On Fuzzy Syntopogenous Structures

A. K. Katsaras and C. G. Petalas

Department of Mathematics, University of Ioannina, Ioannina, Greece

Submitted by L. Zadeh

1. Introduction

In [15] the authors, in their attempt to find a unified theory of fuzzy topologies, fuzzy proximities, and fuzzy uniformities, introduced the fuzzy syntopogenous structures. The concept of a fuzzy syntopogenous structure on a set \( X \) is based on the basic term of order on the family of all fuzzy sets in \( X \). As shown in [15], the fuzzy topologies, the fuzzy proximities, and the fuzzy uniformities are special cases of these structures. In this paper we continue with the investigation of fuzzy syntopogenous structures. We study in particular initial fuzzy syntopogenous structures, products and subspaces of fuzzy syntopogenous structures, and initial fuzzy proximities. We also introduce the concept of continuity of a function between two fuzzy syntopogenous structures and show that such a function is necessarily continuous with respect to the corresponding fuzzy topologies.

2. Preliminaries

In this section we recall some of the definitions related to fuzzy topologies, fuzzy proximities, fuzzy uniformities, and fuzzy syntopogenous structures.

A fuzzy set, in a set \( X \), is an element of the set \( I^X \) of all functions \( \mu \) from \( X \) to the unit interval \( I \). If \( f \) is a function from \( X \) to \( Y \) and \( \mu \in I^X \), then \( f^{-1}(\mu) \) is the fuzzy set in \( X \) defined by \( f^{-1}(\mu)(x) = \mu(f(x)) \). Also, for \( \sigma \in I^X \), \( f(\sigma) \) is the fuzzy set in \( Y \) defined by \( f(\sigma)(y) = 0 \) if \( y \notin f(X) \) and \( f(\sigma)(y) = \sup\{\sigma(x) : x \in f^{-1}[y]\} \) if \( y \in f(X) \). A fuzzy topology \( \tau \) in \( X \) is a subset of \( I^X \) containing the constant fuzzy sets \( 0 \) and \( 1 \) and closed under finite infima and arbitrary suprema. If \( \tau \) is a fuzzy topology on \( X \), then the pair \((X, \tau)\) is called a fuzzy topological space and the members of \( \tau \) are the open fuzzy sets of this space. A fuzzy set \( \mu \) is closed if \( 1 - \mu \) is open. The interior \( \mu^0 \) of a fuzzy set \( \mu \) is the largest open fuzzy set contained in \( \mu \) while the closure \( \bar{\mu} \) of \( \mu \) is the smallest closed fuzzy set containing \( \mu \). We have that \( \mu^0 = 1 - \overline{1 - \mu} \). A function \( f \).
A mapping \( f \) from a fuzzy topological space \( X \) to another one \( Y \) is called continuous if \( f^{-1}(\mu) \) is open in \( X \) for each open fuzzy set \( \mu \) in \( Y \).

A fuzzy proximity on \( X \) (see [12]) is a binary relation \( \delta \) on \( I^X \) which satisfies the following axioms:

1. (FP1) \( \mu \delta \rho \) implies \( \rho \delta \mu \).
2. (FP2) \( (\mu \vee \rho) \delta \sigma \) iff \( \mu \in \sigma \) or \( \rho \delta \sigma \).
3. (FP3) \( \mu \delta \rho \) implies that \( \mu \neq 0 \) and \( \rho \neq 0 \).
4. (FP4) \( \mu \delta \rho \) (\( \delta \) is the negation of \( \delta \)) implies that \( \mu \leq 1 - \rho \).
5. (FP5) \( \mu \delta \rho \) implies the existence of a \( \sigma \in I^X \) such that \( \mu \delta \sigma \) and \( (1 - \sigma) \delta \rho \).

The pair \((X, \delta)\) is called a fuzzy proximity space. To every fuzzy proximity \( \delta \) corresponds a fuzzy topology \( \tau(\delta) \) given by the closure operator \( \mu \rightarrow \bar{\mu} \) on \( I^X \), where \( \bar{\mu} = 1 - \sup \{ \rho : \rho \delta \mu \} \). A mapping \( f \), from a fuzzy proximity space \((X, \delta_X)\) to another one \((Y, \delta_Y)\), is called a proximity mapping or a proximally continuous mapping if \( \mu \delta_X \rho \) implies \( \delta_Y f(\mu) \delta_Y f(\rho) \). Equivalently, \( f \) is a proximity mapping if \( \mu \delta_X \rho \) in \( Y \) implies that \( f^{-1}(\mu) \delta_Y f^{-1}(\rho) \) in \( X \). If \( \mu \delta_X \rho \) implies \( \mu \delta_Y \rho \), then \( \delta_Y \) is said to be finer than \( \delta_X \).

A binary relation \( \ll \) on \( I^X \) is called a fuzzy semi-topogenous order if it satisfies the following axioms:

1. (1) \( 0 \ll 0 \) and \( 1 \ll 1 \),
2. (2) \( \mu \ll \rho \) implies \( \mu \leq \rho \).
3. (3) \( \mu_1 \ll \mu_2 \ll \rho_1 \ll \rho_2 \) implies \( \mu_1 \ll \rho_1 \ll \rho_2 \).

The complement of a fuzzy semi-topogenous order \( \ll \) is the fuzzy semi-topogenous order \( \ll^c \) which is defined by \( \mu \ll^c \rho \) iff \( 1 - \rho \ll 1 - \mu \).

A fuzzy semi-topogenous order \( \ll \) is called:

1. (i) symmetrical if \( \ll = \ll^c \),
2. (ii) topogenous if \( \mu_1 \ll \rho_1 \) and \( \mu_2 \ll \rho_2 \) imply \( \mu_1 \lor \mu_2 \ll \rho_1 \lor \rho_2 \) and \( \mu_1 \land \mu_2 \ll \rho_1 \land \rho_2 \),
3. (iii) perfect if \( \mu_j \ll \rho_j, j \in J \), implies \( \sup \mu_j \ll \sup \rho_j \),
4. (iv) biperfect if \( \mu_j \ll \rho_j, j \in J \), implies \( \inf \mu_j \ll \inf \rho_j \) and \( \sup \mu_j \ll \sup \rho_j \).

A fuzzy semi-topogenous order \( \ll_1 \) is called finer than another one \( \ll_2 \) if \( \mu \ll_2 \rho \) implies \( \mu \ll_1 \rho \). In this case we also say that \( \ll_2 \) is coarser than \( \ll_1 \). Given a fuzzy topogenous order \( \ll \) on \( X \) there exists a perfect fuzzy topogenous order \( \ll^p \) finer than \( \ll \) and coarser than any perfect fuzzy semi-topogenous order on \( X \) which is finer than \( \ll \). It is defined by: \( \mu \ll^p \rho \) iff there is a family \( \{ \mu_\alpha : \alpha \in A \} \) of fuzzy sets such that \( \mu = \sup \mu_\alpha \) and \( \mu_\alpha \ll \rho \) for each \( \alpha \in A \). Similarly, given a fuzzy semi-topogenous order \( \ll \) on \( X \) there
exists a biperfect fuzzy topogenous order \( \ll^b \) finer than \( \ll \) and coarser than any biperfect fuzzy topogenous order on \( X \) which is finer than \( \ll \). It is defined by \( \mu \ll^b \rho \) iff there are families \( \{\mu_\alpha : \alpha \in A\}, \{\rho_\beta : \beta \in B\} \) of fuzzy sets such that \( \mu = \sup \mu_\alpha, \rho = \inf \rho_\beta \) and \( \mu_\alpha \ll \rho_\beta \) for all \( \alpha \in A \) and all \( \beta \in B \).

A fuzzy syntopogenous structure on \( X \) is a nonempty family \( S \) of fuzzy topogenous orders on \( X \) having the following two properties:

1. (FS1) \( S \) is directed in the sense that given any two members of \( S \) there exists a member of \( S \) finer than both.
2. (FS2) For each \( \ll \) in \( S \) there exists \( \ll_1 \) in \( S \) such that \( \ll \ll_1 \mu \) implies the existence of a fuzzy set \( \sigma \) with \( \mu \ll_1 \sigma \ll_1 \rho \).

