Liftingsimplicialcomplexestotheboundaryofconvexpolytopes
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ABSTRACT
A simplicial complex $C$ on a $d$-dimensional configuration of $n$ points is $k$-regular if its faces are projected from the boundary complex of a polytope with dimension at most $d+k$. Since $C$ is obviously $(n-d-1)$-regular, the set of all integers $k$ for which $C$ is $k$-regular is non-empty. The minimum $\delta(C)$ of this set deserves attention because of its link with flip-graph connectivity. This paper introduces a characterization of $\delta(C)$ derived from the theory of Gale transforms. Using this characterization, it is proven that $\delta(C)$ is never greater than $n-d-2$. Several new results on flip-graph connectivity follow. In particular, it is shown that connectedness does not always hold for the subgraph induced by 3-regular triangulations in the flip-graph of a point configuration.

1. Introduction

It is well known that the subgraph induced by regular triangulations in the flip-graph of a point configuration is isomorphic to the 1-skeleton of the secondary polytope [8,9]. Hence, this subgraph is connected. The connectedness of a much larger subgraph of the flip-graph has been established by the author in [15]. Call a triangulation of a $d$-dimensional point configuration $k$-regular if its faces are projected from the boundary complex of a $(d+k)$-dimensional polytope. It is shown in [15] that the subgraph induced in the flip-graph of a point configuration by 2-regular triangulations is connected. This link between flip-graph connectivity and the possibility to lift triangulations to the boundary of low-dimensional polytopes raises a question that will be investigated in this paper: what is the smallest integer $k$ for which a given simplicial complex is $k$-regular?

If $C$ is a simplicial complex on a point configuration, consider the set of all integers $k$ such that $C$ is $k$-regular and let $\delta(C)$ be its minimum. All simplicial complexes on a $d$-dimensional configuration of $n$ points are obviously $(n-d-1)$-regular. The first contribution of this paper is a strict improvement on this bound: it is shown that $\delta(C)$ is never larger than $n-d-2$ and that the following even stronger inequality always holds:

$$\delta(C) \leq n-d-2 - \left\lfloor \frac{n-d-3}{d/2+1} \right\rfloor.$$

It is also proven that all triangulations of the vertex set of the $d$-dimensional cube are $(2^{d-1} - d + 1)$-regular. The other contributions of this paper are new results on flip-graph connectivity. Let $A$ be a $d$-dimensional configuration of $d+5$ points. It is shown here that if $A$ is in general position and if the 1-skeleton of a triangulation $T$ of $A$ is not a complete graph, then there exists a path in the flip-graph of $A$ between $T$ and any regular triangulation. Flip-graph connectedness is further established when $A$ admits 3 collinear points whose middle point is in the relative interior of conv$(A)$, and also when $A$ is the vertex set of a 3-dimensional simplicial polytope. Finally, a question formulated in [15] is settled: using a point

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configuration found by Santos [6,23], it is shown that the subgraph induced by 3-regular triangulations in the flip-graph is not always connected.

This paper is organized as follows: in Section 2, the possible values of $\delta$ are characterized using a decomposition of the space of affine dependences into a complete fan. A new interpretation of Gale transforms is given in this context. Upper bounds on $\delta$ and results on flip-graph connectivity will be found in Sections 3 and 4 respectively. Finally, it is shown in Section 5 that the subgraph induced by 3-regular triangulations in the flip-graph is not always connected. Previous work on the subject will be mentioned within the context rather than in a special section.

2. The space of affine dependences

Let $n$ be a positive integer. Denote by $\mathcal{B}$ the canonical basis of $\mathbb{R}^n$ and consider the hyperplane $H$ of $\mathbb{R}^n$ that contains all the vectors whose coordinates in this basis sum to zero:

$$H = \left\{ x \in \mathbb{R}^n : \sum_{y \in \mathcal{B}} x \cdot y = 0 \right\}.$$ 

Let $P(\mathcal{B})$ denote the set of the orthogonal projections from $\mathbb{R}^n$ onto linear subspaces of $H$ that map distinct vectors of $\mathcal{B}$ to distinct points. The image of $\mathcal{B}$ by a projection in $P(\mathcal{B})$ is a subset of $n$ points of $H$. Now consider a point configuration $A$, that is a finite subset of a vector space over $\mathbb{R}$. The dimension of $A$ will refer to the dimension of its affine hull and will be denoted by $\dim(A)$. One can prove using basic algebra that if $A$ contains exactly $n$ points, then it can be affinely identified with the image of $\mathcal{B}$ by some projection in $P(\mathcal{B})$:

**Proposition 1.** Let $A$ be a configuration of $n$ points and $\xi : \mathcal{B} \to A$ a bijection. There exists a projection $\pi$ in $P(\mathcal{B})$ and an affine bijection $\phi$ from $\pi(\mathbb{R}^n)$ onto $\text{aff}(A)$ so that $\phi \circ \pi$ coincides with $\xi$ on $\mathcal{B}$.

