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Hyers–Ulam stability of linear differential equations of first order[☆]

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Abstract

Using the method of the integral factor, this work proves the Hyers–Ulam stability of linear differential equations of first order and extends the existing results.

Keywords: Hyers–Ulam stability; Integral factor; Differential equation

1. Introduction

In 1941, answering a problem of Ulam (cf. [18,19]) affirmatively, Hyers [4] proved the following result (which is nowadays called the Hyers–Ulam stability (for simplicity, HUs) theorem):

Let $S = (S, +)$ be an Abelian semigroup and assume that a function $f : S \to \mathbb{R}$ satisfies the inequality

$$|f(x + y) - f(x) - f(y)| \le \varepsilon \quad (x, y \in S)$$

for some nonnegative ε. Then there exists an additive function $A : S \to \mathbb{R}$ such that

$$|A(x) - f(x)| \le \varepsilon \quad (x \in S)$$

holds.

Since Hyers' result, a great number of papers on the subject have been published, extending and generalizing Ulam's problem and Hyers' theorem in various directions. An extensive survey demonstrating this activity can be found in a recent paper of Forti [3], in a paper of Rassias [15], and in a paper of Miura [11]. The interested reader can also find further details in the book of Kuczma [8, Chapter XVII].

In this work, we will study the HUs of a class of first-order linear differential equations. Alsina and Ger were the first authors who investigated the HUs of differential equations. In 1998, they proved in [1] the following:

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Assume that a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality

$$|y'(t) - y(t)| \leq \varepsilon,$$

where I is an open subinterval of $\mathbb{R}$. Then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that

$$|f(t) - f_0(t)| \leq 3\varepsilon$$

for any $t \in I$.

Following the same approach as in [1], Miura, Takahasi and Choda [14], Miura [10], Takahasi, Miura and Miyajima [16], and Miura, Jung and Takahasi [12] proved that the HUs holds true for the differential equation

$$y' = \lambda y,$$

while Jung [5] proved a similar result for the differential equation

$$\varphi(t)y' = y.$$

Furthermore, the result of HUs for first-order linear differential equations has been generalized by Miura, Miyajima and Takahasi [13], by Takahasi, Takagi, Miura and Miyajima [17], and also by Jung [6]. They dealt with the nonhomogeneous linear differential equation of first order

$$y' + p(t)y + q(t) = 0.$$

It is worth pointing out that, recently, Jung [7] proved the generalized HUs of differential equations of the form

$$ty'(t) + \alpha y(t) + \beta t^r x_0 = 0$$

and also applied this result to the investigation of the HUs of the Euler (Cauchy) differential equation

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0.$$

For more detailed information on the HUs of differential equations, we refer the reader to Miura [11] and the references therein.

The purpose of this work is to discuss the HUs of the first-order nonhomogeneous linear differential equation

$$p(x)y' - q(x)y - r(x) = 0, \tag{1}$$

where $x \in I = (a, b), \ -\infty \leq a < b \leq +\infty$. We will use the method of the integral factor to prove that Eq. (1) has the HUs under some appropriate conditions. The method of this work is distinctive. It is simpler and clearer, but the result of this work is a generalization of or complement to the existing results. As far as we know, the method has scarcely been used in the available reference materials.

The layout of the work is as follows. Some basic concepts are introduced in Section 2. In Section 3, we present and prove the main result. In Section 4, we illustrate some ideas.

For the sake of convenience, all the integrals in the rest of the work will be viewed as existing.

2. Basic concepts

First of all, we give two basic definitions.

Definition 1 (*Cf. [12,20]*). We say that Eq. (1) has the Hyers–Ulam stability if there exists a constant $K \geq 0$ with the following property: for every $\varepsilon \geq 0$, $y \in C^1(I)$, if

$$|p(x)y' - q(x)y - r(x)| \leq \varepsilon,$$

then there exists some $z \in C^1(I)$ satisfying $p(x)z' - q(x)z - r(x) = 0$ such that

$$|y(x) - z(x)| \leq K\varepsilon.$$

We call such K a HUs constant for Eq. (1).

Definition 2. A function $\mu(x, y)$ is said to be an integral factor of $P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$ in the case where there exists some continuously differentiable function $R(x, y)$ such that

$$\mu P\mathrm{d}x + \mu Q\mathrm{d}y = \mathrm{d}R.$$

It is easy to see that $\frac{1}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}$ (if it exists) is an integral factor of $(-q(x)y - r(x))\mathrm{d}x + p(x)\mathrm{d}y$.

3. Main theorem

Now, the main result of this work is given in the following theorem.

Theorem 1. *Let $p(x)$, $q(x)$ and $r(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $|q(x)| \geq \delta$ for all $x \in I$ and some $\delta > 0$ independent of $x \in I$. Then Eq. (1) has the HUs.*

Proof. Let $\varepsilon > 0$ and $y : I \to \mathbb{R}$ be a continuously differentiable function such that

$$|p(x)y' - q(x)y - r(x)| \leq \varepsilon \tag{2}$$

holds for all $x \in I$. We must show that there exists a constant K independent of ε, y and x such that $|y(x)-z(x)| \leq K\varepsilon$ for all $x \in I$ and some function $z \in C^1(I)$ satisfying $p(x)z' - q(x)z - r(x) = 0$, $x \in I$. To do this, we can without loss of generality assume that $q(x) \geq 1$ for all $x \in I$.

