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The evolution to a steady state for a porous medium model

J.N. Flavin *

Department of Mathematical Physics, National University of Ireland, Galway, Ireland

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Abstract

An initial boundary value problem is considered for a nonlinear diffusion equation, the diffusivity being a function of the dependent variable. Dirichlet boundary conditions, independent of time, are considered and positive solutions are assumed. This paper is mainly concerned with the rate of convergence, in time, of the unsteady to the steady state. This is done by obtaining an upper estimate for a positive-definite, integral measure of the perturbation (i.e., unsteady–steady state) using differential inequality techniques.

A previous result is recalled where the diffusivity $k(\tau) = \tau^n$ (n being a positive constant) appropriate to mass transport, or filtration, in a porous medium. The present paper treats an alternative model, sharing some of the characteristics of the previous one: $k(\tau) = e^\tau - 1$, τ being non-negative.

The paper concludes by considering a "backwards in time" initial boundary value problem for the perturbation (amenable to the same techniques) and establishes that the solution ceases to exist beyond a critical, computable time.

Keywords: Nonlinear diffusion; Porous medium; Initial boundary value problem; Positive solutions; Positive-definite integral measure of the perturbation; Convergence to the steady state; Nonexistence of solution backwards in time

1. Introduction

In previous papers [1,2], unsteady and steady state problems were considered for a nonlinear diffusion equation, where the diffusivity depends on the dependent variable, in the context of

* Fax: 353 91 495527.
E-mail address: james.flavin@nuigalway.ie.

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a prescribed, non-null, Dirichlet boundary condition. Upper estimates were obtained for a novel Liapunov functional, reflecting convergence of the unsteady to the steady state, under various assumptions: in [1], it is supposed that the diffusivity is bounded below by a positive constant, while in [2] a power-law diffusivity (appropriate to a porous medium) is considered in the context of smooth, positive solutions. In the former case exponential decay is recovered while in the latter case the decay is of algebraic type.

In this paper a variant of the porous medium model is considered: we address the issue of convergence of the unsteady to the steady state in the context of smooth positive solutions, for a diffusivity with broadly exponential dependence on the dependent variable, that shares some of the characteristics of the power law model previously considered. In the previous context [2], a pivotal element in the analysis was the establishment of a certain algebraic inequality. Equally well, an inequality of a broadly similar type is pivotal in the present context. The proof of the present inequality is much more complex: the proof—which is lengthy though essentially elementary—is given in Appendix A.

In Section 2, the unsteady, steady, and perturbation problems are outlined for a nonlinear diffusion problem, where the diffusivity depends on the dependent variable (e.g., temperature). Section 3 establishes a global convergence of the unsteady to the steady state in the context of positive, classical solutions for a model diffusivity of the porous medium type. This is done by establishing an upper estimate for a positive-definite integral measure of the perturbation. The section begins by proving the aforementioned positive-definite property (in the context of positive solutions) for a broad class of diffusivities which includes the one specifically examined in this paper. Section 3 concludes with alternative estimates reflecting convergence to the unsteady state:

(a) It is shown that the aforementioned upper estimate for a positive-definite measure of the perturbation implies estimates for related, perhaps more transparent, measures.
(b) The estimate obtained in (1) is shown to be relevant in the current context, and the circumstances are discussed in which this estimate is better or worse than that obtained in the earlier part of Section 3.

Section 4 considers the perturbation problem backwards in time. Using a similar methodology it is proved that the solution of the governing initial boundary value problem fails to exist for sufficiently large, computable times.

The methodology of the paper is that of using differential inequality techniques to obtain bounds for positive-definite measures of the solution. Treatments of various aspects of these techniques are available in [3–6].

