Soft set theory applied to ideals in $d$-algebras

Young Bae Jun $^a$, Kyoung Ja Lee $^{b,*}$, Chul Hwan Park $^c$

$^a$ Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Republic of Korea
$^b$ Department of Mathematics Education, Hannam University, Daejeon 306-791, Republic of Korea
$^c$ Department of Mathematics, University of Ulsan, Ulsan 680-749, Republic of Korea

**Abstract**

The notions of (transitive) soft $d$-algebras, soft edge $d$-algebras, soft $d^+$-algebras, soft $d$-ideals, soft $d^+$-ideals, soft $d^*$-ideals, and $d$-idealistic $(d^+$-idealistic, or $d^*$-idealistic) soft $d$-algebras are introduced. Also, their related properties are surveyed.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical of those problems. There are three theories: the theory of probability, the theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [1]. Maji et al. [2] and Molodtsov [1] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [1] introduced the concept of soft set as a new mathematical tool, for dealing with uncertainties, that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [2] described the application of soft set theory to a decision making problem. Maji et al. [3] also studied several operations on the theory of soft sets. Chen et al. [4] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attribute reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [5]. In 1991, the fuzzy set theory was applied to BCK-algebras (see [6]). The first author of this paper

* Corresponding author.

E-mail addresses: skywine@gmail.com (Y.B. Jun), lsj1109@hotmail.com (K.J. Lee), skyrosemary@gmail.com (C.H. Park).

0898-1221/© 2008 Elsevier Ltd. All rights reserved.
(together with colleagues) applied and studied the fuzzy set theory to BCK-algebras [7,8], BCC-algebras [9], B-algebras [10], hyper-BCK-algebras [11], MTL-algebras [12], hemirings [13], implicative algebras [14], lattice implication algebras [15], and incline algebras [16]. Jun [17] applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras, and introduced the notion of soft BCK/BCI-algebras and soft subalgebras, and then investigated their basic properties. This paper is organized as follows. Sections 2 and 3 present basic definitions of $d$-algebras and soft sets, respectively. In Section 4, we define the notions of soft BCK-algebras, (transitive) soft $d$-algebras, soft edge $d$-algebras, soft $d^*-$algebras, and trivial (or whole) soft $d$-algebras. We investigate their related properties. In Section 5, we consider the notions of soft BCK-ideals, soft sub-$d$-algebras, soft $d$-ideals, soft $d^*$-ideals and soft $d^*-$ideals, and then we investigate their relations. In the final section, we introduce the concepts of $d$-idealistic ($d^*$-idealistic, or $d^*-$idealistic) soft $d$-algebras, and survey several properties.

2. Basic results on $d$-algebras

Let $K(\tau)$ be the class of all algebras of type $\tau$. A $BCK$-algebra is a system $(X, *, 0) \in K(\tau)$, where $\tau = (2, 0)$, such that

\begin{align}
(a1) & \quad (\forall x, y, z \in X)((x * y) * (x * z)) * (z * y) = 0, \\
(a2) & \quad (\forall x, y \in X)((x * (x * y)) * y = 0), \\
(a3) & \quad (\forall x \in X)(x * x = 0), \\
(a4) & \quad (\forall x \in X)(0 * x = 0), \\
(a5) & \quad (\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y).
\end{align}

A $d$-algebra is a system $(X, *, 0) \in K(\tau)$, where $\tau = (2, 0)$, that satisfies $(a3), (a4)$ and $(a5)$. In a $BCK$-algebra $(X, *, 0)$ the following hold:

\begin{align}
(b1) & \quad (\forall x, y \in X)((x * y) * x = 0), \\
(b2) & \quad (\forall x, y, z \in X)(((x * z) * (y * z)) * (x * y)) = 0).
\end{align}

A $d$-algebra $(X, *, 0)$ is called a $d^*$-algebra (see [18]) if it satisfies the identity $(b1)$. Let $(X, *, 0)$ be a $d$-algebra and $x \in X$. Define

$$x * X := \{x * a \mid a \in X\}. \quad (2.1)$$

Then $X$ is said to be edge (see [19]) if it satisfies:

$$x * X = \{0, x\}. \quad (2.2)$$

A non-empty subset $I$ of a $BCK$-algebra $X$ is called a $BCK$-ideal of $X$ if it satisfies the following axioms:

\begin{align}
(11) & \quad 0 \in I, \\
(12) & \quad (\forall x \in X)(\forall y \in I)(x * y \in I \Rightarrow x \in I).
\end{align}

Let $(X, *, 0)$ be a $d$-algebra and $\emptyset \neq I \subseteq X$. Then $I$ is called a $d$-subalgebra of $X$ if $x * y \in I$ whenever $x, y \in I$, and $I$ is called a $BCK$-ideal of $X$ if it satisfies $(11)$ and $(12)$.

A non-empty subset $I$ of a $d$-algebra $(X, *, 0)$ is called a $d$-ideal of $X$ (see [18]) if it satisfies $(12)$ and

$$\forall x, y \in X \ (x \in I \Rightarrow x * y \in I). \quad (2.3)$$

A non-empty subset $I$ of a $d$-algebra $(X, *, 0)$ is called a $d^*$-ideal of $X$ (see [18]) if it is a $d$-ideal of $X$ that satisfies the following axiom:

$$\forall x, y, z \in X \ (x * y \in I, y * z \in I \Rightarrow x * z \in I). \quad (2.4)$$

If a $d^*$-ideal of a $d$-algebra $X$ satisfies:

$$x * y \in I, y * x \in I \Rightarrow (x * z) * (y * z) \in I, (z * x) * (z * y) \in I \quad (2.5)$$

for all $x, y, z \in X$, then we say that $I$ is a $d^*$-ideal of $X$ (see [18]).

3. Basic results on soft sets

Molodtsov [1] defined the soft set in the following way: Let $U$ be an initial universe set and $E$ be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of $U$ and $A \subseteq E$.

Definition 3.1 ([1]). A pair $(\alpha, A)$ is called a soft set over $U$, where $\alpha$ is a mapping given by

$$\alpha : A \rightarrow \mathcal{P}(U).$$

In other words, a soft set over $U$ is a parametrized family of subsets of the universe $U$. For $x \in A$, $\alpha(x)$ may be considered as the set of $x$-approximate elements of the soft set $(\alpha, A)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [1].
Definition 3.2 ([3]). Let \((\alpha, A)\) and \((\beta, B)\) be two soft sets over a common universe \(U\). The intersection of \((\alpha, A)\) and \((\beta, B)\) is defined to be the soft set \((\rho, C)\) satisfying the following conditions:

(i) \(C = A \cap B\),
(ii) \((\forall x \in C)\ (\rho(x) = \alpha(x) \cap \beta(x)),\) (as both are same sets).

In this case, we write \((\alpha, A) \cap (\beta, B) = (\rho, C)\).

Definition 3.3 ([3]). Let \((\alpha, A)\) and \((\beta, B)\) be two soft sets over a common universe \(U\). The union of \((\alpha, A)\) and \((\beta, B)\) is defined to be the soft set \((\rho, C)\) satisfying the following conditions:

(i) \(C = A \cup B\),
(ii) for all \(x \in C\),
\[
\rho(x) = \begin{cases} 
\alpha(x) & \text{if } x \in A \setminus B, \\
\beta(x) & \text{if } x \in B \setminus A, \\
\alpha(x) \cup \beta(x) & \text{if } x \in A \cap B.
\end{cases}
\]

In this case, we write \((\alpha, A) \cap (\beta, B) = (\rho, C)\).

Definition 3.4 ([3]). If \((\alpha, A)\) and \((\beta, B)\) are two soft sets over a common universe \(U\), then \((\alpha, A) \text{ AND } (\beta, B)\) denoted by \((\alpha, A) \cap (\beta, B)\) is defined by \((\alpha, A) \cap (\beta, B) = (\rho, A \times B),\) where \(\rho(x, y) = \alpha(x) \cap \beta(y)\) for all \((x, y) \in A \times B\).

