A HOMOTOPY CLASSIFICATION OF MAPS INTO AN INDUCED FIBRE SPACE

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INTRODUCTION

Let \( P_f \rightarrow A \) be the principal fibre map induced by a map \( f : A \rightarrow B \) from the path space over \( B \). An \( H_0 \)-space is a function space with a restricted \( H \)-structure (see §1.2). Given that \( A^x \) and \( B^x \) are \( H_0 \)-spaces, we define, in §1.2, for each map \( u : X \rightarrow A \), a composite homomorphism

\[
\Delta(f, u) : [X, \Omega A] \approx \pi_1^x(A; u) \xrightarrow{f_u} \pi_1^x(B; fu) \approx [X, \Omega B].
\]

THEOREM 1.3.1. If \( X \) and \( B \) are \( H_0 \)-spaces, there is a (1, 1) correspondence between the set of homotopy classes \([X, P_f]\), and \( \cup \operatorname{coker} \Delta(f, u) \), where the (disjoint) union is taken over all homotopy classes \( \{u\} \) in the kernel of \( f_u : [X, A] \rightarrow [X, B] \).

If \( B \) is a product of Eilenberg–MacLane spaces, the function \( \Delta \) is calculated in terms of the algebroid structure of the cohomology of \( A \) in Theorem 2.4.1. If \( A \) also is a product of Eilenberg–MacLane spaces, explicit formulae for \( \Delta \) are given in terms of cohomology operations in §2.5.

Chapter 3 is devoted to the dual classification problem involving cofibre spaces. The dual of the homomorphism \( \Delta \) is calculated in a general situation in Theorem 3.4.3, using general Whitehead products. As an application, we give, in Theorem 3.5.1, an expansion for the Whitehead product \([\alpha \gamma, \beta]\) in terms of Whitehead products involving \( \alpha \) and \( \beta \) only together with Hopf-invariants of \( \gamma \).

We include, at the end, a synopsis of notation and definitions.

I understand that Dr. I. M. James and Dr. E. Thomas have independently succeeded in obtaining results similar to some of the ones in this paper using a different construction.†

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1. THE CLASSIFICATION THEOREM

1.1 Preliminaries

We consider path connected Hausdorff spaces X with base point usually denoted by *. Given spaces X and Y, the space of base point preserving maps from X to Y, with the compact open topology is denoted by \( Y^X \) or \( \text{Map}(X, Y) \). The unit interval with no base point is denoted by \( I \), and the set of all maps from \( I \) to \( X \) with the compact open topology is denoted by \( X^I \). The preferred base points for \( Y^X \) and \( X^I \) are the trivial maps. The unit interval with base point \( 0 \) is denoted by \( II \). The sphere \( S^1 = \{ e^{2\pi it} : 0 \leq t \leq 1 \} \) has base point given by \( t = 0 \), and \( S^0 = \{ 0, 1 \} \) has 0 as base point. The path and loop spaces on \( X \) are \( PX = X^I \), \( \Omega X = X^{S^1} \) and \( \Omega(X, u) = \{ i \in X^I : i(0) = i(1) = u \} \) for any \( u \) in \( X \).

The wedge \( X \vee Y \), of spaces \( X \) and \( Y \) with base point is the subset \( X \times \ast \cup \ast \times Y \) of \( X \times Y \). The collapsed products \( X \otimes Y \) and \( X \otimes I \) are defined to be the identification spaces \( X \otimes Y = X \times Y / X \vee Y \) and \( X \otimes I = X \times I / \ast \times I \). The reduced cylinder on \( X \) is \( X \otimes I \), and the (reduced) cone \( CX \) and (reduced) suspension \( SX \) of a space \( X \) are, respectively, \( CX = X \otimes II \) and \( SX = X \otimes S^1 \). We thus have a sequence of identification maps \( X \times I \to CX \to SX \). The unreduced suspension of \( X \) is \( S,X \) obtained from \( X \times I \) by identifying \( X \times \{ 0 \} \) and \( X \times \{ 1 \} \) to separate points.

In general, we are only interested in the category \( \mathcal{W} \) of spaces having the (base point preserving) homotopy type of a CW complex. If \( X \) and \( Y \) belong to \( \mathcal{W} \), then so do \( X \times Y \), \( X \otimes Y \) and \( \Omega X \) (cf. Theorem 3 and Proposition 3 of [7]). Also in \( \mathcal{W} \) the projection \( S_0X \to SX \) is a homotopy equivalence.

The exponential map \( e : Y^X \times I \to (Y^I)^X \), defined by \( e(f)(x)(t) = f(x, t) \) is a surjective homeomorphism for all spaces \( X \) and \( Y \).

Definition. The adjoint \( \zeta \) of a map \( \xi : X \times I \to X \otimes Y \) is the map \( \zeta : X \to Y^I \) determined by the exponential bijection.

We may regard \( \zeta \) as a homotopy between maps \( \zeta_0, \zeta_1 : X \to Y \).

The exponential map \( e : Y^{X \otimes I} \to (Y^X)^I \) is a surjective homeomorphism if \( X \) is a CW complex.

Let \( u : X \to Y \), then the \( u \) based track group \( \pi^X_1(Y; u) \) is the set of homotopy classes in the space of functions \( \zeta : X \otimes I \to Y \), satisfying \( \zeta(x, 0) = \zeta(x, 1) = u(x) \), with the obvious group structure (see [2]). Considered as a functor in two variables, this is isomorphic to the group of homotopy classes of functions \( \zeta : X \to Y^I \) satisfying \( \zeta(x)(0) = \zeta(x)(1) = u(x) \). We use the same notation for these functors: in particular \( \pi^X_1(Y; \ast) \cong [X, \Omega Y] \cong [SX, Y] \). If \( F \) is a homotopy \( F : u \sim v \), we define an isomorphism \( F_\# : \pi^X_1(Y; u) \to \pi^X_1(Y; v) \) by \( F_\#(\zeta) = \{-F + \zeta + F\} \); clearly \(-F_\# \) is the inverse isomorphism. This isomorphism corresponds to the change of base point isomorphism in the fundamental group of a path component of the function space \( Y^X \). If \( Y^X \) is a restricted \( H \)-space, we describe, in the next section

\( \dagger \) With certain restrictions on \( X \), for example, if \( X \) is a CW complex, there is a further isomorphism \( \pi^X_1(Y; u) \cong \pi_1(Y^X, u) \).
canonical isomorphisms corresponding to ones between the fundamental groups of the various path components.

1.2 The homomorphism $\Delta(f, u)$

Definition. A path continuous map of function spaces is a map $f : Y^X \to Z^W$ such that $F : Y^X \times I \to Z^W \times I$, defined by $F(l)(w, t) = f(l(w), t)$, is a map. A path continuous homotopy is a homotopy $\phi_t : Y^X \to Z^W$ such that $\Phi_t : Y^X \times I \to Z^W \times I$, defined as above, is a homotopy.

Definition. An $H_0$-space is a function space, $Y^X$, which is an $H$-space† whose structure maps and homotopies are path continuous. A homotopy abelian $H_0$-space is one with a path continuous homotopy abelian structure.

The only $H_0$-spaces which we consider in this and a subsequent paper are described in the following Proposition:

**Proposition 1.2.1.** The $H$-space $Y^X$ is an $H_0$-space if either

(i) the $H$-structure of $Y^X$ is induced by an $H$-structure on $Y$ or an $H'$-structure‡ on $X$, or

(ii) $X$ is a CW complex.

Suppose that $Y^X$ is an $H_0$-space, and $u : X \to Y$. Define $u_\gamma : Y^X \to Y^X$ by $u_\gamma(v) = vu$, then $u_\gamma$ is a path continuous homotopy equivalence with inverse $(u^{-1})_\gamma$. Now define $u_\delta$ to be the composite isomorphism

$$u_\delta : \pi_1(Y; u) \xrightarrow{(u^{-1})_\gamma} \pi_1(Y; u \cdot u^{-1}) \approx \pi_1(Y; *)$$

where the latter isomorphism is induced by a homotopy $u \cdot u^{-1} \sim *$; since $Y^X$ is an $H_0$-space, $\pi_1(Y; *)$ is abelian, and $u_\delta$ is independent of the choice of homotopy. Thus $u_\gamma$ and $u_\gamma^{-1}$ may be defined by $u_\gamma(\xi) = \{K + \xi \cdot u^{-1} - K\}$ and $u_\gamma^{-1}(\xi) = \{L + \xi \cdot u - L\}$ where $K$ and $L$ are homotopies $* \sim u \cdot u^{-1}$ and $u \sim * \cdot u$ respectively.

A proof of the following Lemma is indicated in §1.5.

**Lemma 1.2.2.**

(i) $u_\delta$ is well defined.

(ii) If $F : u \sim v$ is a homotopy, then $u_\delta = v_\delta F_\#$.

(iii) $(*)_\delta$ is the identity.

(iv) Let $X' \to X \to Y \to Y'$ be maps: if $Y$ is an $H$-space then $(ug)_\gamma g^* = g^* u_\gamma$; if $X$ is an $H'$-space then $(hu)_\delta h_* = h u_\gamma$.

(v) If $X$ is an $H'$-space and $Y$ is an $H$-space, then the isomorphisms $u_\gamma$ and $u_\gamma^{-1}$ are independent of the structure used to define them.

**Corollary 1.2.3.** If $Y^X$ is an $H_0$-space, and $u, v : X \to Y$ are homotopies, then the set of homotopy classes of such homotopies, regarded as functions $X \rtimes I \to Y$ is bijective with $[X, \Omega Y] \approx [SX, Y]$.

† A space $G$ with a homotopy associative multiplication $\phi_G : G \times G \to G$, homotopy identity $*$, and homotopy inverse $\psi_G : G \to G$.

‡ A space $M$ with a homotopy associative comultiplication $\phi_M : M \to M \vee M$, homotopy identity $*$, and homotopy inverse $\psi_M : M \to M$. 
Suppose that $A^X$ and $B^X$ are $H_0$-spaces, and let maps $u : X \to A$ and $f : A \to B$ be given.

