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Inherited Groups and Kernels of Derived Translation Planes

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When an affine plane is converted to another plane by derivation, the point permutations which act as collineations of both planes form the inherited group. The full group can be larger than the inherited group. For finite translation planes in which some of the Baer subplanes involved are not vector spaces over the kernel of the original plane then the full collineation group of the derived plane is the inherited group provided the order of the plane is greater than 16.

1. Introduction

Let π be any finite derivable affine plane with derivable net D. Replace (derive) D to obtain the affine plane $\bar{\pi}$, with corresponding derivable net $\bar{D}$. The most basic question regarding the collineation group of $\bar{\pi}$ is whether the full collineation group of $\bar{\pi}$ is the group inherited from the collineation group of π or, equivalently, whether the full group leaves the net $\bar{D}$ invariant.

If the plane π is a strict dual translation plane then the derived plane $\bar{\pi}$ is a strict semi-translation plane, and Ostrom [13] showed that the full group is the inherited group.

If π is a generalized Hall plane then $\bar{\pi}$ is a semi-field plane and, of course, the full group is not the inherited group as it contains an elation group with affine axis acting transitively on the components different than its axis. On the other hand, if the order is greater than 16 the full group of any derived semi-field plane (by a net containing a shears axis) is the inherited group [41]. Note that the kernel of a derived semi-field plane is a subfield of the kernel of the associated semi-field plane [9].

Thus, if π is a derivable translation plane with derivable net D and the Baer subplanes of D incident with the zero vector are all kernel subspaces, then the inherited group of the derived plane is *not* necessarily the full group.

But, what if the Baer subplanes of D incident with the zero vector are *not* all kernel subspaces? Can any general statement be made regarding the inherited group of the derived plane?

If π is a semi-field plane of order 16 and kernel K isomorphic to $GF(4)$ then there is a derivable net D containing the shears axis such that not all Baer subplanes of D incident with the zero vector are K-subspaces. However, the derived plane $\bar{\pi}$ admits $PSL(2, 7)$ and, obviously, the full group of $\bar{\pi}$ is not the full group [5, 8].

If Σ is the affine Desarguesian plane of order 9 and D is a derivable net then the corresponding derived translation plane $\bar{\Sigma}$ is the near-field plane of order 9. Since the kernel of Σ is isomorphic to $GF(9)$ then no Baer subplane of D incident with the zero vector can be a $GF(9)$ subspace, even though the full group of $\bar{\Sigma}$ is not the inherited group.

In spite of the above counterexamples to a potential general statement, we prove the following result:

Theorem A. *Let π be a finite translation plane of order >16. Let D be a derivable net and let K be the kernel of π. If the Baer subplanes of D incident with the zero vector are not all K-subspaces, then the full group of the translation plane obtained by deriving D is the inherited group.*

In [6], one of the authors gave conditions under which the kernel $\bar{K}$ of a derived

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translation plane $\bar{\pi}$ is a subfield of the kernel $\bar{K}$ of the corresponding translation plane π. For example, if π is of order q^2 and kernel k isomorphic to $GF(q)$ and admits an affine elation group of order >2 leaving a derivable net D invariant, then the kernel $\bar{K}$ of the associated derived translation plane $\bar{\pi}$ is a subfield of K.

In this note, we show that $\bar{K}$ is always a subfield of K unless π is a Hall plane.

THEOREM B. *Let π be a derivable translation plane of order q^2 and kernel containing K isomorphic to $GF(q)$. Let $\bar{\pi}$ denote the associated derived translation plane and let $\bar{K}$ denote the kernel of $\bar{\pi}$. Let D denote the derivable net of π. If π is not Hall then the kernel $\bar{K}$ of $\bar{\pi}$ is the maximal subfield L of K such that each Baer subplane of D incident with the zero vector is an L-space.*

2. THEOREM ON PARTIAL SPREADS

(2.1) THEOREM. *Let K and L be any two fields that act as scalar groups on a finite partial spread of order p^{rt} and degree $>p^{rt-1}+1$. Then the ring $\langle K, L\rangle$ generated over the prime field $GF(p)$ is contained in a field.*

PROOF. Choose three components to be represented in the form $x=0$, $y=0$, and $y=x$. Then the partial spread ρ may be represented in the form $y=xA_i$ for $i=0, 1, 2, \ldots k$ for $k>p^{rt-1}$, where A_0 is the zero matrix. Now the mappings $x\rightarrow xA_i$ acting on an rt-dimensional vector subspace over $GF(p)$ for $i>0$ are fixed-point-free and the corresponding matrices and their differences are non-singular (or zero). We consider the group G generated by the matrices A_i for $i>0$. We assert that G acts irreducibly on W. Note that since each element A_i is non-singular and A_i-A_j for $i\neq j$ is non-singular, the G-orbit length of any non-zero vector is at least $k-1$. But if $k-1>p^{rt-1}-1$ then G could fix no proper subspace of W.

