $B \in \mathrm{SL}_n$. This induces an action of R^* on $H_i(\mathrm{SL}_n)$. So by $H_i(\mathrm{SL}_n)_{R^*}$ we mean $H_0(R^*, H_i(\mathrm{SL}_n))$.

Theorem 6.1. *Let k be a field such that* $1/2 \in k$.

(i) $H_3(\mathrm{SL}_2, k)_{R^*} \to H_3(\mathrm{SL}, k)$ *is injective.*
(ii) *If R is an infinite field, then* $H_3(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}])_{R^*} \to H_3(\mathrm{SL}, \mathbb{Z}[\frac{1}{2}])$ *is injective.*
(iii) *If R is either* $\mathbb{R}$ *or an infinite field such that* $R^* = R^{*2}$, *then* $H_3(\mathrm{SL}_2) \to H_3(\mathrm{SL})$ *is injective.*
(iv) *The map* $H_3(\mathrm{SL}_2(\mathbb{H})) \to H_3(\mathrm{SL}_3(\mathbb{H}))$ *is bijective.*

Proof. Part (iv) follows from Theorem 5.4 and by applying the Künneth theorem to $\mathrm{GL}_n(\mathbb{H}) = \mathrm{SL}_n(\mathbb{H}) \times \mathbb{R}^{>0}$, $n \geqslant 1$.

Since $H_3(\mathrm{SL}) \to H_3(\mathrm{GL})$ is injective, to prove (i), (ii) and (iii), by Theorem 5.4 it is sufficient to prove that $H_3(\mathrm{SL}_2, k)_{R^*} \to H_3(\mathrm{GL}_2, k)$, $H_3(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}])_{R^*} \to H_3(\mathrm{GL}_2, \mathbb{Z}[\frac{1}{2}])$ and $H_3(\mathrm{SL}_2) \to H_3(\mathrm{GL}_2)$ are injective.

Set $A := \mathbb{Z}[\frac{1}{2}]$ or k. From the map $\gamma : R^* \times \mathrm{SL}_2 \to \mathrm{GL}_2$, $(a, M) \mapsto aM$, we obtain two short exact sequences

$$1 \to \mu_{2,R} \to R^* \times \mathrm{SL}_2 \to \mathrm{im}(\gamma) \to 1,$$
$$1 \to \mathrm{im}(\gamma) \to \mathrm{GL}_2 \to R^*/R^{*2} \to 1.$$

Writing the Lyndon–Hochschild–Serre spectral sequence of the above exact sequences and carrying out a simple analysis, one gets

$$H_3\big(\mathrm{im}(\gamma), A\big) \simeq H_3\big(R^* \times \mathrm{SL}_2, A\big), \qquad H_3\big(\mathrm{im}(\gamma), A\big)_{R^*/R^{*2}} \simeq H_3(\mathrm{GL}_2, A).$$

Since the action of R^{*2} on $H_3(\mathrm{im}(\gamma), A)$ is trivial,

$$H_3\big(\mathrm{im}(\gamma), A\big)_{R^*} \simeq H_3(\mathrm{GL}_2, A).$$

These imply

$$H_3(\mathrm{GL}_2, A) \simeq H_3\big(R^* \times \mathrm{SL}_2, A\big)_{R^*}.$$

Now the Künneth theorem implies that $H_3(\mathrm{SL}_2, A)_{R^*} \to H_3(\mathrm{GL}_2, A)$ is injective. This proves parts (i) and (ii).

(iii) First let $R^* = R^{*2}$. The map γ induces the short exact sequence

$$1 \to \mu_{2,R} \to R^* \times \mathrm{SL}_2 \to \mathrm{GL}_2 \to 1.$$

From the Lyndon–Hochschild–Serre spectral sequence of this exact sequence, one sees that $H_3(\mathrm{inc}) : H_3(\mathrm{SL}_2) \to H_3(\mathrm{GL}_2)$ has a kernel of order dividing 4. To show that this kernel is trivial we look at the spectral sequence induced by $1 \to \mathrm{SL}_2 \to \mathrm{GL}_2 \to R^* \to 1$,

$$E'^2_{p,q} = H_p\big(R^*, H_q(\mathrm{SL}_2)\big) \Rightarrow H_{p+q}(\mathrm{GL}_2).$$