**Definition.** The homomorphism $\Delta(f, u) : [X, \Omega A] \to [X, \Omega B]$ is the composite function

$\Delta(f, u) : [X, \Omega A] \xrightarrow{f_*} [X, \Omega B]$.

The function $\Delta(f, u)$ depends only on the homotopy classes of $f$ and $u$. In case $fu$ is nullhomotopic, $\Delta(f, u) = F_* f_* u_p^{-1}$, where $F : fu \sim *$ is a homotopy. The homomorphism $F_* f_* u_p^{-1}$ is defined without restricting $B^X$ to be an $H_0$-space, though it will in general depend on the choice of the homotopy $F$.

If $A$ and $B$ are $H$-spaces and also $X$ is an $H'$-space, then $\Delta(f, u)$ is independent of the structures used to define it by 1.2.2.

1.3 The classification theorem

Let $p : P_f \to A$ be the principle fibre map induced from $\pi : PB \to B$ by a map $f : A \to B$; thus $P_f = \{(a, l) \in A \times PB : f(a) = l(1)\}$. The fibre is $\Omega B$. Let $\tilde{p} : P_f \to PB$ denote the projection.

![Diagram](image)

The $H$-space $\Omega B$ acts on the left of $P_f$ by $\lambda : \Omega B \times P_f \to P_f$, where $m(\lambda, (a, l)) = (a, \lambda + l)$. A map $u : X \to A$ can be lifted to $P_f$ if, and only if, $f_* (u) = 0$. Let $[X, P_f : (u)]$ denote the set of homotopy classes, in $[X, P_f]$, of liftings of $u$; clearly it depends only on the homotopy class of $u$. The function $m_* : [X, \Omega B] \times [X, P_f : (u)] \to [X, P_f : (u)]$ determines a transitive abelian group action of $[X, \Omega B]$ on the set $[X, P_f : (u)]$ (see [13] for full details). The action is not generally effective although each element is either the identity or has no fixed points: the kernel of the action is $F_* f_* \pi^X_1 (A; u)$ where $F$ is a homotopy $fu \sim *$ (the proof is similar† to that of Theorem 1 of [9]).

Using Lemma 1.2.1, the following results are immediate.

**Theorem 1.3.1.** If $A^X$ and $B^X$ are $H_0$-spaces, there is a bijection

$$\bigcup_{(u) \in \ker f_*} \coker \Delta(f, u) \leftrightarrow [X, P_f].$$

Let $U : X \to P_f$ be a lifting of $u$. Define $m_U : [X, \Omega B] \to [X, P_f]$ by $m_U[\xi] = m_* [\xi, U]$.

**Corollary 1.3.2.** The following sequence is exact:

$$[X, \Omega A] \xrightarrow{\Delta(f, u)} [X, \Omega B] \xrightarrow{m_U} [X, P_f] \xrightarrow{p_*} [X, A] \xrightarrow{f_*} [X, B].$$

Here, for example, $[X, A]_u$ denotes $[X, A]$ with the class of $u$ distinguished.

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† Nomura's statement differs from ours by an exponential map and may not be valid for arbitrary $X$.  

In case \( u = * \), this is the sequence 2.8' of [13].

1.4 Properties of \( \Delta(f, u) \)

In this section, the homomorphism \( \Delta(f, u) : [X, \Omega A] \to [X, \Omega B] \) is analysed. The results that follow will be applied in \( \S 2 \) to calculate \( \Delta(f, u) \) in case \( A \) and \( B \) are products of Eilenberg–MacLane spaces. Proofs of the following properties are indicated in \( \S 1.5 \).

**Triviality Theorem 1.4.1.** \( \Delta(f, *) = (\Omega f)_* \)

Given two \( H_0 \)-spaces \( A \) and \( C \), we define an \( H_0 \)-structure on \((A \times C)^X\) in the obvious way: the multiplication is the composite
\[
(A \times C)^X \times (A \times C)^X \xrightarrow{1 \times 1 \times 1} (A \times C)^X \times (A \times C)^X \to (A \times C)^X,
\]
where \( \tau \) is the twisting map \( \tau(c, a) = (a, c) \).

**Cartesian Product Theorem 1.4.2.** Let

\[
A, B, C \text{ and } D \text{ be } H_0 \text{-spaces, } X \to A \to B \text{ and } X \to C \to D; \text{ then}
\]

\[\Delta(f_1 \times f_2, (u_1, u_2))(\xi_1, \xi_2) = (\Delta(f_1, u_1)(\xi_1), \Delta(f_2, u_2)(\xi_2)).\]

**Definition.** A class \( \alpha = \{f\} \in [A, B] \) is \( X \)-primitive if \( f : [X, A] \to [X, B] \) is a homomorphism. Further \( \alpha \) is primitive if it is \( X \)-primitive for all spaces \( X \); equivalently \( A \) and \( B \) are \( H \)-spaces and \( \phi^*_\alpha = p^*_1 + p^*_2 \in [A \times A, B] \).

**Primitivity Theorem 1.4.3.** If \( f : A \to B \) is \( X \)-primitive, then

\[\Delta(f, u) = (\Omega f)_* \text{ for } u : X \to A.\]

In particular this implies that \( \Delta(1, u) = 1, \Delta(*, u) = 0, \Delta(p, u) = p \) and \( \Delta(d, u) = d \) for projections \( p \) and diagonal \( d \).

**Corollary 1.4.4.** If \( X \) is an \( H' \)-space, then \( \Delta(f, u) = (\Omega f)_* \).

**Composition Theorem 1.4.5.** Let \( g : A \to B \) and \( f : B \to C \) be maps, then

\[\Delta(g \circ f, u) = \Delta(f, g \circ u) \circ \Delta(g, u) : [X, \Omega A] \to [X, \Omega C].\]

We next wish to consider the possible linearity of \( \Delta \) in its first variable. Given \( u : X \to A \), let \( u_\Box \) be the endomorphism of \([X, \Omega A]\) defined by \( u_\Box(\zeta) = \{F + (u \cdot \zeta) \cdot u^{-1} - F\} \) where \( F \) is a homotopy \( \sim (u \cdot *) \cdot u^{-1} \).

**Lemma 1.4.6.**

(i) \( u_\Box \) depends only on the homotopy class of \( u \).

(ii) Let \( u, v : X \to A \), then \( (u \circ v)_\Box = u_\Box \circ v_\Box \).

(iii) \( u_\Box \) is an automorphism.

(iv) \( (*)_\Box = 1 \).

(v) If \( A \) is homotopy abelian, then \( u_\Box = 1 \).
LEMMA 1.4.7. Let $A$ be an $H$-space, $X^{(u_1, u_2)} \to A \times A \xrightarrow{\phi} A$, and $X^{u} \to A \xrightarrow{\psi} A$, then

(i) $\Delta(\phi, (u_1, u_2))(\xi_1, \xi_2) = \{\xi_1\} + (u_1)\square\{\xi_2\}$

(ii) $\Delta(\psi, u) = -(u^{-1})\square$.

ADDITIVITY THEOREM 1.4.8. Let $B$ be an $H$-space.

(i) $\Delta(f \cdot g, u) = \Delta(\phi(f \cdot g)d, u) = \Delta(f, u) + \Delta(g, u)$.

(ii) $\Delta(f^{-1}, u) = \Delta(\psi f, u) = -((f\cdot u)^{-1})\square\Delta(f, u)$.

(iii) $\Delta(\phi(\xi_1 \times \xi_2), (u_1, u_2))\{\xi_1, \xi_2\} = \{\Delta(f_1, u_1)\{\xi_1\} + (f_1u_1)\square\Delta(f_2, u_2)\{\xi_2\}$.

(iv) If $fu \sim gu$, then $\Delta(f \cdot g^{-1}, u) = \Delta(f, u) - \Delta(g, u)$.

In the applications we are primarily interested in the case that $B$ is homotopy abelian. In this case $\Delta$ is linear in its first variable. It is not generally linear in its second variable (see Theorem 2.4.1.).

Let $A$, $B$ and $C$ be $H$-spaces.

Definition. A bimultiplication is a map $f: A \times B \to C$ for which the following diagrams are homotopy commutative:

(i)

\[
\begin{array}{c}
A \times B \times B \\
\downarrow d \times 1 \times 1 \\
A \times A \times B \times B \\
\downarrow 1 \times 1 \times 1 \\
A \times B \times A \times B \\
\downarrow f \times f \\
C \times C \\
\downarrow \phi_c \\
C
\end{array}
\]

(ii)

\[
\begin{array}{c}
A \times A \times B \\
\downarrow 1 \times 1 \times d \\
A \times A \times B \times B \\
\downarrow 1 \times 1 \times 1 \\
A \times B \times A \times B \\
\downarrow f \times f \\
C \times C \\
\downarrow \phi_c \\
C
\end{array}
\]

It is elementary that $f$ is a bimultiplication if and only if the induced function $f_*: [X, A] \times [X, B] \to [X, C]$ is bilinear for every space $X$. Also if $f$ satisfies (i) and (ii), then the composite $A \vee B \subset A \times B \to C$ is nullhomotopic.
Define $F^A : A \times \Omega B \rightarrow C^I$ and $F^B : \Omega A \times B \rightarrow C^I$ by $F^A(a, l)(t) = f(a, l(t))$ and $F^B(\lambda, b)(t) = f(\lambda(t), b)$. Let $G^A : A \rightarrow C^I$ and $G^B : B \rightarrow C^I$ correspond to nullhomotopies of the composites $A \subset A \times B \rightarrow C$ and $B \subset A \times B \rightarrow C$, respectively. Now define:

$$f^A = G^A + F^A - G^A : A \times \Omega B \rightarrow \Omega C$$

$$f^B = G^B + F^B - G^B : \Omega A \times B \rightarrow \Omega C.$$

The functions $f^A$ and $f^B$ are continuous, and their homotopy classes depend only on the class of $f$.

