Weighted Composition Operators on Hardy Spaces

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Submitted by William F. Ames
Received October 10, 2000

Let \( \varphi, \psi \) be analytic functions defined on \( \mathbb{D} \), such that \( \varphi(\mathbb{D}) \subseteq \mathbb{D} \). The operator given by \( f \mapsto \psi(f \circ \varphi) \) is called a weighted composition operator. In this paper we deal with the boundedness, compactness, weak compactness, and complete continuity of weighted composition operators on Hardy spaces \( H_p \). In particular, we prove that such an operator is compact on \( H_1 \) if and only if it is weakly compact on this space. This result depends on a technique which passes the weak compactness from an operator \( T \) to operators dominated in norm by \( T \).

Key Words: weighted composition operators; Hardy spaces; compact operators; weakly compact operators; completely continuous operators.

1. INTRODUCTION

When does a weighted composition operator map the Hardy space \( H_p \) into itself? A weighted composition operator \( W_{\varphi, \psi} \) is an operator that maps \( f \in \mathcal{H}(\mathbb{D}), \) the space of holomorphic functions on the unit disk \( \mathbb{D}, \) into \( W_{\varphi, \psi}(f)(z) = \psi(z)f(\varphi(z)) \), where \( \varphi \) and \( \psi \) are analytic functions defined in \( \mathbb{D} \) such that \( \varphi(\mathbb{D}) \subseteq \mathbb{D} \). These operators turn up in a natural way. For example, de Leeuw showed that the isometries between the Hardy space \( H_1 \) are weighted composition operators, and Forelli obtained the same result for the Hardy spaces \( H_p \) when \( 1 < p < \infty \), \( p \neq 2 \) (see [7, 9]).

When \( \psi = 1 \), we just have the composition operator \( C_{\varphi} \) defined by \( C_{\varphi}(f) = f \circ \varphi \). In this case, Littlewood’s subordination theorem says that

1 This research has been partially supported by DGESIC project PB97-0706 and by La Consejería de Educación y Ciencia de la Junta de Andalucía.
$C_\phi(f) \in H_p$ whenever $f \in H_p$; that is, $C_\phi: H_p \to H_p$ is a continuous linear map for $1 \leq p < \infty$ [3, Corollary 2.24]. The situation is really different when we consider weighted composition operators $W_{\psi, \phi}$ on $H_p$. It is easy to find examples where $W_{\psi, \phi}(H_p) \not\subseteq H_p$. In Section 2, we characterize the boundedness of $W_{\psi, \phi}$ from $H_p$ into $H_p$.

Once the problem of boundedness is solved, one of the most interesting problems is to analyze the compactness. In Sections 3 and 4, we tackle this problem from three points of view: we study the cases where $W_{\psi, \phi}$ is compact, weakly compact, and completely continuous on $H_p$. Let us recall that an operator $T$ from a Banach space $X$ into another Banach space $Y$ is said to be compact if $T$ maps bounded subsets into relatively norm compact sets; $T$ is said to be weakly compact if it maps bounded subsets into relatively weakly compact sets; and $T$ is said to be completely continuous (or Dunford–Pettis) if it maps weakly compact subsets into compact sets. It is well known that if $T$ is compact, then it is weakly compact and completely continuous. The other implications are not true in general.

In [13], D. Sarason proved that if a composition operator is weakly compact on $H_1$, then it is, in fact, a compact operator on this space. In Theorem 3.4, we prove that $W_{\psi, \phi}$ is compact on $H_1$ if and only if it is weakly compact on this space. Our proof uses ideas different from Sarason’s. Whereas his proof uses the duality between $H_1$ and $BMOA$ and the boundedness of $C_\phi$ as an operator on $L_1(\mathbb{T}, m)$, we use a well-known characterization of weakly compact sets in $L_1(\mathbb{T})$ and a result which passes the weak compactness from one operator $T$ to another operator $S$ such that $\|Sx\| \leq \|Tx\|$ for all $x$ (see Proposition 3.2). A similar result for compact operators is well known and can be seen, for example, in [5, p. 5]. We think that our proof is more elementary. It is worth mentioning that in [2] the first named author and Díaz-Madrigal proved that $W_{\psi, \phi}$ is compact on $H_1$ if and only if it is weakly compact on it.