Definition 3.5 ([3]). If \((\alpha, A)\) and \((\beta, B)\) are two soft sets over a common universe \(U\), then \((\alpha, A) \text{ OR } (\beta, B)\) denoted by \((\alpha, A) \cup (\beta, B)\) is defined by \((\alpha, A) \cup (\beta, B) = (\rho, A \times B),\) where \(\rho(x, y) = \alpha(x) \cup \beta(y)\) for all \((x, y) \in A \times B\).

Definition 3.6 ([3]). For two soft sets \((\alpha, A)\) and \((\beta, B)\) over a common universe \(U\), we say that \((\alpha, A)\) is a soft subset of \((\beta, B)\), denoted by \((\alpha, A) \subseteq (\beta, B)\), if it satisfies:

(i) \(A \subseteq B\),
(ii) For every \(x \in A\), \(\alpha(x)\) and \(\beta(x)\) are identical approximations.

4. Soft \(d\)-algebras

In what follows, let \(X := (X, \star, 0)\) and \(A\) denote a \(d\)-algebra and a non-empty set, respectively, unless specified otherwise.

Definition 4.1. Let \((\alpha, A)\) be a soft set over \(X\). Then \((\alpha, A)\) is called a soft \(BCK\)-algebra over \(X\) if \((\alpha(x), \star, 0)\) is a \(BCK\)-algebra for all \(x \in A\).

Example 4.2. Let \(X = \{0, a, b, c\}\) be a \(d\)-algebra with the following Cayley table:

\[
\begin{array}{cccc}
* & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & a & 0 & 0 \\
c & c & 0 & b & 0 \\
\end{array}
\]

Note that \(X\) is not a \(BCK\)-algebra since \((b \star a) \star (b \star c) \star (c \star a) = a \neq 0\). For \(A = X\), we define a set-valued function \(\alpha : A \rightarrow \mathcal{P}(X)\) by \(\alpha(x) = \{y \in X \mid y \star x = 0\}\) for all \(x \in A\). Then \(\alpha(0) = \{0\}, \alpha(a) = \{0, a, c\}, \alpha(b) = \{0, a, b\}\) and \(\alpha(c) = \{0, b, c\}\), which are all \(BCK\)-algebras. Hence \((\alpha, A)\) is a soft \(BCK\)-algebra over \(X\).

Definition 4.3. Let \((\alpha, A)\) be a soft set over \(X\). Then \((\alpha, A)\) is called a soft \(d\)-algebra over \(X\) if \((\alpha(x), \star, 0)\) is a \(d\)-algebra for all \(x \in A\).

Example 4.4. Let \(X = \{0, a, b, c\}\) be a \(d\)-algebra with the following Cayley table:

\[
\begin{array}{cccc}
* & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & b & 0 & 0 \\
c & c & c & a & 0 \\
\end{array}
\]

Let \((\alpha, A)\) be a soft set over \(X\), where \(A = X\) and \(\alpha : A \rightarrow \mathcal{P}(X)\) is a set-valued function defined by \(\alpha(x) = \{y \in X \mid y \star (y \star x) \in [0, a]\}\) for all \(x \in A\). Then \(\alpha(0) = \alpha(a) = X\) and \(\alpha(b) = \alpha(c) = \{0, a\}\), which are \(d\)-algebras. Hence \((\alpha, A)\) is a soft \(d\)-algebra over \(X\).
Definition 4.5. Let \((\alpha, A)\) be a soft set over \(X\). Then \((\alpha, A)\) is called a soft \(d^s\)-algebra over \(X\) if \((\alpha, A)\) is a soft \(d\)-algebra over \(X\) in which the following assertion is valid:

\[
(\forall x \in A)(\forall a, b \in \alpha(x))(a \ast b \ast a = 0).
\]

(4.1)

Clearly, every soft BCK-algebra over \(X\) is a soft \(d^s\)-algebra (and hence, a soft \(d\)-algebra) over \(X\), but the converse may not be true as seen in the following examples.

Example 4.6. Let \((X, \ast, 0)\) be the \(d\)-algebra which is given in Example 4.2. We can verify that \(X\) is a \(d^s\)-algebra. Let \((\alpha, A)\) be a soft set over \(X\), where \(A = X\) and \(\alpha : A \rightarrow 2^X\) is a set-valued function defined by \(\alpha(x) = \{y \in X \mid y \ast (y \ast x) = 0\}\) for all \(x \in A\). Then \(\alpha(0) = X\), \(\alpha(a) = \alpha(b) = \{0\}\) and \(\alpha(c) = \{0, a, b\}\), which are \(d^s\)-algebras, and hence \((\alpha, A)\) is a soft \(d^s\)-algebra over \(X\). But \((\alpha, A)\) is not a soft BCK-algebra over \(X\) since \(\alpha(0) = X\) is not a BCK-algebra (see Example 4.2).

Example 4.7. Consider the soft \(d\)-algebra \((\alpha, A)\) over \(X\) which is described in Example 4.4. Since \((c \ast b) \ast c = a \neq 0\) and \(((c \ast a) \ast (c \ast b)) \ast (b \ast a) = a = 0\), \(X = \{\alpha(0) = \alpha(a)\}\) is neither a \(d^s\)-algebra nor a BCK-algebra. Hence \((\alpha, A)\) is neither a soft \(d^s\)-algebra over \(X\) nor a soft BCK-algebra over \(X\).

Definition 4.8. Let \((\alpha, A)\) be a soft set over \(X\). Then \((\alpha, A)\) is called a soft edge \(d\)-algebra over \(X\) if \((\alpha(x), \ast, 0)\) is an edge \(d\)-algebra for all \(x \in A\).

Example 4.9. Consider the soft BCK-algebra \((\alpha, A)\) over \(X\) which is described in Example 4.2. Then \((\alpha, A)\) is clearly a soft \(d\)-algebra over \(X\). We can verify that \(\alpha(0) = \{0\}\) and \(\alpha(a) = \{0, a, c\}\) are edge, but \(\alpha(b) = \{0, a, b\}\) and \(\alpha(c) = \{0, b, c\}\). Hence a soft \(d\)-algebra \((\alpha, A)\) over \(X\) is not edge. Now if we take \(B = \{0, a\}\), then \((\alpha, B)\) is obviously a soft edge \(d\)-algebra over \(X\).

Example 4.10. Let \(X = \{0, a, b, c\}\) be a \(d\)-algebra with the following Cayley table:

\[
\begin{array}{cccc}
\ast & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & b & 0 & 0 \\
c & c & c & c & 0 \\
\end{array}
\]

For \(A = X\), we define a set-valued function \(\alpha : A \rightarrow 2^X\) by

\[
\alpha(x) = \{y \in X \mid y \ast (y \ast x) = 0\}
\]

for all \(x \in A\). Then \((\alpha, A)\) is a soft edge \(d\)-algebra over \(X\).

Proposition 4.11. If \((\alpha, A)\) is a soft edge \(d\)-algebra over \(X\), then

(i) \((\forall x \in A)(\forall a \in \alpha(x))(a \ast 0 = a)\).

(ii) \((\forall x \in A)(\forall a, b \in \alpha(x))(a \ast (a \ast b) \ast b = 0)\).

Proof. (i) Let \(x \in A\). Since \((\alpha(x), \ast, 0)\) is an edge \(d\)-algebra, either \(a \ast 0 = a\) or \(a \ast 0 = 0\) for all \(a \in \alpha(x)\). If \(a \neq 0\) and \(a \ast 0 = 0\), then \(a = 0\) by (a5), a contradiction.