If $A$ and $B$ are CW complexes, and $C$ an Eilenberg–MacLane complex, then, by 1.6 of [12], $f$ may be chosen constant on $A \vee B$ and consequently $f^A = F^A$ and $f^B = F^B$; this is the situation we consider in the applications.

**BIMULTIPLICATION THEOREM 1.4.9.** Let $f : A \times B \rightarrow C$ be a bimultiplication, and $(u_1 \times u_2) : X \rightarrow A \times B$, then

$$\Delta(f, (u_1 \times u_2)) = \{f^A(u_1 \times \zeta_2) + f^B(\zeta_1 \times u_2) \}.$$

In §2.3 this Theorem is applied to the case that $f$ is a cup product map (see §2.2); in this case $f^A$ and $f^B$ also correspond to cup product maps after making certain identifications.

### 1.5 Proofs

We now indicate the proofs of 1.2.2, and the results of §1.4. The proofs are all somewhat similar, so the more elementary ones are omitted.

Given an $H_0$-space, $Y^X$, and homotopic maps $u, v : X \rightarrow Y$, then there is a canonical isomorphism (see §1.1) $\lambda : \pi_1(Y; u) \rightarrow \pi_1(Y; v)$, which is independent of the choice of homotopy $J : u \sim v$.

**Definition.** A self homotopy is a homotopy $J_1 : X \rightarrow Y$ with $J_0 = J_1$.

The following Lemma is used implicitly in the remainder of the section, its proof is straightforward:

**LEMMA 1.5.1.** Let $F : X \times I \rightarrow Y^I$ be a homotopy between $F_0$ and $F_1$ where, for each $t$, $F_t = F(\_, t) : X \rightarrow Y^I$ is a self homotopy, then, up to the canonical isomorphism described above, $F_0$ and $F_1$ represent the same element of the track group.

Also we note that, up to the canonical isomorphism, $\Delta(f, u)(\zeta)$ is represented by $(f^I(\zeta, u))(f(u))^{-1}$.

**Proof of 1.2.1.** (i)-(iv) are straightforward, (v) follows on considering homotopy commutative diagrams

$$
\begin{array}{ccc}
X \overset{\phi}{\longrightarrow} X \vee X & \text{and} & Y \vee Y \overset{e}{\longrightarrow} Y \\
\downarrow 1 & & \downarrow 1 \\
X \overset{d}{\longrightarrow} X \times X & & Y \overset{\phi}{\longrightarrow} Y \\
\end{array}
$$

where $d$ is the diagonal and $c$ the collapsing map.
Proof of Primitivity theorem. By primitivity the standard representative may be deformed modulo the canonical isomorphism as follows:

\[ f^I(\xi \cdot u) \cdot (f(u))^{-1} \sim (f^I(\xi) \cdot f(u)) \cdot (f(u))^{-1} \sim f^I(\xi). \]

Proof of Composition theorem.

\[
\Delta(fg, u) = (fgu)_* (fgu)_*^{-1} u^{-1} \\
= (fgu)_* f^I_*(gu)_*^{-1} (gu)_* g_*/u_*^{11} \\
= f^I(g, u) - \Delta(f, g) \cdot \Delta(g, u).
\]

Proof of 1.4.7. (i) The standard representative for \( \Delta(\phi(u_1, u_2)) \{\xi_1, \xi_2\} \) is, modulo the canonical isomorphism,

\[
(\xi_1 \cdot u_1) \cdot (\xi_2 \cdot u_2) \cdot (u_2^{-1} \cdot u_1^{-1}) \sim \xi_1 \cdot u_1 \cdot \xi_2 \cdot u_1^{-1}.
\]

The result follows on writing \( \{\xi_1, \xi_2\} = \{\xi_1, \ast\} + \{\ast, \xi_2\}. \)

Proof of 1.4.8. (i)–(iii) follow from Lemma 1.4.7, by iterating the Composition theorem, and (iv) follows from (i) and (ii) and Lemma 1.4.6.

Proof of Bimultiplication theorem. The standard representative for \( \Delta(f, u) \{\xi_1, \ast\} \) is, modulo the canonical isomorphism,

\[
f^I(\xi_1 \cdot u_1, \ast \cdot u_2) \cdot (f(u_1, u_2))^{-1} \sim f^I(\xi_1, \ast \cdot u_2) \cdot f(u_1, \ast \cdot u_2) \cdot (f(u_1, u_2))^{-1} \\
\sim f^I(\xi_1, u_2).
\]

§2. COHOMOLOGY

2.1

In case \( h : A \to B \) where \( B \) is a product of Eilenberg–MacLane complexes, we give, in §2.5, a method of calculating \( \Delta h \).

The first problem encountered is to relate \( \cup^{K(G, m)} \) and \( \cup^{K(\pi, n)} \), defined as in §1.4 for the cup product map \( \cup : K(G, m) \times K(\pi, n) \to K(G \otimes \pi, m + n) \), to the obvious cup product maps. In §2.3 we show that, after making certain identifications, these maps differ, up to homotopy, only in sign from cup product maps. Using this result and a decomposition of the cohomology of a cartesian product we then show how \( \Delta h \) may be calculated in terms of \( \varphi \{h\} \in [A \times A, B] \). In particular, if \( A \) also is a product of Eilenberg–MacLane spaces, we show how \( \Delta h \) may be calculated explicitly once the decomposition of \( h \) into elementary cohomology operations has been determined. We conclude with some special cases.

2.2 Cohomology and the evaluation map

We first recall some properties of cohomology groups and Eilenberg–MacLane complexes. Let \( \pi \) be a finitely generated abelian group and \( K_m = K(\pi, m) \) an Eilenberg–MacLane complex.

By the universal coefficient theorem, and the Hurewicz theorem, we have

\[ H^m(K(\pi, m); \pi) \cong \text{Hom}(\pi, \pi). \]
The fundamental class $i_m$ of $K(\pi, m)$ is the inverse image of the identity function under the above isomorphism.

Define $\gamma : [X, K(\pi, m)] \to H^m(X; \pi)$ by $\gamma(f) = f^*(i_m)$, then $\gamma$ is a natural isomorphism in case $X \in \mathcal{W}$.†

Let $\text{Op}(\prod_i H^m(\cdot; G_i), H^n(\cdot; \pi))$ denote the set of cohomology operations between the functors $\prod_i H^m(\cdot; G_i)$ and $H^n(\cdot; \pi)$, the former being a finite product. Let $p_j : \prod K_{m_i} \to K_{m_j}$ be the projection. Define $\Xi : \text{Op}(\prod_i H^m(\cdot; G_i), H^n(\cdot; \pi)) \to H^n(\prod K_{m_i}; \pi)$ by $\Xi(T) = T(\ldots, p^*_i i_m, \ldots)$, then $\Xi$ is an isomorphism.

Given $a \in H^m(X; \pi)$ and $b \in H^n(X; G)$, the cup product $a \cup b \in H^{m+n}(X; \pi \otimes G)$ is well defined using the Eilenberg-Zilber Theorem. In particular let $x \in H^m(X; \pi)$ and $y \in H^n(Y; G)$; we denote by $x \cup y$ the cup product $p_1^* x \cup p_2^* y \in H^{m+n}(X \times Y; \pi \otimes G)$ where $p_1, p_2$ are the projections onto the factors of $X \times Y$.

Definition. The cross product of $x \in H^m(X; \pi)$ and $y \in H^n(Y; G)$ is

$$x \times y \in H^{m+n}(X \times Y; \pi \otimes G).$$

By Lemma 1.6 of [12], for $X, Y \in \mathcal{W}$, the collapsing map $k : (X \times Y, X \vee Y) \to (X \ast Y, \ast)$ induces isomorphisms in cohomology, and $k^*$ embeds $H^*(X \ast Y)$ in $H^*(X \times Y)$ as a direct summand. Let $\tilde{t}_m \times \tilde{t}_n \in H^{m+n}(K_m \times K_n; \pi \otimes G)$ be the cross product of the fundamental classes and let $\tilde{\cup} : K(\pi, m) \times K(G, n) \to K(\pi \otimes G, m + n)$ be any map in the homotopy class $\gamma^{-1} k^* \gamma^{-1}(\tilde{t}_m \times \tilde{t}_n)$. Denote by $\cup$ the composite $K_m \times K_n \to K_m \otimes K_n \xrightarrow{\tilde{\cup}} K_{m+n}$.

Definition. A cup product map is any map $\cup : K_m \times K_n \to K_{m+n}$ defined as above.

From the bilinear nature of the cohomology cup product, it follows easily that $\cup$ is a bimultiplication.

Let $p : CX \to X \ast S_1$ be the identification map from the cone on a space $X$ to the suspension. In case $X \in \mathcal{W}$, we define the suspension isomorphism $S^*$ to be the composite

$$S^* : H^{n-1}(X; \pi) \to H^n(CX, X; \pi) \to H^n(X \ast S^1; \pi).$$

The suspension may also be defined in terms of a cross product. An easy calculation yields the following Lemma.

Lemma 2.2.1. Let $\kappa \in H^{n-1}(X; \pi)$ and let $t_1 \in H^1(S^1; \pi)$ be the fundamental class. Then $k^* S^*(\kappa) = (-1)^{n-1}(\tilde{\kappa} \times t_1)$.

The fundamental class $i_n \in H^n(K(\pi, n); \pi)$ defined above depends on a choice of the isomorphism $\pi_0(\pi, n) \cong \pi$. Thus $\gamma$ also depends on this choice. In order to proceed further, we need to choose an equivalence between $\Omega K(\pi, n)$ and $K(\pi, n-1)$. The set of homotopy equivalences $\lambda : \Omega K_n \to K_{n-1}$ fall into several homotopy classes, one for each automorphism of $\pi$. Following Brown [4], we distinguish one of these by defining

† $\mathcal{W}$ is the category of spaces having the (base point preserving) homotopy type of a CW complex.
\[ \lambda : \Omega K_n \to K_{n-1} \] to be a homotopy equivalence chosen such that the following diagram of isomorphisms is commutative:

**Diagram 2.2.2.**

\[
\begin{array}{ccc}
[\Omega K_n \times S^1, K_n] & \xrightarrow{\gamma} & H^n(\Omega K_n \times S^1; \pi) \\
\downarrow{\mu} & & \downarrow{\phi} \\
[\Omega K_n, \Omega K_n] & \xrightarrow{\lambda} & H^{n-1}(\Omega K_n; \pi)
\end{array}
\]

Here \( \mu(f) = \{g\} \) where \( g(x)(t) = f(x, t) \). Clearly \( \lambda \) is in the class \( \gamma^{-1} S^1 \gamma \mu^{-1}(\chi_n) \) where \( \chi_n : \Omega K_n \to \Omega K_n \) is the identity map. Define \( \lambda^{-1} \) to be an inverse equivalence to \( \lambda \).

