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Geometric Classification of Triangulations and Their Enumeration in a Convex Polygon

S. SEN GUPTA, K. MUKHOPADHYAYA,
B. B. BHATTACHARYA AND B. P. SINHA
Electronics Unit, Indian Statistical Institute
203, B. T. Road, Calcutta 700 035, India

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Abstract—Triangulation of polygons is a classical problem in computational geometry. For an arbitrary polygon, the triangulation may depend on its shape. In this paper, we describe a new geometric classification of triangulations of a convex polygon and then derive expressions for counting the number of nonisomorphic triangulations in each class.

Keywords—Convex polygons, Triangulations, Planar graphs, Catalan numbers.

1. INTRODUCTION

Triangulation of a simple polygon is a well-known problem in two-dimensional computational geometry [1]. It has many applications in computer graphics, and in general, it is used in the preprocessing phase of a number of nontrivial operations on simple polygons [1,2]. Triangulating a polygon means partitioning it into triangles by inserting internal diagonals that are nonintersecting except at the vertices. Several fast algorithms are known for triangulating simple polygons. Until 1986, the best known algorithm for triangulating a simple polygon was of complexity $O(n \log n)$ [3]. An outstanding work in this direction was due to Tarjan and Van Wyk [4], who proposed an $O(n \log \log n)$ algorithm. Recently, Chazelle [5] suggested a very tricky linear time triangulation algorithm. Triangulation of an n-vertex polygon requires $(n-3)$ nonintersecting internal diagonals. Thus, to count the total number of triangulations of a simple polygon, the shape of the polygon must also be considered. This makes the counting problem very difficult. However, the problem reduces to a simpler one if we restrict ourselves to convex polygons.

For convex polygons, all diagonals are internal diagonals. Thus, the number of triangulations of a convex n-gon is independent of the shape, and therefore, it can uniquely be characterized by the number of vertices n. Let $\delta(n)$ be the total number of triangulations of a convex n-gon. In [6], Knuth *et al.* have shown that

$$\delta(n) = C_{n-2} = \frac{\binom{2n-4}{n-2}}{(n-1)},$$

where C_n is the n^{th} Catalan number.

A triangulated convex polygon may be represented by a maximal outerplanar graph (see [7]) $G_T(V, \varepsilon)$, where $v_i \in V$ corresponds to a vertex of the polygon, and $(v_i, v_j) \in \varepsilon$ if v_i, v_j are joined either by a polygon edge or a diagonal. The triangulated polygon itself is a natural outerplane embedding, where all the vertices lie on the outer face. Two triangulations are said

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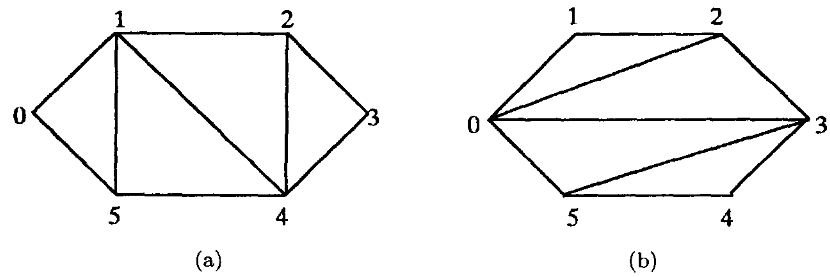

Figure 1. Two isomorphic triangulations of a hexagon.

to be *isomorphic* to each other if the corresponding planar graphs are isomorphic. Figures 1a and 1b represent two isomorphic triangulations of a hexagon.

The problem of counting the total number ($\delta^N(n)$) of triangulations, nonisomorphic up to reflection and rotation, has been solved in [8–10] for convex polygons. The following counting formula is due to Moon and Moser [10]:

$$\begin{aligned}\delta^N(n) &= \frac{f(n)}{2n} + \frac{f(n/3+1)}{3} + \frac{3f(n/2+1)}{4}, && \text{if } n \text{ is even;}\\ &= \frac{f(n)}{2n} + \frac{f(n/3+1)}{3} + \frac{f((n+1)/2)}{2}, && \text{if } n \text{ is odd,}\end{aligned}$$

where

$$\begin{aligned}f(n) &= \delta(n), && \text{if } n \geq 2,\\ &= 0, && \text{for fractional and/or negative values of } (n-2).\end{aligned}$$

Recently, studies on certain special kinds of triangulations, such as min-max length triangulations, thin triangulations, bushy triangulations, etc., have received attention in the area of computational geometry, pattern recognition, and related fields. A quadratic time algorithm has been proposed for min-max length triangulations [11], and the counting problem has been solved for thin and bushy triangulations [12].

In this paper, we first introduce a new geometric classification of triangulations of a convex polygon, and then derive expressions for the number of such triangulations, nonisomorphic up to reflection and rotation, for each of these special classes.

With each edge (i,j) in G_T, we associate a function d, the length of the edge (i,j). The formal definition of length appears in Section 2. If the length of the longest edge in G_T is denoted by Δ, then $\lceil (n/3) \rceil \leq \Delta \leq \lfloor (n/2) \rfloor$. If Δ is equal to $\lceil (n/2) \rceil = \lfloor (n/2) \rfloor$, then we call the corresponding triangulation a *bisector triangulation.* Otherwise, the triangulation is a *nonbisector triangulation.* Next, we show that for each nonbisector triangulation, there exists a unique triangle (face), called the *central triangle*, whose perimeter is equal to n. The notion of the central triangle allows us to partition the set of all nonbisector triangulations into three disjoint classes. Depending on the type of the central triangle, the nonbisector triangulations are classified as

(i) *scalene*,
(ii) *isosceles*, or
(iii) *equilateral.*

This classification is mutually exclusive and collectively exhaustive.

Such special type of triangulations may be of interest in pattern recognition, image processing, robotics, etc. Based on this classification, we derive formulae for the number of nonisomorphic triangulations in each class.

2. NEW CLASSIFICATION OF POLYGON TRIANGULATIONS

Throughout the paper, we will consider a convex polygon with n vertices (n is any positive integer, $n \geq 3$). The vertices are numbered from 0 to $n-1$ in the clockwise direction. The polygon will be denoted by $P(0,1,2,\dots,n-1)$. The notation $T(0,1,\dots,n-1)$, or just T in short, will be used to denote a particular triangulation of the polygon $P(0,1\dots,n-1)$. The corresponding maximal outerplanar graph is denoted by G_T. An edge joining two vertices i and j is denoted by the pair (i,j). Nodes of the graph G_T are also labeled in the same fashion as in P.