Assume that $p(x) > 0$, $x \in I$. In view of (2), we have that

$$-\varepsilon \leq p(x)y' - q(x)y - r(x) \leq \varepsilon.$$

Multiplying the formula by the function $\frac{1}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}$, we obtain

$$\begin{aligned}
-\varepsilon\frac{1}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} &\leq y'\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} - y\frac{q(x)}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} \\
&\quad - \frac{r(x)}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} \\
&\leq \varepsilon\frac{1}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}.
\end{aligned}$$

Noticing that $q(x) \geq 1$, we have that

$$\begin{aligned}
-\varepsilon\frac{q(x)}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} &\leq y'\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} - y\frac{q(x)}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} \\
&\quad - \frac{r(x)}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} \\
&\leq \varepsilon\frac{q(x)}{p(x)}\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}.
\end{aligned} \tag{3}$$

Select $b_1 \in [a, b]$ such that $y(b_1)$ is finite. For any $x \in (a, b_1]$, integrating (3) from x to b_1, we get

$$\begin{aligned}
&-\varepsilon\left(\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} - \exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\}\right) \\
&\quad\leq y(b_1)\exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\} - y(x)\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} - \int_x^{b_1}\frac{r(s)}{p(s)}\exp\left\{-\int_a^s \frac{q(t)}{p(t)}\mathrm{d}t\right\}\mathrm{d}s \\
&\quad\leq \varepsilon\left(\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} - \exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\}\right).
\end{aligned}$$

Then

$$\begin{aligned}
-\varepsilon \exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} &\le (y(b_1)-\varepsilon)\exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\} - y(x)\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} \\
&\quad - \int_x^{b_1} \frac{r(s)}{p(s)} \exp\left\{-\int_a^s \frac{q(t)}{p(t)}\mathrm{d}t\right\}\mathrm{d}s \\
&\le \varepsilon\left(\exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} - 2\exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\}\right) \\
&\le \varepsilon \exp\left\{-\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
-\varepsilon \le \exp\left\{\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}\Bigg[(y(b_1)-\varepsilon)\exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\} \\
- \int_x^{b_1} \frac{r(s)}{p(s)} \exp\left\{-\int_a^s \frac{q(t)}{p(t)}\mathrm{d}t\right\}\mathrm{d}s\Bigg] - y(x) \le \varepsilon. \qquad (4)
\end{aligned}$$

Similarly, for any $x \in [b_1, b)$, integrating (3) from b_1 to x, we have that

$$\begin{aligned}
\varepsilon &\le y(x) - \exp\left\{\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}\Bigg[(y(b_1)-\varepsilon)\exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\} \\
&\quad - \int_x^{b_1} \frac{r(s)}{p(s)} \exp\left\{-\int_a^s \frac{q(t)}{p(t)}\mathrm{d}t\right\}\mathrm{d}s\Bigg] \\
&\le \varepsilon\left(2\exp\left\{\int_{b_1}^x \frac{q(s)}{p(s)}\mathrm{d}s\right\} - 1\right) \le \varepsilon\left(2\exp\left\{\int_a^b \frac{q(s)}{p(s)}\mathrm{d}s\right\} - 1\right) \le \varepsilon(2A-1), \qquad (5)
\end{aligned}$$

where $A \equiv \exp\left\{\int_a^b \frac{q(s)}{p(s)}\mathrm{d}s\right\}$. Summing (4) and (5),

$$|y(x) - z_1(x)| \le (2A-1)\varepsilon, \quad x \in I,$$

where

$$z_1(x) \equiv \exp\left\{\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}\left[(y(b_1)-\varepsilon)\exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\} - \int_x^{b_1} \frac{r(s)}{p(s)} \exp\left\{-\int_a^s \frac{q(t)}{p(t)}\mathrm{d}t\right\}\mathrm{d}s\right].$$

Obviously,

$$p(x)z_1' - q(x)z_1 - r(x) \equiv 0, \quad x \in I.$$

By an argument similar to the above, for the case of $p(x) < 0$, $x \in I$, we can show that

$$|y(x) - z_2(x)| \le (2B-1)\varepsilon, \quad x \in I,$$

where $B \equiv \exp\left\{-\int_a^b \frac{q(s)}{p(s)}\mathrm{d}s\right\}$, and

$$z_2(x) \equiv \exp\left\{\int_a^x \frac{q(s)}{p(s)}\mathrm{d}s\right\}\left[(y(b_1)-\varepsilon)\exp\left\{-\int_a^{b_1} \frac{q(s)}{p(s)}\mathrm{d}s\right\} - \int_x^{b_1} \frac{r(s)}{p(s)} \exp\left\{-\int_a^s \frac{q(t)}{p(t)}dt\right\}\mathrm{d}s\right].$$

Obviously,

$$p(x)z_2' - q(x)z_2 - r(x) \equiv 0, \quad x \in I.$$

Thus, the proof is completed. □

Remark. If an initial condition, say $y(b_1)$, is known, then it is easy to evaluate the lower and upper bounds for $y(x)$ from Theorem 1.

4. Ideas

(1) Considerable attention has been given to the study of the HUs of functional equations. But there are very few results on the study of the HUs of ordinary differential equations. Moreover, all the existing results, including this work, concern linear ordinary differential equations under some special conditions. Then, how about the HUs of the general linear ordinary differential equations or even the nonlinear ordinary differential equations?

(2) From the definition of HUs, the HUs of some equation means that the solution of the little perturbed equation has little change. One basic idea of the theory of perturbation (cf. Refs. [2,9] and the references therein) is just that the solution of a perturbed problem may approximate sufficiently that of its reduced problem. Thus, is the theory of HUs subject to the theory of perturbation? In other words, may we use the theory of perturbation to study the theory of HUs?

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