In other words, one can assume without loss of generality that any configuration of $n$ points is the image of $\mathcal{B}$ by some projection in $P(\mathcal{B})$. Now consider a projection $\pi$ in $P(\mathcal{B})$, and observe that the vectors of $\pi(\mathcal{B})$ sum to 0. As a consequence, the affine hull of $\pi(\mathcal{B})$ is precisely $\pi(\mathbb{R}^n)$ and the dimension of $\pi(\mathcal{B})$ is the rank of $\pi$. This setting provides a particularly elegant way to define the space of all affine dependences of a given point configuration. Indeed, the space of affine dependences of $\pi(\mathcal{B})$ is made up of vectors whose coordinates in a given basis are precisely the coefficients of the affine dependences of $\pi(\mathcal{B})$. If this basis is $\mathcal{B}$ itself, one obtains the following definition:

**Definition 1.** Consider a projection $\pi$ in $P(\mathcal{B})$. The space of all affine dependences of point configuration $\pi(\mathcal{B})$ is the following linear subspace of $H$:

$$D(\pi) = \left\{ x \in H : \sum_{y \in \mathcal{B}} (x \cdot y)\pi(y) = 0 \right\}.$$ 

The space of affine dependences plays a particularly important role in Gale diagrams and Gale transforms [6,10,25]. While the Gale transform of a point configuration is defined as a set of vectors that belong to its space of affine dependences, the derived tools described in this section do not focus on such vectors but rather on the space of affine dependences itself and on its linear subspaces. This machinery turns out to be very general and, in addition to the results obtained in this paper, it may well lead to further insights on polytope projections and on the structure of flip-graphs. The two propositions given hereafter are borrowed from the theory of Gale transforms. The first one is essentially Lemma 4.1.34. in [6]. It states that for every projection $\pi$ in $P(\mathcal{B})$, the orthogonal complement in $H$ of the vector space spanned by $\pi(\mathcal{B})$ is precisely $D(\pi)$:

**Proposition 2.** For any projection $\pi$ in $P(\mathcal{B})$, the orthogonal complement of $\pi(\mathbb{R}^n)$ in $H$ is $D(\pi)$.

The following proposition is another classical observation found in the theory of Gale transforms (see for example Lemma 4.1.34 in [6]) whose straightforward proof is therefore also omitted:

**Proposition 3.** For any projection $\pi$ in $P(\mathcal{B})$, $D(\pi)$ has dimension $n - \dim(\pi(\mathcal{B})) - 1$.

Before going further into the study of the space of affine dependences, the other mathematical objects central to this paper are now introduced. Consider a $d$-dimensional point configuration $A$ and denote by $\mathcal{P}(A)$ its power set. An abstract simplicial complex on $A$ is a set $C \subseteq \mathcal{P}(A)$ so that for all $s \in C$, $\mathcal{P}(s) \subseteq C$. The elements of $C$ are referred to as its faces. A simplicial complex on $A$ is an abstract simplicial complex $C$ on $A$ so that the convex hulls of two distinct faces of $C$ necessarily have disjoint relative interiors. A triangulation of $A$ is a simplicial complex $T$ on $A$ so that every point in the convex hull of $A$ belongs to the convex hull of some face of $T$. If $C$ is an abstract simplicial complex on $A$, then the set $\{a \in A : \{a\} \in C\}$ is denoted by $\nu(C)$ in the following and is referred to as the vertex set of $C$. It turns out that the notion of projection already used to define the space of affine dependences of a point configuration also provides a classification of its simplicial complexes.

Let $p \subseteq \mathbb{R}^m$ be a polytope and $\pi : \mathbb{R}^m \to \mathbb{R}^d$ an affine map. Call $V$ the vertex set of $p$. The pair $(p, \pi)$ is a polytope projection if the restriction of $\pi$ to $V$ is injective. In this case, there exists a unique map $\phi : \pi(V) \to V$ so that $\pi \circ \phi$ maps
every element of $\pi(V)$ to itself. An abstract simplicial complex $C$ on $\pi(V)$ is induced by $\pi$ from $p$ if for all $s \in C$, the convex hull of $\phi(s)$ is a face of $p$. This terminology has been introduced in [3,2] in the context of polyhedral subdivisions. Note that when using polytope projections, it is not ordinarily assumed that the restriction of the projection to the vertex set of the polytope is injective. However, if one uses polytope projections to lift simplicial complexes to higher dimensional spaces, this assumption turns out to be simplifying and can be made without loss of generality. The notion of $k$-regularity, first defined for polyhedral subdivisions in [15], is now stated in the case of abstract simplicial complexes:

**Definition 2.** An abstract simplicial complex $C$ on a $d$-dimensional point configuration is $k$-regular if there exists a polytope projection $(p, \pi)$ so that $p$ has dimension at most $d + k$ and $C$ is induced by $\pi$ from $p$. 