(ii) Let \(x, y \in A\). Assume that \(a \neq 0\) and \((a \ast (a \ast b)) \ast b = 0\) for some \(b \in \alpha(x)\). Then \(w := a \ast (a \ast b)\). Then \(w \ast b \neq 0\) and \(w \neq 0\). This means that \(a \neq a \ast b \in a \ast \alpha(x) = \{0, a\}\) and hence \(a \ast b = 0\). It follows from (i) that

\[
(a \ast (a \ast b)) \ast b = (a \ast 0) \ast b = a \ast b = 0,
\]

a contradiction. \(\square\)

Definition 4.12. Let \((\alpha, A)\) be a soft set over \(X\). Then \((\alpha, A)\) is called a transitive soft \(d\)-algebra over \(X\) if \((\alpha(x), \ast, 0)\) is a \(d\)-algebra for all \(x \in A\), in which the following implication is valid:

\[
(\forall x \in A)(\forall a, b, c \in \alpha(x))(a \ast c = 0, c \ast b = 0, \Rightarrow a \ast b = 0).
\]

(4.2)

Theorem 4.13. Every transitive soft edge \(d\)-algebra over \(X\) is a soft BCK-algebra over \(X\).

Proof. Let \((\alpha, A)\) be a transitive soft edge \(d\)-algebra over \(X\). It is enough to show that \(\alpha(x)\) satisfies the condition (a1) for all \(x \in A\). Now, let \(x \in A\) and assume that there exist \(a, b, c \in \alpha(x)\) such that \(((a \ast b) \ast (a \ast c)) \ast (c \ast b) \neq 0\). Since \((a \ast b) \ast (a \ast c) \in (a \ast b) \ast \alpha(x) = \{0, a \ast b\}\),

\[
(a \ast b) \ast (a \ast c) = a \ast b.
\]

(4.3)
If \( a \ast b = 0 \), then \( 0 = 0 \ast (c \ast b) = (0 \ast (a \ast c)) \ast (c \ast b) = ((a \ast b) \ast (a \ast c)) \ast (c \ast b) \neq 0 \), which is a contradiction. Hence
\[
a \ast b = a. \tag{4.4}
\]
It follows from (4.3) and (4.4) that
\[
a = a \ast b = (a \ast b) \ast (a \ast c) = a \ast (a \ast c),
\]
that is,
\[
a = a \ast (a \ast c). \tag{4.5}
\]
If \( a \ast c 
eq 0 \), then \( a \ast c = a \) since \( \alpha(x) \) is an edge \( d \)-algebra. By using (a3), (a4) and (4.4), we get
\[
0 = 0 \ast (c \ast b) = (a \ast a) \ast (c \ast b) = ((a \ast b) \ast (a \ast c)) \ast (c \ast b) \neq 0,
\]
which is a contradiction. Thus we conclude
\[
a \ast c = 0. \tag{4.6}
\]
We claim that \( c \ast b = 0 \). If \( c \ast b = c \), then
\[
0 = a \ast c = (a \ast b) \ast c = ((a \ast b) \ast 0) \ast c = ((a \ast b) \ast (a \ast c)) \ast (c \ast b) \neq 0
\]
by using (4.4) and (4.6) and Proposition 4.11(i). This is a contradiction. Hence we have \( a \ast c = 0 \) and \( c \ast b = 0 \). From (4.2), we have \( a \ast b = 0 \) and so
\[
0 \neq ((a \ast b) \ast (a \ast c)) \ast (c \ast b) = 0,
\]
which is a contradiction. This proves the theorem. \( \square \)

In Theorem 4.13, both conditions, i.e., to have the transitive and edge properties, are necessary for a soft \( d \)-algebra over \( X \) of this type to be a soft BCK-algebra over \( X \).

**Example 4.14.** Let \( X = \{0, a, b, c, d\} \) be a set with the following Cayley table:

\[
\begin{array}{c|cccc}
* & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & a \\
b & b & b & 0 & 0 \\
c & c & c & c & 0 \\
d & c & c & a & a \\
\end{array}
\]

Then \((X, \ast, 0)\) is a \( d \)-algebra. Let \( \alpha : A \rightarrow \mathcal{P}(X) \) be defined by
\[
(\forall x \in A)(\alpha(x) = \{0, a, b, c\}).
\]
Then \((\alpha, A)\) is a soft edge \( d \)-algebra over \( X \) which is not transitive, since \( a \ast b = 0 \) and \( b \ast c = 0 \), but \( a \ast c = a \). Note that
\[
((a \ast c) \ast (a \ast b)) \ast (b \ast c) = a \neq 0,
\]
Hence \((\alpha, A)\) is not a soft BCK-algebra over \( X \).

**Example 4.15.** Let \((\alpha, A)\) be the soft edge \( d \)-algebra over \( X \) which is given in Example 4.10. Since \( a \ast b = 0, b \ast c = 0 \) and \( a \ast c = a \neq 0 \), \( \alpha(0)(= X) \) is not transitive, and so \((\alpha, A)\) is a non-transitive soft edge \( d \)-algebra over \( X \). Note that \( \alpha(0)(= X) \) is not a BCK-algebra. Hence \((\alpha, A)\) is not a soft BCK-algebra over \( X \).

**Example 4.16.** Let \( X = \{0, 1, 2, \ldots\} \) and let the binary operation \( \ast \) be defined as follows:
\[
x \ast y := \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}
\]
Then \((X, \ast, 0)\) is a \( d \)-algebra (see [19]). For \( A = \{5, 6, 7, 8, 9\} \), let \( \alpha : A \rightarrow \mathcal{P}(X) \) be defined by \( \alpha(k) = N_k \) for all \( k \in A \), where \( N_k = \{0, 1, 2, \ldots, k\} \). Then \((\alpha, A)\) is a transitive soft non-edge \( d \)-algebra over \( X \). Since
\[
(2 \ast (2 \ast 0)) \ast 0 = (2 \ast 1) \ast 0 = 1 \ast 0 = 1,
\]
\((\alpha, A)\) is not a soft BCK-algebra over \( X \).

We give a method to make a soft edge \( d \)-algebra over \( X \) from a soft \( d \)-algebra over \( X \). Let \((\alpha, A)\) be a soft non-edge \( d \)-algebra over \( X \). For any \( x \in A \), define a binary operation \( \oplus \) on \( \alpha(x) \) by
\[
a \oplus b := \begin{cases} a & \text{if } a \ast b \neq 0, \\ 0 & \text{otherwise} \end{cases}
\]
for all \( a, b \in \alpha(x) \). Then we can check easily that \( (\alpha(x), \odot, 0) \) is a \( d \)-algebra. Suppose now that \( a \odot \alpha(x) = \{0\} \) for every \( a \in \alpha(x) \). Then \( a \odot b = 0 \) for all \( b \in \alpha(x) \). In particular, \( a \odot 0 = 0 = 0 \odot a \), which implies that \( a = 0 \). Hence if \( a \neq 0 \), then \( a \odot \alpha(x) = \{0, a\} \). Therefore \( (\alpha, A) \) is a soft edge \( d \)-algebra over \( X \) under the operation \( \odot \). We summarize:

**Theorem 4.17.** Given a soft \( d \)-algebra over \( X \), we can induce a soft edge \( d \)-algebra over \( X \).

**Proposition 4.18.** Let \((\alpha, A)\) be a soft \( d \)-algebra over \( X \). Then \((\alpha, A)\) is transitive if and only if the induced soft edge \( d \)-algebra over \( X \) is transitive.

**Proof.** Straightforward.  \( \square \)

Combining Theorem 4.13 and Proposition 4.18, we have the following corollary.

**Corollary 4.19.** Given a transitive soft \( d \)-algebra over \( X \), we can induce a soft BCK-algebra over \( X \).

**Definition 4.20.** A soft \( d \)-algebra \((\alpha, A)\) over \( X \) is said to be trivial (resp. whole) if \( \alpha(x) = \{0\} \) (resp. \( \alpha(x) = X \)) for all \( x \in A \).