In Section 3, we also characterize the case where $W_{\psi, \phi}$ is compact on $H_p$ ($1 < p < \infty$). Note that when $1 < p < \infty$, $W_{\psi, \phi}$ is always weakly compact on $H_p$.

Although the classes of completely continuous and compact weighted composition operators agree on $H_p$ for $1 < p \leq \infty$ (this result is obvious for $1 < p < \infty$, and it can be seen in [2] for $p = \infty$), they are not the same on $H_1$. This was pointed out for composition operators by Cima and Matheson [1] by showing that the composition operator $C_\phi$, with $\phi(z) = z(z - 1)$, is completely continuous on $H_1$, but it is not compact. In Section 4, we study the case where $W_{\psi, \phi}$ is completely continuous on $H_1$. For composition operators this result was obtained by Cima and Matheson [1].

In what follows we denote by $\mathbb{T}$ the unit circle, by $m$ the normalized Lebesgue measure on $\mathbb{T}$, and by $\|f\|_p$ the usual norm of a function.
2. BOUNDEDNESS

In this section we characterize the boundedness of $W_{\varphi, \psi}$ on $H_p$ in terms of a Carleson measure criterion. This criterion has been used to characterize boundedness of composition operators in different papers (see, for example, [10, 11]).

**Definition 2.1.** A positive measure $\mu$ on $\mathbb{D}$ is called a Carleson measure (in $\mathbb{D}$) if there is a constant $M < \infty$ such that $\mu(S(b, r)) \leq Mr$ for all $b \in \mathbb{T}$ and $0 < r < 1$, where $S(b, r) = \{z \in \mathbb{D} : |z - b| \leq r\}$.

Most of the information we are going to obtain about weighted composition operators will be given in terms of a certain measure, which we turn to next. Given an analytic function $\varphi$ of the unit disk into itself, it is well known from Fatou’s theorem that the radial limits $\lim_{r \to 1^-} \varphi(re^{it})$ exist almost everywhere. So, we can consider $\varphi$ as a function belonging to $L_p(\mathbb{T}, m)$. Thus, taking $\psi \in H_p$, we can define the measure $\mu_{\varphi, \psi, p}$ on $\mathbb{D}$ by

$$\mu_{\varphi, \psi, p}(E) := \int_{\varphi^{-1}(E) \cap \mathbb{T}} |\psi|^p \, dm,$$

where $E$ is a measurable subset of the unit closed disk $\overline{\mathbb{D}}$.

The next lemma will be crucial in what follows. In fact, it is a slight generalization of [8, p. 163].

**Lemma 2.1.** Fixing $1 \leq p < \infty$ and given $\varphi, \psi \in H_p$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, we have

$$\int_{\mathbb{D}} gd\mu_{\varphi, \psi, p} = \int_{\mathbb{T}} |\psi|^p (g \circ \varphi) \, dm,$$

where $g$ is an arbitrary measurable positive function in $\mathbb{D}$.

**Proof.** If $g$ is a measurable simple function defined on $\mathbb{D}$ given by $g = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, we have that

$$\int_{\mathbb{D}} gd\mu_{\varphi, \psi, p} = \sum_{i=1}^{n} \alpha_i \mu_{\varphi, \psi, p}(E_i) = \sum_{i=1}^{n} \alpha_i \int_{\varphi^{-1}(E_i) \cap \mathbb{T}} |\psi|^p \, dm$$

$$= \int_{\mathbb{T}} |\psi|^p \left( \sum_{i=1}^{n} \alpha_i \chi_{\varphi^{-1}(E_i)} \cap \mathbb{T} \right) \, dm = \int_{\mathbb{T}} |\psi|^p (g \circ \varphi) \, dm.$$
Now, if $g$ is a measurable positive function in $\mathbb{D}$, we take an increasing sequence $(g_n)$ of positive and simple functions such that $(g_n(z)) \to g(z)$ for all $z \in \mathbb{D}$. Then, we have $\int |g_n|^p dm \to \int |g|^p dm$. On the other hand, $(|\psi|^p g_n \circ \varphi)$ is an increasing sequence such that $(|\psi|^p(g_n(\varphi(z)))) \to |\psi|^p(\varphi(z))$ for all $z \in \mathbb{D}$, so $\int |g_n|^p dm \to \int |\psi|^p g \circ \varphi dm$.