By Proposition 2.6 and the fact that the action of R^* on $H_i(\mathrm{SL}_2)$ is trivial, we get the following E'^2-terms:

$$\begin{array}{ccccc}
* & * & & & \\
H_3(\mathrm{SL}_2) & * & * & & \\
K_2^M(R) & R^* \otimes K_2^M(R) & E'^2_{2,2} & * & \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & H_1(R^*) & H_2(R^*) & H_3(R^*) & H_4(R^*).
\end{array}$$

Here $E'^2_{2,2} = H_2(R^*) \otimes K_2^M(R) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mu_R, K_2^M(R))$, which is 2-divisible as $K_2^M(R)$ is uniquely 2-divisible. Hence

$$H_3(\mathrm{SL}_2)/\operatorname{im}\big(d'^2_{2,2}\big) \simeq E'^{\infty}_{0,3} \subseteq H_3(\mathrm{GL}_2),$$

which is induced by $\mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2$. Thus, $\operatorname{im}(d'^2_{2,2}) \subseteq \ker(H_3(\mathrm{inc}))$. This means that $\operatorname{im}(d'^2_{2,2})$ is 2-divisible of order dividing 4. This is possible only if $\operatorname{im}(d'^2_{2,2})$ is trivial.

Now let $R = \mathbb{R}$. Consider the following exact sequences

$$0 \to \mathbb{Z}/4\mathbb{Z} \to H_3\big(\mathrm{SL}_2(\mathbb{R})\big) \to H_3\big(\mathrm{PSL}_2(\mathbb{R})\big) \to 0,$$
$$0 \to H_3\big(\mathrm{PSL}_2(\mathbb{R})\big) \to H_3\big(\mathrm{PGL}_2(\mathbb{R})\big) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

(see [15, App. C, C.10, Theorem C.14]). In the first exact sequence $\mathbb{Z}/4\mathbb{Z}$ is mapped onto the subgroup of order 4 generated by $w := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (see [15, p. 207]). Set $\alpha : H_3(\mathrm{SL}_2(\mathbb{R})) \to H_3(\mathrm{GL}_2(\mathbb{R}))$. From the diagram

$$\begin{tikzcd}
H_3(\mathrm{SL}_2(\mathbb{R})) \arrow[r] \arrow[d] & H_3(\mathrm{GL}_2(\mathbb{R})) \arrow[d] \\
H_3(\mathrm{PSL}_2(\mathbb{R})) \arrow[r] & H_3(\mathrm{PGL}_2(\mathbb{R}))
\end{tikzcd}$$

and the above exact sequences, one sees that $\ker(\alpha)$ is of order dividing 4. Here we describe the E_2-terms $E^2_{1,2}$ and $E^2_{2,2}$ of the spectral sequence

$$E^2_{p,q} = H_p\big(\mathbb{R}^*, H_q\big(\mathrm{SL}_2(\mathbb{R})\big)\big) \Rightarrow H_{p+q}\big(\mathrm{GL}_2(\mathbb{R})\big),$$

which is associated to $1 \to \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^* \to 1$. It is well-known that

$$H_2\big(\mathrm{SL}_2(\mathbb{R})\big) \simeq K_2^M(\mathbb{R})^\circ \oplus \mathbb{Z},$$

where $K_2^M(\mathbb{R})^\circ$ is the uniquely divisible part of $K_2^M(\mathbb{R})$. The action of $\mathbb{R}^*$ on $K_2^M(\mathbb{R})^\circ$ is trivial and its action on $\mathbb{Z}$ is through multiplication by $\mathrm{sign}(r)$, where $r \in \mathbb{R}^*$ (see the proof of Proposition 2.15 in [17, p. 288]). Let $\bar{\mathbb{Z}}$ be $\mathbb{Z}$ with this new action of $\mathbb{R}^*$. Thus for $p = 1, 2$,

$$E_{p,2}^2 = H_p\big(\mathbb{R}^*\big) \otimes K_2^M(\mathbb{R})^\circ \oplus H_p\big(\mathbb{R}^*, \bar{\mathbb{Z}}\big).$$