If \( S \) is a fuzzy syntopogenous structure on \( X \), then the pair \((X, \delta)\) is called a fuzzy syntopogenous space. A fuzzy syntopogenous structure \( S \) consisting of a single topogenous order is called a topogenous structure and the pair \((X, S)\) a fuzzy topogenous space. \( S \) is called perfect (resp. biperfect) if each member of \( S \) is perfect (resp. biperfect). A fuzzy syntopogenous structure \( S_1 \) is called finer than another one \( S_2 \) if for each \( \ll \) in \( S_2 \) there exists a member of \( S_1 \) finer than \( \ll \). In this case we also say that \( S_2 \) is coarser than \( S_1 \). If \( S_1 \) is finer than \( S_2 \) and \( S_2 \) finer than \( S_1 \), then \( S_1, S_2 \) are called equivalent. To every fuzzy syntopogenous structure corresponds a fuzzy topology \( \tau(S) \) given by the interior operator

\[
\mu^0 = \sup \{\rho : \rho \ll \mu \text{ for some } \ll \in S\}.
\]

If \( \{\ll_\alpha : \alpha \in A\} \) is a family of fuzzy semi-topogenous order on \( X \), then \( \ll = \bigcup_{\alpha \in A} \ll_\alpha \) is the fuzzy semi-topogenous order defined by: \( \mu \ll \rho \) iff \( \mu \ll_\alpha \rho \) for some \( \alpha \in A \). If \( S \) is a fuzzy syntopogenous structure, then it is easy to see that \( \ll_S = \bigcup \{\ll : \ll \in S\} \) is a fuzzy topogenous order and that \( \{\ll_S\} \) is a topogenous structure. Moreover, \( \mu \in \tau(S) \) iff \( \mu \ll_\mu^2 \mu \). To every fuzzy topology \( \tau \) on \( X \) corresponds a perfect fuzzy topogenous structure \( S = \{\ll\} \), where \( \mu \ll \rho \) iff there exists \( \sigma \in \tau \) with \( \mu \ll \sigma \ll \rho \). Moreover, \( \tau = \tau(S) \). Conversely, to every perfect fuzzy topogenous structure \( S = \{\ll\} \) corresponds the fuzzy topology \( \tau = \tau(S) \), where \( \mu \in \tau \) iff \( \mu \ll \mu \). To two different fuzzy topologies correspond different perfect fuzzy topogenous structures. Similarly, to every symmetrical fuzzy topogenous structure \( S = \{\ll\} \) on \( X \) corresponds a fuzzy proximity \( \delta = \delta(\ll) \), defined by \( \mu \ll_\delta \rho \) iff \( \mu \ll 1 - \rho \). Moreover, the mapping \( \{\ll\} \to \delta(\ll) \), from the family of all symmetrical fuzzy topogenous structures on \( X \) to the family of all fuzzy proximities on \( X \), is one-to-one and onto.

Now let \( \Omega_x \) denote the family of all functions \( \alpha : I^X \to I^X \) with the following properties:

1. \( \alpha(0) = 0 \) and \( \mu \ll \alpha(\mu) \) for all \( \mu \in I^X \),
2. \( \alpha(\sup \mu_i) = \sup \alpha(\mu_i) \),
For $\alpha \in \Omega_X$, the function $\alpha^{-1} : I^X \to I^X$ defined by

$$\alpha^{-1}(\rho) = \inf\{\rho : \alpha(1 - \rho) \leq 1 - \mu\}$$

belongs to $\Omega_X$ (see [7]). A base for a fuzzy quasi-uniformity on $X$ is a nonempty subset $B$ of $\Omega_X$ having the following two properties:

(FU1) Given $\alpha_1, \alpha_2 \in B$ there exists $\alpha \in B$ with $\alpha \leq \alpha_1, \alpha_2$.

(FU2) Given $\alpha \in B$ there exists $\alpha_i \in B$ such that $\alpha_i \circ \alpha_i \leq \alpha$.

A base for a fuzzy uniformity on $X$ is a base $B$ for a fuzzy quasi-uniformity which has also the following property:

(FU3) For each $\alpha \in B$ there exists $\alpha_1 \in B$ with $\alpha_1 \leq \alpha^{-1}$.

$B$ is called a fuzzy quasi-uniformity (resp. uniformity) if $B$ is a base for a fuzzy quasi-uniformity (resp. uniformity) such that if $\alpha_i \in B$ and $\alpha \in \Omega_X$ with $\alpha \geq \alpha_i$, then $\alpha \in B$.

To every biperfect fuzzy topogenous order $\preceq$ on $X$ corresponds an element $\alpha = \alpha_{\preceq} \in \Omega_X$ defined by

$$\alpha(\rho) = \inf\{\rho : \mu \preceq \rho\}.$$ 

Thus we get a mapping $\omega = \omega_X$, $\omega(\preceq) = \alpha_{\preceq}$, from the set $\Omega_X$ of all biperfect fuzzy topogenous orders on $X$ to the set $\Omega_X$. Some of the properties of the mapping $\omega$ are the following:

1. $\omega$ is one-to-one and into.
2. $\omega(\preceq^c) = [\omega(\preceq)]^{-1}$.
3. $\preceq_1$ is finer than $\preceq_2$ iff $\omega(\preceq_1) \leq \omega(\preceq_2)$.
4. If $\alpha = \omega(\preceq)$, then $\mu \preceq \rho$ iff $\alpha(\mu) \leq \rho$.
5. Let $\alpha = \omega(\preceq)$ and $\alpha_i = \omega(\preceq_i)$, $i = 1, 2$. Then $\alpha_1 \circ \alpha_2 \leq \alpha$ iff the following condition is satisfied: If $\mu \preceq \rho$, then there exists $\sigma \in I^X$ with $\mu \preceq \sigma \preceq \rho$.

If $S$ is a biperfect fuzzy syntopogenous structure on $X$, then the family $\omega(S) = \{\omega(\preceq) : \preceq \in S\}$ is a base for a fuzzy quasi-uniformity on $X$. Conversely, if $B$ is a base for a fuzzy quasi-uniformly on $X$, then $\omega^{-1}(B)$ is a biperfect fuzzy syntopogenous structure on $X$.

If $S_1$ and $S_2$ are biperfect fuzzy syntopogenous structures on $X$, then $S_1$ is equivalent to $S_2$ iff $\omega(S_1)$ and $\omega(S_2)$ generate the same quasi-uniformity. Moreover, $\omega(S)$ is a base for a fuzzy uniformity iff for every $\preceq \in S$ there exists $\preceq_s \in S$ finer than $\preceq^c$. Finally, to every base $B$ for a fuzzy quasi-uniformity corresponds a fuzzy topology $\tau(B)$ given by the interior operator

$$\mu^0 = \sup\{\rho : \alpha(\rho) \leq \mu \text{ for some } \alpha \in B\}.$$
Clearly, if $S = \omega^{-1}(\mathcal{B})$ is the corresponding biperfect fuzzy syntopogenous structure, then $\tau(S) = \tau(\mathcal{B})$.

3. Operations on Fuzzy Syntopogenous Structures

**Theorem 3.1.** If $\preceq$ is a fuzzy semi-topogenous order on $X$, then there exists a fuzzy topogenous order $\preceq^q$ on $X$ finer than $\preceq$ and coarser than all fuzzy topogenous orders on $X$ which are finer than $\preceq$. This fuzzy topogenous order can be defined as follows:

For $\mu, \rho \in I^X$, we have $\mu \preceq^q \rho$ iff there are natural numbers $m, n$ and fuzzy sets $\mu, \rho$, $i = 1, \ldots, m$, and $j = 1, \ldots, n$, such that $\mu = \bigvee_{i=1}^{m} \mu_i$, $\rho = \bigwedge_{j=1}^{n} \rho_j$ and $\mu_i \preceq \rho_j$ for all $i = 1, \ldots, m$ and all $j = 1, \ldots, n$.