DEFINITION 1. *In G_T, the length $d(i,j)$ of an edge (i,j) is defined as the minimum path length between i and j along the outermost cycle. In other words,*

$$d(i,j) = d(j,i) = \min\{|i-j|, n-|i-j|\}.$$

EXAMPLE. In Figure 1a, $d(1,4) = 3$, $d(1,5) = 2$.

For any triangulation T, the length of the longest edge in G_T will be denoted by Δ. It is easy to verify that $\Delta \leq \lfloor n/2 \rfloor$.

2.1. Classification

DEFINITION 2. *A triangulation is said to be a bisector triangulation if $\Delta = \lfloor n/2 \rfloor = \lceil n/2 \rceil$. Otherwise, T will be a nonbisector triangulation.*

Thus, for odd values of n, we cannot have any bisector triangulation, whereas for even n, we may have both types of triangulations.

DEFINITION 3. *Since each internal face in G_T is enclosed by three vertices, any two vertices corresponding to an internal face are said to be the consecutive neighbors of the third vertex.*

THEOREM 1. *For any triangulation T, $\lceil n/3 \rceil \leq \Delta \leq \lfloor n/2 \rfloor$.*

PROOF. For a bisector triangulation, $\Delta = \lceil n/2 \rceil = \lfloor n/2 \rfloor$. So we need to consider only the nonbisector triangulations. Let T be a nonbisector triangulation, and without any loss of generality, let us assume that G_T contains the edge $(0,x)$, such that $d(0,x) = x = \Delta < \lceil n/2 \rceil$. Let i and x be two consecutive neighbors of vertex 0, where $i > x$, as shown in Figure 2.

Now, $d(x,i) \leq \Delta$. Therefore,

$$i - x \leq \Delta. \tag{1}$$

Also, $d(0,i) \leq \Delta$. As $i > x$ and $d(0,i) = n - i$,

$$n - i \leq \Delta. \tag{2}$$

Adding (1) and (2), we get $2\Delta \geq n - x$. As $x = \Delta$, we have $3\Delta \geq n$, i.e., $\lceil n/e \rceil \leq \Delta$. Hence, the proof.

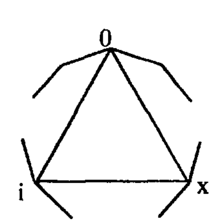

Figure 2. Illustration of Theorem 1.

LEMMA 1. *In any nonbisector triangulation T, if there are more than one longest edge, then every two of them must have one end-point in common.*

PROOF. Without loss of generality, let $(0, x)$ be one such edge, where $d(0, x) = x = \Delta < \lceil n/2 \rceil$. Let there be another longest edge $(i, i+x)$ such that $i > x$ and $i + x \neq 0 \pmod n$. The situation is shown in Figure 3. For proper triangulation of the original polygon $P(0, 1, \dots, n-1)$, we now have to triangulate the polygonal structure $(0, x, x+1, x+2, \dots, i, i+x, i+x+1, \dots, n-1)$, which is also convex. Thus, we have to join at least one vertex from the set $(i+x, i+x+1, \dots, n-1, 0)$ to one of $(x, x+1, x+2, \dots, i)$ by an edge other than $(0, x)$ and $(i, i+x)$. But the length of that edge will be at least $\Delta + 1$. This contradicts our hypothesis.

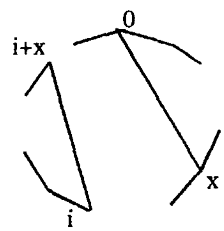

Figure 3. Illustration of Lemma 1.

THEOREM 2. *For any triangulation T, G_T will satisfy the following properties:*

(1) *If $\Delta = \lfloor n/2 \rfloor = \lceil n/2 \rceil$, there will be exactly one longest edge in T.*
(2) *If $\Delta = \lfloor n/3 \rfloor = \lceil n/3 \rceil$, there will be exactly three longest edges in T.*
(3) *If $\lfloor n/3 \rfloor < \Delta < \lceil n/2 \rceil$, there will at most two longest edges in T.*

PROOF. Follows from Theorem 1 and Lemma 1.

THEOREM 3. *For any nonbisector triangulation T, there is a unique triangle (face) in G_T whose perimeter is equal to n.*

PROOF. Without loss of generality, let us assume that G_T contains the edge $(0, x)$, such that $d(0, x) = x = \Delta < \lceil n/2 \rceil$. Let i and x be two consecutive neighbors of the vertex 0, where $x + 1 \leq i \leq n - 1$. Now,

$$
\begin{aligned}
d(0, x) &= \Delta \\
d(x, i) &= i - x = i - \Delta \\
d(i, 0) &= n - i \qquad [\text{as } \Delta \text{ is the maximum length}].
\end{aligned}
$$

Therefore, the perimeter of the triangle $(0, x, i)$ is n.

To prove uniqueness, let us assume that there exists another triangle in G_T, whose perimeter is also n. From planarity, the second triangle must be contained in either of the three portions of the original polygon (Figure 4). These three portions (regions) are named as α, β, and γ.

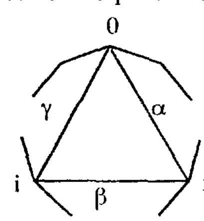

Figure 4. A unique face with perimeter n.

Now, the perimeter of any triangle contained in the portion α, will be at most 2Δ, which cannot be equal to n, as $\Delta < \lceil n/2 \rceil$. Similarly, the perimeter of any triangle contained in the portion β and γ will be at most $2(i - \Delta)$ and $2(n - i)$, respectively. But none of these can be equal to n, as $\Delta < \lceil n/2 \rceil$.

DEFINITION 4. *In any nonbisector triangulation, the unique triangle with perimeter n will be called the central triangle of that triangulation. The position of the central triangle will be specified by its three vertices.*

EXAMPLE. The triangulation of the polygon, as shown in Figure 5, has the central triangle $(0, 3, 6)$.

The central triangle may be equilateral, isosceles, or scalene.

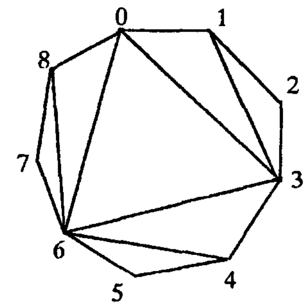

Figure 5. The central triangle.