An obvious necessary condition for $W_{\varphi, \psi}$ to be bounded on $H_p$ is that $\psi = W_{\varphi, \psi}(1) \in H_p$. Whereas this condition is trivially sufficient for $p = \infty$, it is not sufficient for $p < \infty$.

**Theorem 2.2.** Fixing $1 \leq p < \infty$ and given $\varphi, \psi \in H_p$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, we have that $W_{\varphi, \psi}$ is bounded on $H_p$ if and only if $\mu_{\varphi, \psi, \rho}$ is a Carleson measure in $\mathbb{D}$.

**Proof.** On the one hand, by [3, Theorem 2.35], $\mu_{\varphi, \psi, \rho}$ is a Carleson measure in $\mathbb{D}$ if and only if there is a constant $C > 0$ so that

$$\int_{\mathbb{D}} |f|^p dm \leq C\|f\|_p^p$$

for all $f \in H_p$. On the other hand, by Lemma 2.1, taking $g = |f|^p$, we have that

$$\int_{\mathbb{D}} |f|^p dm \mu_{\varphi, \psi, \rho} = \int_{\mathbb{D}} |\psi|^p |f \circ \varphi|^p dm = \|W_{\varphi, \psi}(f)\|_p.$$ 

Hence, $\mu_{\varphi, \psi, \rho}$ is a Carleson measure in $\mathbb{D}$ if and only if there is a constant $C > 0$ so that $\|W_{\varphi, \psi}(f)\|_p \leq C\|f\|_p$ for all $f \in H_p$.

In [12], Mirzakarimi and Seddighi got a sufficient condition of the boundedness of $W_{\varphi, \psi}$ on $H_2$. Namely, they proved that if the measures given by

$$\mu(E) := \int_E |\psi'(z)|^2 (1 - |z|^2)dA(z) \quad \text{and} \quad \nu(E) := \int_E |\psi(z)|^2 |\psi'(z)|^2 (1 - |z|^2)dA(z)$$

for every measurable subset $E$ of $\mathbb{D}$, where $A$ denotes the Lebesgue measure on $\mathbb{D}$, satisfy

$$\sup_{0 < r < 1, b \in T} \frac{\mu(S(b, r))}{r^3} < \infty \quad \text{and} \quad \sup_{0 < r < 1, b \in T} \frac{\nu(S(b, r))}{r^3} < \infty,$$

then $W_{\varphi, \psi}$ is bounded on $H_2$. 
3. COMPACTNESS AND WEAK COMPACTNESS

In this section, we present the main result of this paper, namely, every weakly compact weighted composition operator on $H_1$ is compact on this space. Its proof leans on the following preliminary results. The first one can be found in [4, Corollary 1].

**Lemma 3.1.** Let $(x_n)$ be a bounded sequence in a Banach space $X$. Then $(x_n)$ is weakly null if and only if for each subsequence $(x_{n_k})$ there is a sequence of convex combinations of $(x_{n_k})$, that we denote by $(y_{n_k})$, such that $\|y_{n_k}\| \to 0$.

**Proposition 3.2.** Let $X, Y, Z$ be Banach spaces, and let $T: X \to Y$ and $S: X \to Z$ be bounded operators such that $\|Sx\| \leq \|Tx\|$ for all $x \in X$. Suppose that there are two linear topologies $\tau_1$ on $X$ and $\tau_2$ on $Y$ such that $T$ is $\tau_1 - \tau_2$ continuous, $(B_X, \tau_1)$ is metrizable and compact, and the weak topology of $Y$ is finer than $\tau_2$. If $T$ is weakly compact, then so is $S$.