**Proof.** It is easy to see that $\preceq^q$, as defined in the statement of the theorem, is a fuzzy semi-topogenous order on $X$. To show that $\preceq^q$ is topogenous, let $\mu \preceq^q \rho$ and $\mu' \preceq^q \rho'$. There are fuzzy sets $\mu, \mu', \rho, \rho'$, $k = 1, \ldots, m'$, $j = 1, \ldots, n$, and $\rho'_s$, $s = 1, \ldots, n'$, such that

$$
\mu = \bigvee_{i=1}^{m} \mu_i, \quad \mu' = \bigvee_{k=1}^{m'} \mu'_k, \quad \rho = \bigwedge_{j=1}^{n} \rho_j, \quad \rho' = \bigwedge_{s=1}^{n'} \rho'_s,
$$

$\mu_i \preceq \rho_j$ and $\mu'_k \preceq \rho'_s$. We have

$$
\mu \land \mu' = \sup\{\mu_i \land \mu'_k : 1 \leq i \leq m, 1 \leq k \leq m'\}
$$

and

$$
\rho \land \rho' = \bigwedge_{t=1}^{n+n'} \sigma_t,
$$

where

$$
\sigma_t = \rho_t \quad \text{if} \quad 1 \leq t \leq n,
$$

$$
= \rho'_{t-n} \quad \text{if} \quad n < t \leq n + n'.
$$

For $1 \leq t \leq n$ we have $\mu_i \land \mu'_k \preceq \mu_j \preceq \rho_t = \sigma_t$ and thus $\mu_i \land \mu'_k \preceq \sigma_t$. If $n < t \leq n + n'$, then $\mu_i \land \mu'_k \preceq \mu'_k \preceq \rho'_{t-n} = \sigma_t$ and so we have again that $\mu_i \land \mu'_k \preceq \sigma_t$. From the definition of $\preceq^q$, it follows that $\mu \land \mu' \preceq^q \rho \land \rho'$. Analогously we show that $\mu \lor \mu' \preceq^q \rho \lor \rho'$ and so $\preceq^q$ is a fuzzy topogenous order. It is clear that $\mu \preceq \rho$ implies that $\mu \preceq^q \rho$ and hence $\preceq^q$ is finer than $\preceq$. Finally let $\preceq'$ be a fuzzy topogenous order on $X$ finer than $\preceq$ and let $\mu = \bigvee_{i=1}^{m} \mu_i$, $\rho = \bigwedge_{j=1}^{n} \rho_j$, $\mu_i \preceq \rho_j$. Since $\preceq'$ is finer than $\preceq'$, we have that $\mu_i \preceq' \rho_j$. Since $\preceq'$ is topogenous, we have $\mu = \bigvee_{i=1}^{m} \mu_i \preceq' \rho_j$ for each $j$. 

409-991 1 15
\[ j = 1, \ldots, n, \text{ and hence } \mu \leq \' \land \rho_j = \rho. \] This proves that \( \leq' \) is coarser than \( \leq' \) and the proof is complete.

**Corollary 3.2.** If \( \leq \) is a fuzzy semi-topogenous order on \( X \), then:

(a) \( \leq \) is topogenous iff \( \leq = \leq^q \).

(b) \( \leq^q = \leq^q \).

Proposition 3.3 gives another description of the fuzzy topogenous order \( \leq^q \).

**Proposition 3.3.** For \( \leq \) a fuzzy semi-topogenous order on \( X \), \( \mu \leq^q \rho \) holds iff there are natural numbers \( m \) and \( n_i, i = 1, \ldots, m \), and fuzzy sets \( \mu_i, \rho_i, \rho_{ij}, i = 1, \ldots, m, j = 1, \ldots, n_i \), such that

\[
\begin{align*}
\mu &= \bigvee_{i=1}^{m} \mu_i, \\
\rho &= \bigvee_{i=1}^{m} \rho_i, \\
\mu_i &= \bigwedge_{j=1}^{n_i} \mu_{ij}, \\
\rho_i &= \bigwedge_{j=1}^{n_i} \rho_{ij}, \\
\mu_{ij} &\leq \rho_{ij}
\end{align*}
\]

for \( i = 1, \ldots, m; j = 1, \ldots, n_i \).

**Proof.** If \( \mu_i, \rho_i, \mu_{ij}, \) and \( \rho_{ij} \) are as in the statement of the proposition, then putting \( Z_i = \{1, 2, \ldots, n_i\}, Z = \prod_{i=1}^{m} Z_i \) and, for \( t \in Z \), \( \sigma_t = \bigvee_{i=1}^{m} \rho_{it} \), we have

\[
\rho = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n_i} \rho_{ij} = \bigwedge_{t \in Z} \sigma_t.
\]

For \( i = 1, \ldots, m \) and \( t \in Z \), we have

\[
\mu_i \leq \mu_{it} \leq \rho_{it} \leq \sigma_t,
\]

and thus \( \mu_i \leq \sigma_t \). From the definition of \( \leq^q \), it follows that \( \mu \leq^q \rho \). Conversely, let \( \mu \leq^q \rho \) and let \( \mu_i, i = 1, \ldots, m, \rho_j, j = 1, \ldots, n \), be fuzzy sets such that

\[
\begin{align*}
\mu &= \bigvee_{i=1}^{m} \mu_i, \\
\rho &= \bigwedge_{j=1}^{n} \rho_j, \\
\mu_i &\leq \rho_j.
\end{align*}
\]

Take \( n_i = n, \mu_{ij} = \mu_i (1 \leq j \leq n), \rho_{ij} = \rho_j (1 \leq i \leq m) \), \( \sigma_i = \rho_j (i = 1, \ldots, m) \). We have \( \mu = \bigvee_{i=1}^{m} \mu_i, \rho = \bigvee_{i=1}^{m} \sigma_i, \mu_i = \bigwedge_{j=1}^{n_i} \mu_{ij}, \sigma_i = \bigwedge_{j=1}^{n_i} \rho_{ij}, \) and \( \mu_{ij} \leq \rho_{ij} \). The result follows.

We omit the proof of the easily established.
**Proposition 3.4.** Let $\preceq$ be a fuzzy semi-topogenous order. Then:

1. $\preceq \circ \preceq = \preceq \circ \preceq$.
2. If $\preceq$ is symmetrical, so is $\preceq$.

Now let $\preceq_1$ and $\preceq_2$ be fuzzy semi-topogenous orders on a set $X$. The composition $\preceq = \preceq_1 \circ \preceq_2$ is defined by

$$\mu \preceq p \iff \text{there exists } \sigma \in I^X \text{ such that } \mu \preceq_1 \sigma \preceq_1 p.$$ 

It is easy to see that $\preceq$ is a fuzzy semi-topogenous order on $X$. For a fuzzy semi-topogenous order $\preceq$, we will usually write $\preceq^2$ for the composition $\preceq \circ \preceq$.

We have the easily established

**Proposition 3.5.** Let $\preceq_1$, $\preceq_2$, be fuzzy semi-topogenous orders on $X$ and $\preceq = \preceq_1 \circ \preceq_2$. Then:

1. $\preceq^q$ is coarser than $\preceq_1 \circ \preceq_2^q$.
2. $\preceq^a$ (resp. $\preceq^b$) is coarser than $\preceq_1^a \circ \preceq_2^a$ (resp. $\preceq_1^b \circ \preceq_2^b$).
3. If both $\preceq_1$ and $\preceq_2$ are topogenous, so is $\preceq$.
4. If $\preceq_1$ and $\preceq_2$ are both perfect (resp. biperfect), then $\preceq$ is perfect (resp. biperfect).
5. $\preceq^c = \preceq_1 \circ \preceq_2^c$.

**Proposition 3.6.** Let $S$ be a fuzzy syntopogenous structure on $X$ and let $S^p = \{ \preceq^p : \preceq \in S \}$ and $S^b = \{ \preceq^b : \preceq \in S \}$. Then $S^p$ (resp. $S^b$) is a perfect (resp. biperfect) fuzzy syntopogenous structure on $X$ finer than $S$ and coarser than any perfect (resp. biperfect) fuzzy syntopogenous structure which is finer than $S$.

**Proof:** Since $\preceq_1$ coarser than $\preceq_2$ implies that $\preceq_1^p$ (resp. $\preceq_2^b$) is coarser than $\preceq_2^p$ (resp. $\preceq_1^b$), it follows easily, using Proposition 3.5(2), that $S^p$ and $S^b$ are fuzzy syntopogenous structures which are finer than $S$ since $\preceq^p$ and $\preceq^b$ are both finer than $\preceq$. If $S_1$ is a perfect (resp. biperfect) fuzzy syntopogenous structure finer than $S$, then given $\preceq \in S$, there exists $\preceq_1 \in S_1$ finer than $\preceq$. Since $\preceq_1$ is perfect (resp. biperfect), it follows that $\preceq^p$ (resp. $\preceq^b$) is coarser than $\preceq_1$, which proves that $S_1$ is finer than $S^p$ (resp. $S^b$).