**Example 4.21.** Let \( X = \{0, a, b, c\} \) be a \( d \)-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>(*)</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For \( A = \{a, b, c\} \), we define two set-valued functions \( \alpha : A \rightarrow \mathcal{P}(X) \) and \( \beta : A \rightarrow \mathcal{P}(X) \) by

\[
\alpha(x) = \{y \in X \mid x * y \in \{0, c\}\}
\]

\[
\beta(x) = \{y \in X \mid x * y = x\}
\]

for all \( x \in A \). Then \((\alpha, A)\) is a whole soft \( d \)-algebra over \( X \) and \((\beta, A)\) is a trivial soft \( d \)-algebra over \( X \).

Let \( f : X \rightarrow Y \) be a mapping of \( d \)-algebras. For a soft set \((\alpha, A)\) over \( X \), \((f(\alpha), A)\) is a soft set over \( Y \) where \( f(\alpha) : A \rightarrow \mathcal{P}(Y) \) is defined by \( f(\alpha)(x) = f(\alpha(x)) \) for all \( x \in A \).

**Lemma 4.22.** Let \( f : X \rightarrow Y \) be a homomorphism of \( d \)-algebras. If \((\alpha, A)\) is a soft \( d \)-algebra over \( X \), then \((f(\alpha), A)\) is a soft \( d \)-algebra over \( Y \).

**Proof.** Straightforward.  \( \square \)

Using Definition 4.20 and Lemma 4.22, we have the following theorem.

**Theorem 4.23.** Let \( f : X \rightarrow Y \) be a homomorphism of \( d \)-algebras and let \((\alpha, A)\) be a soft \( d \)-algebra over \( X \).

(i) If \( \alpha(x) \subseteq \ker(f) \) for all \( x \in A \), then \((f(\alpha), A)\) is the trivial soft \( d \)-algebra over \( Y \).

(ii) If \( f \) is onto and \((\alpha, A)\) is whole, then \((f(\alpha), A)\) is the whole soft \( d \)-algebra over \( Y \).

5. Soft \( d \)-ideals

**Definition 5.1.** Let \( S \) be a \( d \)-subalgebra of \( X \). A subset \( I \) of \( S \) is called a BCK-ideal of \( X \) related to \( S \) (briefly, \( S \)-BCK-ideal of \( X \)) if it satisfies (11) and

\[
(\forall x \in S) \ (\forall y \in I) \ (x * y \in I \Rightarrow x \in I).
\]  \hspace{1cm} (5.1)

Note that if \( S \) is a \( d \)-subalgebra of \( X \) and \( I \) is a subset of \( X \) that contains \( S \), then \( I \) is an \( S \)-BCK-ideal of \( X \).

**Example 5.2.** Let \( (X, \ast, 0) \) be the \( d \)-algebra which is described in Example 4.10. Note that \( X \) is not a BCK-algebra (see [19]). Consider a \( d \)-subalgebra \( S = \{0, a\} \) of \( X \). Obviously, \( I = \{0, a, b\} \) is an \( S \)-BCK-ideal of \( X \). But \( J = \{0, b\} \) is not an \( S \)-BCK-ideal of \( X \) since \( a \ast b = 0 \in J \) and \( a \not\in J \). On the other hand, let \( S = \{0, b, c\} \) be a \( d \)-subalgebra of \( X \) and \( I = \{0, b\} \). Then \( I \) is an \( S \)-BCK-ideal of \( X \).

**Definition 5.3.** Let \( S \) be a \( d \)-subalgebra of \( X \). A non-empty subset \( I \) of \( S \) is called a \( d \)-ideal of \( X \) related to \( S \) (briefly, \( S \)-\( d \)-ideal of \( X \), denoted by \( I \subset_d S \), if it satisfies (5.1) and

\[
(\forall x, y \in S) \ (x \in I \Rightarrow x * y \in I).
\]  \hspace{1cm} (5.2)

Note that if \( S \) is a \( d \)-subalgebra of \( X \) and \( I \) is a subset of \( X \) that contains \( S \), then \( I \) is an \( S \)-\( d \)-ideal of \( X \).
Example 5.4. Let $X = \{0, a, b, c\}$ be a $d$-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>$0$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$0$</td>
<td>$0$</td>
<td>$a$</td>
</tr>
<tr>
<td>$b$</td>
<td>$c$</td>
<td>$b$</td>
<td>$0$</td>
<td>$c$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$b$</td>
<td>$b$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Then $X$ is not a BCK-algebra. For any $d$-subalgebra $S$ of $X$, we can verify that $\{0, a\}$ is an $S$-$d$-ideal of $X$. Now if we take subsets $I = \{0, b\}$ and $J = \{0, c\}$ of $X$, then they are not an $S$-$d$-ideal of $X$, where $S = \{0, a\}$ is a $d$-subalgebra of $X$. For, $I$ is not an $S$-BCK-ideal of $X$ and $J$ does not satisfy (5.2), i.e., $c * a = b \notin I$.

Example 5.5. Let $X = \{0, a, b, c, d\}$ be a $d$-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>$0$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$0$</td>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b$</td>
<td>$b$</td>
<td>$0$</td>
<td>$c$</td>
<td>$0$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$c$</td>
<td>$b$</td>
<td>$0$</td>
<td>$c$</td>
</tr>
<tr>
<td>$d$</td>
<td>$d$</td>
<td>$c$</td>
<td>$a$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Then $X$ is not a BCK-algebra. Consider a $d$-subalgebra $S = \{0, a, c, d\}$ of $X$. If $I_1 = \{0, c\}, I_2 = \{0, d\}, I_3 = \{0, c, d\}$ and $I_4 = \{0, a\}$, then $I_1$ and $I_2$ are not $S$-$d$-ideals of $X$ since $I_1$ is not an $S$-BCK-ideal of $X$ and $d * 0 = c \notin I_2$. But, $I_1$ and $I_4$ are $S$-$d$-ideals of $X$. On the other hand, if we take $S = X$, then $I_4 \lhd_{gr} S$, but $I_3$ is not an $S$-$d$-ideal of $X$ since $I_3$ is not an $S$-BCK-ideal of $X$.

Definition 5.6. Let $S$ be a $d$-subalgebra of $X$. A non-empty subset $I$ of $X$ is called a $d^2$-ideal of $X$ related to $S$ (briefly, $S$-$d^2$-ideal of $X$), denoted by $I \lhd_{d^2} S$, if it is an $S$-$d$-ideal of $X$ that satisfies the following axiom:

$$(\forall x, y, z \in S) \ (x * y \in I, y * z \in I \Rightarrow x * z \in I).$$ \hspace{1cm} (5.3)

Note that if $S$ is a $d$-subalgebra of $X$ and $I$ is a subset of $X$ that contains $S$, then $I$ is an $S$-$d^2$-ideal of $X$.

Example 5.7. Consider the $d$-algebra $(X, *, 0)$ which is given in Example 5.2. If $S$ is any $d$-subalgebra of $X$, then $\{0, a\}$ is an $S$-$d^2$-ideal of $X$.

Example 5.8. Let $(X, *, 0)$ be the $d$-algebra which is described in Example 5.5. Consider two $S$-$d$-ideals $I = \{0, c, d\}$ and $J = \{0, a\}$ of $X$, where $S = \{0, a, c, d\}$ is a $d$-subalgebra of $X$. We can verify that $I \lhd_{d^2} S$ and $J \lhd_{d^2} S$. On the other hand, for a $d$-subalgebra $S = X$ of $X$, we obtain that $I$ and $J$ are not $S$-$d^2$-ideals of $X$, since $I$ is not an $S$-$d$-ideal of $X$ (see Example 5.5) and $J$ does not satisfy (5.3), i.e., $b * d \in I, d * c \in I$ and $b * c \in J$.