2. Steady, unsteady, and perturbation problems

We consider a fixed, bounded spatial region Ω with smooth boundary $\partial\Omega$, in a three-dimensional context (say). Suppose $T(\mathbf{x}, t)$ satisfies the diffusion equation

$$\frac{\partial T}{\partial t} = \nabla \cdot \{k(T)\nabla T\} \quad \text{in } \Omega \tag{1}$$

(where $k(T)$ is the assigned diffusivity) subject to

$$T(\mathbf{x}, t) = \bar{T}(\mathbf{x}) \quad \text{on } \partial\Omega, \tag{2}$$

and subject to

$$T(\mathbf{x}, 0) = f(\mathbf{x}) \quad \text{in } \Omega. \tag{3}$$

This is referred to as the *unsteady state* problem. The corresponding *steady state* problem is: $U(\mathbf{x})$ satisfies

$$\nabla \cdot \{k(U)\nabla U\} = 0 \quad \text{in } \Omega \tag{4}$$

subject to

$$U(\mathbf{x}) = \bar{T}(\mathbf{x}) \quad \text{on } \partial\Omega. \tag{5}$$

The perturbation defined by

$$u = T - U \tag{6}$$

satisfies, on defining

$$\Phi(u,U) = \int_0^u d\bar{u} \int_0^{\bar{u}} k(\tau + U)\, d\tau, \tag{7}$$

the following initial boundary value problem:

$$\frac{\partial u}{\partial t} = \nabla^2 \Phi_u \quad \text{in } \Omega, \tag{8}$$

subject to

$$u(\mathbf{x}, t) = 0 \quad \text{on } \partial\Omega \tag{9}$$

and

$$u(\mathbf{x}, 0) = f(\mathbf{x}) - U(\mathbf{x}) \quad \text{in } \Omega. \tag{10}$$

In (8) and subsequently, subscripts denote partial differentiation with respect to the appropriate variable.

We shall suppose that we are dealing with *positive, classical solutions* (T,U) throughout, and our general/principal aim is to establish a result reflecting convergence of the unsteady to the steady state, i.e., the convergence to zero of the perturbation. This was done in [2] for the model diffusivity $k(\tau) = \tau^n$, n being a positive constant, and τ being positive—appropriate to a porous medium. By contrast, this paper considers a model diffusivity—sharing some of the characteristics of the aforementioned one:

$$k(\tau) = e^{\tau} - 1, \tag{11}$$

τ being positive.

It was established in [2] that the function $\Phi(u,U)$ is positive-definite in the perturbation u for the diffusivity considered therein (i.e., $k(\tau) = \tau^n$). We now show that this property continues to hold for a general class of diffusivities, including the diffusivity (11) considered in this paper.

Lemma 1. *In the context of the diffusivity $k(\tau)$ which satisfies*

(a) $k(\tau)$ *is continuous,*
(b) $k(\tau) > 0$ *for* $\tau > 0$,
(c) $k(0) = 0$,

and supposing that T,U are both positive, then $\Phi(u,U)$—defined by (7)*—is positive-definite in u.*

Proof. Using Taylor's theorem (remainder form) together with (7), we obtain

$$\Phi(u,U) = \Phi_{uu}(\theta u, U)u^2/2 = k\big[\theta(U+u) + (1-\theta)U\big]u^2/2, \tag{12}$$

where θ is a number (independent of u) such that $0 < \theta < 1$. In view of the assumptions, k is always positive and the lemma is therefore established. □

We define the functional

$$E(t) = \int_{\Omega} \Phi(u,U)\, d\Omega, \tag{13}$$

where Φ is defined by (7). In view of the positive definite property of Φ with respect to u, for the class of diffusivities, etc., mentioned in Lemma 1, the functional is an appropriate global measure of the magnitude of the perturbation u. In particular, convergence to zero, as $t \to \infty$, of $E(t)$ represents a type of global convergence of the unsteady to the steady state.