Now let $(S_A)_{A \in \Lambda}$ be a family of fuzzy syntopogenous structures on $X$. Consider the family $S$ of all fuzzy topogenous orders of the form $S_k = \bigcup_{k=1}^n \preceq_{\lambda_k}$, where $\lambda_k \in \Lambda$ and $\preceq_{\lambda_k} \in S_{\lambda_k}$. It is easy to see that $S$ is a fuzzy syntopogenous structure on $X$ finer than each $S_A$ and coarser than every fuzzy syntopogenous structure on $X$ which is finer than each $S_A$. We will
call $S$ the supremum of the family $\{S_\alpha\}_{\alpha \in \Lambda}$ and we will denote it by $\bigvee_{\alpha \in \Lambda} S_\alpha$.

**Theorem 3.7.** Let $(S_\alpha)_{\alpha \in \Lambda}$ be a family of fuzzy syntopogenous structures on $X$ and let $S = \bigvee_{\alpha \in \Lambda} S_\alpha$. Then $\tau(S) = \sup_{\alpha \in \Lambda} \tau(S_\alpha)$.

**Proof:** Let $\tau_0 = \tau(S)$, $\tau_\alpha = \tau(S_\alpha)$, and $\tau = \sup \tau_\alpha$. Since each $S_\alpha$ is coarser than $S$, the topology $\tau_\alpha$ is coarser than $\tau_0$ and hence $\tau$ is coarser than $\tau_0$. Now let $\mu$ be $\tau_0$-open and let $x_0 \in X$. If $\mu(x_0) > \theta$, there exist $\rho \in \mathcal{I}^X$ and $\rho \in S$ with $\rho \ll \mu$ and $\rho(x_0) > \theta$. Let $\rho^{1_k} \in S_\alpha$, $k = 1, \ldots, n$, such that

$$
\rho^{1_k} = \bigcup_{k=1}^n \rho^{1_k}.
$$

Let $\rho = \bigcup_{k=1}^n \rho^{1_k}$. From the definition of $\rho^{1_k}$, there fuzzy sets $\rho_i$, $\mu_j$, $i = 1, \ldots, m$, $j = 1, \ldots, N$, such that $\rho = \bigvee_{i=1}^m \rho_i$, $\mu = \bigwedge_{j=1}^N \mu_j$, $\rho_i \ll \mu_j$. Since $\rho(x_0) > \theta$, there exists an $i$ with $\rho_i(x_0) > \theta$. For a fixed $j$, $1 \leq j \leq N$, there exists some $k$ such that $\rho_i \ll \rho^{1_k}_j$. It follows that $\rho_i$ is contained in the $\tau_0$-interior of $\mu_j$ and hence in the $\tau$-interior of $\mu_j$. Therefore $\rho_i \ll \bigwedge_{j=1}^n \mu^{0r}_j = \mu^{0r}$. Thus $\mu^{0r}(x_0) \geq \rho_i(x_0) > \theta$. Since this is true for each $\rho(x_0) > \theta$, we have that $\mu(x_0) = \mu^{0r}(x_0)$. Thus $\mu = \mu^{0r}$ and so $\mu$ is $\tau$-open. This completes the proof.

**4. Inverse Image of a Fuzzy Semitopogenous Order**

Let $f$ be a function from a set $X$ to a set $Y$ and let $\ll$ be a fuzzy semitopogenous order on $Y$. Define a binary relation $\ll_1$ on $I^X$ by

$$
\mu \ll_1 \rho \iff f(\mu) \ll f(1 - f(1 - \rho)).
$$

It is easy to see that $\ll_1$ is a fuzzy semi-topogenous order on $X$. We will call $\ll_1$ the inverse image of $\ll$ by the mapping $f$ and we will denote it by $f^{-1}(\ll)$.

**Proposition 4.1.** Let $f : X \rightarrow Y$ be a function and let $\ll$, $\ll'$ be fuzzy semitopogenous orders on $Y$. Then:

1. For $\mu, \rho \in \mathcal{I}^Y$, we have $\mu^{-1}(\ll) \rho$ iff there are fuzzy sets $\mu_i$, $\rho_i$ in $Y$ such that $\mu_i \ll \rho_i$, $\mu \ll f^{-1}(\mu_i)$ and $f^{-1}(\rho_i) \ll \rho$.

2. $f^{-1}(\ll)$ is the coarsest fuzzy semi-topogenous order $\ll_1$ on $X$ for which $\mu' \ll \rho'$ in $I^X$ implies that $f^{-1}(\mu') \ll f^{-1}(\rho')$.

3. If $\ll$ is coarser than $\ll'$, then $f^{-1}(\ll)$ is coarser than $f^{-1}(\ll')$. If we assume in addition that $f$ is onto, then the converse is also true.
(4) If $\{\preceq: \lambda \in \Lambda\}$ is a family of fuzzy semi-topogenous orders on $Y$, then
\[ f^{-1} \bigcup_{\lambda \in \Lambda} \preceq_{\lambda} = \bigcup_{\lambda \in \Lambda} f^{-1}(\preceq_{\lambda}). \]

(5) $[f^{-1}(\preceq)]^q = f^{-1}(\preceq^q)$.

(6) $[f^{-1}(\preceq)]^p = f^{-1}(\preceq^p)$ and $[f^{-1}(\preceq)]^b = f^{-1}(\preceq^b)$.

(7) $[f^{-1}(\preceq)]^c = f^{-1}(\preceq^c)$.

(8) If $\preceq$ is topogenous, so is $f^{-1}(\preceq)$.

(9) If $\preceq$ is perfect (resp. biperfect), then $f^{-1}(\preceq)$ is perfect (resp. biperfect).

(10) If $\preceq$ is symmetrical, then $f^{-1}(\preceq)$ is also symmetrical.

(11) Let $\preceq_1 = f^{-1}(\preceq)$, $\preceq_2 = f^{-1}(\preceq')$ and $\preceq_3 = f^{-1}(\preceq_0 \preceq')$. Then $\preceq_3$ is coarser than $\preceq_1 \preceq_2$. If $f$ is onto, then $\preceq_3 = \preceq_1 \preceq_2$.

**Proof.** (1) If $\mu f^{-1}(\preceq) \rho$, then $f(\mu) \preceq 1 - f(1 - \rho)$. Putting $\mu_1 = f(\mu)$ and $\rho_1 = 1 - f(1 - \rho)$, we have $\mu_1 \preceq \rho_1$, $\mu_1 \preceq f^{-1}(\mu_1)$ and $\rho_1 \preceq f^{-1}(\rho_1)$. Conversely, suppose that there exist $\mu_1, \rho_1$ in $I^Y$ such that $\mu \preceq f^{-1}(\mu_1)$, $f^{-1}(\rho_1) \preceq \rho$ and $\mu_1 \preceq \rho_1$. Since $f(\mu) \preceq f(f^{-1}(\mu_1)) \preceq \mu_1 \preceq \rho_1 \preceq 1 - f(1 - f^{-1}(\rho_1)) \preceq 1 - f(1 - \rho)$, we have $f(\mu) \preceq 1 - f(1 - \rho)$ and so $\mu f^{-1}(\preceq) \rho$.

(2), (4) and the first part of (3) follow easily from (1). To prove the converse in (3), suppose that $f$ is onto and that $f^{-1}(\preceq)$ is coarser than $f^{-1}(\preceq')$. Set $\preceq_1 = f^{-1}(\preceq)$ and $\preceq'_1 = f^{-1}(\preceq')$. Let $\mu', \rho'$ in $I^Y$ with $\mu' \preceq \rho'$. Then $f^{-1}(\mu') \preceq f^{-1}(\rho')$ and so $f^{-1}(\mu') \preceq f^{-1}(\rho')$. Hence $f(f^{-1}(\mu')) = 1 - f(1 - f^{-1}(\rho'))$. Since $f$ is onto, we have $f(f^{-1}(\mu')) = \mu'$ and $f(1 - f^{-1}(\rho')) = 1 - \rho'$. Therefore $\mu' \preceq \rho'$.