Before proving this proposition, it is worth mentioning that we plan to apply it to the spaces $X = Y = H_1$, $\tau_1$ the topology of uniform convergence on compact sets, $\tau_2$ the topology of the pointwise convergence, and $T = W_{\psi, \theta}$.

**Proof.** Let $(x_n)$ be a sequence in $B_X$. We have to find a subsequence $(x_{n_k})$ of $(x_n)$ such that $(Sx_{n_k})$ converges in the weak topology of $Z$.

Since $(B_X, \tau_1)$ is metrizable and compact, there is a subsequence $(x_{n_k})$ of $(x_n)$ and a point $x \in B_X$ such that $(x_{n_k} - x)$ converges to zero in the topology $\tau_1$. This is the subsequence we are looking for. Now, using Lemma 3.1, we are going to prove that $(S(x_{n_k} - x))$ is a weakly null sequence. Bearing in mind that $T$ is $\tau_1 - \tau_2$ continuous, the weak topology of $Y$ is finer than $\tau_2$, and $T$ is weakly compact, we have that $(T(x_{n_k} - x))$ converges to zero in the weak topology. Let us take a subsequence $(y_{n_k})$ of $(x_{n_k})$. Then there is a sequence $(z_k)$ of convex combinations of the $y_{n_k}$ such that $\|T(z_k - x)\| \to 0$. Since $\|S(z_k - x)\| \leq \|T(z_k - x)\|$, we have that $\|S(z_k - x)\| \to 0$. Summing up, for each subsequence $(y_{n_k})$ of $(x_n)$, we have found a sequence $(z_k)$ of convex combinations of the $y_{n_k}$ such that $\|S(z_k - x)\| \to 0$. By Lemma 3.1, $(S(x_{n_k} - x))$ converges to zero in the weak topology.

The proof of the following lemma can be obtained by adapting the proof of [3, Proposition 3.11].

**Lemma 3.3.** For $1 \leq p < \infty$ and $\varphi, \psi \in H_p$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $W_{\varphi, \psi}$ is continuous on $H_p$, we have that $W_{\varphi, \psi}$ is compact on $H_p$ if and only if whenever $f_n$ is bounded on $H_p$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|W_{\varphi, \psi}(f_n)\|_p \to 0$. 
Theorem 3.4. Given \( \varphi, \psi \in H_1 \) such that \( \varphi(\mathbb{D}) \subseteq \mathbb{D} \) and \( W_{\varphi, \psi} \) is continuous on \( H_1 \), we have that the following assertions are equivalent:

1. The operator \( W_{\varphi, \psi} \) is compact on \( H_1 \).
2. The operator \( W_{\varphi, \psi} \) is weakly compact on \( H_1 \).
3. The measure \( \mu_{\varphi, \psi, 1} \) satisfies

\[
\lim_{r \to 0} \sup_{b \in \mathbb{T}} \frac{\mu_{\varphi, \psi, 1}(S(b, r))}{r} = 0.
\]

Proof. (1) \( \Rightarrow \) (2). This is obvious.

(2) \( \Rightarrow \) (3). We apply Proposition 3.2 with \( X = Y = H_1 \), \( \tau_1 \) the topology of the uniform convergence on compact sets, \( \tau_2 \) the topology of the pointwise convergence, and, of course, \( T = W_{\varphi, \psi} \). It is clear that \( W_{\varphi, \psi} \) is \( \tau_1 \)-\( \tau_2 \) continuous. Consider the map \( S: H_1 \to L_1(\mathbb{T}, \mu_{\varphi, \psi, 1}) \) given by \( S(f) = f \). By Lemma 2.1, we have that \( ||W_{\varphi, \psi}(h)||_1 = ||S(h)||_{L_1(\mathbb{T}, \mu_{\varphi, \psi, 1})} \) for all \( h \in H_1 \). Since \( W_{\varphi, \psi} \) is weakly compact on \( H_1 \), by Proposition 3.2, \( S \) is also weakly compact.