Recall that the suspension of a cohomology operation \( T \) is the cohomology operation \( \sigma T \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
\prod_i H^m(X \times S^1; G) & \xrightarrow{T} & H^m(X \times S^1; \pi) \\
\downarrow{S^*} & & \downarrow{S^*} \\
\prod_i H^{m-1}(X, G) & \xrightarrow{\sigma T} & H^{n-1}(X; \pi)
\end{array}
\]

Suppose that \( \gamma(t) = T(\ldots, p_{t}^* i_{m}, \ldots) \) and \( \gamma(\sigma t) = \sigma T(\ldots, p_{t}^* i_{m-1}, \ldots) \), where \( t \in \text{Map}(\Pi K_m, K_n) \) and \( \sigma t \in \text{Map}(\Pi K_m, K_n) \).

**Lemma 2.2.3.** \( \{\sigma t\} = \{\lambda(\Omega t)\lambda^{-1}\} \).

The proof is omitted (cf. Theorem 4 of [4]).

Consider now the commutative diagram of isomorphisms:

**Diagram 2.2.4.**

\[
\begin{array}{ccc}
[K_{n-1} \times S^1, K_n] & \xrightarrow{\gamma} & H^n(K_{n-1} \times S^1) \\
\downarrow{(\lambda \ast 1)^*} & & \downarrow{(\lambda \ast 1)^*} \\
[\Omega K_n \times S^1, K_n] & \xrightarrow{\gamma} & H^n(\Omega K_n \times S^1) \\
\downarrow{\mu} & & \downarrow{\phi} \\
[\Omega K_n, \Omega K_n] & \xrightarrow{\lambda} & H^{n-1}(\Omega K_n) \\
\downarrow{\lambda} & & \downarrow{\lambda} \\
[\Omega K_n, K_{n-1}] & \xrightarrow{\gamma} & H^{n-1}(\Omega K_n) \\
\downarrow{(\lambda \ast 1)^*} & & \downarrow{(\lambda \ast 1)^*} \\
[K_{n-1}, K_{n-1}] & \xrightarrow{\gamma} & H^{n-1}(K_{n-1})
\end{array}
\]
Let $\tilde{\cup} : K_{n-1} \times S^1 \to K_n$ be a cup product map; then $\gamma\{\tilde{\cup}\} = (-1)^{n-1} S^n(\iota_{n-1})$. Recall that the evaluation map $\varepsilon : \Omega K_n \times S^1 \to K_n$ is given by $\varepsilon(\iota, t) = \iota(t)$. We denote by $\bar{\varepsilon}$ the composite

$$\Omega K_n \times S^1 \to \Omega K_n \times S^1 \to K_n.$$  

Since $\lambda$ is a homotopy equivalence,

$$\gamma \lambda^{*-1} \iota \mu (\varepsilon) = \gamma \lambda^{*-1} \lambda \{1\} = \gamma\{1\} = \iota_{n-1} \in H^{n-1}(K_{n-1}).$$

We have immediately the Lemma:

**Lemma 2.2.5.** $(\lambda \times 1)^* \{ \cup \} = (-1)^{n-1}\{ \bar{\varepsilon} \}$.

### 2.3 Properties of $A$ involving cup products

Given a cup product map $\cup : K_m \times K_n \to K_{m+n}$, we define $\cup^K : K_m \times \Omega K_n \to \Omega K_{m+n}$ by $\cup^K(\iota, x, l)(t) = \cup(\iota, l(t))$ as in §1.4. The inner figures of the following diagram are commutative:

![Diagram](attachment:image.png)

By simple geometry

$$\cup(\iota \times 1)(1 \times \lambda \times 1) \sim (-1)^n \cup(\lambda \times 1)(\cup^K \times 1),$$

which is equivalent to

$$(\iota_m \cup \lambda \cup 1) \cup \iota_1 = (-1)^n(\gamma(\lambda \cup^K 1)) \cup \iota_1.$$  

Now by Lemma 2.2.1, $\varphi : H^{q-1}(X) \to H^q(X \times S^1)$, given by $\varphi(\kappa) = \kappa^{*-1}(\kappa \cup 1)$, is an isomorphism for $q \geq 2$. From this we deduce that $\cup(1 \times \lambda) \sim (-1)^n \lambda \cup^K : K_m \times \Omega K_n \to K_{m+n-1}$ for $n \geq 2$.

Assume now that $(u_1 \times u_2) : X \to K_m \times K_n$, and that $X$ is path connected, then, by 1.4.9 we have the Lemma:

**Lemma 2.3.1.**

$$\Delta(\cup, (u_1 \times u_2) d)\{0, \zeta_2\} = (-1)^n \lambda^{*-1} \{ \cup(u_1 \times \lambda \zeta_2) d \} \quad n \geq 2$$

$$= 0 \quad n = 1.$$
In case $n = 1$ or $m = 1$, we now agree to identify respectively $\lambda^{-1}_* \{ (u_1 \times \lambda \zeta_2) d \}$ or $\lambda^{-1}_* \{ \cup (\lambda \zeta_1 \times u_2) d \}$ with 0.

Let $\tau : K_m \times K_n \to K_n \times K_m$ be the twisting map, $t : G \otimes \pi \to \pi \otimes G$ the twisting homomorphism, and $\cup' : K(G, n) \times K(\pi, m) \to K(G \otimes \pi, n + m)$ a cup product map. Then we have

$$\Delta(t \cup \tau + (-1)^{m+1} \cup, (u_1 \times u_2) d) = 0.$$ 

Expanding this by using the properties of $\Delta$ gives

$$\Delta((u_1 \times u_2) d) \{ \xi_1, * \} = \lambda^{-1}_* \{ \cup (\lambda \zeta_1 \times u_2) d \}.$$ 

Collecting the above results together, we have the Theorem:

**CUP PRODUCT THEOREM 2.3.2.** If $\cup : K_m \times K_n \to K_{m+n}$, then

$$\Delta((u_1 \times u_2) d) \{ \xi_1, * \} = \lambda^{-1}_* \{ \cup (\lambda \zeta_1 \times u_2) d \}.$$ 

This result is now generalized to multiple cup products. Let $\cup^p : K_{m_1} \times \ldots \times K_{m_p+1} \to K_{m_1 + \ldots + m_p + 1}$ be a map representing a multiple cup product. We assume in fact that $\cup^p$ has been defined inductively by the formula $\cup^p = \cup(1 \times \ldots \times 1)$.

**COROLLARY 2.3.3.**

$$\Delta((x_1 \times \ldots \times x_{p+1}) d^p) \{ \xi_1, \ldots, \xi_{p+1} \} = \sum_{r=1}^{p+1} (-1)^{m+\ldots+mr-1} \lambda^{-1}_* \{ \cup^p (u_1 \times \ldots \times (\lambda \zeta_r) \times \ldots \times u_{p+1}) d^p \}.$$ 

The proof proceeds by induction on $p$, and is straightforward.

As an easy consequence of this and the anticommutativity of the cup product, we have the Corollary:

**COROLLARY 2.3.4.** Let $\cup : K(G, n) \times K(G, n) \to K(G \otimes G, 2n)$ be a cup product map, and $u : X \to K_n$, then

$$\Delta(\cup^p d^p, u) \{ \zeta \} = \begin{cases} 
(p + 1) \lambda^{-1}_* \{ \cup^p ((\lambda \zeta) \times u \times \ldots \times u) d^p \} & \text{for } n \text{ even} \\
\lambda^{-1}_* \{ \cup^p ((\lambda \zeta) \times u \times \ldots \times u) d^p \} & \text{for } n \text{ odd } p \text{ even} \\
0 & \text{for } n \text{ odd } p \text{ odd}.
\end{cases}$$

### 2.4 The calculation of $\Delta h$

We now obtain a method of calculating $\Delta h$ in case $h : A \to B$ where $B$ is a product of Eilenberg–MacLane spaces. The expansion for $\Delta h$ is determined by the multiplication in the space $A$. We suppose that $B = K_n$ is an Eilenberg–MacLane space; the extension to the more general case is immediate.

Let $h : A \to K_n$ be a map and $\varphi : A \times A \to A$ the $H$-space multiplication. Then $\varphi^* : [A, K_n] \to [A \times A, K_n]$ and from the Appendix of [10] we see that

$$h \varphi \sim (h \varphi_1 + h \varphi_2 + \Sigma f_\# \cup (a \times b) + \Sigma \delta^\# \cup (c \times c),$$
A HOMOTOPY CLASSIFICATION OF MAPS INTO AN INDUCED FIBRE SPACE

where the $f_\#$ are maps induced by homomorphisms of coefficient groups, the $\delta_\#$ are Bockstein coboundary maps, and $a, b, c$ and $e$ are maps into Eilenberg-MacLane spaces, of positive dimension (see below).