(5) Set $\preceq = f^{-1}(\preceq)$ and $\preceq' = f^{-1}(\preceq')$. If $\mu \preceq \rho$, then there exist $\mu', \rho'$ in $I^Y$ with $\mu' \preceq f^{-1}(\mu')$ and $f^{-1}(\rho') \preceq \rho$. By the definition of $\preceq$, there are fuzzy sets $\mu_1', \mu_2', \ldots, \mu_m', \rho_1', \ldots, \rho_n'$ in $Y$ with $\mu' = \bigvee_{i=1}^m \mu_i'$, $\rho' = \bigwedge_{j=1}^n \rho_j'$, $\mu_i' \preceq \rho_j'$. Set $\mu_i = f^{-1}(\mu_i')$ and $\rho_j = f^{-1}(\rho_j')$. Then $f^{-1}(\mu') = \bigvee_{i=1}^m \mu_i$, $f^{-1}(\rho') = \bigwedge_{j=1}^n \rho_j$ and $\mu_i \preceq \rho_j$. Hence
\[ \mu \preceq f^{-1}(\mu') \preceq f^{-1}(\rho') \preceq \rho \]
and so $\mu \preceq_1 \rho$. Analogously, we prove that $\mu \preceq_1 \rho$ implies that $\mu \preceq_2 \rho$.

(6) For the proof we use the definitions of $\preceq^p$ and $\preceq^b$ and an argument analogous to that of (5).

(7) It is a direct consequence of the definitions.

(8), (9), and (10) follow from (5), (6), and (7), respectively.

(11) Let $\mu, \rho \in I^Y$ with $\mu \preceq \rho$. There are $\mu_1, \rho_1$ in $I^Y$ with $\mu_1 \preceq 0 \preceq \rho_1$, $\mu \preceq f^{-1}(\mu_1)$ and $\rho \preceq f^{-1}(\rho_1)$. Let $\sigma_1 \in I^Y$ be such that $\mu_1 \preceq \sigma_1 \preceq \rho_1$. If $\sigma = f^{-1}(\sigma_1)$, then $f^{-1}(\mu_1) \preceq_2 \sigma \preceq f^{-1}(\rho_1)$ and hence $f^{-1}(\mu_1) \preceq_1 0 \preceq_2 f^{-1}(\rho_1)$. 
Therefore, \( \mu \preceq_1 0 \preceq_2 \rho \) since \( \mu \preceq f^{-1}(\mu_1) \) and \( \rho \geq f^{-1}(\rho_1) \). Finally, suppose that \( f \) is onto and let \( \mu \preceq_2 \sigma \preceq_1 \rho \). Then
\[
f(\mu) \preceq f(1 - \sigma) \quad \text{and} \quad f(\sigma) \preceq 1 - f(1 - \rho).
\]
Since \( f \) is onto, we have that \( 1 - f(1 - \sigma) \preceq f(\sigma) \). Hence \( f(\mu) \preceq f(\sigma) \preceq 1 - f(1 - \rho) \) which implies that \( f(\mu) \preceq 0 \preceq 1 - f(1 - \rho) \) and so \( \mu \preceq_1 \rho \).

We also have the easily established

**Proposition 4.2.** Let \( f : X \to Y \) and \( g : Y \to Z \) be functions and let \( \preceq \) be a fuzzy semi-topogenous order on \( Z \). Then \( (g \circ f)^{-1}(\preceq) = f^{-1}(g^{-1}(\preceq)) \).

5. **Inverse Image of a Fuzzy Syntopogenous Structure**

**Proposition 5.1.** Let \( f : X \to Y \) be a function and let \( S \) be a fuzzy syntopogenous structure on \( Y \). Then:

1. \( f^{-1}(S) = \{ f^{-1}(\preceq) : \preceq \in S \} \) is a fuzzy syntopogenous structure on \( X \).
2. If \( S \) is perfect, biperfect or symmetrical, then \( f^{-1}(S) \) is perfect, biperfect or symmetrical, respectively.

**Proof.** It follows easily from Proposition 4.1.

**Proposition 5.2.** If \( (S_\lambda)_{\lambda \in \Lambda} \) is a family of fuzzy syntopogenous structures on \( Y \) and \( f : X \to Y \) a mapping, then \( f^{-1}(\bigvee_{\lambda \in \Lambda} S_\lambda) = \bigvee_{\lambda \in \Lambda} f^{-1}(S_\lambda) \).

**Proof.** If \( \preceq_k, k = 1, \ldots, n \), are fuzzy semi-topogenous orders on \( Y \), then
\[
f^{-1}\left(\bigcup_{k=1}^{n} \preceq_k\right)^q = \left[f^{-1}\left(\bigcup_{k=1}^{n} \preceq_k\right)^q\right] = \left[\bigcup_{k=1}^{n} f^{-1}(\preceq_k)^q\right]
\]
by proposition 4.1. Hence the result follows directly from the definitions.

**Proposition 5.3.** Let \( f : X \to Y \) be a function, \( S \) a fuzzy syntopogenous structure on \( Y \), \( \tau = \tau(S) \), \( \tau_1 = \tau(f^{-1}(S)) \) and \( \tau_2 = f^{-1}(\tau) \). Then \( \tau_1 = \tau_2 \).

Set \( \preceq_S = \bigcup \{ \preceq : \preceq \in S \} \) and \( \preceq = \bigcup \{ f^{-1}(\preceq) : \preceq \in S \} = f^{-1}(\preceq_S) \). Then \( \mu \in \tau_1 \) iff \( \mu \preceq_1 \mu \). Similarly, \( \mu \in \tau \) iff \( \mu \preceq_\tau \mu \). Also, \( \preceq_2 = f^{-1}(\preceq_S) \) by Proposition 4.1. If now \( \mu' \in \tau \), then \( \mu' \preceq_2 \mu' \) and so \( f^{-1}(\mu') \preceq_2 f^{-1}(\mu') \) which implies that \( f^{-1}(\mu') \in \tau_1 \). Conversely, if \( \mu \in \tau_1 \), then \( \mu \preceq_1 \mu \) and so
f(\mu) \leq \frac{\alpha}{\beta} 1 - f(1 - \mu). \text{ Hence, there exists } \sigma \in \tau \text{ with } f(\mu) \leq \sigma \leq 1 - f(1 - \mu). \text{ Thus }

\mu \leq f^{-1}(f(\mu)) \leq f^{-1}(\sigma) \leq f^{-1}(1 - f(1 - \mu)) \leq \mu

and so \mu = f^{-1}(\sigma) \in \tau_2.

**Proposition 5.4.** Let \(\{(Y_\lambda, S_\lambda) : \lambda \in \Lambda\}\) be a family of fuzzy syntopenous spaces, \(X\) a set and, for each \(\lambda \in \Lambda, f_\lambda : X \to Y_\lambda\) a function. If \(S = \bigvee_{\lambda \in \Lambda} f^{-1}(S_\lambda)\), then \(\tau(S)\) is the weakest fuzzy topology \(\tau\) on \(X\) for which each \(f_\lambda : (X, \tau) \to (Y_\lambda, \tau(S_\lambda))\) is continuous.

**Proof:** It follows from Proposition 5.3 and from Theorem 3.7.

Now let \(f\) be a function from \(X\) to \(Y\) and let \(\beta \in \Omega_Y\). The function \(\alpha = f^{-1} \circ \beta \circ f (a(\mu)(x)) = \beta(f(\mu))(f(x))\) for each \(\mu \in I^X\) and each \(x \in X\) belongs to \(\Omega_X\). We will call \(\alpha\) the inverse image of \(\beta\) by the function \(f\) and we will denote it by \(f^{-1}(\beta)\).

**Proposition 5.5.** Let \(f : X \to Y\) be a function and let \(\beta, \beta_1, \beta_2 \in \Omega_Y\). Then:

1. \(\beta_1 \leq \beta_2\) implies that \(f^{-1}(\beta_1) \leq f^{-1}(\beta_2)\).
2. If \(\alpha_1 = f^{-1}(\beta_1)\), \(\alpha_2 = f^{-1}(\beta_2)\) and \(\alpha = f^{-1}(\beta_1 \circ \beta_2)\), then \(\alpha_1 \circ \alpha_2 \leq \alpha\). If \(f\) is onto, then \(\alpha = \alpha_1 \circ \alpha_2\).
3. \(f^{-1}(\beta^{-1}) = [f^{-1}(\beta)]^{-1}\).