REMARK. If $n = 3$, the polygon itself is the central triangle. For $n > 3$, the central triangle may involve at most one polygonal edge. If it involves a polygonal edge, the central triangle would be isosceles.

If n (> 3) is even, the central triangle cannot involve any polygonal edge. If n (> 3) is odd, the central triangle may involve a polygonal edge, and in that case, the length of the other two edges will be $(n - 1)/2$.

DEFINITION 5. *A triangulation will be called scalene or isosceles or equilateral if the central triangle is scalene or isosceles or equilateral, respectively.*

The notions of bisector triangulation and the central triangle allow us to classify the various triangulations of a convex polygon into the following cases, as shown in Figure 6.

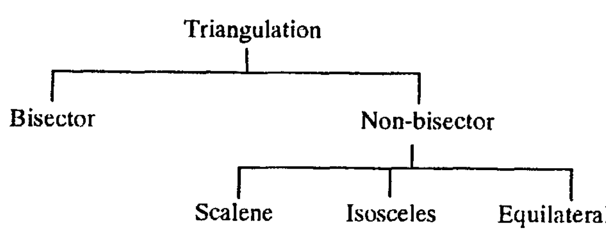

Figure 6. Classification of triangulations.

REMARK. For $n = 6r + 1$ or $6r + 5$, $(r \geq 0)$, bisector and equilateral triangulations are not possible.

For $n = 6r + 2$ or $6r + 4$, $(r \geq 0)$, equilateral triangulations are not possible.

For $n = 6r + 3$, $(r \geq 0)$, bisector triangulations are not possible.

2.2. Reflectional Symmetry of Triangulations

In this section, we are going to define reflectional symmetry (a) about a vertex, and (b) about the perpendicular bisector of an edge.

DEFINITION 6. *The reflection of a triangulation $T(0, 1, \dots, n - 1)$ about the vertex 0 is defined as $T'(0, 1 \dots, n - 1) = T(0, n - 1, n - 2, \dots, 1)$.*

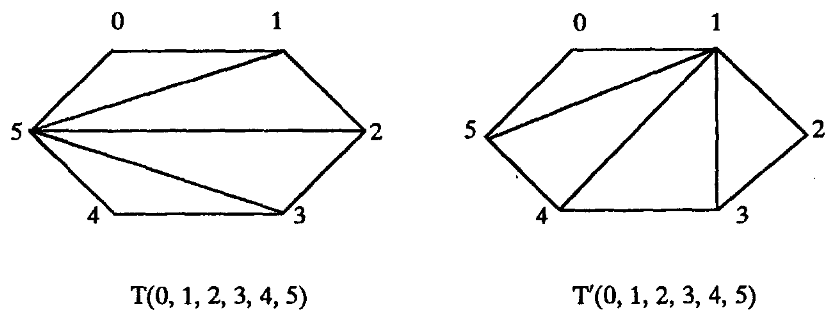

Figure 7. Reflection of a triangulation about the vertex 0.

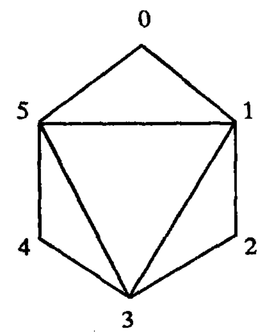

Figure 8. V-symmetric triangulation about the vertex 0.

Figure 7 shows a triangulation of a convex hexagon and its reflection about vertex 0.

REMARK. $(T')' = T$.

DEFINITION 7. *A triangulation* $T(0,1,\dots,n-1)$ *is called* V*-symmetric triangulation about vertex 0, if* $T(0,1,\dots,n-1) = T'(0,1,\dots,n-1)$.

Figure 8 shows an example of a V-symmetric triangulation about vertex 0.

DEFINITION 8. *Reflection of the vertices of a convex polygon* $P(0,1,\dots,n-1)$ *about the perpendicular bisector of the edge* $(0,n-1)$ *will be defined by the mapping* $E : V \to V$, *where* $E(v) = n - v - 1$, $v \in V$.

DEFINITION 9. *Reflection of a triangulation* $T(0,1,\dots,n-1)$ *about the perpendicular bisector of the edge* $(0,n-1)$ *will be denoted by* $T^E(0,1,\dots,n-1)$ *and defined as*

$$T^E(0,1,\dots,n-1) = T(E(0),E(1),E(2),\dots,E(n-1)) = T(n-1,n-2,\dots,1,0).$$

Figure 9 shows an example of a triangulation and its reflection about the perpendicular bisector of an edge.

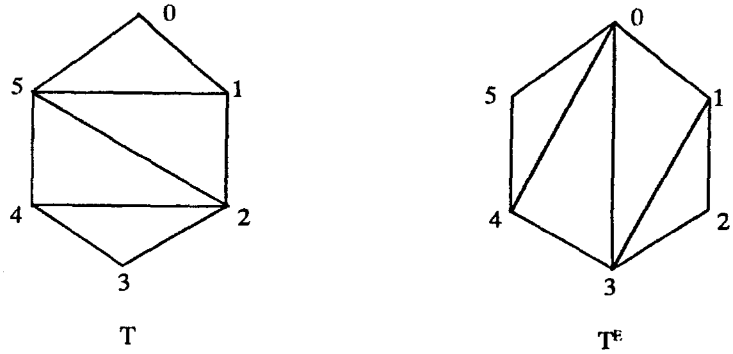

Figure 9. Reflection about the perpendicular bisector of the edge $(0,5)$.

REMARK. $\left(T^E\right)^E = T$.

DEFINITION 10. *A triangulation $T(0,1,\ldots,n-1)$ is called an E-symmetric triangulation if $T(0,1,\ldots,n-1) = T^E(0,1,\ldots,n-1)$.*

Figure 10 displays an example.

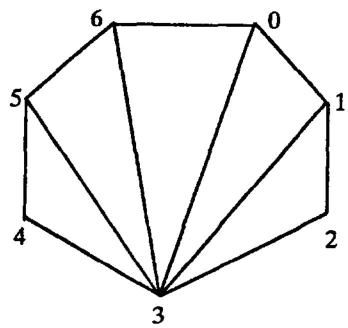

Figure 10. An E-symmetric triangulation about the perpendicular bisector of the edge $(0,6)$.