**Definition.** The dimension of $a \in \text{Map}(A, K_i)$ is $t$.

**Theorem 2.41.** Let $X$ be path connected, then

$$
\Delta(h, u)(\zeta) = (\Omega h)_\#(\zeta) + \sum (-1)^{\dim a} \lambda_\#^{-1} f_\# \{ \cup (au \times \lambda(\Omega b)(\zeta))d \}
+ \sum (-1)^{1+\dim c} \lambda_\#^{-1} \delta_\# \{ \cup (cu \times \lambda(\Omega e)(\zeta))d \}.
$$

**Proof.** Let $(u \times *)d : X \to A \times A$ and $(* \times *)d : X \to \Omega A \times \Omega A$, then we have

$$
\Delta(hp_1 + hp_2 + \sum f_\# \cup (a \times b) + \sum \delta_\# \cup (c \times e), (u \times *)d)(\zeta)
= \Delta(h, u)(\zeta) + \Delta(h, u)(\zeta)
= \Delta(h, \varphi(u \times *)d) \circ \Delta(h, u)(\zeta).
$$

On the other hand, since $\varphi(\delta_\#) = -\delta_\#$ and $f_\# = f_\#$, we have, on using 2.2.3,

$$
\Delta(hp_1, (u \times *)d)(\zeta) = \Delta(h, u)(\zeta) = 0,
$$

$$
\Delta(hp_2, (u \times *)d)(\zeta) = \Delta(h, u)(\zeta) = (\Omega h)_\#(\zeta),
$$

$$
\Delta(f_\# \cup (a \times b), (u \times *)d)(\zeta) = (-1)^{\dim a} (\Omega f_\#)_\# \lambda_\#^{-1} \{ \cup (au \times \lambda(\Omega b)(\zeta))d \},
$$

and

$$
\Delta(\delta_\# \cup (c \times e), (u \times *)d)(\zeta) = (-1)^{\dim c} (\Omega \delta_\#)_\# \lambda_\#^{-1} \{ \cup (cu \times \lambda(\Omega e)(\zeta))d \},
$$

where for example $\lambda(b, *)(\zeta)$ is a representative for $\Delta(b, *)(\zeta)$. The Theorem now follows easily.

**2.5 The cohomology interpretation**

Consider now the case that $X \in \mathcal{W}$ and $A$ is a finite product of Eilenberg-MacLane spaces; we replace $\Delta(h, u) : \prod_i [X, \Omega K_i] \to [X, \Omega K_i]$ by a function involving cohomology groups. Let $T$ be the cohomology operation satisfying $\gamma^{-1}\Xi(T) = \{ h \} \in \prod K_i$. Given $s \geq 2$, define

$$
\nabla(T, \gamma[u]) : \Pi H^{s-1}(X; A_s) \to H^{s-1}(X; \pi)
$$

so as to make the following diagram commutative:

$$
\begin{array}{ccc}
\prod_i [X, \Omega K_i] & \xrightarrow{\Delta(h, u)} & [X, \Omega K_i] \\
\downarrow \lambda_\# & & \downarrow \lambda_\# \\
\prod_i [X, K_{s-1}] & \xrightarrow{\gamma} & [X, K_{s-1}] \\
\downarrow \gamma & & \downarrow \gamma \\
\prod_i H^{s-1}(X; A_s) & \xrightarrow{\nabla(T, \gamma[u])} & H^{s-1}(X; \pi)
\end{array}
$$
Then if $T$ is a $k$-variable cohomology operation,

$$VT : \prod_i H^n(\ , A_i) \times \prod_i H^{n-1}(\ , A_i) \to H^{n-1}(\ , \pi),$$

defined by $VT(\mu, v) = \nabla(T, \mu)(v)$, is a $2k$-variable operation.

The properties of $\nabla$ can easily be translated from those of $\Delta$ by using the correspondences $\gamma$ and $\Xi$.

The cohomology operation $T$ satisfies the identity

$$T(p_1^* + p_2^*) - T p_1^* - T p_2^* = \Sigma f_\alpha(A \cup B) + \Sigma \delta^\beta(C \cup E),$$

where $A, B, C,$ and $E$ are the operations corresponding to $a, b, c, \text{ and } e$. The right-hand side of this identity is the deviation from additivity of $T$. Let $\text{dim } A$ denote the dimension of the image cohomology functor of $A$. Corresponding to Theorem 2.41 we have the following Theorem:

**Theorem 2.5.1.** Let $T$ be a $k$-variable cohomology operation, and $X$ a path connected CW complex, then†

$$\nabla(T, \mu)(v) = \omega T(v) + \Sigma (-1)^{\text{dim } A} f_\alpha(A(\mu) \cup \omega B(v))$$

$$+ \Sigma (-1)^{1 + \text{dim } C} \delta^\beta(C(\mu) \cup \omega E(v)).$$

We now single out a class of operations which is to provide the basis for the calculation of $\nabla T$ in the general case. These are the **elementary operations**; they are of three types.

(a) **Group manipulations.** Let $\Theta$ and $\Phi$ be two functors on the category of spaces with base point, each determining a product of cohomology groups. The operations we require are diagonal and addition

$$d : \Theta \to \Theta \times \Theta, \quad a : \Theta \times \Theta \to \Theta$$

and the identity and zero

$$1 : \Theta \to \Theta, \quad 0 : \Theta \to \Phi.$$

(b) **Steenrod one-variable operations.** These are:

(i) the coefficient homomorphism

$$f_* : H^*(\ , G) \to H^*(\ , G')$$

associated with any homomorphism $f : G \to G'$ of coefficient groups;

(ii) the Bockstein–Whitney coboundary

$$\delta^* : H^r(\ , G) \to H^{r+1}(\ , G)$$

associated with any exact sequence $0 \to G \to G' \to G'' \to 0$ of coefficient groups;

(iii) the cyclic reduced powers [15]

$$\mathcal{P}^i : H^q(\ , Z_p) \to H^{q+2i(p-1)}(\ , Z_p)$$

and the Pontrjagin powers ([16] and [17])

$$\mathcal{P}_p : H^{2q}(\ , Z_{p^k}) \to H^{2pq}(\ , Z_{p^{k+1}})$$

for each prime number $p$.

† In case the final dimension of a cohomology operation is 1, we identify its suspension with the trivial operation.
(c) *Cup products.* These are of the form
\[ \cup : H^p(\ ; A) \times H^q(\ ; \pi) \to H^{p+q}(\ ; A \otimes \pi). \]

Using results of Dold (5.5 of [5]), Steenrod (5.1 of [14]) and Steenrod and Thomas ([15]), it follows (c.f. §6 of [10]) that any many variable cohomology operation is generated, using composition and product, by the above elementary operations.

This result provides us with two slightly different methods of calculating \( \nabla T \): the first is to exhibit \( T \) in terms of elementary operations and to use the known deviations from additivity of these (see below) to calculate the deviation of \( T \), then to apply Theorem 2.5.1; the second is to calculate \( \nabla \) of each elementary operation, and then to write \( \nabla T \) in terms of these known results using the decomposition of \( T \) into elementary operations, and the properties of \( \nabla \). We now calculate \( \nabla \) for the elementary operations.

**Lemma 2.5.2.** The operations of group manipulations, coefficient homomorphism, Bockstein coboundary, and Steenrod squares and cyclic \( p^k \) powers are primitive operations and hence satisfy
\[ T(p_1^* + p_2^*) = Tp_1^* + Tp_2^*. \]

**Corollary 2.5.3.** The above operations satisfy \( \nabla T = \hat{\delta} T \).

**Lemma 2.5.4** (2.7 of [3] and 4.6 of [17]). The Pontrjagin \( k^\text{th} \) powers satisfy
\[ \Psi_k(p_1^* + p_2^*) = \Psi_k p_1^* + \Psi_k p_2^* + f_\sum_{0 < i < k} \cup((\cup^i 1 d^i 1 p_1^*) \times (\cup^i 1 d^i 1 p_2^*))d, \]
where \( \cup^i \) and \( d^i \) are the cup product and diagonal map iterated \( i \) times, and \( f : \otimes Z_{p^r} \to Z_{p^r} \to Z_{p^r+1} \) is the coefficient homomorphism.

Noting that \( \Psi_k = 0 \) in case \( k \) is an odd prime (2.7 of [16]) and that \( \Psi(\cup^0) = 0 \) for \( q \geq 1 \) (9.2 of [10]), we have the Corollary:

**Corollary 2.5.5.** For \( k \) an odd prime,
\[ \nabla(\Psi_k, \mu)(v) = f_* (\mu^{k-1} \cup v). \]
Since \( \Psi_2 = \Psi \) the Postnikov square (5.5 of [18]), we have the Corollary:

**Corollary 2.5.6.** \( \nabla(\Psi_2, \mu)(v) = \Psi(v) + f_*(\mu \cup v). \)

2.6 Some examples

(a) Let \( f : K(\pi, m) \times K(G, n) \to K(\pi \otimes G, m + n) \) be a cup product map. We determine \([S^p \times S^q, P_f] \) for several values of \( p, q, m \) and \( n \).

\[
\begin{array}{ccc}
S^p \vee S^q & \overset{u}{\longrightarrow} & P_f \\
S^p \times S^q & \overset{f}{\longrightarrow} & K_{m+n}
\end{array}
\]

(i) Let \( p = m, q = n - 1 \). First suppose \( m \neq 1, n, n - 1 \); we may take \( u = u_1 \times \ast \). Then \( \ker f_\ast \cong \pi \), and
\[
[S^m \times S^{n-1}, P_f : (u)] \leftrightarrow H^{m+n-1}(S^m \times S^{n-1}; \pi \otimes G)/\gamma(u_1) \cup H^{n-1}(S^{n-1}; G) \leftrightarrow \pi \otimes G/J(u_1) \otimes G.
\]
Here $J(u_i)$ denotes the image of $\gamma(u_i)$ under the isomorphism $H^m(S^m; \pi) \cong \pi$. For $m = 1$, $m \neq n$, $n - 1$, the only difference is ker $f_* = \pi \times G$.

Now let $m = n \geq 2$, then we may take $u = (u^1, u^2) : S^m \times S^{m-1} \to K^1_n \times K^2_n$. Here ker $f_* = \pi \times G$ and

$$[S^m \times S^{m-1}, P_f : (u)] \mapsto H^{2m-1}(S^m \times S^{m-1}; \pi \otimes G) \mapsto \gamma(u^2) \circ H^{m-1}(S^{m-1}; \pi) + (-1)^m \gamma(u^1) \circ H^{m-1}(S^{m-1}; G)$$

$$\mapsto \pi \otimes G \mapsto J(u^1) \otimes G + \pi \otimes J(u^2).$$

Here $t_*$ is the coefficient homomorphism induced by the twisting homomorphism $t : G \otimes \pi \to \pi \otimes G$.