**Proof:**

1. It is clear from the definitions.
2. If \(\mu \in I^X\), then \(\alpha_1(\mu) = f^{-1}[\beta_2(f(\mu))]\) and hence \(f(\alpha_1(\mu)) = f[f^{-1}[\beta_2(f(\mu))]] \leq \beta_2(f(\mu))\). Since \(\beta_1\) is increasing, we have \(\beta_1[f(\alpha_2(\mu))] \leq (\beta_1 \circ \beta_2)(f(\mu))\). Hence

\[
(\alpha_1 \circ \alpha_2)(\mu) = \alpha_1(\alpha_2(\mu)) = f^{-1}[\beta_2(f(\alpha_2(\mu)))] \\
\leq f^{-1}[(\beta_1 \circ \beta_2)(f(\mu))] = \alpha(\mu).
\]

If \(f\) is onto, then \(f(\alpha_2(\mu)) = \beta_2(f(\mu))\) from which follows that \((\alpha_1 \circ \alpha_2)(\mu) = \alpha(\mu)\).
3. We have

\[
f^{-1}(\beta^{-1})(\mu)(x) = \inf\{\rho'(f(x)) : \beta(1 - \rho') \leq 1 - f(\mu)\}
\]

and

\[
[f^{-1}(\beta)]^{-1}(\mu)(x) = \inf\{\rho(x) : f^{-1}(\beta)(1 - \rho) \leq 1 - \mu\}.
\]
Now let \( \rho' \in I' \) with \( \beta(1 - \rho') \leq 1 - f(\mu) \) and take \( \rho = f^{-1}(\rho') \). Since \( f(1 - \rho) \leq 1 - \rho' \), we have
\[
f^{-1}(\beta)(1 - \rho)(x) = \beta(f(1 - \rho))(f(x)) \leq \beta(1 - \rho')(f(x)) \leq |1 - f(\mu)|(f(x)) \leq 1 - \mu(x).\]
Thus \( f^{-1}(\beta)(1 - \rho) \leq 1 - \mu \) which implies that
\[
f^{-1}(\beta^{-1})(\mu) \geq |f^{-1}(\beta)|^{-1}(\mu).\]

For the converse inequality, let \( \rho \in I^x \) with \( f^{-1}(\beta)(1 - \rho) \leq 1 - \mu \). Let \( y \in Y \). If \( y = f(x) \), then
\[
1 - \mu(x) \geq f^{-1}(\beta)(1 - \rho)(x) = \beta(f(1 - \rho))(f(x)) = \beta(f(1 - \rho))(y).
\]
Thus, for \( y \in f(X) \), we have
\[
|1 - f(\mu)|(y) = \inf \{ 1 - \mu(x) : f(x) = y \} \geq \beta(f(1 - \rho))(y).
\]
Also, for \( y \notin f(X) \), \( |1 - f(\mu)|(y) = 1 \geq \beta(f(1 - \rho))(y) \). Thus, taking \( \rho' = 1 - f(1 - \rho) \), we have \( 1 - f(\mu) \geq \beta(1 - \rho') \) and so \( f^{-1}(\beta^{-1})(\mu) \leq f^{-1}(\rho') \leq \rho \). This proves that
\[
f^{-1}(\beta^{-1})(\mu) \leq |f^{-1}(\beta)|^{-1}(\mu)
\]
and the result follows.

**Proposition 5.6.** Let \( f : X \to Y \) be a function, \( \omega = \omega_X : O_Y \to \Omega_Y \), \( \omega' = \omega_Y : O_Y \to \Omega_Y \) and \( \mu \in O_Y \). Then \( \omega(f^{-1}(\mu)) = f^{-1}(\omega(\mu)) \).

**Proof.** Let \( \mu \in I^x \). It is easy to that we have
\[
\omega(f^{-1}(\mu))(\mu) = \inf \{ \rho : \rho \in I^x \text{ and } \mu f^{-1}(\mu) \rho \}
\]
and
\[
f^{-1}(\omega'(\mu))(\mu) = \inf \{ f^{-1}(\rho') : \rho' \in I^y \text{ and } f(\mu) \leq \rho' \}.
\]
If \( f(\mu) \leq \rho' \) and \( \rho = f^{-1}(\rho') \), then \( \mu f^{-1}(\mu) \rho \) since \( \mu \leq f^{-1}(f(\mu)) \). Conversely, let \( \rho \in I^x \) with \( \mu f^{-1}(\mu) \rho \). There are \( \mu' \), \( \rho' \in I^y \) with \( \mu' \leq \rho' \), \( \mu \leq f^{-1}(\mu') \) and \( \rho \geq f^{-1}(\rho') \). Since \( f(\mu) \leq f(f^{-1}(\mu')) \leq \mu' \leq \rho' \), we have \( f(\mu) \leq \rho' \). Thus given \( \rho \) with \( \mu f^{-1}(\mu) \rho \) there exists \( \rho' \in I^y \) with \( f^{-1}(\rho') \leq \rho \) and \( f(\mu) \leq \rho' \). Combining the preceding remarks we get the result.

Using Propositions 5.5 and 5.6 we get

**Proposition 5.7.** Let \( f \) be a function from \( X \) to \( Y \) nd let \( B \subset \Omega_Y \). Then:
(1) If \( B \) is a base for a fuzzy quasi-uniformity (resp. uniformity), then \( f^{-1}(B) = \{ f^{-1}(\beta) : \beta \in B \} \) is a base for a fuzzy quasi-uniformity (resp. uniformity).

(2) If \( B \) is a base for a fuzzy quasi-uniformity and \( S \) the corresponding biperfect fuzzy syntopogenous structure, then \( f^{-1}(S) \) is the biperfect fuzzy syntopogenous structure which corresponds to \( f^{-1}(B) \).

### 6. Continuity

**Definition 6.1.** Let \( S, S' \) be fuzzy syntopogenous structures on \( X, Y \), respectively, and let \( f \) be a function from \( X \) to \( Y \). Then \( f \) is said to be \((S, S')\)-continuous if \( f^{-1}(S') \) is coarser than \( S \), i.e. for each \( \ll' \in S' \) there exists \( \ll \in S \) finer than \( f^{-1}(\ll') \).

We have the easily established

**Proposition 6.2.** Let \( \{ (Y_{\lambda}, S_{\lambda}) : \lambda \in \Lambda \} \) be a family of fuzzy syntopogenous spaces, \( X \) a set and, for each \( \lambda \in \Lambda \), \( f_{\lambda} \) a function from \( X \) to \( Y_{\lambda} \). If \( S = \bigvee_{\lambda \in \Lambda} f_{\lambda}^{-1}(S_{\lambda}) \), then each \( f_{\lambda} \) is \((S, S_{\lambda})\)-continuous. Moreover, \( S \) is coarser than any fuzzy syntopogenous structure \( S' \) on \( X \) for which each \( f_{\lambda} \) is \((S', S_{\lambda})\)-continuous.

**Proposition 6.3.** Let \( (X, S_{1}), (Y, S_{2}), \) and \( (Z, S_{3}) \) be fuzzy syntopogenous spaces and let \( f : X \to Y \) be \((S_{1}, S_{2})\)-continuous and \( g : Y \to Z \) an \((S_{2}, S_{3})\)-continuous function. Then \( g \circ f : X \to Z \) is \((S_{1}, S_{3})\)-continuous.

**Proof.** It follows from the equality

\[
(g \circ f)^{-1}(S_{3}) = f^{-1}(g^{-1}(S_{3}))
\]

**Proposition 6.4.** Let \( X, Y_{\lambda}, S_{\lambda}, f_{\lambda}, \lambda \in \Lambda \), and \( S \) be as in Proposition 6.2 and let \( (Y, S') \) be a fuzzy syntopogenous space. Then, a function \( g : Y \to X \) is \((S', S)\)-continuous iff each \( f_{\lambda} \circ g : Y \to Y_{\lambda} \) is \((S', S_{\lambda})\)-continuous.