Now, suppose assertion (3) is not satisfied. Then there are \( \beta > 0 \), \( r_n \to 0 \) (\( 0 < r_n < 1 \)), and \( b_n \in \mathbb{T} \) such that \( \mu_{\varphi, \psi, 1}(S(b_n, r_n)) \geq \beta r_n \). Let us denote \( a_n = (1 - r_n)b_n \) and \( f_n(z) = 1/(1 - \overline{a_n}z)^4 \). Then \( f_n \in H_1 \) and

\[
||f_n||_1 = \frac{1}{r_n^3} \left( 1 + \frac{(1 - r_n)^2}{2 + r_n} \right).
\]

Now we take \( g_n = f_n/||f_n||_1 \). To get a contradiction, we are going to show that for each subsequence \( (g_{n_k}) \), the sequence \( S(g_{n_k}) \) is not weakly convergent. By [14, p. 137], it will be enough to get that the set \( \{S(g_{n_k}) : k \in \mathbb{N}\} \) is not uniformly integrable, i.e., there is \( \varepsilon > 0 \) such that for every \( \eta > 0 \) there exists a measurable subset \( A \) of \( \mathbb{D} \) and \( k \in \mathbb{N} \) such that \( \mu_{\varphi, \psi, 1}(A) \leq \eta \) and \( \int_A |g_{n_k}| \, d\mu_{\varphi, \psi, 1} \geq \varepsilon \). Take \( \varepsilon = \beta/4 \) and let us fix an arbitrary \( \eta \). Since \( \mu_{\varphi, \psi, 1} \) is a Carleson measure, there is a constant \( M \) such that \( \mu_{\varphi, \psi, 1}(S(b, r)) \leq Mr \) for all \( b \in \mathbb{T} \) and \( 0 < r < 1 \). So, we can take \( k \) such that \( \mu_{\varphi, \psi, 1}(S(b_n, r_n)) \leq \eta \). On the other hand, bearing in mind that \( |f_{n_k}(z)| \geq (2r_{n_k})^{-4} \) whenever \( z \in S(b_n, r_n) \), we have that

\[
\int_{S(b_n, r_n)} |g_{n_k}| \, d\mu_{\varphi, \psi, 1} \geq \frac{(2r_{n_k})^{-4}}{||f_{n_k}||_1} \mu_{\varphi, \psi, 1}(S(b_n, r_n)) \geq \frac{(2r_{n_k})^{-4}}{||f_{n_k}||_1} \beta r_{n_k} \geq \frac{\beta}{4}.
\]
(3) \implies (1). We will apply Lemma 3.3. Before doing this, we have to introduce an auxiliary Carleson measure \( \mu \). By (3),

\[
\lim_{r \to 0} \sup_{b \in \mathbb{T}} \frac{\mu_{\varphi, \psi, 1}(S(b, r))}{r} = 0.
\]

Then we also have that

\[
\lim_{r \to 0} \sup_{b \in \mathbb{T}} \frac{\mu_{\varphi, \psi, 1}(W(b, r))}{r} = 0,
\]

where \( W(b, r) \) are the Carleson windows in \( \mathbb{D} \) given by

\[
W(b, r) = \{ \varrho e^{i\theta} \in \overline{\mathbb{D}} : 1 - r \leq \varrho \leq 1, |\theta - t| \leq r \}
\]

where \( b = e^{it} \). Given \( \varepsilon > 0 \), we may find \( r_0 \) such that \( \mu_{\varphi, \psi, 1}(W(b, r)) \leq 2\varepsilon r \) for all \( b \in \mathbb{T} \) and \( r \leq r_0 \). Let us define the measure \( \mu \) given by

\[
\mu(E) := \mu_{\varphi, \psi, 1}(E \cap \{ z \in \overline{\mathbb{D}} : 1 - r_0 \leq |z| \leq 1 \}).
\]

Then \( \mu \) is a Carleson measure on \( \overline{\mathbb{D}} \) with \( \mu(W(b, r)) \leq 2\varepsilon r \) for \( 0 < r < 1 \) (see [3, p. 130]). So, by [3, p. 43], there is a constant \( C \) (independent of \( \varepsilon \)) such that

\[
\int_{\mathbb{T}} |f|d\mu \leq C\varepsilon \| f \|_1.
\]

for all \( f \in H_1 \).