Finally for $m = n - 1 \geq 2$, we may take $u = (u_1 + u_2, \ast) : S^{m-1} \times S^{m-2} \to K_m \times K_{m+1}$. Then ker $f_* = \pi \times \pi$ and

$$[S^{m-1} \times S^{m-2}, P_f : (u)] \mapsto H^{2m-3}(S^{m-1} \times S^{m-2}; \pi \otimes G) \mapsto \gamma(u_1) \circ H^{m}(S^{m-1}; G) + (-1)^m \gamma(u_2)$$

$$\mapsto \pi \otimes G \mapsto J(u_1) \otimes G + J(u_2) \otimes G.$$

Collecting these results together, we have

**Example 2.6.1.**

$$[S^n \times S^{n-1}, P_f : (u)] \leftarrow \bigcup_{x \in \pi} \pi \otimes G / x \otimes G \quad m \neq 1, n, n - 1$$

$$\leftarrow G \times \bigcup_{x \in \pi} \pi \otimes G / x \otimes G \quad m = 1, m \neq n, n - 1$$

$$\leftarrow \bigcup_{(x, y) \in \pi \times G} \pi \otimes G / x \otimes G + \pi \otimes G \quad m \geq 2$$

$$\leftarrow \bigcup_{(x_1, x_2) \in \pi \times \pi} \pi \otimes G / x_1 \otimes G + x_2 \otimes G \quad m = n - 1 \geq 2.$$

(iii) Let $p = m$, $q = n$. First suppose that $m \neq n$; we may take $u = u_1 \times u_2$ where $u_1 : S^n \to K_m$ and $u_2 : S^n \to K_n$. Then

ker $f_* \cong \Gamma = \ker$ (projection : $\pi \times G \to \pi \otimes G$).

If $fu \sim \ast$, then

$$[S^n \times S^n, P_f : (u)] \leftrightarrow [S^n \times S^n, \Omega K_{m+n}] / \Delta(f, u)[S^n \times S^n, \Omega K_m \times \Omega K_n].$$

In case $m, n \geq 2$ this set is trivial; in case $m = 1, n \geq 2$ it is (using 2.3.2 in case $n = 2$) in correspondence with $[S^n, \pi \otimes G, n]$.

Suppose now that $m = n$, then $u : S^n \times S^n \to K^1_n \times K^2_n$, and we may take $u$ to be $(u^1_1 + u^1_2, u^2_1 + u^2_2)$. Then ker $f_* \cong \Theta = \ker$ (addition : $\pi \times \pi \times G \to \pi \times G \to \pi \otimes G$). If $fu \sim \ast$, then

$$[S^n \times S^n, P_f : (u)] \leftrightarrow [S^n \times S^n, \Omega K_{2n}] / \Delta(f, u)[S^n \times S^n, \Omega K_n \times \Omega K_n].$$
Again this set is trivial if $n \geq 2$, and if $n = 1$ it is in correspondence with $[S^1 \times S^1, K(\pi \otimes G, 1)]$. Collecting the results together, we have

**Example 2.6.2.**

$$[S^m \times S^n, P_f] \leftrightarrow \Gamma$$

$$m \neq n; \quad m, n \geq 2$$

$$\leftrightarrow \Omega \times (\pi \otimes G)$$

$$m = 1, \ n \geq 2 \ or \ m \geq 2, \ n = 1$$

$$\leftrightarrow \Theta \times (\pi \otimes G) \times (\pi \otimes G)$$

$$m = n = 1.$$

(b) Let $f = \cup^p d^p : K(G, n) \to K(G_{p+1}, (p + 1)n)$ be given by a multiple cup product. Here $G_{p+1}$ denotes the tensor product $G \otimes \ldots \otimes G$ having $(p + 1)$ factors. Let $\Theta = \ker f_\#: [X, K_n] \to [X, K_{(p+1)n}]$. Using 2.3.4, we have

**Example 2.6.3.** If $n$ is even and $(p + 1)G = 0$, or if $n$ and $p$ are both odd, then

$$[X, P_f] = \Theta \times H^{m+n-1}(X; G_{p+1}).$$

If $n$ is odd and $p$ is even, then

$$[X, P_f] = \bigcup_{x \in \Theta} H^{m+n-1}(X; G_{p+1}) \cup H^{n-1}(X; G).$$

### §3. DUALITY

#### 3.1 Introduction

Let $A \cup_f CB$ be the principal cofibre space induced by map $f : B \to A$ from the cofibration $B \hookrightarrow CB$. The dual problem to that considered in §§ 1 and 2 is to determine $[A \cup_f CB, X]$. We present here the dual classification theorem and the properties of the dual of the function $\Delta$. The proofs are generally omitted. As an application, we obtain, in 3.5.1, an expansion for a Whitehead product $[\alpha \gamma, \beta]$ in terms of Whitehead products involving $\alpha$ and $\beta$ only, together with certain Hilton–Hopf invariants of $\gamma$.

#### 3.2 The dual classification theorem

Let $i : A \hookrightarrow C_f$ be the principal cofibre map induced from $i : B \hookrightarrow CB$ by a map $f : B \to A$; thus $C_f$ is the space obtained from the topological sum $A + CB$ by identifying $(b, 1)$ with $f(b)$ for each $b$ in $B$. The cofibre is $SB$. Let $i : CB \to C_f$ denote the projection

$$C_f \xleftarrow{i} CB$$

$$\xleftarrow{u} X \xleftarrow{j} A \xleftarrow{j} B$$

The $H'$-space $SB$ coacts on the left of $C_f$ by $\mu : C_f \to SB \vee C_f$, where $\mu(a) = a$, $\mu(b, t) = (b, 2t) \in SB$ for $0 \leq t \leq \frac{1}{2}$ and $\mu(b, t) = (b, 2t - 1) \in C_f$ for $\frac{1}{2} \leq t \leq 1$. A map $u : A \to X$ can be extended to $C_f$ if, and only if, $f^*\{u\} = 0$. Let $[C_f, X : (u)]$ denote the set of homotopy classes, in $[C_f, X]$, of the extensions of $u$; clearly it depends only on the homotopy class.
The group action of \([SB, X]\) on \([C_f, X: (u)]\) induced by \(\mu^*: [SB, X] \times [C_f, X: (u)] \to [C_f, X: (u)]\) determines a bijection

\[
\begin{array}{c}
[SB, X] \\
\downarrow \quad F \circ f \circ \pi^*_f(X; u)
\end{array} \quad \leftrightarrow \quad [C_f, X: (u)]
\]

where \(F : uf \sim *\) is a homotopy.

Suppose that \(X^A\) and \(X^B\) are \(H_0\)-spaces, recall that this is true in particular if \(A\) and \(B\) are \(H^*\)-spaces, and let maps \(f : B \to A\) and \(u : A \to X\) be given.

**Definition.** The homomorphism \(\Gamma(u, f) : [SA, X] \to [SB, X]\) is the composite function:

\[
[SA, X] \overset{u^{-1}}{\longrightarrow} \pi^*_1(X, u) \overset{f^*}{\longrightarrow} \pi^*_1(X; uf) \overset{(uf)_*}{\longrightarrow} [SB, X].
\]

In case \(uf\) is nullhomotopic, \(\Gamma(u, f) = F_* f_* u_*^{-1}\) where \(F : fu \sim *\) is a homotopy.

**Theorem 3.2.1.** If \(X^A\) and \(X^B\) are \(H_0\)-spaces, then there is a bijection

\[
\bigcup\limits_{[u] \in \ker f^*} \Gamma(u, f) \leftrightarrow [C_f, X].
\]

Let \(U : C_f \to X\) be an extension of \(u\). Define \(\mu_U : [SB, X] \to [C_f, X]\) by \(\mu_U(\zeta) = \mu^*(\zeta, U)\).

**Corollary 3.2.2.** The following sequence is exact:

\[
\begin{array}{cccc}
[SA, X] & \overset{\Gamma(u, f)}{\longrightarrow} & [SB, X] & \overset{\mu_U}{\longrightarrow} [C_f, X]_u \overset{i^*}{\longrightarrow} [A, X]_u \overset{f_*}{\longrightarrow} [B, X]_u.
\end{array}
\]

Here, for example, \([A, X]_u\) denotes \([A, X]\) with the class of \(u\) distinguished.

In case \(u = *\), this is the exact sequence 2.8 of [13].

### 3.3 Properties of \(\Gamma(u, f)\)

We now list the properties of \(\Gamma(u, f) : [SA, X] \to [SB, X]\).

**Triviality Theorem 3.3.1.** \(\Gamma(*) = (Sf)^*\)

**Wedge Product Theorem 3.3.2.** Let \(X^A, X^B, X^C,\) and \(X^D\) be \(H_0\)-spaces,

\[
f_1 \quad u_1 \quad f_2 \quad u_2
\]

\(B \to A \to X\) and \(D \to C \to X\), then

\[
\Gamma(c(u_1 \vee u_2), f_1 \vee f_2)\{\zeta_1, \zeta_2\} = (\Gamma(u_1, f_1)\{\zeta_1\}, \Gamma(u_2, f_2)\{\zeta_2\}).
\]

**Definition.** A class \(\alpha = \{f\} \in [B, A]\) is \(X\)-coprimitive if \(f^* : [A, X] \to [B, X]\) is a homomorphism. Further \(\alpha\) is coprimitive if it is \(X\)-coprimitive for all spaces \(X\); equivalently \(A\) and \(B\) are \(H^*\)-spaces and \((\phi_A)_* \alpha = i_{1*}\alpha + i_{2*}\alpha \in [B, A \vee A]\).

**Coprimativity Theorem 3.3.3.** If \(\{f\}\) is \(X\)-coprimitive, then \(\Gamma(u, f) = (Sf)^*\) for \(B \to A \to X\).