**Proof.** The necessity follows from Proposition 6.3. Conversely, suppose that each \( f_{\lambda} \circ g \) is \((S', S_{\lambda})\)-continuous and let \( \ll \in S \). By the definition of \( S \), there are \( \lambda_{1}, \ldots, \lambda_{n} \in \Lambda \) and \( \ll_{k_{1}}^{1}, \ldots, \ll_{k_{n}}^{1} \in S_{\lambda_{k}}, k = 1, \ldots, n \), such that \( \ll = \bigcup_{k=1}^{n} f_{\lambda_{k}}^{-1}(\ll_{k_{1}}^{1}) \). Since each \( f_{\lambda_{k}} \circ g \) is \((S', S_{\lambda_{k}})\)-continuous and since \( S' \) is directed, there exists \( \ll' \in S' \) finer than each \( f_{\lambda_{k}}^{-1}(\ll_{k_{1}}^{1}) \). Since

\[
g^{-1}(\ll) = \left[ \bigcup_{k=1}^{n} (f_{\lambda_{k}} \circ g)^{-1}(\ll_{k_{1}}^{1}) \right]^{q},
\]

it follows that \( g^{-1}(\ll) \) is coarser than \( \ll' \). This completes the proof.
Proposition 6.5 tells us that continuity of a function $f$ with respect to fuzzy syntopogenous structures implies the continuity of $f$ with respect to the corresponding fuzzy topologies.

**PROPOSITION 6.5.** Let $S$, $S'$ be fuzzy syntopogenous structures on $X$ and $Y$, respectively, and let $f : X \to Y$ be a function. If $f$ is $(S, S')$-continuous, then $f$ is continuous with respect to the corresponding fuzzy topologies.

**Proof:** Let $\preceq_1 = \bigcup \{\preceq : \preceq \in S\}$, $\preceq_2 = f^{-1}(\preceq_1)$ and $\preceq_3 = \bigcup \{\preceq : \preceq \in S\}$. Since $S$ is finer than $f^{-1}(S')$, it is clear that $\preceq_2$ is finer than $\preceq_1$. Hence $\preceq_2$ is finer than $\preceq_3 = f^{-1}(\preceq_1)$. If now $\mu \in \tau(S')$, then $\mu \preceq_3 \mu$ and so $f^{-1}(\mu) \preceq_3 f^{-1}(\mu)$. Thus $f^{-1}(\mu) \preceq_3 f^{-1}(\mu)$ which implies that $f^{-1}(\mu) \in \tau(S)$.

Let now $\preceq$ be a fuzzy semi-topogenous order on a set $X$ and let $Y$ be a subset of $X$. The restriction of $\preceq$ to $Y$ is defined to be the fuzzy semi-topogenous order $\preceq_1 = f^{-1}(\preceq)$, where $f$ is the canonical mapping of $Y$ into $X$. We will denote $\preceq_1$ by $\preceq|_Y$. It is easy to see that, for $\mu, \rho \in I^X$, we have $\mu \preceq|_Y \rho$ iff $\mu_1 \preceq \rho_1$, where $\mu_1, \rho_1 \in I^Y$ are defined by $\mu_1(x) = \mu(x)$ and $\rho_1(x) = \rho(x)$ if $x \in Y$ while for $x \notin Y$ we have $\mu_1(x) = 0$ and $\rho_1(x) = 1$.

**DEFINITION 6.6.** Let $S$ be a fuzzy syntopogenous structure on $X$ and let $Y \subseteq X$. Then the restriction of $S$ to $Y$ is the fuzzy syntopogenous structure

$$S|_Y = \{\preceq|_Y : \preceq \in S\}.$$ 

Using Proposition 6.4, we get easily

**PROPOSITION 6.7.** Let $(X, S)$ and $(Y, S')$ be fuzzy syntopogenous spaces and let $f$ be a function from $X$ to $Y$. Then $f$ is $(S, S')$-continuous iff $f : X \to f(X)$ is $(S, S'|_Y)$-continuous.

7. **PRODUCT OF FUZZY SYNTOPOGENOUS SPACES**

**DEFINITION 7.1.** Let $\{(X_\lambda, S_\lambda) : \lambda \in \Lambda\}$ be a family of fuzzy syntopogenous spaces and let $X = \prod_{\lambda \in \Lambda} X_\lambda$. If $\pi_\lambda$ denotes the canonical projection of $X$ into $X_\lambda$, then the fuzzy syntopogenous structure $\bigvee_{\lambda \in \Lambda} \pi_\lambda^{-1}(S_\lambda)$ is called the product of the family of fuzzy syntopogenous structures $(S_\lambda)_{\lambda \in \Lambda}$ and it is denoted by $\prod_{\lambda \in \Lambda} S_\lambda$. The set $X$ equipped with the product fuzzy syntopogenous structure is called the product of the family $\{(X_\lambda, S_\lambda) : \lambda \in \Lambda\}$.

Using Propositions 5.4 and 6.4, we get
PROPOSITION 7.2. Let \( X, X_\lambda, S_\lambda, \lambda \in \Lambda \), be as in Definition 7.1. Then:

1. The fuzzy topology which corresponds to the product fuzzy syntopogenous structure \( S = \prod_{\lambda \in \Lambda} S_\lambda \) is the product of the fuzzy topologies \( \tau(S_\lambda), \lambda \in \Lambda \).

2. If \( g \) is a function from a fuzzy syntopogenous space \((Y, S')\) to \( X\), then \( g \) is \((S', S)\)-continuous iff each \( \pi_\lambda \circ g \) is \((S', S_\lambda)\)-continuous.

8. Initial Fuzzy Proximities

Let \( f \) be a function from \( X \) to \( Y \) and let \( \delta \) be a fuzzy proximity on \( X \). To \( \delta \) corresponds (see [15]) a unique symmetrical topogenous structure \( S = \{ \langle \rangle \} \) where \( \mu \preceq \rho \iff \mu \tilde{\delta}(1 - \rho) \). If \( \langle \rangle = f^{-1}(\langle \rangle) \), then \( f^{-1}(S) = \{ \langle \rangle \} \) is a symmetrical topogenous structure on \( X \). Let \( \delta_i \) be the fuzzy proximity on \( X \) which corresponds to \( f^{-1}(S) \). We will call \( \delta_i \) the inverse image by \( f \) of the fuzzy proximity \( \delta \) and we will denote it by \( f^{-1}(\delta) \). For \( \mu, \rho \in I^X \), we have

\[
\mu \tilde{\delta}_i \rho \iff \mu \preceq 1 - \rho \iff f(\mu) \preceq 1 - f(\rho) \iff f(\mu) \tilde{\delta} f(\rho).
\]

Thus \( \mu \tilde{\delta}_i \rho \iff f(\mu) \tilde{\delta} f(\rho) \). It follows that the mapping \( f : (X, f^{-1}(\delta)) \rightarrow (Y, \delta) \) is a proximity mapping and clearly \( f^{-1}(\delta) \) is the coarsest fuzzy proximity on \( X \) for which \( f \) is a proximity mapping.

Summarizing the above remarks, we have

PROPOSITION 8.1. If \( f : X \rightarrow Y \) is a function and \( \delta \) a fuzzy proximity on \( Y \), then the binary relation \( f^{-1}(\delta) \) on \( I^X \) defined by

\[
\mu f^{-1}(\delta) \rho \iff f(\mu) \tilde{\delta} f(\rho),
\]

is a fuzzy proximity on \( X \). Moreover, \( f^{-1}(\delta) \) is the coarsest of all fuzzy proximities \( \delta_i \) on \( X \) for which \( f \) is \((\delta_i, \delta)\)-proximally continuous.