Then, by Schur's lemma the centralizer of G is a field. Since both k and l centralize G, we have the proof to (2.1). □

3. THE ISOMORPHISM THEOREM

Our result, Theorem A, stated in the introduction, is actually a corollary to the following isomorphism theorem:

(3.1) THEOREM. *Let π and ρ be derivable translation planes of the same order $q^2>16$, with derivable nets D and R respectively. Let the kernels of π and ρ be K and L respectively, and assume that not all of the Baer subplanes of D (or R) incident with the zero vector are K (or L respectively) subspaces. Let $\bar{\pi}$ and $\bar{\rho}$ denote the corresponding derived planes from π and ρ respectively. If $\bar{\pi}$ and $\bar{\rho}$ are isomorphic then π and ρ are isomorphic by an isomorphism mapping D onto R.*

We first state a lemma:

(3.2) LEMMA. *For $q=p^r>4$ then $q^2-2q-1>p^{2r-1}+1$.*

PROOF. The inequality we wish to establish is equivalent to $p^{2r}-2p^r>p^{2r-1}+2$. First of all (neglecting the 2), note that $p^{2r}-2p^r>p^{2r-1}$ is equivalent, on dividing by p^{2r-1}, to $p>1+2p^{1-r}$, which is true if $p>2$ and $r>1$ or $p=2$ and $r>2$. Now suppose that $p^{2r}-2p^r\leqslant p^{2r-1}+2$ and $p^{2r}-2p^r>p^{2r-1}$, with p satisfying the conditions of the lemma. Then $p^r(p^r-2)=p^{2r-1}+1$ or $p^{2r-1}+2$, This can only occur if p^r divides 2, contrary to the condition that $p^r>4$.

(3.3) LEMMA. *Let $\bar{D}$ and $\bar{R}$ refer to the derived nets of D and R respectively. Let f be any isomorphism from $\bar{\pi}$ onto $\bar{\rho}$. If $\bar{D}f \cap \bar{R}$ contains* $\geqslant(c-1)$ components then $\bar{D}f = \bar{R}$.

PROOF. Let f be any isomorphism from $\bar{\pi}$ onto $\bar{\rho}$. $\bar{D}(\bar{R})$ is a Desarguesian net contained in a Desarguesian affine plane $\Sigma(\Sigma^*)$. (See Foulser [2] for finite translation planes. This result is also valid for arbitrary finite derivable nets: see Johnson [7].) Thus, Σf is also a Desarguesian affine plane defined on the same points as Σ, the components of which are f-images of the components of Σ. Since $q-1>2$, the degree of $\Sigma^* \cap \Sigma f = 1+p^s$ by *Ostrom* [15] (for $q=p^r$ and some integer s) so that $i+p^s \geqslant q-1$. If $1+p^s \leqslant q$ then $1+p^s = q-1$ and $p^s = q-2$, so that q is even. But, clearly, this implies that $p^s=2$ and $q=4$. But, for $q^2>16$, $1+p^s \geqslant 1+q$. This implies that $\Sigma^* \cap \Sigma f$ is either a regulus in the spread of Σ^* or $\Sigma^* = \Sigma f$. In the initial case, let T denote the common regulus. Since $\bar{D}f \cap T$ is a net with at least $q-1$ components and $\bar{D}f$ is a regulus in Σ^*, clearly $\bar{D}f = T$. Similarly, $\bar{R} = T$. In the latter case, $\Sigma^* = \Sigma f$, so that $\bar{D}f$ and $\bar{R}$ are both reguli in the same Desarguesiona spread and share at least $q-1$ components, so that $\bar{D}f = \bar{R}$. □