Definition 5.9. Let $S$ be a $d$-subalgebra of $X$. A non-empty subset $I$ of $X$ is called a $d^a$-ideal of $X$ related to $S$ (briefly, $S$-$d^a$-ideal of $X$), denoted by $I \lhd_{d^a} S$, if it is an $S$-$d^a$-ideal of $X$ that satisfies the implication (2.5) for all $x, y, z \in S$.

Note that if $S$ is a $d$-subalgebra of $X$ and $I$ is a subset of $X$ that contains $S$, then $I$ is an $S$-$d^a$-ideal of $X$.

Example 5.10. Let $S$ be any $d$-subalgebra of $X$ which is given in Example 5.7. Then $\{0, a\}$ is an $S$-$d^a$-ideal of $X$ if

$S \in \{\{0\}, \{0, a\}, \{0, b\}, \{0, c\}, \{0, b, c\}\}.$

But if $S \in \{\{0, a, b\}, \{0, a, c\}, \{0, a, b, c\}\}$, then $\{0, a\}$ is not an $S$-$d^a$-ideal of $X$ since $0 * a = 0 \in \{0, a\}, a * 0 = a \in \{0, a\}, (b * 0) * (b * a) = c * b = b \notin \{0, a\}$ and $(c * 0) * (c * a) = c * b = b \notin \{0, a\}$.

Example 5.11. In Example 5.8, consider $S$-$d^a$-ideals $I = \{0, c, d\}$ and $J = \{0, a\}$ of $X$, where $S = \{0, a, c, d\}$ is a $d$-subalgebra of $X$. We can verify that $I \lhd_{d^a} S$ and $J \lhd_{d^a} S$.

Definition 5.12. Let $(\alpha, A)$ be a soft $d$-algebra over $X$. A soft set $(\beta, I)$ over $X$ is called a soft BCK-ideal of $(\alpha, A)$ if it satisfies:

(i) $I \subseteq A$,
(ii) $(\forall x \in I) \ (\beta(x) \text{ is an } \alpha(x)\text{-BCK-ideal of } X)$.

Definition 5.13. Let $(\alpha, A)$ be a soft $d$-algebra over $X$. A soft set $(\beta, B)$ over $X$ is called a soft sub-$d$-algebra of $(\alpha, A)$, denoted by $(\beta, B) \lesssim (\alpha, A)$, if it satisfies:

(i) $B \subseteq A$,
(ii) $(\forall x \in B) (\beta(x) \text{ is a } d\text{-subalgebra of } \alpha(x))$. 


373
Example 5.14. Consider the soft d-algebra \((\alpha, A)\) over \(X\) which is described in Example 4.4. Let \((\beta, B)\) be a soft set over \(X\), where \(B = \{a, b\}\) and \(\beta : B \rightarrow \mathcal{P}(X)\) is a set-valued function defined by \(\beta(x) = \{y \in X \mid y \ast x = 0\}\) for all \(x \in B\). Then \(\beta(a) = \{0, a\}\), which is a \(d\)-subalgebra of \(\alpha(a) = X\), and hence \((\beta, B) \subseteq_d (\alpha, A)\).

Definition 5.15. Let \((\alpha, A)\) be a soft \(d\)-algebra over \(X\). A soft set \((\beta, I)\) over \(X\) is called a soft \(d\)-ideal of \((\alpha, A)\), denoted by \((\beta, I) \subseteq_d (\alpha, A)\), if it satisfies:

(i) \(I \subseteq A\),
(ii) \((\forall x \in I)(\beta(x) \subseteq_d \alpha(x))\).

Example 5.16. Let \((X, \ast, 0)\) be the \(d\)-algebra which is given in Example 5.2. For \(A = \{a, b, c\} \subset X\), we define a set-valued function \(\alpha : A \rightarrow \mathcal{P}(X)\) by \(\alpha(x) = \{y \in X \mid y \ast (y \ast x) \in \{0, b\}\}\) for all \(x \in A\). Then \((\alpha, A)\) is a soft \(d\)-algebra over \(X\). Now let \((\beta, I)\) be a soft set over \(X\), where \(I = \{a, b, c\} \subseteq A\) and \(\beta : I \rightarrow \mathcal{P}(X)\) is a set-valued function defined by

\[
\beta(x) = \{y \in X \mid y \ast (y \ast x) \in \{0, a\}\}
\]

for all \(x \in I\). Then \(\beta(a) = \beta(b) = \{0, a\} \subseteq_d \alpha(a) = \alpha(b) = \{0, b, c\}\) (see Example 5.4) and \(\beta(c) = \{0, a, c\} \subseteq_d \alpha(c) = \{0, a\}\), and hence \((\beta, I) \subseteq_d (\alpha, A)\).

Note that every \(d\)-ideal is a \(d\)-subalgebra in a \(d\)-algebra. But we know that a soft \(d\)-ideal may not be a soft sub-\(d\)-algebra in a soft \(d\)-algebra over \(X\). In fact, in Example 5.16, \((\beta, I) \subseteq_d (\alpha, A)\). But \((\beta, I)\) is not a soft sub-\(d\)-algebra of \((\alpha, A)\) since \(\beta(c) = \{0, a, c\}\) is not a \(d\)-subalgebra of \(\alpha(c) = \{0, a\}\). Of course, a soft sub-\(d\)-algebra may not be a soft \(d\)-ideal in a soft \(d\)-algebra over \(X\) as seen in the following example.

Example 5.17. Consider the \(d\)-algebra \((X, \ast, 0)\) which is described in Example 5.4. If we define a set-valued function \(\alpha : \{c\} \rightarrow \mathcal{P}(X)\) by \(\alpha(c) = \{y \in X \mid y \ast (y \ast c) \in \{0, c\}\}\), then \(\alpha(c) = X\), and so \((\alpha, \{c\})\) is a soft \(d\)-algebra over \(X\). Now let \(\beta : \{c\} \rightarrow \mathcal{P}(X)\) be a set-valued function defined by \(\beta(c) = \{y \in X \mid y \ast c = 0\}\). Then \(\beta(c) = \{0, c\}\), which is a \(d\)-subalgebra of \(\alpha(c) = X\). Hence \((\beta, \{c\}) \subseteq (\alpha, \{c\})\). But \((\beta, \{c\})\) is not a soft \(d\)-ideal of \((\alpha, \{c\})\) since \(c \ast a = b \notin \beta(c)\), i.e., \(\beta(c)\) is not an \(\alpha(c)\)-\(d\)-ideal of \(X\).

Since every \(S\)-\(d\)-ideal of \(X\) is an \(S\)-BCK-ideal of \(X\), where \(S\) is a \(d\)-subalgebra of \(X\), the following proposition is straightforward.

Proposition 5.18. In a soft \(d\)-algebra over \(X\), every soft \(d\)-ideal is a soft BCK-ideal.

Generally, we know that a soft BCK-ideal may not be a soft \(d\)-ideal in a soft \(d\)-algebra over \(X\), i.e., the converse of Proposition 5.18 is not true as seen in the following example.

Example 5.19. Consider the \(d\)-algebra \((X, \ast, 0)\) which is described in Example 5.2. Let \((\alpha, A)\) be a soft set over \(X\), where \(A = \{b, c\}\) and \(\alpha : A \rightarrow \mathcal{P}(X)\) is a set-valued function defined by \(\alpha(x) = \{y \in X \mid y \ast (y \ast x) = 0\}\) for all \(x \in A\). Then \((\alpha, A)\) is a soft \(d\)-algebra over \(X\). If \(I = \{c\} \subseteq A\) and \(\beta : B \rightarrow \mathcal{P}(X)\) is a set-valued function defined by \(\beta(x) = \{y \in X \mid y \ast x = 0\}\) for all \(x \in I\), then \(\beta(c) = \{0, c\}\) is an \(\alpha(c)\)-\(d\)-BCK-ideal of \(X\), but \(\beta(c)\) is not an \(\alpha(c)\)-\(d\)-ideal of \(X\) (see Example 5.4). Hence \((\beta, I)\) is a soft BCK-ideal of \((\alpha, A)\), but \((\beta, I)\) is not a soft \(d\)-ideal of \((\alpha, A)\).