Differentiating (13), using (8), (9) and the divergence theorem, together with the variational characterization of the lowest "fixed membrane" eigenvalue λ_1 of the region Ω (i.e., $\nabla_1^2\chi + \lambda\chi = 0$ in Ω, $\chi = 0$ on $\partial\Omega$), one obtains, as in [2],

$$\frac{dE}{dt} \leqslant -\lambda_1 \int_{\Omega} \Phi_u^2\, d\Omega. \tag{14}$$

We recall, en passant, that since $E(t)$ is a positive definite measure of u, non-increasing in time, it is termed a *Liapunov functional*, e.g., [3].

To make progress, confining attention to the model diffusivity (11), we need the inequality (valid for positive U and positive $U+u$):

$$\Phi_u^2 \geqslant K\Phi^2, \tag{15}$$

where

$$K = \min\big[1, \{r(U_M)\}^2\big], \tag{16}$$

wherein

$$r(U) = \big(e^U - U - 1\big)/\big(1 - e^U + Ue^U - U^2/2\big), \tag{17}$$

and U_M is the maximum value of U (or $\bar{T}$) prescribed on the boundary $\partial\Omega$. The proof of this inequality—which is lengthy yet elementary—is given in Appendix A.

Applying inequality (15)–(17) to (14) and applying Schwarz's inequality, one obtains

$$\frac{dE}{dt} \leqslant -\lambda_1 K V^{-1} E^2, \tag{18}$$

where K is given by (16), (17), etc., and V is the volume of the region Ω. Integration of (18) gives

$$E(t) \leqslant E(0)\big[1 + \lambda_1 K\{E(0)V^{-1}\}t\big]^{-1}. \tag{19}$$

We embody this result in the following theorem:

Theorem. *The evolution in time of the unsteady to the steady state, defined by* (1)–(5)*, in the context of positive, smooth solutions and a diffusivity given by* (11)*, satisfies the inequality* (19)*, wherein the various quantities arising are defined by* (13)*,* (16)–(17)*, etc.,* λ_1 *being the lowest "fixed membrane" eigenvalue of* Ω*, and* V *the volume of the region.*

Remark 1. In particular, it follows from (19) that

$$E(t) \to 0 \quad \text{as } t \to \infty,$$

i.e., the unsteady state T tends to the steady state U in the measure E.

Remark 2. In the context of the diffusivity

$$k(\tau) = \tau^n,$$

n being a positive constant, and for positive, classical solutions, it was established in [2] that

$$E(t) \leqslant E(0)\big[1 + \lambda_1 K_n \sigma \big\{E(0)V^{-1}\big\}^{\sigma} t\big]^{-\sigma^{-1}},$$

where

$$K_n = (n+2)^{2(n+1)(n+2)^{-1}} n^{-2}, \qquad \sigma = n(n+2)^{-1}.$$

This result is seen to have a formal resemblance to (19) in the limit $n \to \infty$.

Remark 3. The estimate (19) also implies convergence to the steady state in measure other than $E(t)$-measures that are arguably more tangible: defining (for $U > 0$) the positive quantity (bounded for finite U)

$$J(U) = U^{-2}\big(1 + Ue^U - e^U - U^2/2\big), \tag{20}$$

one may prove that

$$\Phi(u, U) \geqslant J(U)u^2. \tag{21}$$

See Appendix A for a proof of this and proofs of other relevant properties of $J(U)$. Hence a weighted L^2 estimate for u follows from

$$\int J(U)u^2\, d\Omega \leqslant E(t). \tag{22}$$

Now since $J(U)$ is an increasing function of U (see Appendix A) and in view of the maximum/minimum principle for U, one may deduce an L^2 estimate for u from

$$\int u^2 d\Omega \leqslant \big[J(U_m)\big]^{-1} E(t),$$

where U_m (> 0) is the minimum prescribed value of U (or $\bar{T}$) on $\partial\Omega$. Also, (22) together with the Schwarz inequality gives an L_1 estimate for u through