LEMMA 2. *Let $\delta_E(n)$ denote the total number of E-symmetric triangulations of a polygon $P(0,1,\ldots,n-1)$. Then $\delta_E(2)=1$, and for $n>2$,*

$$\begin{aligned} \delta_E(n) &= 0, && \text{if } n \text{ is even;} \\ &= \delta\left(\frac{(n+1)}{2}\right), && \text{if } n \text{ is odd.} \end{aligned}$$

PROOF. We prove the result by induction on n. It is easy to verify the result for $n = 3$, 4, and 5. Let the result be true for $n = 1,2,\ldots,n-1$. Let $T(0,1,2,\ldots,n-1)$ be any E-symmetric triangulation. Because of E-symmetry, for every edge (i,j) in T, the edge $(n-j-1, n-i-1)$ should also be present in T. But, due to planarity, for $i < (n-1)/2 < j$, it is not possible for any triangulation T to have both of these edges, unless $j = n-i-1$ (i.e., when (i,j) and $(n-j-1,n-i-1)$ represent the same edge). An edge of the form $(i,n-i-1)$, $1 \le i \le n-2$, divides a polygon into two polygons of smaller size and both of them have to be triangulated in an E-symmetric fashion to get an E-symmetric triangulation of the original polygon. The polygon, containing the edge $(0,n-1)$ and $(i,n-i-1)$ will have even number of vertices. But by induction hypothesis, it is not possible to triangulate it in an E-symmetric way. So, T cannot have an internal edge of the form $(i,n-i-1)$.

From the above discussion, it follows that if n is even, the polygon $(0,n/2-1,n/2,n-1)$ will remain intact in T. Thus, no E-symmetric triangulation exists, for even values of n.

For odd values of n, there must be the central triangle $(0,(n-1)/2,n-1)$ in T, as edges (i,j) for $i < (n-1)/2 < j$ cannot exist in T. This divides the polygon into two smaller polygons $(0,1,\ldots,(n-1)/2)$ and $((n-1)/2,\ldots,n-2,n-1)$. Referring to Figure 11, for T to be E-symmetric, we must have $T^E(0,1,\ldots,(n-1)/2) = T((n-1)/2,\ldots,n-1)$. Thus, the total number of E-symmetric triangulations will be $\delta_E(n) = \delta((n+1)/2)$.

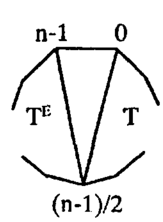

Figure 11. Illustration of Lemma 2.

REMARK. If we redefine $\delta(n)$ as

$$\begin{aligned} \delta(n) &= D_{n-2} = C_{n-2}, && \text{if } n \geq 2, \\ &= 0, && \text{for fractional and/or negative values of } (n-2), \end{aligned}$$

then the above result can be restated as: $\delta_E(2) = 1$, and for $n > 2$, $\delta_E(n) = D_{(n-3)/2}$.

3. COUNTING NONISOMORPHIC TRIANGULATIONS IN EACH CLASS

Having classified the triangulations of a convex polygon as above, we now show that two triangulations belonging to two different classes cannot be isomorphic. Then, we proceed to count the number of nonisomorphic triangulations in each of the above classes. Summing them up, we could also find the overall expression for the total number of nonisomorphic triangulations of a convex polygon.

3.1. Properties of Isomorphic Triangulations

Many results on isomorphism of planar graphs appeared in [13]. In this subsection, we point out a special property of outerplanar graphs, which in turn characterizes nonisomorphic triangulations.

LEMMA 3. *The maximal outerplanar graph G_T, corresponding to any triangulation $T(0, 1, \dots, n-1)$, has a unique Hamiltonian cycle $(0, 1 \dots, n-1, 0)$, along the outermost cycle.*

PROOF. The sequence of vertices $(0, 1, 2, \dots, n-1, 0)$ along the outermost face of G_T gives us one Hamiltonian cycle.

If possible, let there be another Hamiltonian cycle which includes an internal edge (i, j), as in Figure 12. Let us try to traverse this Hamiltonian cycle starting from the vertex i and moving towards j along the edge (i, j). The edge divides the polygon into two halves, and from vertex j we must go to a vertex in one of the halves. But once we enter one of the halves, there is no internal edge which can take us to the other half (from planarity). The other two edges into the other half also cannot be used, as i and j both have already been traversed. Thus, we cannot traverse all the vertices. Hence the contradiction.

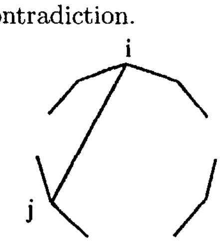

Figure 12. Uniqueness of the Hamiltonian cycle.

THEOREM 4. *Two triangulations $T_1(0, 1, \dots, n-1)$ and $T_2(0, 1, \dots, n-1)$ are isomorphic iff one can be obtained from the other by reflection and/or rotation.*

PROOF. If one of the triangulations can be obtained from the other by reflection and/or rotation, it is easy to see that they are isomorphic.

To prove the necessity, let $T_2 = f(T_1)$. Now, $(0, 1, \dots, n-1, 0)$ is a Hamiltonian cycle in T_1. As f is an isomorphism, $(f(0), f(1), \dots, f(n-1), f(0))$ will also be a Hamiltonian cycle in T_2. But, by Lemma 3, the only Hamiltonian cycle in T_2 is $(0, 1, \dots, n-1, 0)$. So, $\{f(0), f(1), \dots, f(n-1), f(0)\}$ is nothing but a reflection and/or rotation of $(0, 1, \dots, n-1, 0)$. Thus, T_2 can be obtained from T_1 by reflection and/or rotation.

COROLLARY 1. *If T_1 and T_2 are two isomorphic triangulations, then the central triangle of T_1 is identical to that of T_2.*

PROOF. If T_1 and T_2 are isomorphic to each other, then T_1 can be obtained from T_2 by rotation and/or reflection. Such an operation can only affect the orientation of the central triangle of a triangulation, but not its shape and size. Hence the result.

COROLLARY 2. *Two triangulations belonging to two different classes cannot be isomorphic to each other.*

PROOF. Immediate from Corollary 1.

Corollary 2 allows us to calculate the number of nonisomorphic triangulations for each class separately.

3.2. Counting

We will now derive expressions for enumerating nonisomorphic triangulations in each of the classes.

3.2.1. Counting scalene triangulations

THEOREM 5. *The total number of nonisomorphic scalene triangulations of a convex n-gon is*

$$N_{ST} = \sum_{\substack{a+b+c=n \\ a<b<c<\lceil n/2 \rceil}} [\delta(a+1)\delta(b+1)\delta(c+1)].$$

PROOF. Assume that the three sides of the central triangle are of length a, b, c (a, b, c are distinct), where $a+b+c=n$. Let us first fix the position of the central triangle in the polygon as $(0, a, a+b)$ as shown in Figure 13. With this fixed position of the central triangle, the total number of possible triangulations is $\delta(a+1)\delta(b+1)\delta(c+1)$.