It is not difficult to see that $H_1(\mathbb{R}^*, \bar{\mathbb{Z}}) = 0$ and $H_2(\mathbb{R}^*, \bar{\mathbb{Z}}) = \mathbb{Z}/2\mathbb{Z}$. Now by an easy analysis of the above spectral sequence, one sees that $\ker(\alpha)$ is of order diving 2. Since $w^2 = -I_2 \in \mathrm{GL}_2(\mathbb{R})$, $\ker(\alpha)$, if not trivial, must be generated by $x = [-I_2| - I_2| - I_2]$. But $\alpha(x) = [-I_2| - I_2| - I_2] \in H_3(\mathrm{GL}_2(\mathbb{R}))$ is non-trivial. Therefore $\ker(\alpha) = 0$. Note that here one has to use the fact that the action of $\mathbb{R}^*$ on $H_3(\mathrm{SL}_2(\mathbb{R}))$ is trivial (see [15, App. C.14] and [6, 2.10, p. 230]). Therefore $E_{0,3}^2 = H_3(\mathrm{SL}_2(\mathbb{R}))$. □

Corollary 6.2. *Let k be a field such that $1/2 \in k$.*

(i) *We have the split exact sequence*

$$0 \to H_3(\mathrm{SL}_2, k)_{R^*} \to H_3(\mathrm{SL}, k) \to K_3^M(R) \otimes k \to 0.$$

(ii) *If R is an infinite field, then we have the split exact sequence*

$$0 \to H_3\left(\mathrm{SL}_2, \mathbb{Z}\left[\frac{1}{2}\right]\right)_{R^*} \to H_3\left(\mathrm{SL}, \mathbb{Z}\left[\frac{1}{2}\right]\right) \to K_3^M(R) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to 0.$$

(iii) *If R is an infinite field such that $R^* = R^{*2}$, then*

$$0 \to H_3(\mathrm{SL}_2) \to H_3(\mathrm{SL}) \to K_3^M(R) \to 0$$

is split exact.

(iv) *We have the split exact sequence*

$$0 \to H_3\big(\mathrm{SL}_2(\mathbb{R})\big) \to H_3\big(\mathrm{SL}(\mathbb{R})\big) \to K_3^M(\mathbb{R})^\circ \to 0,$$

where $K_3^M(\mathbb{R}) \simeq \langle\{-1, -1, -1\}\rangle \oplus K_3^M(\mathbb{R})^\circ$.

Proof. First we prove (iv). The injectivity follows from Theorem 6.1. From the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{SL}_2(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{GL}_2(\mathbb{R}) \arrow[r] \arrow[d] & \mathbb{R}^* \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{SL}(\mathbb{R}) \arrow[r] & \mathrm{GL}(\mathbb{R}) \arrow[r] & \mathbb{R}^* \arrow[r] & 1,
\end{tikzcd}$$

we obtain a map of spectral sequences

$$\begin{tikzcd}
E^2_{p,q} = H_p(\mathbb{R}^*, H_q(\mathrm{SL}_2(\mathbb{R}))) \arrow[r, Rightarrow] \arrow[d] & H_{p+q}(\mathrm{GL}_2(\mathbb{R})) \arrow[d] \\
E'^2_{p,q} = H_p(\mathbb{R}^*, H_q(\mathrm{SL}(\mathbb{R}))) \arrow[r, Rightarrow] & H_{p+q}(\mathrm{GL}(\mathbb{R})),
\end{tikzcd}$$

which give us a map of filtration

$$\begin{tikzcd}
0 = F_{-1} \arrow[r, phantom, "\subseteq"] & F_0 \arrow[r, phantom, "\subseteq"] \arrow[d] & F_1 \arrow[r, phantom, "\subseteq"] \arrow[d] & F_2 \arrow[r, phantom, "\subseteq"] \arrow[d] & F_3 = H_3(\mathrm{GL}_2(\mathbb{R})) \arrow[d] \\
0 = F'_{-1} \arrow[r, phantom, "\subseteq"] & F'_0 \arrow[r, phantom, "\subseteq"] & F'_1 \arrow[r, phantom, "\subseteq"] & F'_2 \arrow[r, phantom, "\subseteq"] & F'_3 = H_3(\mathrm{GL}(\mathbb{R})).
\end{tikzcd}$$