$$\|u\|_1 \leqslant \sqrt{\int \big[J(U)\big]^{-1}\, d\Omega . E(t)}.$$

Remark 4. Suppose that the boundary and initial values are strictly positive, i.e., that there exists a positive number δ such that $\bar{T}(\mathbf{x}) > \delta$, $f(\mathbf{x}) > \delta$. Then, by the maximum principle, one has for $0 \leqslant \alpha < 1$,

$$U + \alpha u = (1-\alpha)U + \alpha(U+u) \geqslant \delta.$$

It follows that the diffusivity $k(\cdot)$ satisfies

$$k(U + \alpha u) \geqslant e^{\delta} - 1.$$

It follows from the previous work [1,2], that, in these circumstances, we have

$$E(t) \leqslant E(0)\exp\bigl[-2\bigl(e^{\delta}-1\bigr)\lambda_1 t\bigr]. \tag{23}$$

Remark 5. The question arises as to when the upper bound given by (19) is better than the one given by (23). It is easily verified that this is, in fact, so provided

$$KE(0)V^{-1} > 2\bigl(e^{\delta}-1\bigr) \tag{24}$$

and provided that the time interval is sufficiently small, i.e., the time is less than the (positive) root of

$$2\bigl(e^{\delta}-1\bigr)\lambda_1 t = \log\bigl[1+\lambda_1 KE(0)V^{-1}t\bigr]. \tag{25}$$

Note that (24) means that, in a sense, the initial state $(T(\mathbf{x},0))$ is sufficiently far from equilibrium. If however, $E(0)$ satisfies the strict inequality complementary to (24), then the exponential bound (23) is always better than that given by (19).

Remark 6. Suppose that the boundary and/or initial value of $T(\mathbf{x})$ take(s) the value zero at some points. One could use a limiting process to prove that the inequality (19) remains valid in these circumstances: replace $f(\mathbf{x})$ and $\bar{T}(\mathbf{x})$ by $f(\mathbf{x})+\epsilon$, $\bar{T}(\mathbf{x})+\epsilon$, respectively, where ϵ is a sufficiently large positive quantity to ensure that the latter two functions are positive; denote the relevant quantities arising in (19) by $E_\epsilon(t)$, $E_\epsilon(0)$, and let $\epsilon \to 0$. In these circumstances, the constant δ arising in Remark 4 is zero, and Remarks 4 and 5 are then redundant.

If we now consider the backwards in time version of the initial boundary value problem in the perturbation u, i.e., formally replacing (8)

$$-\frac{\partial u}{\partial t} = \nabla^2 \Phi_u, \tag{26}$$

the remaining specifications unchanged (i.e., u is then the perturbation at time t prior to the initial instant). A virtual repetition of the argument leading to (19) establishes that, in the present context,

$$\bigl[E(0)\bigr]^{-1} - \lambda_1 K V^{-1} t \geqslant \bigl[E(t)\bigr]^{-1}. \tag{27}$$

In view of the non-negativity of $E(t)$, it is clear that there is non-existence for times t such that

$$t > \bigl[\bigl\{E(0)V^{-1}\bigr\}\lambda_1 K\bigr]^{-1}. \tag{28}$$

This result, together with a recasting of (27) for sufficiently small time t, are embodied in the following theorem:

Theorem. *For the backwards in time initial boundary value problem in the perturbation u, specified by* (26), (9), (10), *etc., the solution fails to exist for times t satisfying* (28). *For times less than that given by the right-hand side of* (28), *one has*

$$E(t) \geqslant E(0)\bigl[1-\lambda_1 K\bigl\{E(0)V^{-1}\bigr\}t\bigr]^{-1}. \tag{29}$$

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Appendix A

A.1. *Proof of inequality* (15)–(17)

We now prove the inequality (15)–(17), proceeding in five steps:

(a) This comprises some pertinent, preliminary remarks.
(b) It is proved that

$$\Phi_u \geqslant \Phi \quad \text{for } u \geqslant 0. \tag{A.1}$$

(c) It is proved, for fixed U, and for $-U < u < 0$,
  (i) that $-\Phi_u/\Phi$ (> 0) is an increasing function of u,
  (ii) and thus, that for the stated u, U,

$$-\Phi_u/\Phi > \left(e^U - U - 1\right)/\left(1 - e^U + Ue^U - U^2/2\right).$$

(d) It is proved that

$$r(U) \stackrel{\text{defn}}{=} \left(e^U - U - 1\right)/\left(1 - e^U + Ue^U - U^2/2\right) \tag{A.2}$$

is a decreasing function of U $(U > 0)$, and that, for $U > 0$,

$$r(U) \geqslant r(U_M), \tag{A.3}$$

where U_M is the maximum prescribed value of U on the boundary.
(e) Here (b), (c), (d) are combined to give the required inequality (15).

We now proceed as outlined.
(a) We note that

$$\Phi(u, U) = e^{U+u} - e^U - ue^U - u^2/2,$$
$$\Phi_u(u, U) = e^{U+u} - e^U - u,$$

we recall that the former is positive definite in u, and note that

$$\Phi_u \geqslant 0 \quad \text{for } u \geqslant 0, \qquad \Phi_u < 0 \quad \text{for } u < 0.$$

(b) Consider

$$\Phi_u/\Phi = \left(e^{U+u} - e^U - u\right)/\left(e^{U+u} - e^U - ue^U - u^2/2\right)$$

for $u > 0$, noting, *en passant*, that $\Phi_u/\Phi \to \infty$ as $u \to 0+$. Since $\Phi_u/\Phi \to 1$ as $u \to \infty$, we seek to establish that, for $u \geqslant 0$,

$$\Phi_u \geqslant \Phi$$

or equivalently,

$$\left(e^U - 1\right)u \geqslant -u^2/2.$$

Since $U > 0$ and $u > 0$ (here), the result (A.1) follows.

(c) We prove that $-\Phi_u/\Phi$ is an increasing function of u for $-U < u < 0$, i.e., that, for $-U < u < 0$, U being regarded as fixed,

$$f(u) = \left(u + e^U - e^{U+u}\right)/\left(e^{U+u} - e^U - ue^U - u^2/2\right),$$

one has

$$f'(u) > 0, \tag{A.4}$$

where primes denote differentiation with respect to u, both here and subsequently. Differentiation establishes that (A.4) holds provided that

$$h(u) \stackrel{\text{def}}{=} (u-1)e^{U+u} + \big(u^2/2 - 2u + 1\big)e^u + \big(u^2/2\big)e^{-U} + u - 1 + e^U > 0$$

(for the stated range of u, U).

Further differentiation establishes that

$$\begin{aligned}
h'(u) &= ue^{U+u} + \big(u^2/2 - u - 1\big)e^u + ue^{-U} + 1,\\
h''(u) &= (u+1)e^{U+u} + \big(u^2/2 - 2\big)e^u + e^{-U},\\
h'''(u) &= (u+2)e^{U+u} + \big(u^2/2 + u - 2\big)e^u,\\
\big[h'''(u)e^{-u}\big]' &= e^U + u + 1\\
&> e^U + 1 - U \quad (\text{for } u > -U)\\
&> 0 \quad (\text{since } U > 0).
\end{aligned}$$

Since $h'''(0) > 0$ (as $U > 0$) in addition to the foregoing, it follows that $h'''(u) > 0$. Noting that $h''(0) > 0, h'(0) = h(0) = 0$, it follows successively that $h''(u), h'(u)$ are positive, and (most importantly) that (A.4) holds (for $-U < u < 0$).

Thus $f(u)$ is an increasing function of u, i.e., $-\Phi_u/\Phi$ is an increasing function of u (for $-U < u < 0$).

(d) We proceed to prove that $r(U)$, defined by (A.3), is a decreasing function of U for $U > 0$ (noting, *en passant*, that $r(U) \to \infty$ as $U \to 0$).