† Theorems 3.3 and 3.4 of [1] are, essentially, special cases of this bijection.
In particular $\Gamma(u, 1) = 1$, $\Gamma(u, *) = 0$, $\Gamma(u, c) = d$ and $\Gamma(u, i_j) = p_j$ for the inclusion $i_j$ into one of the factors of a wedge.

**Corollary 3.3.4.** If $X$ is an H-space, then $\Gamma(u, f) = (Sf)^*$. 

**Composition Theorem 3.3.5.** Let $A$, $B$, and $C$ be $H'$-spaces $g : B \to A$ and $f : C \to B$. Then $\Gamma(u, gf) = \Gamma(u, g) \cdot \Gamma(u, f)$.

**Additivity Theorems 3.3.6.** Let $B$ be an $H'$-space.

(i) $\Gamma(u, f \cdot g) = \Gamma(u, f) + (uf) \cdot \Gamma(u, g)$.

(ii) $\Gamma(u, f^{-1}) = -((uf)^{-1}) \cdot \Gamma(u, f)$.

(iii) $\Gamma(c(u_1 \vee u_2), (f_1 \vee f_2) \phi)\{\xi_1, \xi_2\} = \Gamma(u_1, f_1)\{\xi_1\} + (u_1 f_1) \cdot \Gamma(u_2, f_2)\{\xi_2\}$.

(iv) If $uf \sim ug$, then $\Gamma(u, f \cdot g^{-1}) = \Gamma(u, f) - \Gamma(u, g)$.

**Definition.** Let $A, B, \text{ and } D$ be $H'$-spaces. A cobimultiplication is a map $f : D \to A \vee B$ for which the following diagrams are homotopy commutative:

(i) 

```
\begin{align*}
D & \xrightarrow{\phi} D \vee D \\
& \downarrow f \circ f \quad \downarrow f \circ f \\
A \vee B \vee A \vee B & \xrightarrow{1 \vee 1 \vee 1} A \vee A \vee B \vee B \\
& \downarrow c \vee 1 \vee 1 \\
A \vee B & \xrightarrow{1 \vee \phi_B} A \vee B \vee B \\
\end{align*}
```

(ii) 

```
\begin{align*}
D & \xrightarrow{\phi_D} D \vee D \\
& \downarrow f \circ f \quad \downarrow f \circ f \\
A \vee B \vee A \vee B & \xrightarrow{1 \vee 1 \vee 1} A \vee A \vee B \vee B \\
& \downarrow c \vee 1 \vee 1 \\
A \vee B & \xrightarrow{\phi_{A \vee 1}} A \vee A \vee B \\
\end{align*}
```

It is elementary that $f$ is a co-bimultiplication, if, and only if, the induced function $f^* : [A, X] \times [B, X] \to [D, X]$.
is bilinear for every space $X$. Also if $f$ satisfies (i) and (ii), then the composite $D \xrightarrow{f} A \vee B \to A \times B$ is nullhomotopic.

Define $F_A : D \times I \to A \vee SB$ by $F_A(d, t) = (f_1(d), (f_2(d), t))$ where $f_1$ and $f_2$ are the components of $f$. Let $G_A : CD \to A$ be determined by a nullhomotopy of $D \to A \vee B \to A$. Now define

$$f_A = G_A + F_A - G_A : SD \to A \vee SB,$$

$$f_B = G_B + F_B - G_B : SD \to SA \vee B.$$

**CO-BIMULTIPLICATION THEOREM 3.3.7.** Let $f : D \to A \vee B$ be a co-bimultiplication and $c(u_1 \vee u_2) : A \vee B \to X$, then

$$\Gamma(c(u_1 \vee u_2), f) \{\zeta_1, \zeta_2\} = \{c(u_1 \vee \zeta_2)f_A\} + \{c(\zeta_1 \vee u_2)f_B\}.$$

3.4 Whitehead products

Let $A$ and $B$ be CW complexes. Then for any space $X$, the following sequence is exact:

$$0 \to [S(A \times B), X] \xrightarrow{\pi^*} [S(A \times B), X] \to [SA \vee SB, X] \to 0.$$

Let $p_1 : S(A \times B) \to SA \subset SA \vee SB$ and $p_2 : S(A \times B) \to SB \subset SA \vee SB$ be the projections and $X = SA \vee SB$. Choose $W : S(A \times B) \to SA \vee SB$ such that

$$\pi^*(W) = \{p_1 + p_2 - p_1 - p_2\}.$$

**Definition.** A Whitehead product map is a map $W$ defined as above. The Whitehead product of $\alpha : SA \to X$ and $\beta : SB \to X$ is $[\alpha, \beta] = W^*((\{\alpha\}, \{\beta\})) \in [S(A \times B), X]$.

Let $\omega_1 : S(A \times B) \to A \vee SB$ and $\omega_2 : S(A \times B) \to SA \vee B$, be the continuous bijections; and let $p_1 : SS(A \times B) \to SSA \subset SSA \vee SB$, $p_2 : SS(A \times B) \to SSB \subset SA \vee SSB$, $q_1 : S(A \times B) \to SA \subset SA \vee SSB$ and $q_2 : S(A \times B) \to SB \subset SSA \vee SB$ be the projections. Consider the diagrams:

$$\begin{align*}
[S(SA \times B), SSA \vee SB] &\xrightarrow{\pi^*} [S(SA \times B), SSA \vee SB] \\
[S(SA \times B), SSA \vee SB] &\xrightarrow{\pi^*} [SS(A \times B), SSA \vee SB] \\
[S(A \times SB), SA \vee SSB] &\xrightarrow{\pi^*} [S(A \times SB), SA \vee SSB] \\
[S(SA \times B), SSA \vee SB] &\xrightarrow{\pi^*} [SS(A \times B), SA \vee SSB].
\end{align*}$$
The vertical maps are isomorphisms† and, by Theorem 2 of [11],

\[
(S\omega_1)^* \pi^{-1} \{(p_1) - q_2 \Box (p_1)\} = \{W\} \quad \text{and} \\
(S\omega_2)^* \pi^{-1} (q_1 \Box (p_2) - \{p_2\}) = \{W\}.
\]

**Whitehead Product Theorem 3.4.1.** Let \(A, B\) be CW complexes,

\(W : S(A \boxtimes B) \to SA \vee SB\) a Whitehead product, \(u_1 : SA \to X, u_2 : SB \to X, \zeta_1 : SSA \to X\) and \(\zeta_2 : SSB \to X\); then in \([SS(A \boxtimes B), X]\) we have

\[
\Gamma(c(u_1 \vee u_2), W)\{(\zeta_1, \zeta_2) = (S\omega_1)^* \{\zeta_1, u_1 \cdot u_2 \cdot u_1^{-1}\} - (S\omega_2)^* u_1 \Box [u_2 \cdot u_1 \cdot u_2^{-1}, \zeta_2].
\]

**Proof.** On iterating the Additivity theorem, we have

\[
\Gamma(u_1 \cdot p_1 \cdot p_2 \cdot p_2^{-1})\{(\zeta_1, \zeta_2) = \{\zeta_1\} - (\zeta_2) = \{\zeta_1\} + u_1 \Box (\zeta_2) - (u_2 \cdot u_1 \cdot u_2^{-1}) \Box (\zeta_2).
\]

The Theorem now follows on using Theorem 2 of [11] (quoted above) as a universal example.

We note that \(u_1 \Box = 1\) if \(S(A \boxtimes B)\) is homotopy abelian.

Taking \(SA = S^m\) and \(SB = S^n\), we have the Corollary (c.f. Theorem 4.2 of [1]).

**Corollary 3.4.2.**

\[
\Gamma(u, W)\{(\zeta) = (-1)^{m-1}\{\zeta_1, u_2\} - u_1 \cdot (\zeta_2) \quad m, n > 1
\]

\[
= + [\zeta_1, u_2] - [u_2 \cdot u_1 \cdot u_2^{-1}, \zeta_2] \quad m > 1, n = 1.
\]

\[
= (-1)^{m-1}\{\zeta_1, u_1 \cdot u_2 \cdot u_1^{-1}\} - \{u_1, \zeta_2\} \quad m = 1, n > 1.
\]

\[
= + [\zeta_1, u_1 \cdot u_2 \cdot u_1^{-1}] - u_1 \Box [u_2 \cdot u_1 \cdot u_2^{-1}, \zeta_2] \quad m = n = 1.
\]

To each basic symbolic pair, \(\mu\), on the symbols 1 and 2 and each CW complex \(A\), Milnor [8] associates a collapsed product \(A_{\mu}\). Let \(W_{\mu} : SA_{\mu} \to SA_1 \vee SA_2\) be the composite Whitehead product map. Given \(\alpha : SA_1 \to X\) and \(\beta : SA_2 \to X\), then \([\alpha, \beta]_{\mu}\) will denote \(W_{\mu}^\ast(\alpha, \beta)\) in \([SA_{\mu}, X]\). It follows from Theorem 4 of [8] that

\[
[SB, SA_1 \vee SA_2] = \sum_{\mu} W_{\mu}[SB, SA_{\mu}].
\]

Given \(h : SB \to SA\), then \(\{\phi' h\} = \Sigma W_{\mu}^\ast H_{\mu}(h)\). The functions \(H_{\mu}\) are the Hilton-Hopf invariants (see [6]).

Let \(x_1, x_2, \ldots\) be those non-trivial basic pairs involving the symbol 2 only once, thus \(x_1 = (1, 2)\), and generally \(x_{n+1} = (1, x_n)\); and let \(\omega : (A \boxtimes (A \boxtimes (A \ldots (A \boxtimes SA) \ldots))) \to S(A \boxtimes (A \boxtimes (A \ldots (A \boxtimes A) \ldots)))\) be the canonical homotopy equivalence.