Since $H_3(\mathrm{SL}_2(\mathbb{R})) \to H_3(\mathrm{GL}_2(\mathbb{R}))$ is injective, $F_0 = E^\infty_{0,3} \simeq H_3(\mathrm{SL}_2(\mathbb{R}))$. It is easy to see that $E^\infty_{p,1} = E'^\infty_{p,1} = 0$, $F'_0 = E'^\infty_{0,3} \simeq H_3(\mathrm{SL}(\mathbb{R}))$ and $E^\infty_{3,0} \simeq E'^\infty_{3,0}$. Since

$$H_2\big(\mathrm{SL}_2(\mathbb{R})\big) = \mathbb{Z} \oplus K_2^M(\mathbb{R})^\circ \to \mathbb{Z}/2\mathbb{Z} \oplus K_2^M(\mathbb{R})^\circ = H_2\big(\mathrm{SL}(\mathbb{R})\big)$$

is surjective, $E^\infty_{2,2} \hookrightarrow E'^\infty_{2,2}$ with $\operatorname{coker}(E^\infty_{2,2} \to E'^\infty_{2,2}) \simeq \mathbb{Z}/2\mathbb{Z}$ (see the proof of Theorem 6.1(iii)). By an easy analysis of the above filtration, one gets the exact sequence

$$0 \to H_3\big(\mathrm{SL}(\mathbb{R})\big)/H_3\big(\mathrm{SL}_2(\mathbb{R})\big) \to H_3\big(\mathrm{GL}(\mathbb{R})\big)/H_3\big(\mathrm{GL}_2(\mathbb{R})\big) \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Therefore $H_3(\mathrm{SL}(\mathbb{R}))/H_3(\mathrm{SL}_2(\mathbb{R})) \simeq K_3^M(\mathbb{R})^\circ$. A splitting map can be constructed using the composition $K_3^M(\mathbb{R})^\circ \to H_3(\mathrm{GL}(\mathbb{R})) \to H_3(\mathrm{SL}(\mathbb{R}))$.

The proof of (i), (ii) and (iii) are similar. In the proof of (iii) we need the homology stability $H_2(\mathrm{SL}_2) = H_2(\mathrm{SL})$, and in the proof of (i) and (ii) we need the isomorphism

$$H_1\left(R^*, H_2\left(\mathrm{SL}_2, \mathbb{Z}\left[\frac{1}{2}\right]\right)\right) \simeq H_1\left(R^*, H_2\left(\mathrm{SL}, \mathbb{Z}\left[\frac{1}{2}\right]\right)\right).$$

To prove the latter, consider the exact sequence

$$1 \to R^{*2} \to R^* \to R^*/R^{*2} \to 1.$$

This induces a map of Lyndon–Hochschild–Serre spectral sequences, with coefficients in $H_2(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}])$ and $H_2(\mathrm{SL}, \mathbb{Z}[\frac{1}{2}])$ respectively, from which one easily obtains the commutative diagram

$$\begin{tikzcd}
H_1(R^{*2}, H_2(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}]))_{R^*} \arrow[r, "\simeq"] \arrow[d] & H_1(R^*, H_2(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}])) \arrow[d] \\
H_1(R^{*2}, H_2(\mathrm{SL}, \mathbb{Z}[\frac{1}{2}])) \arrow[r, "\simeq"] & H_1(R^*, H_2(\mathrm{SL}, \mathbb{Z}[\frac{1}{2}])).
\end{tikzcd}$$