Once we have built the measure \( \mu \), we are going to apply Lemma 3.3 to get that \( W_{\varphi, \psi} \) is compact on \( H_1 \). Take \( (f_n) \) a sequence in \( H_1 \) such that \( (f_n) \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) and \( \| f_n \|_1 \leq 1 \). Then, by Lemma 2.1,

\[
\| W_{\varphi, \psi}(f_n) \|_1 = \int_{\mathbb{T}} |\psi| |f_n \circ \varphi| dm = \int_{\mathbb{T}} |f_n| d\mu_{\varphi, \psi, 1}
\]

\[
= \int_{\mathbb{T} \setminus (1 - r_0)\mathbb{D}} |f_n| d\mu_{\varphi, \psi, 1} + \int_{(1 - r_0)\mathbb{D}} |f_n| d\mu_{\varphi, \psi, 1}.
\]

Since \( (f_n) \to 0 \) uniformly on compact subsets of \( \mathbb{D} \), there is \( n_0 \) such that if \( n \in \mathbb{N} \) and \( n \geq n_0 \) we have that \( |f_n(z)| \leq \varepsilon / \mu_{\varphi, \psi, 1}((1 - r_0)\mathbb{D}) \) for all \( z \in (1 - r_0)\mathbb{D} \). So

\[
\int_{(1 - r_0)\mathbb{D}} |f_n| d\mu_{\varphi, \psi, 1} \leq \frac{\varepsilon}{\mu_{\varphi, \psi, 1}((1 - r_0)\mathbb{D})} \mu_{\varphi, \psi, 1}((1 - r_0)\mathbb{D}) = \varepsilon.
\]
On the other hand, we have that
\[
\int_{\mathbb{D} \setminus (1-r)\mathbb{D}} |f_n| d\mu_{\varphi, \psi, 1} = \int_{\mathbb{D} \setminus (1-r)\mathbb{D}} |f_n| d\mu
\]
\[
= \int_{\mathbb{D}} |f_n| d\mu \leq C \|f_n\|_1 \leq C.
\]
Hence \(\|W_{\varphi, \psi}(f_n)\|_1 \leq (1 + C)\varepsilon\).

**Theorem 3.5.** Fixing \(1 < p < \infty\) and given \(\varphi, \psi \in H_p\) such that \(\varphi, \psi \in H_p\), we have that \(W_{\varphi, \psi}\) is compact on \(H_p\) if and only if

\[
\lim_{r \to 0} \sup_{b \in \mathbb{T}} \frac{\mu_{\varphi, \psi, p}(S(b, r))}{r} = 0.
\]

**Proof.** Suppose that \(W_{\varphi, \psi}\) is compact on \(H_p\) and that there are \(\beta > 0, r_n \to 0 (0 < r_n < 1)\), and \(b_n \in \mathbb{T}\) such that \(\mu_{\varphi, \psi, p}(S(b_n, r_n)) \geq \beta r_n\). Let us denote \(a_n = (1 - r_n)b_n\) and \(f_n(z) = 1/(1 - \bar{a}_nz)^2\). Then \(f_n \in H_p\) and

\[
\|f_n\|_p^p = \frac{1}{r_n^2} \frac{1 + (1 - r_n)^2}{(2 + r_n)^2}.
\]