**Theorem‡ 3.4.3.** If \(SA\) and \(SB\) are homotopy abelian and \(h : SB \to SA\), then

\[
\Gamma(u, h)\{(\zeta) = (Sh)^\ast \{\zeta\} + \sum_{\mu} (-1)^{m}(S\omega)^\ast \{u, \zeta\} [S_{\mu} H_{x_{\mu}}(h)].
\]

† The collapsed product is 'homotopy associative' for CW complexes in that there is a canonical natural homotopy equivalence class \((A \boxtimes B) \ast C \to A \ast (B \ast C)\) (see Lemma 1 of [11]).

‡ If \(SA\) and \(SB\) are not restricted to be homotopy abelian, a more complicated version of this theorem is valid. The proof is similar.
Proof. Let \(c(* \vee u) : SA \vee SA \to X\) and \(c(\zeta \vee *) : SSA \vee SSA \to X\) then we have

\[
\Gamma(c(* \vee u), i_1 h + i_2 h + \Sigma W_{\mu} H_{\mu}(h))\{*, \zeta\} = \Gamma(c(* \vee u), \phi' h)\{*, \zeta\} = \Gamma(u, h)\{\zeta\}.
\]

On the other hand

\[
\Gamma(c(* \vee u), i_1 h)\{*, \zeta\} = (Sh)^*\{\zeta\}
\]

and

\[
\Gamma(c(* \vee u), i_2 h)\{*, \zeta\} = 0
\]

Now each product \(W_{\mu}\) is a composite

\[
\begin{array}{c}
S((\otimes A) \otimes (\otimes A)) \\
\downarrow \cong \\
S((\otimes A) \vee (\otimes A)) \\
\downarrow W^1 \vee W^2 \\
S(V) \\
\end{array}
\]

and

\[
\Gamma(c(u \vee *), W_{\mu})\{*, \zeta\} = -(S\omega_2)^* \cdots [c(u \vee *)W^1, \Gamma(c(u \vee *), W^2)\{*, \zeta\}].
\]

It follows by induction that this is zero if \(\mu\) contains the symbol 2 more than once. The proof is now straightforward.

In case \(A\) and \(B\) are spheres, Theorem 3.4.3. is equivalent to Theorem 7.1 of [1].

3.5 An application

We now use the results of the previous paragraph to generalize a theorem of Barcus and Barratt (7.4 of [1]).

Given \(\gamma : SSC \to SSA\) and \(\alpha : SSA \to X\) and \(\beta : SSB \to X\), define \(\gamma_D : S(SC \otimes D) \to S(SA \otimes D)\) to correspond to \(\gamma \otimes 1 : SSC \otimes D \to SSA \otimes D\).

THEOREM 3.5.1.

\[
[x, \beta] = y_{SB}^* [x, \beta] + \sum_n [x, \beta]_{n+1} \circ S_\ast H_{x_n}(\gamma_B).
\]

Proof. For complexes \(Y\) and \(Z\) we have the commutative diagram:

\[
\begin{array}{ccc}
[S(SY \otimes Z), SSY \vee SZ] & \xrightarrow{\pi^*} & [S(SY \times Z), SSY \vee SZ] \\
\downarrow \cong \downarrow \cong \downarrow \cong \\
[S(S_0 Y \times Z)^{1 \times I} \times (*)], SSY \vee SZ] & \xrightarrow{(S_0)^*} & [S(SY \times Z), SSY \vee SZ]
\end{array}
\]

where the vertical homomorphisms are induced by the projections\(\dagger\) (see §3 of [11]) and \(S_0\)

\(\dagger\) In this proof, products of \(CW\) complexes are given the weak topology in order that the maps involved are continuous: this does not change homotopy type.
is the unreduced suspension functor. Let $\overline{W} : S(S_0C \times B) \to SSC \vee SB$ and
$W : S(S_0A \times B) \to SSA \vee SB$
be maps determined by the Whitehead products and consider the (non commutative) diagram:

\[
\begin{array}{ccc}
S(S_0C \times B) & \xrightarrow{\overline{W}} & SSC \vee SB \\
\downarrow \gamma & & \downarrow \gamma \vee 1 \\
S(S_0A \times B) & \xrightarrow{W} & SSA \vee SB \\
\end{array}
\]

Now

\[
(\gamma \vee 1)(\overline{W} - \overline{p}_1) \sim -(\gamma \vee 1)(\overline{\varphi}_2)(\overline{p}_1)
\]

Also

\[
\Gamma(c(\alpha \vee \beta))(\gamma \vee 1)(\overline{W} - \overline{p}_1)](\alpha, \beta) = \Gamma(c(\gamma \vee \beta), \overline{W})(\gamma, \beta) - \Gamma(c(\gamma \vee \beta), p_1)(\alpha, \beta)
\]

\[
= - (Sv_2)^*[\gamma, \beta].
\]

On the other hand

\[
\Gamma(c(\alpha \vee \beta), (W - p_1)](\alpha, \beta) = \Gamma(c(\alpha \vee \beta)(W - p_1), \gamma) \circ \Gamma(c(\gamma \vee \beta), W - p_1)(\gamma, \beta)
\]

\[
= \Gamma(c(-\alpha \vee \beta), p_1, \gamma)(-Sv_2)^*[\alpha, \beta]
\]

\[
= (Sv_2)^*[-(S\gamma)^*[\alpha, \beta] + \Sigma(-1)^i(-1)^{i+1}c[\alpha, \beta]_x, l, SH_{\alpha, \beta}(\gamma)].
\]

The theorem now follows.

SYNOPSIS OF NOTATIONS AND DEFINITIONS

1.1 $I$ unit interval with no base point.
$\Pi$ unit interval with base point.
$W$, category of spaces having homotopy type of a CW complex.
[adjoint map.]
u-based track group $\pi\gamma(Y; u).$
$F : u \sim v$ induces $F_u : \pi\gamma(Y; u) \to \pi\gamma(Y; v).$

1.2 $\phi_o : G \times G \to G, \psi_o : G \to G$, structure maps for $H$-space $G$.
path continuous maps and homotopies.
$H_0$-space : $Y^X$ $H$-space with structure maps and homotopies path continuous, homotopy abelian if homotopy path continuous.
u$_r : Y^X \to Y^X$, right multiplication given by $u_r(v) = v.u.$
u$_p : \pi\gamma(Y; u) \to \pi\gamma(Y; uu^{-1})\to \pi\gamma(Y; \ast).$
\[
\Delta(f, u) : \{X, \Omega A\} \to \pi\gamma(A; u) \to \pi\gamma(B; fu) \to \{X, \Omega B\}.
\]

1.3 $P_f$ is principal fibre space induced by $f : A \to B$ from $\pi : PB \to B.$
projections $p : P_f \to A, \overline{p} : P_f \to PB$, multiplication $m : \Omega B \times P_f \to P_f.$
$[X, P_f : (u)] \subset [X, P_f]$ set of homotopy classes of liftings of $u.$
1.4 \(X\)-primitive, \(\alpha = \{f\} \in [A, B]\) such that \(f_* : [X, A] \to [X, B]\) is homomorphism.
\(u \Box\) automorphism of \([X, \Omega A]\) induced by inner automorphism \(\{\xi\} \to \{u \cdot \xi \cdot u^{-1}\}\).
Bimultiplication \(f : A \times B \to C\) such that \(f_* : [X, A] \times [X, B] \to [X, C]\) bilinear for all \(X\).
\(f^A : A \times \Omega B \to \Omega X, f^B : \Omega A \times B \to \Omega C\) induced by \(f : A \times B \to C\).

1.5 self homotopy, \(u \sim u\) determines element of \(\pi^n(Y; u)\).

2.2 \(K_\pi, or K(\pi, t), an Eilenberg-MacLane space.
Fundamental class, \(t_m \in H^m(K(\pi, m); \pi)\)
\(\gamma : [X, K(\pi, m)] \to H^n(X; \pi)\) by \(\gamma[f] = f^* t_m\).
\(\text{Op}(\Pi H^m(\ ; G_i), H^n(\ ; \pi))\), set of cohomology operations.
\(\Xi : \text{Op}(\Pi H^m(\ ; G_i), H^n(\ ; \pi)) \to H^n(\Pi K_m; \pi)\).
\(\cup, \cup\) cup product and cross product.
\(k^* : H^*(X \times Y) \to H^*(X \times Y)\) monomorphism onto direct summand.
\(\cup, \cup\) cup product maps.
\(S^* : H^{n-1}(X; \pi) \to H^n(X \times S^1; \pi)\) suspension isomorphism.
\(\lambda : \Omega K(\pi, n) \to K(\pi, n - 1)\) canonical equivalence class.
\(\sigma T, \sigma t\) suspension of cohomology operation and of map.
\(\delta : \Omega K_n \times S^1 \to K_n, the evaluation map.

2.3 \(\cup^p\) multiple cup product map.

2.4 \(f_*\), coefficient homomorphism map.
\(\delta^p\), Bockstein–Whitney coboundary map.
\(\dim a, a : A \to K(\pi, t), is t\).

2.5 \(V(T, \mu)\), function of cohomology corresponding to \(\Delta(h, u)\).
\(\nabla T\), cohomology operation it determines.

3.2 \(C_f = A \cup_f CB\), principal cofibre space induced by \(f : B \to A\) from \(B \subset CB\).
\(i : A \subset C_f \; ; i : CB \to C_f, \mu : C_f \to SB \vee C_f\) coaction.
\([C_f, X; (u)] \subset [C_f, X]\), homotopy classes of extensions of \(u : A \to X\).
\(\Gamma(u, f) : [SA, X]^{2^n-1} \to [\pi^1(X; u)]^{2^n-1} \oplus [\pi^0(X; u)]^{2^n-1} \cup [SB, X]\).

3.3 \(X\)-coprimitive, \(\alpha = \{f\} \in [B, A]\) such that \(f_* : [A, X] \to [B, X]\) is a homomorphism.
Cobimultiplication, \(f : D \to A \vee B\) such that \(f_* : [A, X] \times [B, X] \to [D, X]\) is bilinear for all \(X\).
\(f_A : SD \to A \vee SB, f_B : SD \to SA \vee B\) induced by \(f : D \to A \vee B\).

3.4 \(W : S(A \times B) \to SA \vee SB\), a Whitehead product map.

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