Now we give a condition for a soft BCK-ideal to be a soft \(d\)-ideal.

Theorem 5.20. In a soft \(d^*\)-algebra over \(X\), every soft BCK-ideal is a soft \(d^*\)-ideal.

Proof. Let \((\beta, I)\) be a soft BCK-ideal of a soft \(d^*\)-algebra \((\alpha, A)\) over \(X\). Then \(I \subseteq A\) and \(\beta(x)\) is an \(\alpha(x)\)-BCK-ideal of \(X\) for all \(x \in I\). For \(x \in X\), let \(a, b \in \alpha(x)\). If \(a \in \beta(x)\), it follows from (4.1) and (11) that \((a \ast b) \ast a = 0 \in \beta(x)\) since \((\alpha, A)\) is a soft \(d^*\)-algebra over \(X\). Hence \(a \ast b \in \beta(x)\) by (5.1). Therefore \(\beta(x)\) is an \(\alpha(x)\)-\(d\)-ideal of \(X\) for all \(x \in I\). Consequently, \((\beta, I)\) is a soft \(d\)-ideal of \((\alpha, A)\).

Definition 5.21. Let \((\alpha, A)\) be a soft \(d\)-algebra over \(X\). A soft set \((\beta, I)\) over \(X\) is called a soft \(d^*\)-ideal of \((\alpha, A)\), denoted by \((\beta, I) \subseteq_{d^*} (\alpha, A)\), if it satisfies:

(i) \(I \subseteq A\),
(ii) \((\forall x \in I)(\beta(x) \subseteq_{d^*} \alpha(x))\).

Example 5.22. In Example 5.16, consider the soft \(d\)-ideal \((\beta, I)\) of \((\alpha, A)\). We can verify that \((\beta, I) \subseteq_{d^*} (\alpha, A)\) (see Example 5.7).

Definition 5.23. Let \((\alpha, A)\) be a soft \(d\)-algebra over \(X\). A soft set \((\beta, I)\) over \(X\) is called a soft \(d^*\)-ideal of \((\alpha, A)\), denoted by \((\beta, I) \subseteq_{d^*} (\alpha, A)\), if it satisfies:

(i) \(I \subseteq A\),
(ii) \((\forall x \in I)(\beta(x) \subseteq_{d^*} \alpha(x))\).
Example 5.24. Let \((X, \ast, 0)\) be the \(d\)-algebra which is given in Example 5.5. Consider \(A = \{0, a, c\} \subseteq X\) and a set-valued function \(\alpha : A \rightarrow \mathcal{P}(X)\) defined by \(\alpha(x) = \{y \in X \mid y \ast x \in \{0, a\}\}\) for all \(x \in A\). Then \((\alpha, A)\) is a soft \(d\)-algebra over \(X\). Now let us take \(I = \{a, c\} \subseteq A\) and define a set-valued function \(\beta : I \rightarrow \mathcal{P}(X)\) by \(\beta(x) = \{y \in X \mid y \ast x = 0\}\) for all \(x \in I\). We can verify that \(\beta(a) = \{0, a\} \triangleleft d \alpha(a) = \{0, a\}\) and \(\beta(c) = \{0, c, d\} \triangleleft d \alpha(c) = \{0, a, c, d\}\) (see Example 5.11). Hence \((\beta, I) \triangleleft d \alpha(a)\).

Since every \(S-d^2\)-ideal of \(X\) is an \(S-d\)-ideal of \(X\), where \(S\) is a \(d\)-subalgebra of \(X\), the following proposition is straightforward.

Proposition 5.25. In a soft \(d\)-algebra over \(X\), every soft \(d^2\)-ideal is a soft \(d\)-ideal.

The following example shows that the converse of Proposition 5.25 is not true in general.

Example 5.26. Consider the \(d\)-algebra \((X, \ast, 0)\) which is described in Example 5.5. Let \((\alpha, A)\) be a soft set over \(X\), where \(A = \{a, b\} \subseteq X\), and \(\alpha : A \rightarrow \mathcal{P}(X)\) is a set-valued function defined by \(\alpha(x) = \{y \in X \mid y \ast x \in \{0, a\}\}\) for all \(x \in A\). Then \((\alpha, A)\) is a soft \(d\)-algebra over \(X\). Let \(I = \{a\} \subseteq A\) and \(\beta : B \rightarrow \mathcal{P}(X)\) be a set-valued function defined by \(\beta(x) = \{y \in X \mid y \ast x = 0\}\) for all \(x \in I\). Then \(\beta(a) = \{0, a\} \triangleleft d \alpha(a) = X\) (see Example 5.5), but \(\beta(a)\) is not an \(X-d^2\)-ideal of \(X\) (see Example 5.8). Hence \((\beta, I) \triangleleft d \alpha(a)\), but \((\beta, I)\) is not a soft \(d\)-ideal of \(\alpha, A\).

Since every \(S-d^2\)-ideal of \(X\) is an \(S-d\)-ideal of \(X\), where \(S\) is a \(d\)-subalgebra of \(X\), the following proposition is straightforward.

Proposition 5.27. In a soft \(d\)-algebra over \(X\), every soft \(d^2\)-ideal is a soft \(d\)-ideal.

The following example shows that the converse of Proposition 5.27 is not true in general.

Example 5.28. Let \((X, \ast, 0)\) be the \(d\)-algebra which is given in Example 5.2. For \(A = \{a, b\} \subseteq X\), we define a set-valued function \(\alpha : A \rightarrow \mathcal{P}(X)\) by \(\alpha(x) = \{y \in X \mid y \ast x \in \{0, b\}\}\) for all \(x \in A\). Then \((\alpha, A)\) is a soft \(d\)-algebra over \(X\). Now let \((\beta, I)\) be a soft set over \(X\), where \(I = \{a\} \subseteq A\) and \(\beta : B \rightarrow \mathcal{P}(X)\) is a set-valued function defined by \(\beta(x) = \{y \in X \mid y \ast x = 0\}\) for all \(x \in I\). Then \(\beta(a) = \{0, a\} \triangleleft d \alpha(a) = X\) (see Example 5.7), but \(\beta(a)\) is not an \(X-d^2\)-ideal of \(X\) (see Example 5.10). Hence \((\beta, I) \triangleleft d \alpha(a)\), but \((\beta, I)\) is not a soft \(d\)-ideal of \(\alpha, A\).

Lemma 5.29 ([18]). If \((X, \ast, 0)\) is a \(BCK\)-algebra, then every \(BCK\)-ideal of \(X\) is a \(d^2\)-ideal of \(X\).

Theorem 5.30. Let \((\alpha, A)\) be a soft \(BCK\)-algebra over \(X\). Then every \(BCK\)-ideal of \((\alpha, A)\) is a soft \(d^2\)-ideal of \((\alpha, A)\).

Proof. Let \((\beta, I)\) be a soft \(BCK\)-ideal of \((\alpha, A)\). Then \(\beta(x)\) is an \((\alpha, x)\)-\(BCK\)-ideal of \(X\) for all \(x \in I \subseteq A\). Since \(\alpha(x)\) is a \(BCK\)-algebra for all \(x \in A\), it follows from Lemma 5.29 that \(\beta(x)\) is an \((\alpha, x)\)-\(d^2\)-ideal of \(X\) for all \(x \in I\). Therefore \((\beta, I)\) is a soft \(d^2\)-ideal of \((\alpha, A)\).