Suppose next that \((X_\alpha, \delta_\alpha)_{\alpha \in \Lambda} \) is a family of fuzzy proximity spaces, \( X \) a set and for each \( \alpha \in \Lambda \) let \( f_\alpha : X \rightarrow X_\alpha \) be a mapping. For each \( \alpha \in \Lambda \), let \( S_\alpha = \{ \langle \alpha \rangle \} \) be the symmetrical topogenous structure which corresponds to \( \delta_\alpha \). Let \( S = \bigvee_{\alpha \in \Lambda} f_\alpha^{-1}(S_\alpha) \). Since each \( f_\alpha^{-1}(S_\alpha) \) is symmetrical, \( S \) is a symmetrical fuzzy syntopogenous space. Let \( \langle \rangle_S = \{ \langle \rangle : \langle \rangle \in S \} \). It is easy to see that \( S_\alpha = \{ \langle \alpha \rangle \} \) is a symmetrical fuzzy topogenous structure. Let \( \delta_0 \) be the fuzzy proximity which corresponds to \( S_0 \). We will obtain a direct construction of \( \delta_0 \) by means of the given fuzzy proximities \( \delta_\alpha \). To this end, we observe that \( \mu \delta_0 \rho \) is equivalent to the existence of \( \alpha_1, \ldots, \alpha_n \in \Lambda \) such that

\[
\mu \left[ \bigcup_{k=1}^n f_{\alpha_k}^{-1}(\langle \alpha_k \rangle) \right] (1 - \rho).
\]
By Theorem 3.1, (*) is equivalent to the existence of fuzzy sets \( \mu_1, \ldots, \mu_m, \sigma_1, \ldots, \sigma_N \) such that \( \mu = \bigvee_{i=1}^m \mu_i \), \( 1 - \rho = \bigwedge_{j=1}^N \sigma_j \) and for each pair \((i,j)\) there exists \( k = k(i,j), 1 \leq k \leq n \), such that

\[
\mu_i \delta_{a_k}^{-1} (\leq_{a_k}) \sigma_j.
\]

But (**) means that \( f_{a_k}^\rho (\mu_i) \leq_{a_k} 1 - f_{a_k}^\rho (1 - \sigma_j) \) or equivalently \( f_{a_k}^\rho (\mu_i) \delta_{a_k} f_{a_k} (1 - \sigma_j) \). We also have \( \rho = \bigvee_{j=1}^n (1 - \sigma_j) \). Combining the preceding remarks we see easily that \( \mu \delta_\rho \delta_0 \) iff there are fuzzy sets \( \mu_1, \ldots, \mu_m, \rho_1, \ldots, \rho_N \) such that \( \mu = \bigvee_{i=1}^m \mu_i, \rho = \bigwedge_{j=1}^N \rho_j \) and for each pair \((i,j)\) there exists an \( a \in A \) such that \( f_{a_k}^\rho (\mu_i) \delta_{a_k} f_{a_k} (\rho_j) \). Next we see that each \( f_a \) is \( (\delta_0, \delta_a) \)-proximally continuous. In fact, if \( \mu, \rho \in \mathcal{F}^X \) with \( \mu \delta_0 \rho, \rho \), then \( f_a^{-1}(\mu) \delta_0 f_a^{-1}(\rho) \) since \( f_a(f_a^{-1}(\mu)) \leq \mu \) and \( f_a(f_a^{-1}(\rho)) \leq \rho \). Let now \( \delta_1 \) be a fuzzy proximity on \( X \) such that each \( f_a \) is \( (\delta_1, \delta_0) \)-proximally continuous. We will show that \( \delta_1 \) is finer than \( \delta_0 \). So let \( \mu \delta_1 \rho \). We need to show that \( \mu \delta_0 \rho \). In fact, if \( \mu = \bigvee_{i=1}^m \mu_i, \rho = \bigwedge_{j=1}^n \rho_j, \) then there exist \( i, j \) such that \( \mu_i, \delta_1, \rho_j \). Since each \( f_a \) is \( (\delta_1, \delta_0) \)-proximally continuous, we have \( f_a(\mu_i) \delta_0 f_a(\rho_j) \) for each \( a \in A \). This proves that \( \mu \delta_0 \rho \).

Finally, let \( (Y, \delta_1) \) be a fuzzy proximity space and let \( f : (Y, \delta_1) \rightarrow (X, \delta_0) \). If \( f \) is a proximity mapping, then each \( h_a = f_a \circ f \) is \( (\delta_1, \delta_0) \)-proximally continuous. Conversely, suppose that each \( h_a \) is proximally continuous and let \( \mu \delta_0 \rho \). We will show that \( f(\mu) \delta_0 f(\rho) \). In fact, let \( f(\mu) = \bigvee_{i=1}^m \mu_i, f(\rho) = \bigwedge_{j=1}^n \rho_j, \). Let \( \mu_0 = \bigvee_{i=1}^m f_a^{-1}(\mu_i) \geq \mu \) and \( \rho_0 = \bigwedge_{j=1}^n f_a^{-1}(\rho_j) \geq \rho \). Since \( \mu_0, \rho_0 \), we have \( \mu_0 \delta_1 \rho_0 \). Hence there are \( i, j \) such that \( f_a^{-1}(\mu_i) \delta_1 f_a^{-1}(\rho_j) \). Since \( h_a \) is \( (\delta_1, \delta_0) \)-proximally continuous and since \( h_a(f_a^{-1}(\mu_i)) \leq f_a(\mu_i) \) and \( h_a(f_a^{-1}(\rho_j)) \leq f_a(\rho_j) \), we have \( f_a(\mu_i) \delta_0 f_a(\rho_j) \) for all \( a \in A \). This proves that \( f(\mu) \delta_0 f(\rho) \). Hence \( f : (Y, \delta_1) \rightarrow (X, \delta_0) \) is proximally continuous.

Summarizing, we have

**Theorem 8.2.** Let \( \{(X_a, \delta_a) : a \in A\} \) be a nonempty family of fuzzy proximity spaces, \( X \) a set and for each \( a \in A \) let \( f_a : X \rightarrow X_a \) be a mapping. Define a binary relation \( \delta \) on \( X \) by \( \mu \delta_0 \rho \) iff the following condition is satisfied: If \( \mu = \bigvee_{i=1}^m \mu_i \) and \( \rho = \bigwedge_{j=1}^n \rho_j \), then there are \( i, j \) such that \( f_a(\mu_i) \delta_a f_a(\rho_j) \) for all \( a \in A \). Then:

1. \( \delta \) is the coarsest fuzzy proximity on \( X \) with respect to which each \( f_a \) is a proximity mapping.
2. A mapping \( f \) from a fuzzy proximity space \( (Y, \delta_1) \) to \( (X, \delta) \) is proximally continuous iff each \( f_a \circ f : (Y, \delta_1) \rightarrow (X_a, \delta_a) \) is proximally continuous.

Let now \( \delta \) be a fuzzy proximity on \( X \) and let \( S = \{ \leq \} \) be the...
corresponding symmetrical topogenous structure. Let $\tau = \tau(\delta)$ be the fuzzy topology which correspond to $\delta$. The $\tau$-interior of a fuzzy set $\mu$ is given by

$$\mu^0 = 1 - \overline{1 - \mu} = \sup \{ \rho : \rho \delta(1 - \mu) \} = \sup \{ \rho : \rho \ll \mu \}$$

Thus $\tau(\delta) = \tau(S)$.

**Theorem 8.3.** Under the hypotheses of Theorem 8.2, the fuzzy topology $\tau(\delta)$ is the coarsest of all fuzzy topologies $\tau$ on $X$ for which each $f_a : (X, \tau) \to (X_a, \tau(\delta_a))$ is continuous.

**Proof.** For each $a \in A$, let $S_a = \{ \ll_a \}$ be the symmetric fuzzy topogenous structure on $X_a$ which corresponds to $\delta_a$ and let $S = \bigvee_{a \in A} f_a^{-1}(S_a)$. If $\ll_0 = \bigcup \{ \ll : \ll \in S \}$, then $S_0 = \{ \ll_0 \}$ is the symmetric fuzzy topogenous structure on $X$ which corresponds to $\delta$ and thus $\tau(\delta) = \tau(S_0)$. Since $\tau(S) = \tau(S_0)$, the result follows from Theorem 3.7.

### 9. Product of Fuzzy Proximity Spaces

**Definition 9.1.** Let $\{(X_a, \delta_a) : a \in A \}$ be a nonempty family of fuzzy proximity spaces and let $X = \bigcap_{a \in A} X_a$. Then, the product fuzzy proximity $\prod_{a \in A} \delta_a$ on $X$ is defined to be the coarsest fuzzy proximity on $X$ with respect to which each canonical projection $\pi_a : X \to X_a$ is a proximity mapping.

Using Theorems 8.2 and 8.3, we get

**Theorem 9.2.** (a) If $\delta = \prod_{a \in A} \delta_a$, then $\mu \delta \rho$ iff the following condition is satisfied: If $\mu - \bigvee_{i=1}^m \mu_i$ and $\rho - \bigvee_{j=1}^n \rho_j$, then there are $i, j$ such that $\pi_a(\mu_i) \delta_a \pi_a(\rho_j)$ for all $a \in A$.

(b) A function $f$ from a fuzzy proximity space $(Y, \delta_1)$ to $(X, \delta)$ is proximally continuous iff each $\pi_a \circ f$ is $(\delta_1, \delta_a)$-proximally continuous.

(c) $\tau(\delta) = \prod_{a \in A} \tau(\delta_a)$.

**References**