Now we take \(g_n = f_n/\|f_n\|_p\). By [3, p. 130], \(g_n\) converges to zero uniformly on compact subsets of \(\mathbb{D}\). By Lemma 3.3, to get that \(W_{\varphi, \psi}\) is not compact, we have just to prove that \(\|W_{\varphi, \psi}(g_n)\|_p\) does not converge to zero. Arguing as in the proof of Theorem 3.4, we have that

\[
\|W_{\varphi, \psi}(g_n)\|_p^p = \int_{\mathbb{T}} |\psi|^p |g_n \circ \varphi|^p dm = \int_{\mathbb{T}} |g_n|^p d\mu_{\varphi, \psi, p}
\]
\[
\geq \int_{S(b_n, r_n)} |g_n|^p d\mu_{\varphi, \psi, p} \geq \frac{(2r_n)^{-4}}{\|f_n\|_p} \mu_{\varphi, \psi, p}(S(b_n, r_n))
\]
\[
\geq \frac{(2r_n)^{-4}}{\|f_n\|_p^p} \beta r_n \geq \frac{\beta}{4}.
\]

The other implication can be obtained by following the same steps as in the proof of Theorem 3.4.
4. COMPLETE CONTINUITY

In this section we characterize the case where \( W_{\varphi, \psi} \) is a completely continuous operator. Its proof is a slight generalization of [1, Proposition 1].

**Theorem 4.1.** Given \( \varphi, \psi \in H_1 \) such that \( \varphi(\mathbb{D}) \subseteq \mathbb{D} \) and \( W_{\varphi, \psi} \) is continuous on \( H_1 \), we have \( W_{\varphi, \psi} \) is completely continuous on \( H_1 \) if and only if \( \psi = 0 \) almost everywhere in \( \{ e^{it} \in \mathbb{T} : \varphi(e^{it}) \in \mathbb{T} \} \).

**Proof.** Let \( f \) be a function in \( L_\phi(\mathbb{T}, m) \). By the Riemann–Lebesgue lemma, the sequence given by its Fourier coefficients is in \( c_0 \), so we have that \( \int_{\mathbb{T}} f(z) z^n \, dm \to 0 \) as \( n \to \infty \). Equivalently, the sequence \( (z^n) \) converges to 0 in the weak topology of \( L_\phi(\mathbb{T}, m) \) and, hence, in \( H_1 \). Therefore, \( \| W_{\varphi, \psi}(z^n) \|_1 \to 0 \).

Moreover,

\[
\int_{\{ e^{it} \in \mathbb{T} : \varphi(e^{it}) \in \mathbb{T} \}} |\psi| \, dm = \int_{\{ e^{it} \in \mathbb{T} : \varphi(e^{it}) \in \mathbb{T} \}} |\psi| |\varphi|^n \, dm \\
\leq \int_{\mathbb{T}} |\psi| |\varphi|^n \, dm = \| W_{\varphi, \psi}(z^n) \|_1.
\]

Hence \( \int_{\{ e^{it} \in \mathbb{T} : \varphi(e^{it}) \in \mathbb{T} \}} |\psi| \, dm = 0 \), and we get that \( \psi = 0 \) almost everywhere on the set \( \{ e^{it} \in \mathbb{T} : \varphi(e^{it}) \in \mathbb{T} \} \).

Conversely, let \( (f_n) \) be a weakly null sequence in \( H_1 \). Since \( (f_n(z)) \to 0 \) for all \( z \in \mathbb{D} \) and \( \psi = 0 \) almost everywhere in \( \{ e^{it} \in \mathbb{T} : \varphi(e^{it}) \in \mathbb{T} \} \), we have that \( W_{\varphi, \psi}(f_n) \) goes to zero pointwise almost everywhere on the unit circle. In particular, the sequence \( W_{\varphi, \psi}(f_n) \) converges in measure to zero in \( L_\phi(\mathbb{T}, m) \). Moreover, \( W_{\varphi, \psi}(f_n) \) goes to zero in the weak topology of \( H_1 \) and, so, in the weak topology of \( L_\phi(\mathbb{T}, m) \). Finally, bearing in mind that a sequence in \( L_\phi(\mathbb{T}, m) \) converges to zero in the norm topology whenever it converges to zero in measure and in the weak topology (see, for example, [6, p. 295]), we have that \( \| W_{\varphi, \psi}(f_n) \|_1 \to 0 \).  

**Acknowledgments**

The authors thank Santiago Díaz-Madrigal for some fruitful discussions on the content of this paper.

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