Theorem 5.31. Let \((\alpha, A)\) be a soft \(d\)-algebra over \(X\). For any soft sets \((\beta_1, I_1)\) and \((\beta_2, I_2)\) over \(X\) where \(I_1 \cap I_2 \neq \emptyset\), we have

(i) \((\beta_1, I_1) \triangleleft d (\beta_2, I_2) \Rightarrow (\beta_1, I_1) \triangleleft d (\beta_2, I_2) \triangleleft d (\alpha, A)\).

(ii) \((\beta_1, I_1) \triangleleft d (\alpha, A), (\beta_2, I_2) \triangleleft d (\alpha, A) \Rightarrow (\beta_1, I_1) \triangleleft d (\beta_2, I_2) \triangleleft d (\alpha, A)\).

(iii) \((\beta_1, I_1) \triangleleft d (\alpha, A), (\beta_2, I_2) \triangleleft d (\alpha, A) \Rightarrow (\beta_1, I_1) \triangleleft d (\beta_2, I_2) \triangleleft d (\alpha, A)\).

Proof. (i) Using Definition 3.2, we can write

\[(\beta_1, I_1) \triangleleft d (\beta_2, I_2) = (\beta, I)\]

where \(I = I_1 \cap I_2\) and \(\beta(x) = \beta_1(x)\) or \(\beta_2(x)\) for all \(x \in I\). Obviously, \(I \subseteq A\) and \(\beta : I \rightarrow \mathcal{P}(X)\) is a mapping. Hence \((\beta, I)\) is a soft set over \(X\). Since \((\beta_1, I_1) \triangleleft d (\alpha, A)\) and \((\beta_2, I_2) \triangleleft d (\alpha, A)\), we know that \(\beta(x) = \beta_1(x) \leq d \alpha(x)\) or \(\beta(x) = \beta_2(x) \leq d \alpha(x)\) for all \(x \in I\). Hence

\[(\beta_1, I_1) \triangleleft d (\beta_2, I_2) = (\beta, I) \triangleleft d (\alpha, A)\).

Similarly, we have (ii) and (iii). This completes the proof.

Corollary 5.32. Let \((\alpha, A)\) be a soft \(d\)-algebra over \(X\). For any soft sets \((\beta, I)\) and \((\delta, I)\) over \(X\), we have

(i) \((\beta, I) \triangleleft d (\alpha, A), (\delta, I) \triangleleft d (\alpha, A) \Rightarrow (\beta, I) \triangleleft d (\delta, I) \triangleleft d (\alpha, A)\).

(ii) \((\beta, I) \triangleleft d (\alpha, A), (\delta, I) \triangleleft d (\alpha, A) \Rightarrow (\beta, I) \triangleleft d (\delta, I) \triangleleft d (\alpha, A)\).

(iii) \((\beta, I) \triangleleft d (\alpha, A), (\delta, I) \triangleleft d (\alpha, A) \Rightarrow (\beta, I) \triangleleft d (\delta, I) \triangleleft d (\alpha, A)\).

Proof. Straightforward.

Theorem 5.33. Let \((\alpha, A)\) be a soft \(d\)-algebra over \(X\). For any soft sets \((\beta, I)\) and \((\gamma, J)\) over \(X\) in which \(I\) and \(J\) are disjoint, we have

(i) \((\beta, I) \triangleleft d (\alpha, A), (\gamma, J) \triangleleft d (\alpha, A) \Rightarrow (\beta, I) \triangleleft d (\gamma, J) \triangleleft d (\alpha, A)\).

(ii) \((\beta, I) \triangleleft d (\alpha, A), (\gamma, J) \triangleleft d (\alpha, A) \Rightarrow (\beta, I) \triangleleft d (\gamma, J) \triangleleft d (\alpha, A)\).

(iii) \((\beta, I) \triangleleft d (\alpha, A), (\gamma, J) \triangleleft d (\alpha, A) \Rightarrow (\beta, I) \triangleleft d (\gamma, J) \triangleleft d (\alpha, A)\).
Proof. (i) Assume that \((\beta, I)\tilde{\gamma}_d(\alpha, A)\) and \((\gamma, J)\tilde{\gamma}_d(\alpha, A)\). By means of Definition 3.3, we can write \((\beta, I)\tilde{\bigcup}(\gamma, J) = (\delta, U)\) where \(U = I \cup J\) and for every \(x \in U\),
\[
\delta(x) = \begin{cases} 
\beta(x) & \text{if } x \in I \setminus J, \\
\gamma(x) & \text{if } x \in J \setminus I, \\
\beta(x) \cup \gamma(x) & \text{if } x \in I \cap J.
\end{cases}
\]
Since \(I \cap J = \emptyset\), either \(x \in I \setminus J\) or \(x \in J \setminus I\) for all \(x \in U\). If \(x \in I \setminus J\), then \(\delta(x) = \beta(x) \circ \alpha(x)\) since \((\beta, I)\tilde{\gamma}_d(\alpha, A)\). If \(x \in J \setminus I\), then \(\delta(x) = \gamma(x) \circ \alpha(x)\) since \((\gamma, J)\tilde{\gamma}_d(\alpha, A)\). Thus \(\delta(x) \circ \alpha(x)\) for all \(x \in U\), and so \((\beta, I)\tilde{\bigcup}(\gamma, J) = (\delta, U)\tilde{\gamma}_d(\alpha, A)\).

Similarly, we have (ii) and (iii). \(\square\)

If \(I\) and \(J\) are not disjoint in Theorem 5.33, then Theorem 5.33 is not true in general as seen in the following example.

Example 5.34. Let \(X = \{a, b, c, d\}\) be a \(d\)-algebra with the following Cayley table:
\[
\begin{array}{cccc}
* & 0 & a & b & c & d \\
0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a & a \\
b & b & c & 0 & a & b \\
c & c & c & 0 & c & c \\
d & d & c & c & 0 & 0 \\
\end{array}
\]
Then \(X\) is not a BCK-algebra. For \(A = \{a, b, c\}\), let \(\alpha : A \rightarrow \mathcal{P}(X)\) be a set-valued functions defined by \(\alpha(x) = \{y \in X \mid y \ast x \in \{0, a\}\}\) for all \(x \in A\). Then \((\alpha, A)\) is a soft \(d\)-algebra over \(X\). Now we take \(I := \{a, c\}\) and let \((\beta, I)\) be a soft set over \(X\) which is given by \(\beta(x) = \{y \in X \mid y \ast x = 0\}\) for all \(x \in I\). Then \(\beta(a) = \{0, a\} \triangleleft_{d^*} \alpha(a) = \{0, a\}\) and \(\beta(c) = \{0, c, d\} \triangleleft_{d^*} \alpha(c) = X\), and so \((\beta, I)\tilde{\gamma}_{d^*}(\alpha, A)\) which implies that \((\beta, I)\tilde{\gamma}_{d^*}(\alpha, A)\) and \((\beta, I)\tilde{\gamma}_d(\alpha, A)\). Now, let \(J := \{c\}\) which is not disjoint with \(I\), and let \(\gamma : J \rightarrow \mathcal{P}(X)\) be a set-valued function defined by \(\gamma(x) = \{y \in X \mid y \ast (y \ast x) = 0\}\) for all \(x \in J\). Then \(\gamma(c) = \{0, d\} \triangleleft_{d^*} \alpha(c) = X\), and hence \((\gamma, J)\tilde{\gamma}_{d^*}(\alpha, A)\) which implies that \((\gamma, J)\tilde{\gamma}_{d^*}(\alpha, A)\) and \((\gamma, J)\tilde{\gamma}_d(\alpha, A)\).