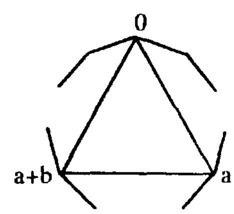

Figure 13. Counting scalene triangulations.

We claim that the triangulations which are included in the above count are nonisomorphic to each other. To prove this, we assume the converse. Let T_1 and T_2 be two isomorphic triangulations present in the above count. Then, by Theorem 4, T_1 can be obtained from T_2 by reflection and/or rotation. As the central triangle is scalene, reflection or rotation will change the orientation of the central triangle. But this contradicts our hypothesis that the position of the central triangle is fixed at $(0, a, a+b)$.

For a given set of values a, b, c, $a+b+c=n$ (a, b, c are all distinct), it is enough to consider any one of the six possible central triangles, viz., $(0, a, a+b)$, $(0, a, a+c)$, $(0, b, a+b)$, $(0, b, b+c)$, $(0, c, a+c)$, $(0, c, b+c)$. All triangulations for the remaining position of the central triangle will be isomorphic to some member of the set already considered. So let us assume $a < b < c$, and consider the central triangle $(0, a, a+b)$. Thus, the total number of nonisomorphic scalene triangulations of a convex n-gon will be:

$$\sum_{\substack{a+b+c=n \\ a<b<c<\lceil n/2 \rceil}} [\delta(a+1)\delta(b+1)\delta(c+1)].$$

REMARK. With our notation defined above, the above result can further be simplified as:

$$N_{ST} = \frac{1}{3} D_{n-2} - \frac{1}{12} D_{n-1} - \frac{1}{4} D^2_{n/2-1} - \frac{1}{6} D^3_{n/3-1} - \frac{1}{2} \sum_{\substack{2a+b=n \\ a\neq b<\lceil n/2\rceil}} [D^2_{a-1} D_{b-1}],$$

(see Appendix).

3.2.2. Counting isosceles triangulations

THEOREM 6. *The total number of nonisomorphic isoceles triangulations of a convex n-gon is*

$$N_{IT} = \sum_{\substack{2a+b=n \\ a\neq b,\, a,b<\lceil n/2\rceil}} \frac{1}{2}\{[\delta(a+1)]^2\delta(b+1) + \delta(a+1)\delta_E(b+1)\}.$$

PROOF. Let the two equal sides of the central triangle be of length a, and the other side be of length b. Thus, $2a + b = n$, $a \neq b$, and $a, b < \lceil n/2\rceil$. Let us first fix the position of the central triangle as $(0, a, a + b)$. With this fixed position of the central triangle, the total number of possible triangulations will be $[\{\delta(a+1)\}^2\delta(b+1)]$.

Since an isosceles triangle has a twofold symmetry, the above triangulations are nonisomorphic up to rotation only. In other words, the above set contains both T and T' (if they are not identical). Let us consider an isosceles triangulation $T(0, 1, \dots, n-1)$, with central triangle $(0, a, a + b)$. The central triangle divides the original polygon into three smaller polygons, $(0, 1, \dots, a)$, $(a, a + 1, \dots, a + b)$, and $(a + b, a + b + 1, \dots, n - 1, 0)$ as shown in Figure 14. In any triangulation, these three smaller polygons are also triangulated in some fashion. This is symbolically represented as:

$$T(0,1,\dots,n-1) = \big(A(0,1,\dots,a)\, B(a,a+1,\dots,a+b)\, C(a+b,a+b+1,\dots,n-1,0)\big),$$

where A and C are any two triangulations of a polygon of size $(a+1)$ and B is that of a polygon of size $(b + 1)$. In short, we also denote this by $T \equiv (ABC)$.

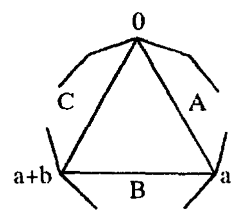

Figure 14. Counting isosceles triangulations.

Now,

$$\begin{aligned} T'(0,1,\dots,n-1) &= T(0,n-1,n-2,\dots,1) \\ &= A(0,n-1,\dots,a+b)\, B(a+b,a+b+1,\dots,a)\, C(a,a-1,\dots,0) \\ &= C^E(0,1,\dots,a)\, B^E(a,a+1,\dots,a+b)\, A^E(a+b,\dots,n-1,0) \\ &\equiv (C^E B^E A^E). \end{aligned}$$

If T is a V-symmetric triangulation, then

$$T(0,1,\dots,n-1) = T'(0,1,\dots,n-1) = T(0,n-1,n-2,\dots,1).$$

Therefore, $C = A^E$ and $B = B^E$. Thus, for T to be a V-symmetric triangulation, B should be E-symmetric and C should be A^E. Therefore, the total number of V-symmetric triangulations will be $\delta(a+1)\delta_E(b+1)$ which can be comupted using Lemma 2.

Thus, for the fixed position of the central triangle, the number of nonisomorphic triangulations will be

$$\frac{1}{2}\left\{[\delta(a+1)]^2\delta(b+1)+\delta(a+1)\delta_E(b+1)\right\}.$$

Any change in the position of the central triangle will generate a triangulation isomorphic to one that has already been considered in the above count.

Therefore, the total number of nonisomorphic isosceles triangulations will be

$$N_{IT} = \sum_{\substack{2a+b=n\\ a\neq b,\, a,b<\lceil n/2\rceil}} \frac{1}{2}\left\{[\delta(a+1)]^2\delta(b+1)+\delta(a+1)\delta_E(b+1)\right\}.$$

REMARK. The above result can be simplified as:

$$N_{IT} = \frac{1}{2}\sum_{\substack{2a+b=n\\ a\neq b,\, a,b<\lceil n/2\rceil}} \left\{[D_{a-1}]^2 D_{b-1}\right\} + \frac{1}{2}D_{(n-1)/2-1} + \frac{1}{4}D_{n/2-1} - \frac{1}{4}D^2_{n/4-1} - \frac{1}{2}D_{n/3-1}D_{n/6-1},$$

(see Appendix).