PROOF OF (3.2). By (2.2), we may assume that f is an isomorphism from $\bar{\pi}$ to $\bar{\rho}$ and $\bar{D}f \cap \bar{R}$ share less than $q-1$ components. Let H_{K^*} denote the homology group of π, which acts as a collineation group of $\bar{\pi}$ fixing all components of the net $\pi - \bar{D}$. By Biliotti and Lunardon [1, Theorem 3], H_{K^*} fixes either 0 or 2 components of $\bar{D}$. Similarly, let H_{L^*} denote the homology group of ρ, which acts as a collineation group of $\bar{\rho}$ fixing all components of $\bar{\rho} - \bar{R}$ and 0 or 2 components of $\bar{R}$. Note that $|\bar{\rho} - \bar{D}f \cap \bar{R}| = q^2+1-|\bar{D}f \cap \bar{R}| > q^2+1-(q-1) = q^2-q+2$. Now assume that $H^f_{K^*} = H_{L^*}$. Then $H^f_{K^*}$ would fix each component of $\bar{\rho}$ not in $\bar{D}f$ and H_{L^*} would fix each component of $\bar{\rho}$ not in $\bar{R}$, so that H_{L^*} would fix at least q^2-q+3 components. However, as above, H_{L^*} fixes at most q^2-q+2 components of $\bar{\rho}$. Thus, $H_{L^*} \neq H^f_{K^*}$. Now H_{L^*} acts on $\bar{\rho} - \bar{R}$ as a scalar group and $H^f_{K^*}$ acts on $\bar{\rho} - \bar{D}f$ as a scalar group. Hence, $H^f_{K^*}$ and H_{L^*} both act on the partial spread $\bar{\rho} - \bar{D}f \cup \bar{R}$ as a scalar group. Since $|\bar{\rho} - \bar{D}f \cup \bar{R}| \geqslant q^2+1-2(q+1)$ and by Theorem (2.1), $q^2+1-2(q+1) > p^{2r-1}+1$, and since H_{L^*} and $H^f_{K^*}$ are subgroups isomorphic to the multiplicative groups L^* and K^* respectively, we have by Theorem (2.1) that $\langle H_{L^*}, H^f_{K^*}\rangle$ is contained in a finite field. However, since both are cyclic groups of the same order, it must be that $H_{L^*} = H^f_{K^*}$, which is a contradiction. Thus, any isomorphism must map $\bar{D}$ onto $\bar{R}$ and thus induce an isomorphism $\bar{f}$ from D onto R, and hence from π onto ρ, which proves (3.2). □

4. ANNIHILATING THE KERNEL

Assume the conditions in the statement of Theorem B. Assume that $\bar{K}$ is not contained in K. We consider the partial spread $\bar{\pi} - \bar{D}$ of degree q^2-q. $\bar{K}$ and K act as scalar groups on this partial spread. By Theorem (3.1) and Theorem (2.1), $\langle \bar{K}, K\rangle$ is contained in an finite field F. However, if $\bar{K}$ is not contained in K this forces F to be isomorphic to $GF(q^2)$, so that the net $\bar{\pi} - \bar{D}$ is Desarguesian and is contained in a Desarguesian affine plane Σ. The deficiency of the net $\bar{\pi} - \bar{D}$ is $(q+1)$, i.e. 'critical'. By Ostrom [11], either π is equal to Σ or π is derived from Σ so that π is Hall. This completes the proof of Theorem B.

(4.1) COROLLARY. *Let π be a non-Hall translation plane of order q^2, $q>4$ and kernel K isomorphic to $GF(q)$. If D is a derivable net but D does not correspond to a regulus in $PG(3, K)$ then the kernel $\bar{K}$ of the derived plane $\bar{\pi}$ is the subfield of K which fixes each Baer subplane of D incident with the zero vector and the full collineation group of $\bar{\pi}$ is the inherited group of π.*

PROOF. Apply Theorems A and B. □

5. COLLINEATION GROUPS OF DERIVED TRANSLATION PLANES

For more details on the following remarks, see Ostrom [16]. The Class of groups which are known to occur in the translation complement of some finite translation plane is very limited. Here we use the convention that the plane has order q^d and kernel $GF(q) = K$, so that the spread is defined on a vector space of dimension $2d$. Since larger-dimensional vector spaces admit larger classes of groups, it has been suggested that we should look at planes where d is large. Johnson [6] has shown how to produce planes with very large values of d by derivation. If $\bar{K} = GF(\bar{q})$ is a subfield of $K = GF(q)$ and the new plane (derived) is a vector space over $\bar{K}$ then the effect is to replace d by $\bar{d}$, where $\bar{d}$ is larger than d.

However, Theorem A tells us that the full collineation group of the plane $\bar{\pi}$ is the inherited group in this case and thus is a subgroup of the collineation group of the original plane. Hence, even though d has been increased to $\bar{d}$, we do not obtain any new groups.

However, in some cases, we can say something about the original groups which act on $\bar{\pi}$ (again see Ostrom [16]).

When $d = 2$, the non-solvable composition factors which can occur are known. They are also known when d is odd (note that the case where A_7 occurs is still open). Hering and Ho [3] have given an explicit list of possibilities for the case in which $d \equiv 2 \bmod 4$ and q is even. These results are independent of any assumptions about derivation.