Differentiation with respect to U, establishes that this is so provided that (for $U > 0$)

$$1 + U + U^2/2 - e^U < 0,$$

which is clearly true.

Now, in view of the maximum principle for Laplace's equation, one has

$$U \leqslant U_M,$$

where U_M is the maximum value of U prescribed on the boundary. Thus (A.4) holds.

(e) The required inequality now follows, on combining (A.1), (A.2) and (A.4), that

$$\Phi_u^2 \geqslant \min\big[1, \{r(U_M)\}^2\big]\Phi^2. \tag{A.5}$$

Finally, note that the constant arising in the inequality is, in a sense, optimal: this is implicit in the analysis.

A.2. *Properties of $J(U)$ including inequality* (21)

We now prove relevant properties of $J(U)$, defined in Remark 3, including inequality (21).

Firstly, it is clear that $J(U)$ (defined for $U > 0$) is bounded for finite U as

$$\lim_{U\to 0} 3J(U)U^{-1} = 1. \tag{A.6}$$

One may also prove that $J(U)$ is a non-decreasing function of U: one easily shows that the sign of the derivative of $J(U)$ is that of

$$a(U) = U^2 + 2 - 2U - 2e^{-U}. \tag{A.7}$$

That this is positive is evident from the facts that

$$a(0) = \dot{a}(0) = 0$$

and

$$\ddot{a}(U) = 2\bigl(1 - e^{-U}\bigr)$$

which is positive for $U > 0$. (Differentiation with respect to U in the foregoing, and subsequently, is denoted by superposed dots.)

We now turn to the proof of (21). This is equivalent to proving that

$$b(u) = e^{U+u} - e^U - ue^U - u^2/2 - J(U)u^2 \geqslant 0. \tag{A.8}$$

The proof of (A.8) is accomplished in two broad, general steps: (a) for the case $u \geqslant 0$, (b) for the case $-U \leqslant u < 0$.

First consider case (a). Henceforward let us denote differentiation with respect to u by primes. It is easily proved that

$$b(0) = b'(0) = 0, \qquad b''(0) = e^U - 1 - J(U). \tag{A.9}$$

Case (a) is thus established on proving $b''(0) > 0$. This is seen to be equivalent to proving (for $U > 0$) that $a(U)$, defined in (A.7), is positive—which has been done. Thus case (a) of inequality (21) has been proved.

We turn now to proving inequality (21) in case (b), i.e., for $-U \leqslant u \leqslant 0$.

This is done by proving that

(i) $b(-U) = 0$,
(ii) $b'(-U) > 0$,
(iii) $b''(u) \leqslant 0$ for $u \leqslant q(U)$, $b''(u) > 0$ for $u > q(U_c)$,

where U_q is a uniquely defined quantity such that $-U < q < 0$. (Recall primes denote differentiation with respect to u.)

Proving (ii) above is equivalent to proving that

$$c(U) = U + Ue^U + 2 - 2e^U > 0.$$

This follows on noting that

$$c(0) = \dot{c}(U) = 0, \qquad \ddot{c}(U) = Ue^U > 0.$$

We now prove (iii) above. This is seen to be equivalent to proving that

$$b''(u) = e^{U+u} - 2\bigl(1 - e^U + Ue^U\bigr)U^{-2}$$

is negative for $u < q(U)$, and positive for $u > q(U)$. As e^{U+u} is an increasing function of u, it suffices to prove that

$$b''(-U) < 0, \qquad b''(0) > 0.$$

As the latter has been proved already, it remains to prove the former. This is equivalent to proving that

$$d(U) = 2\left(1 - e^U + Ue^U\right) - U^2 > 0.$$

This follows by noting that

$$d(0) = 0, \qquad d^{\cdot}(U) = 2U\left(e^U - 1\right) > 0 \quad (\text{for } U > 0).$$

Thus the proof of inequality (21) has been completed.

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