3.2.3. Counting equilateral triangulations

THEOREM 7. *The total number of nonisomorphic equilateral triangulations of a convex n-gon is*

$$N_{ET} = \frac{1}{6}\left\{\left[\delta\left(\frac{n}{3}+1\right)\right]^3 + 2\delta\left(\frac{n}{3}+1\right) + 3\delta\left(\frac{n}{3}+1\right)\delta_E\left(\frac{n}{3}+1\right)\right\}.$$

PROOF. Let n be a multiple of 3. Without loss of generality, we fix the central triangle at $(0, n/3, 2n/3)$. With this central triangle, the total number of possible triangulations will be $[\delta(n/3+1)]^3$. Among these, let us consider a particular triangulation T such that

$$T(0,1,\ldots,n-1) = \left(A\left(0,1,\ldots,\frac{n}{3}\right)\, B\left(\frac{n}{3},\frac{n}{3}+1,\ldots,\frac{2n}{3}\right)\, C\left(\frac{2n}{3},\frac{2n}{3}+1,\ldots,n-1,0\right)\right),$$

i.e., $T \equiv (ABC)$, where A, B, C are any three triangulations of a convex polygon of size $(n/3+1)$, as shown in Figure 15.

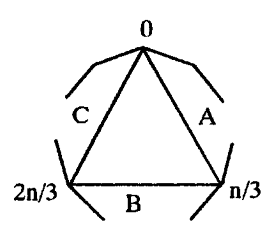

Figure 15. Counting equilateral triangulations.

As the position of the central triangle is fixed, there are at most five triangulations, namely, $T_1 = (BCA)$, $T_2 = (CAB)$, $T_3 = (C^E B^E A^E)$, $T_4 = (B^E A^E C^E)$, $T_5 = (A^E C^E B^E)$, all of which are isomorphic to T. But the triangulation T_i $(1 \le i \le 5)$ may be identical to T. If so, T_i has no contribution in the above count. Otherwise, it has been considered separately in the count.

If T_1 (or T_2) $= T$, then $A = B = C$. The number of such triangulations, for which T_1 (or T_2) is identical to T, is $\delta(n/3+1)$.

If $T_3 = T$, then $A = C^E$, $B = B^E$, $C = A^E$. The number of such triangulations will be $\delta(n/3+1)\delta_E(n/3+1)$.

In a similar way, the number of triangulations for which T is identical to T_4 or T_5 will be $\delta(n/3+1)\delta_E(n/3+1)$.

Thus, the total number of nonisomorphic equilateral triangulations, with the central triangle $(0, n/3, 2n/3)$ will be $1/6\{[\delta(n/3+1)]^3 + 2\delta(n/3+1) + 3\delta(n/3+1)\delta_E(n/3+1)\}$.

Any triangulation generated by changing the position of the central triangle has already been included in the above count.

Thus, the total number of nonisomorphic equilateral triangulations of $P(0, 1, \ldots, n-1)$ will be

$$N_{ET} = \frac{1}{6}\left\{\left[\delta\left(\frac{n}{3}+1\right)\right]^3 + 2\delta\left(\frac{n}{3}+1\right) + 3\delta\left(\frac{n}{3}+1\right)\delta_E\left(\frac{n}{3}+1\right)\right\}.$$

When n is not a multiple of 3, $N_{ET} = 0$. Therefore, for any n-gon,

$$N_{ET} = \frac{1}{6}D^3_{n/3-1} + \frac{1}{3}D_{n/3-1} + \frac{1}{2}D_{n/3-1}D_{n/6-1}.$$

3.2.4. Counting bisector triangulations

THEOREM 8. *The total number of nonisomorphic bisector triangulations of a convex n-gon is*

$$N_{BT} = \frac{1}{4}\left[\left\{\delta\left(\frac{n}{2}+1\right)\right\}^2 + 2\delta\left(\frac{n}{2}+1\right) + \left\{\delta_E\left(\frac{n}{2}+1\right)\right\}^2\right].$$

PROOF. In this case, the length of the longest edge is $\lfloor n/2 \rfloor = \lceil n/2 \rceil$, that is, n must be even. Let us first fix the position of this edge as $(0, n/2)$.

With this fixed position of the longest edge, the total number of possible triangulations is $[\delta(n/2+1)]^2$.

Let us consider a triangulation T such that

$$T(0,1,\ldots,n-1) = \left(A\left(0,1,\ldots,\frac{n}{2}\right) B\left(\frac{n}{2},\frac{n}{2}+1,\ldots,n-1,0\right)\right),$$

i.e., $T \equiv (AB)$, where A and B are any two triangulations of a convex polygon of size $(n/2+1)$, as shown in Figure 16.

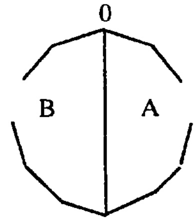

Figure 16. Counting bisector triangulations.

With the fixed position of the longest edge, the triangulation T has at most three isomers, namely, $T_1 \equiv (BA)$, $T_2 \equiv (A^E B^E)$, and $T_3 \equiv (B^E A^E)$. But these triangulations may be identical to T. If they are not, then they must have been considered separately in the above count.

If $T = T_1$, then $A = B$. The number of such triangulations, (AA), is $\delta(n/2+1)$.

If $T = T_2$, then $A = A^E$ and $B = B^E$. The number of such triangulations is $[\delta_E(n/2+1)^2]$.

If $T = T_3$, then $A = B^E$ and $B = A^E$. The number of such triangulations, $(A^E A)$, is $\delta(n/2+1)$.

Thus, the total number of nonisomorphic bisector triangulations with the fixed position of the longest edge will be $1/4[\{\delta(n/2+1)\}^2 + 2\delta(n/2+1) + \delta_E(n/2+1)^2]$.

Triangulations obtained by changing the position of the longest edge will be isomorphic to one which has already been included in the above count.

Therefore, the total number of nonisomorphic bisector triangulations of a convex n-gon will be

$$N_{BT} = \frac{1}{4}\left[\left\{\delta\left(\frac{n}{2}+1\right)\right\}^2 + 2\delta\left(\frac{n}{2}+1\right) + \left\{\delta_E\left(\frac{n}{2}+1\right)\right\}^2\right].$$

For odd values of n, $N_{BT} = 0$. Therefore, for all values of n,

$$N_{BT} = \frac{1}{4}D^2_{n/2-1} + \frac{1}{2}D_{n/2-1} + \frac{1}{4}D^2_{n/4-1}.$$

3.2.5. Counting total number of nonisomorphic triangulations

THEOREM 9. *The total number of nonisomorphic triangulations of a convex polygon $P(0,1,\dots, n-1)$ is given by*

$$\delta^N(n) = \frac{1}{12}\left[4D_{n-2} + 9D_{n/2-1} + 4D_{n/3-1} + 6D_{(n-1)/2-1} - D_{n-1}\right].$$

PROOF. See Appendix.