But if \((\delta, U) := (\beta, I)\tilde{\bigcup}(\gamma, J)\), then \(\delta(c) = \beta(c) \triangledown \gamma(c) = \{a, c, d\}\) is not an \(\alpha(c)\)-ideal of \(X\) since \(b \ast a = c \in \delta(c)\) and \(\delta(c) \subseteq \delta(c)\). Hence \((\delta, U) = (\beta, I)\tilde{\bigcup}(\gamma, J)\) is not a soft \(d\)-ideal of \((\alpha, A)\), which implies that \((\delta, U)\) is neither a soft \(d^*\)-ideal of \((\alpha, A)\) nor a soft \(d^*\)-ideal of \((\alpha, A)\).

Remark 5.35. (1) In a soft \(d\)-algebra over \(X\), we have the following diagram in which reverse implications are not valid.

(2) In a soft \(d^*\)-algebra over \(X\), the concepts of soft \(d\)-ideal and soft BCK-ideal coincide.

(3) In a soft BCK-algebra over \(X\), the concepts of soft \(d\)-ideal, soft \(d^*\)-ideal, soft \(d^*\)-ideal and soft BCK-ideal coincide.

6. Idealistic soft \(d\)-algebras

Definition 6.1. Let \((\alpha, A)\) be a soft set over \(X\). Then \((\alpha, A)\) is called a \(d\)-idealistic (resp. \(d^*\)-idealistic and \(d^*\)-idealistic) soft \(d\)-algebra over \(X\) if \(\alpha(x)\) is a \(d\)-ideal (resp. \(d^*\)-ideal and \(d^*\)-ideal) of \(X\) for all \(x \in A\).

Let us illustrate this definition using the following examples.

Example 6.2. Let \((X, \ast, 0)\) be the \(d\)-algebra which is given in Example 5.34. For \(A = X\), we define a set-valued function \(\alpha : A \rightarrow \mathcal{P}(X)\) by \(\alpha(x) = \{y \in X \mid y \ast x \in \{0, c\}\}\) for all \(x \in A\). Then \((\alpha, A)\) is a \(d^*\)-idealistic soft \(d\)-algebra over \(X\).
Example 6.3. Consider the $d$-algebra $(X, *, 0)$ which is described in Example 5.2. Let $(\alpha, A)$ be a soft set over $X$, where $A = \{0, a, b\}$ and $\alpha : A \to \mathcal{P}(X)$ is a set-valued function defined by

$$\alpha(x) = \{y \in X \mid y \ast (y \ast x) \in \{0, a\}\}$$

for all $x \in A$. Then $(\alpha, A)$ is a $d^2$-idealistic soft $d$-algebra over $X$, but $(\alpha, A)$ is not a $d^*$-idealistic soft $d$-algebra over $X$ (see Examples 5.7 and 5.10).

Proposition 6.4. Let $(\alpha, A)$ and $(\alpha, B)$ be soft sets over $X$ where $B \subseteq A \subseteq X$. If $(\alpha, A)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$, then so is $(\alpha, B)$.

Proof. Straightforward. □

The converse of Proposition 6.4 is not true in general as seen in the following example.

Example 6.5. Consider the $d$-algebra $(X, *, 0)$ which is described in Example 5.34. For $A = X$, let $\alpha : A \to \mathcal{P}(X)$ be a set-valued function defined by $\alpha(x) = \{y \in X \mid y \ast x = 0\}$ for all $x \in A$. Then $\alpha(d)$ is not a $d$-ideal of $X$ since $d \ast 0 = c \notin \alpha(d)$, and hence $(\alpha, A)$ is not a $d$-idealistic soft $d$-algebra over $X$. But if we take $B := \{0, a, b, c\} \subseteq A$, then $(\alpha, B)$ is a $d$-idealistic soft $d$-algebra over $X$.

Theorem 6.6. Let $(\alpha, A)$ and $(\beta, B)$ be two $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebras over $X$. If $A \cap B \neq \emptyset$, then the intersection $(\alpha, A) \cap (\beta, B)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$.

Proof. Using Definition 3.2, we can write $(\alpha, A) \cap (\beta, B) = (\delta, C)$, where $C = A \cap B$ and $\delta(x) = \alpha(x) \lor \beta(x)$ for all $x \in C$. Note that $\delta : C \to \mathcal{P}(X)$ is a mapping, and therefore $(\delta, C)$ is a soft set over $X$. Since $(\alpha, A)$ and $(\beta, B)$ are $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebras over $X$, it follows that $\delta(x) = \alpha(x)$ is a $d$-ideal (resp. $d^2$-ideal and $d^*$-ideal) of $X$, or $\delta(x) = \beta(x)$ is a $d$-ideal (resp. $d^2$-ideal and $d^*$-ideal) of $X$ for all $x \in C$. Hence $(\delta, C) = (\alpha, A) \cap (\beta, B)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$. □

Corollary 6.7. Let $(\alpha, A)$ and $(\beta, A)$ be two $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebras over $X$. Then their intersection $(\alpha, A) \cap (\beta, A)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$.

Proof. Straightforward. □

Theorem 6.8. Let $(\alpha, A)$ and $(\beta, B)$ be two $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebras over $X$. If $A$ and $B$ are disjoint, then the union $(\alpha, A) \cup (\beta, B)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$.

Proof. Using Definition 3.3, we can write $(\alpha, A) \cup (\beta, B) = (\rho, C)$, where $C = A \cup B$ and for every $x \in C$,

$$\rho(x) = \begin{cases} \alpha(x) & \text{if } x \in A \setminus B, \\ \beta(x) & \text{if } x \in B \setminus A, \\ \alpha(x) \cup \beta(x) & \text{if } x \in A \cap B. \end{cases}$$

Since $A \cap B = \emptyset$, either $x \in A \setminus B$ or $x \in B \setminus A$ for all $x \in C$. If $x \in A \setminus B$, then $\rho(x) = \alpha(x)$ is a $d$-ideal (resp. $d^2$-ideal and $d^*$-ideal) of $X$ since $(\alpha, A)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$. If $x \in B \setminus A$, then $\rho(x) = \beta(x)$ is a $d$-ideal (resp. $d^2$-ideal and $d^*$-ideal) of $X$ since $(\beta, B)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$. Hence $(\rho, C) = (\alpha, A) \cup (\beta, B)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$. □

Theorem 6.9. If $(\alpha, A)$ and $(\beta, B)$ are $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebras over $X$, then $(\alpha, A) \cap (\beta, B)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$.

Proof. By means of Definition 3.4, we know that

$$(\alpha, A) \cap (\beta, B) = (\rho, A \times B),$$

where $\rho(x, y) = \alpha(x) \cap \beta(y)$ for all $(x, y) \in A \times B$. Since $\alpha(x)$ and $\beta(y)$ are $d$-ideals (resp. $d^2$-ideals and $d^*$-ideals) of $X$, the intersection $\alpha(x) \cap \beta(y)$ is also a $d$-ideal (resp. $d^2$-ideal and $d^*$-ideal) of $X$. Hence $\rho(x, y)$ is a $d$-ideal (resp. $d^2$-ideal and $d^*$-ideal) of $X$ for all $(x, y) \in A \times B$, and therefore $(\alpha, A) \cap (\beta, B) = (\rho, A \times B)$ is a $d$-idealistic (resp. $d^2$-idealistic and $d^*$-idealistic) soft $d$-algebra over $X$. □

7. Conclusions

Soft sets are a new mathematical tool to deal with uncertainties. We applied the theory of soft sets to an algebraic structure, the so-called $d$-algebras. We introduced the notions of (transitive) soft $d$-algebras, soft edge $d$-algebras, soft $d^*$-algebras, soft $d$-ideals, soft $d^2$-ideals, soft $d^*$-ideals, and $d$-idealistic ($d^2$-idealistic, or $d^*$-idealistic) soft $d$-algebras. Many related properties were surveyed.

Acknowledgements

The authors are very grateful to the referees for their valuable comments and suggestions for improving this paper.
References