This leaves the cases where $d \equiv 2 \bmod 4$ and q is odd and $d \equiv 0 \bmod 4$.

The rest of this section is devoted to the case in which Theorem A applies and where d is increased by derivation to $\bar{d} \equiv 0 \bmod 4$. The result is that the groups involved are essentially two-dimensional over some field.

(5.1) THEOREM. *Let π denote a derivable translation plane of order $q^2 > 16$ with derivable net D and kernel K. Assume that not all of the Baer subplanes incident with the zero vector are K-subspaces. Let $GF(\bar{q})$ denote the kernel of the derived plane $\bar{\pi}$, where $q^2 = \bar{q}^{\bar{d}}$ and $\bar{d} \equiv 0 \bmod 4$. If G is the full translation complement of the derived plane $\bar{\pi}$ then there is a normal series $\mathcal{G}_2 \leq \mathcal{G}_1 \leq \mathcal{G}$ such that the index of $\mathcal{G}_1$ is 1 or 2, $\mathcal{G}_2$ is solvable of order dividing $(q-1)$ and $\mathcal{G}_1/\mathcal{G}_2$ is isomorphic to a subgroup of $\Gamma L(2, q)$.*

PROOF. $\mathcal{G}$ is a subgroup of $\Gamma L(\pi, K)$ and normalizes the kernel homology group H_{K^*} of π. Since $4 \mid \bar{d}$ then if $q = p^r$ and K isomorphic to $GF(p^k)$, $4r = 2k\bar{d}$. By Biliotti and Lunardon [1, Theorem 3], there are 0 or 2 Baer subplanes of D incident with the zero vector which are k-subspaces. First assume that there are 0 Baer subplanes which are K-subspaces. K^* acts fixed-point-free on the components of π so that $p^k - 1$ divides $q^2 - 1$ or k divides $2r$. If k divides r then $p^k - 1$ divides $p^r - 1$ so that K^* fixes two Baer subplanes incident with the zero vector. Hence, k divides $2r$ but k does not divide r. However, $r = kd/2$ and d is even, which is a contradiction. Thus, H_{K^*} fixes exactly two Baer subplanes π_0 and π_1 of D incident with the zero vector so that $\mathcal{G}$ fixes the set $\{\pi_0, \pi_1\}$. Let $\mathcal{G}_1$ denote the subgroup of $\mathcal{G}$ which fixes both subplanes. Thus, the index of $\mathcal{G}_1$ in $\mathcal{G}$ is 1 or 2. Let $\mathcal{G}_2$ be the subgroup of $\mathcal{G}_1$ which fixes π_0 pointwise. By Foulser [2], $\mathcal{G}_2$ must be a subgroup of a 1-dimensional affine group and thus is solvable of order dividing $(q-1)$, as this subgroup must fix the second Baer subplane π_1. Then $\mathcal{G}_1/\mathcal{G}_2$ acts faithfully on the Desarguesian affine plane π_0 and thus is a subgroup of $\Gamma L(2, q)$. □

(5.2) COROLLARY. *Let π denote a derivable translation plane of odd order $q^2 > 9$ with derivable net D and kernel K. Assume that not all of the Baer subplanes of D incident with the zero vector are K-subspaces. Then the derived plane $\bar{\pi}$ cannot admit non-Abelian simple groups in the translation complement.*

PROOF. Let G be a non-Abelian simple collineation group in the translation complement of π. By Theorem A, G must leave the derived net $\bar{D}$ invariant. By Ostrom [14], if $\bar{\pi}$ has vector dimension $2\bar{d}$ then $4 \mid \bar{d}$. By (5.1), $\mathscr{G}_1 = \mathscr{G}$ and $\mathscr{G}_2 = \langle 1 \rangle$. By the argument to (5.1), there are exactly two Baer subplanes, say π_0 and π_1, of D incident with the zero vector which are K-subspaces. Thus, $\mathscr{G}$ fixes the two Baer subplanes incident with the zero vector which are K-subspades. However, π_0 is a Desarguesian subplane and, by Foulser [2], the group which fixes π_0 pointwise is a subgroup of a 1-dimensional affine group of order $q(q-1)$. Therefore, G acts faithfully on a Desarquesian plane of order q, which is clearly a contradiction as, in particular, by Ostrom [14] $4 \mid$ (the vector dimension of $\pi_0)/2 = 1$ as G is forced to act in the general linear group of π_0 over its kernel $GF(q)$.

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