4. CONCLUSION

In this paper, we have presented a new classification scheme for triangulating a convex polygon, namely: bisector, scalene, isosceles, and equilateral. As there is a bijection between maximal outerplanar graphs and planar embedding of triangulations of convex polygons, this classification scheme is valid for maximal outerplanar graphs as well. Then, we enumerated the number of nonisomorphic triangulations in each class separately. Numerical results up to $n = 20$ are given in Table 1.

Table 1.

n	N_{ST}	N_{IT}	N_{ET}	N_{BT}	$\delta^N(n)$
3	0	0	1	0	1
4	0	0	0	1	1
5	0	1	0	0	1
6	0	0	1	2	3
7	0	4	0	0	4
8	0	3	0	9	12
9	10	15	2	0	27
10	0	26	0	56	82
11	70	158	0	0	228
12	140	105	25	463	733
13	1008	1274	0	0	2282
14	1176	1930	0	4422	7528
15	12180	12192	462	0	24834
16	20328	17339	0	46231	83898
17	150414	134943	0	0	285357
18	280962	177938	12404	511940	983244
19	1826682	1585738	0	0	3412420
20	3804372	2228001	0	5912241	11944614

The proposed characterization of triangulations can be extended further to explore more properties of triangulations. For example, if $D(n)$ denotes the total length of the diagonals in a convex n-gon, then what are the lower and upper bounds of $D(n)$?

Similar geometric classification of triangulations in a general polygon remains open.

APPENDIX

From Theorems 5–8, we get

$$\delta^N(n) = N_{ST} + N_{IT} + N_{ET} + N_{BT}. \tag{A}$$

From the recurrence relation [8] of Catalan numbers, we have

$$D_n = \sum_{k=0}^{n-1} D_{n-k-1} D_k.$$

One can easily verify that

$$\sum_{k=0}^{\lceil n/2-1 \rceil} D_{n-k-1} D_k = \frac{1}{2}(D_n + D^2_{(n-1)/2}). \tag{3}$$

Now, for any three positive integers a, b, and c and any function $f(a,b,c)$, we can write:

$$\sum_{a+b+c=n} f = \sum_{\substack{a+b+c=n \\ a,b,c<n/2}} f + 3 \sum_{\substack{a+b+c=n \\ a\geq n/2}} f,$$

i.e.,

$$\sum_{a+b+c=n} f = \sum_{\substack{a+b+c=n \\ a,b,c \text{ are distinct} \\ a,b,c<n/2}} f + 3 \sum_{\substack{a+b+c=n \\ a=b\neq c \\ a,b,c<n/2}} f + \sum_{\substack{a+b+c=n \\ a=b=c=n/2}} f + 3 \sum_{\substack{a+b+c=n \\ a\geq n/2}} f.$$

Therefore,

$$\sum_{a+b+c=n} f = 6 \sum_{\substack{a+b+c=n \\ a<b<c<n/2}} f + 3 \sum_{\substack{a+b+c=n \\ a=b\neq c \\ a,b,c<n/2}} f + \sum_{\substack{a+b+c=n \\ a=b=c=n/3}} f + 3 \sum_{\substack{a+b+c=n \\ a\geq n/2}} f.$$

Therefore,

$$\begin{aligned} \sum_{\substack{a+b+c=n \\ a<b<c<n/2}} [\delta(a+1)\delta(b+1)\delta(c+1)] = \frac{1}{6} & \sum_{a+b+c=n} [\delta(a+1)\delta(b+1)\delta(c+1)] \\ & - \frac{1}{2} \sum_{\substack{2a+b=n \\ a\neq b<n/2}} [\delta(a+1)^2\delta(b+1)] - \frac{1}{6}\left[\delta\left(\frac{n}{3}+1\right)\right]^3 \\ & - \frac{1}{2} \sum_{\substack{a+b+c=n \\ a\geq n/2}} [\delta(a+1)\delta(b+1)\delta(c+1)]. \end{aligned} \tag{4}$$

Now,

$$\begin{aligned} \sum_{a+b+c=n} [\delta(a+1)\delta(b+1)\delta(c+1)] &= \sum_{a=0}^{n} \delta(a+1) \sum_{b+c=n-a} \delta(b+1)\delta(c+1) \\ &= \sum_{a=0}^{n} \delta(a+1) f(n-a), \end{aligned}$$

where

$$\begin{aligned} f(k) &= \sum_{b+c=k} \delta(b+1)\delta(c+1), && \text{for } k > 1 \\ &= 0, && \text{for } k = 1 \\ &= \sum_{b+c=k} D_{b-1}D_{c-1}, && \text{for } k > 1 \\ &= 0, && \text{for } k = 1. \end{aligned}$$

Therefore,

$$\begin{aligned} f(k) &= D_{k-1}, && \text{for } k > 1 \\ &= 0, && \text{for } k = 1. \end{aligned} \tag{5}$$

Hence,

$$\begin{aligned} \sum_{a+b+c=n} [\delta(a+1)\delta(b+1)\delta(c+1)] &= \sum_{a=0}^{n-2} [\delta(a+1)f(n-a)] + \delta(n)f(1) \\ &= \sum_{a=0}^{n-2} D_{a-1}D_{n-a-1} \qquad [\text{as } f(1) = 0] \\ &= \sum_{a=0}^{n-1} [D_{a-1}D_{n-a-1}] - D_{n-2} \\ &= D_{n-1} - D_{n-2}. \end{aligned} \tag{6}$$

Now,

$$\begin{aligned} \sum_{\substack{a+b+c=n \\ a \geq n/2}} [\delta(a+1)\delta(b+1)\delta(c+1)] &= \sum_{a=\lceil n/2 \rceil}^{n} [\delta(a+1)f(n-a)] \\ &= \sum_{a=\lceil n/2 \rceil}^{n-1} [D_{a-1}D_{n-a-1}] - D_{n-2} \qquad [\text{using (5)}] \\ &= \sum_{k=\lceil (n-1)/2 \rceil}^{n} [D_k D_{n-k-2}] - D_{n-2} \\ &= \frac{1}{2}(D_{n-1} + D_{n/2-1}^2) - D_{n-2} \qquad [\text{using (3)}]. \end{aligned} \tag{7}$$