The action of R^{*2} on $H_2(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}])$ is trivial, so

$$\begin{aligned} H_1\left(R^{*2}, H_2\left(\mathrm{SL}_2, \mathbb{Z}\left[\frac{1}{2}\right]\right)\right)_{R^*} &\simeq \left(H_1\left(R^{*2}, \mathbb{Z}\left[\frac{1}{2}\right]\right) \otimes H_2\left(\mathrm{SL}_2, \mathbb{Z}\left[\frac{1}{2}\right]\right)\right)_{R^*} \\ &\simeq H_1\left(R^{*2}, \mathbb{Z}\left[\frac{1}{2}\right]\right) \otimes H_2\left(\mathrm{SL}_2, \mathbb{Z}\left[\frac{1}{2}\right]\right)_{R^*} \\ &\simeq H_1\left(R^{*2}, \mathbb{Z}\left[\frac{1}{2}\right]\right) \otimes H_2\left(\mathrm{SL}, \mathbb{Z}\left[\frac{1}{2}\right]\right) \\ &\simeq H_1\left(R^{*2}, H_2\left(\mathrm{SL}, \mathbb{Z}\left[\frac{1}{2}\right]\right)\right). \end{aligned}$$

Thus the left-hand column map in the above diagram is an isomorphism. This implies the isomorphism of the right-hand column map. □

Remark 6.3. Let $R = \mathbb{R}$, $R = \mathbb{H}$ or R be an infinite field such that $R^* = R^{*2}$. Then $H_3(\mathrm{SL}_2) \to H_3(\mathrm{SL}_3)$ is injective. This follows from Theorem 6.1, and the commutativity of the following diagram

$$\begin{tikzcd} H_3(\mathrm{SL}_2) \arrow[rr] \arrow[dr] & & H_3(\mathrm{SL}_3) \arrow[dl] \\ & H_3(\mathrm{SL}). & \end{tikzcd}$$

This generalizes the main theorem of Sah in [17, Theorem 3.0].

Let $K_3^M(R) \to K_3(R)$ be the natural map from the Milnor K-group to the Quillen K-group. Define $K_3(R)^{\mathrm{ind}} := \mathrm{coker}(K_3^M(R) \to K_3(R))$. This group is called the indecomposable part of $K_3(R)$.

Proposition 6.4. *Let k be a field such that $1/2 \in k$.*

(i) $K_3(R)^{\mathrm{ind}} \otimes k \simeq H_3(\mathrm{SL}_2, k)_{R^*}$.
(ii) *If R is an infinite field, then* $K_3(R)^{\mathrm{ind}} \otimes \mathbb{Z}[\frac{1}{2}] \simeq H_3(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}])_{R^*}$.
(iii) *If R is either $\mathbb{R}$, or an infinite field such that $R^* = R^{*2}$, then* $K_3(R)^{\mathrm{ind}} \simeq H_3(\mathrm{SL}_2)$.

Proof. Let $A = \mathbb{Z}[\frac{1}{2}]$, $\mathbb{Z}$ or k. By Corollary 6.2, we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K_3^M(R) \otimes A \arrow[r] \arrow[d] & K_3(R) \otimes A \arrow[r] \arrow[d, "h_3"] & K_3(R)^{\mathrm{ind}} \otimes A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_3^M(R) \otimes A \arrow[r] & H_3(\mathrm{SL}, A) \arrow[r] & H_3(\mathrm{SL}_2, A)_{R^*} \arrow[r] & 0. \end{tikzcd}$$

Here h_3 is the Hurewicz map $K_3(R) = \pi_3(B\,\mathrm{SL}^+) \to H_3(\mathrm{SL})$ and it is surjective with two torsion kernel [17, Proposition 2.5]. In case $R^* = R^{*2}$, h_3 is an isomorphism. The snake lemma implies (i), (ii) and the second part of (iii). If $R=\mathbb{R}$, we look at the following commutative diagram

$$\begin{tikzcd}
 & K_3^M(\mathbb{R}) \arrow[r] \arrow[d] & K_3(\mathbb{R}) \arrow[r] \arrow[d, "h_3"] & K_3(\mathbb{R})^{\mathrm{ind}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_3^M(\mathbb{R})^{\circ} \arrow[r] & H_3(\mathrm{SL}(\mathbb{R})) \arrow[r] & H_3(\mathrm{SL}_2(\mathbb{R})) \arrow[r] & 0.
\end{tikzcd}$$

The claim follows from the snake lemma using the fact that $\ker(K_3(\mathbb{R}) \xrightarrow{h_3} H_3(\mathrm{SL}(\mathbb{R}))) = \mathbb{Z}/2\mathbb{Z}$ [17, 2.17]. □

Remark 6.5. Theorem 6.4 generalizes Theorem 4.1 in [17], where three torsion is not treated.