Using (4), (6), and (7) we get

$$\begin{aligned} N_{ST} &= \sum_{\substack{a+b+c=n \\ a<b<c<n/2}} [\delta(a+1)\delta(b+1)\delta(c+1)] \\ &= \frac{1}{6}(D_{n-1} - D_{n-2}) - \frac{1}{2} \sum_{\substack{2a+b=n \\ a \neq b < n/2}} [\delta(a+1)^2\delta(b+1)] \\ &\quad - \frac{1}{6}\left[\delta\left(\frac{n}{3}+1\right)\right]^3 - \frac{1}{2}\left[\frac{1}{2}(D_{n-1} + D_{n/2-1}^2) - D_{n-2}\right]. \end{aligned}$$

Now,

$$\sum_{2a+b=n} \delta(a+1)\delta_E(b+1) = \sum_{b\geq 0} \delta\left(\frac{n}{2}-\frac{b}{2}+1\right)\delta_E(b+1)$$
$$= \sum_{b=0}^{n} D_{n/2-b/2-1}\delta_E(b+1)$$
$$= D_{(n-1)/2-1}\delta_E(2) + \sum_{b/2=1}^{n/2-1} D_{n/2-b/2-1}D_{b/2-1}$$
$$= D_{(n-1)/2-1} + D_{n/2-1}. \tag{8}$$

Again,

$$\sum_{2a+b=n} \delta(a+1)\delta_E(b+1) = \sum_{\substack{a+b/2=n/2\\ b/2<n/4}} \delta(a+1)\delta_E(b+1) + \sum_{\substack{a+b/2=n/2\\ b/2\geq n/4}} \delta(a+1)\delta_E(b+1), \quad \text{or}$$

$$\sum_{2a+b=n} \delta(a+1)\delta_E(b+1) = \sum_{\substack{a+b/2=n/2\\ b/2<n/4}} \delta(a+1)\delta_E(b+1) + \sum_{\substack{a+b/2=n/2\\ b/2\geq n/4}} \delta(a+1)\delta\left(\frac{b}{2}+1\right)$$
$$= \sum_{\substack{a+b/2=n/2\\ b/2<n/4}} \delta(a+1)\delta_E(b+1) + \sum_{k=0}^{\lfloor n/4\rfloor-1} D_k D_{n/2-k-2}$$
$$= \sum_{\substack{a+b/2=n/2\\ b/2<n/4}} [\delta(a+1)\delta_E(b+1)] + \frac{1}{2}(D_{n/2-1} + D^2_{n/4-1}) \qquad \text{[using (3)]}. \tag{9}$$

Using (8) and (9), we get

$$\sum_{\substack{2a+b=n\\ b<n/2}} \delta(a+1)\delta_E(b+1) = D_{(n-1)/2-1} + D_{n/2-1} - \frac{1}{2}(D_{n/2-1} + D^2_{n/4-1}).$$

Therefore,

$$\sum_{\substack{2a+b=n\\ b<n/2,\, a\neq b}} \delta(a+1)\delta_E(b+1) = D_{(n-1)/2-1} + D_{n/2-1} - \frac{1}{2}(D_{n/2-1} + D^2_{n/4-1}) - D_{n/3-1}\delta_E\left(\frac{n}{3}+1\right).$$

Now,

$$N_{IT} = \sum_{\substack{2a+b=n\\ a\neq b,\, a,b<\lceil n/2\rceil}} \frac{1}{2}\left[\delta(a+1)\right]^2 \delta(b+1) + \delta(a+1)\delta_E(b+1)$$
$$= \sum_{\substack{2a+b=n\\ a\neq b,\, a,b<\lceil n/2\rceil}} \left[\delta(a+1)\right]^2 \delta(b+1) + \frac{1}{2}D_{(n-1)/2-1} - \frac{1}{4}(D_{n/2-1} + D^2_{n/4-1})$$
$$- \frac{1}{2}D_{n/3-1}\delta_E\left(\frac{n}{3}+1\right).$$

Using simplified expressions obtained for N_{ST} and N_{IT}, Expression (A) can further be simplified as

$$\delta^N(n) = \frac{1}{3}D_{n-2} - \frac{1}{12}D_{n-1} + \frac{3}{4}D_{n/2-1} + \frac{1}{3}D_{n/3-1} + \frac{1}{2}D_{(n-1)/2-1}$$
$$= \frac{1}{12}\left(4D_{n-2} + 9D_{n/2-1} + 4D_{n/3-1} + 6D_{(n-1)/2-1} - D_{n-1}\right).$$

REFERENCES

1. F.P. Preparata and M.I. Shamos, *Computational Geometry—An Introduction*, Springer-Verlag, Berlin, (1988).
2. J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, New York, (1987).
3. M.R. Garey, D.S. Johnson, F.P. Preparata and R.E. Tarjan, Triangulating a simple polygon, *Inform. Process. Lett.* **7**, 175–180 (1978).
4. R.E. Tarjan, C.J. Vanwyk, An $O(n \log\log n)$-time algorithm for triangulation of a simple polygon, *SIAM J. Comput.* **17**, 143–178 (1988).
5. B. Chazelle, Triangulating a simple polygon in linear time, *Discrete and Computational Geometry* **6**, 485–524 (1991).
6. R.L. Graham, D.E. Knuth and O. Potashnik, *Concrete Mathematics*, Addison-Wesley, Reading, MA, (1988).
7. F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, (1969).
8. R.K. Guy, Dissecting a polygon into triangles, Note 90, *Bull. Malayan Math. Soc.* **5**, 57–60 (1968).
9. F. Harary, M. Palmer and R.C. Read, On the cell-growth problem for arbitrary polygons, *Discrete Mathematics* **11**, 371–389 (1975).
10. J.W. Moon and L. Moser, Triangular dissection of n-gons, *Canad. Math. Bull.* **6**, 175–178 (1963).
11. H. Edelsbrunner and T.S. Tan, A quadratic time algorithm for the minmax length triangulation, *Proc. 32nd FOCS*, pp. 414–423, (1991).
12. S. Chattapadhyay and P.P. Das, Counting thin and bushy triangulations of convex polygons, *Pattern Recognition Letters* **12**, 139–144 (March 1991).
13. J.E. Hopcroft and J.K. Wong, Linear time algorithm for isomorphism of planar graphs, *Proc. 6th Annual ACM Symp. on Theory of Computings*, pp. 172–184, (1974).