We can offer the following non-commutative version of the above results.

Proposition 6.6.

(i) *Let R be a quaternion algebra. Then*

$$0 \to H_3\left(\mathrm{SL}_2, \mathbb{Z}\left[\frac{1}{2}\right]\right)_{R^*} \to H_3\left(\mathrm{SL}, \mathbb{Z}\left[\frac{1}{2}\right]\right) \to K_3^M(R) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to 0$$

is exact.

(ii) *If R is an Azumaya R-algebra, where R is a commutative local ring with an infinite residue field, then*

$$0 \to H_3(\mathrm{SL}_2, \mathbb{Q})_{R^*} \to H_3(\mathrm{SL}, \mathbb{Q}) \to K_3^M(R) \otimes \mathbb{Q} \to 0$$

is exact.

Proof. (i) From the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{SL}_2 \arrow[r] \arrow[d] & \mathrm{GL}_2 \arrow[r] \arrow[d] & K_1(R) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{SL} \arrow[r] & \mathrm{GL} \arrow[r] & K_1(R) \arrow[r] & 1,
\end{tikzcd}$$

we obtain a map of spectral sequences

$$\begin{tikzcd}
E^2_{p,q} = H_p(K_1(R), H_q(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}])) \arrow[r, Rightarrow] \arrow[d] & H_{p+q}(\mathrm{GL}_2, \mathbb{Z}[\frac{1}{2}]) \arrow[d] \\
E'^2_{p,q} = H_p(K_1(R), H_q(\mathrm{SL}, \mathbb{Z}[\frac{1}{2}])) \arrow[r, Rightarrow] & H_{p+q}(\mathrm{GL}, \mathbb{Z}[\frac{1}{2}]).
\end{tikzcd}$$

Since the map $Z(R)^* \times \mathrm{SL}_2 \to \mathrm{GL}_2$, $(a, B) \mapsto aB$, has two torsion kernel and cokernel (use Example 5.2), $H_i(\mathrm{SL}_2, \mathbb{Z}[\frac{1}{2}])_{R^*} \hookrightarrow H_i(\mathrm{GL}_2, \mathbb{Z}[\frac{1}{2}])$ (see the proof of Theorem 6.1(i)). By Lemma 5.1, $H_i(Z(R)^*, \mathbb{Z}[\frac{1}{2}]) \hookrightarrow H_i(\mathrm{GL}_2, \mathbb{Z}[\frac{1}{2}])$, and it is easy to prove the injectivity of $H_i(\mathrm{SL}, \mathbb{Z}[\frac{1}{2}]) \hookrightarrow H_i(\mathrm{GL}, \mathbb{Z}[\frac{1}{2}])$. By an easy analysis of the above spectral sequences, as in the proof of Corollary 6.2, we get the desired result. The proof of (ii) is similar. □

Corollary 6.7. *Let D be a finite-dimensional F-division algebra. Let*

$$K_3^M(F, D) := \ker\big(K_3^M(F) \to K_3^M(D)\big).$$

Then we have the following exact sequence

$$0 \to H_3\big(\mathrm{SL}_2(F), \mathbb{Q}\big)_{F^*} \to H_3\big(\mathrm{SL}_2(D), \mathbb{Q}\big)_{D^*} \to K_3^M(F, D) \otimes \mathbb{Q} \to 0.$$

Proof. By Corollary 2.3 from [9], $K_3(F) \otimes \mathbb{Q} \simeq K_3(D) \otimes \mathbb{Q}$. Therefore,

$$H_3\big(\mathrm{SL}(F), \mathbb{Q}\big) \simeq H_3\big(\mathrm{SL}(D), \mathbb{Q}\big)$$

(see [17, Theorem 2.5]). Now the claim follows from Corollary 6.2 and Proposition 6.6. □

Acknowledgments

I would like to thank W. van der Kallen for his interest in this work and for his valuable comments. The last draft of this article was written during my stay at mathematics department of Queen's University Belfast which I was supported by EPSRC R1724PMR. I would like to thank them for their support and hospitality.

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