Note that in the case of triangulations, 1-regularity coincides with the usual notion of regularity (see Proposition 1. in [15]). Let $C$ be an abstract simplicial complex on a $d$-dimensional configuration $A$ of $n$ points. Since $A$ is the image by some affine map of an affinely independent $(n - 1)$-dimensional point configuration, $C$ is necessarily $(n - d - 1)$-regular. As a consequence, the set of integers $k$ for which $C$ is $k$-regular is non-empty, and it admits a minimum denoted by $\delta(C)$ in this paper. Now observe that, following Proposition 1, $C$ can be identified with the image of a simplicial complex $K$ on $B$ by a projection $\pi$ in $P(B)$. Further consider another projection $\sigma$ in $P(B)$ so that $\pi(\mathbb{R}^n) \subset \sigma(\mathbb{R}^n)$ and assume that the convex hull of every face of $\sigma(K)$ belongs to the boundary complex of $\text{conv}(v(\sigma(K)))$. This special case is of particular importance. Indeed, the image of $K$ under $\pi$ can be alternatively obtained as the image of $\sigma(K)$ under the orthogonal projection $\pi' : \sigma(\mathbb{R}^n) \to \pi(\mathbb{R}^n)$ so that $\pi = \pi' \circ \sigma$. It immediately follows that $\pi(K)$ is induced by $\pi'$ from the convex hull of $\sigma(K)$ and that $\delta(\pi(K))$ is not greater than the difference $\dim(\sigma(\mathbb{B})) - \dim(\pi(\mathbb{B}))$. The following lemma states that one can always find a projection $\sigma$ satisfying these properties and whose rank is, in addition, equal to the sum of $\delta(\pi(K))$ with the rank of $\pi'$:

**Lemma 1.** Let $\pi$ be a projection in $P(B)$ and $K$ a simplicial complex on $B$. There exists a projection $\sigma \in P(\mathbb{B})$ of rank $\dim(\pi(\mathbb{B})) + \delta(\pi(K))$ so that $\sigma(\mathbb{R}^n) \subset \pi(\mathbb{R}^n)$ and for all $s \in \sigma(K)$, $\text{conv}(s)$ is a face of $\text{conv}(\sigma(K))$.

**Proof.** Respectively denote by $d$ and $d'$ the rank of $\pi$ and the dimension of $v(\pi(K))$. Further denote $k = \delta(\pi(K))$. By definition, there exists a polytope projection $(p, \pi')$ so that $p$ has dimension at most $d' + k$ and $\pi(K)$ is induced by $\pi'$ from $p$. One can assume without loss of generality that $\text{aff}(p)$ is a linear subspace of $\mathbb{R}^{d+k}$ and that $\pi'$ is a linear map from $\mathbb{R}^{d+k}$ onto $\pi(\mathbb{R}^n)$, by translating polytope $p$ if necessary. It is also possible to require that the vertex set of $p$ is projected onto $v(\pi(K))$ by $\pi'$. Since $\pi'(\mathbb{R}^{d+k})$ and $\pi(\mathbb{R}^n)$ coincide, one can find a $(d + k)$-dimensional configuration $A \subset \mathbb{R}^{d+k}$ of $n$ points that admits the vertex set of $p$ as a subset and so that $\pi(A) = \pi(B)$. Let $\xi : \mathbb{B} \to A$ be the bijection so that for all $y \in \mathbb{B}$, $\pi' \circ \xi(y) = \pi(y)$.

According to Proposition 1, there exists a projection $\sigma \in P(\mathbb{B})$ and an affine bijection $\phi : \sigma(\mathbb{R}^n) \to \mathbb{R}^{d+k}$ so that for all $y \in \mathbb{B}$, $\phi \circ \sigma(y) = \xi(y)$. One therefore obtains that $\pi' \circ \phi \circ \sigma$ and $\pi$ are two affine maps from $\mathbb{R}^n$ onto $\pi(\mathbb{R}^n)$ that coincide on the canonical basis of $\mathbb{R}^n$. Since both $\sigma(\mathbb{B})$ and $\pi(\mathbb{B})$ sum to 0, it follows that $\pi' \circ \phi(0) = 0$. Hence, $\pi' \circ \phi \circ \sigma$ and $\pi$ also coincide in 0, and these two maps are necessarily identical. Moreover, as $\pi$ and $\sigma$ are linear, then so is $\pi' \circ \phi$. This shows in particular that $\ker(\sigma)$ is a subset of $\ker(\pi)$. As $\sigma$ and $\pi$ both are orthogonal projections onto linear subspaces of $\mathbb{R}^n$, then $\pi(\mathbb{R}^n) \subset \sigma(\mathbb{R}^n)$. In addition, since $\phi$ is bijective, the dimensions of $\sigma(\mathbb{R}^n)$ and $\mathbb{R}^{d+k}$ are equal and as a consequence, $\sigma$ has rank $d + k$.

Now observe that $\pi' \circ \phi$ projects $\sigma(K)$ precisely onto $\pi(K)$. Hence, for every $s \in K$, $\phi \circ \sigma(s)$ is the unique subset of $A$ whose image under $\pi'$ is $\pi(s)$. Since $\pi(K)$ is induced by $\pi'$ from $p$ and since the vertex set of $p$ is exactly the subset of $A$ projected onto $v(\pi(K))$ by $\pi'$, this shows that $\phi$ projects the vertex set of $\sigma(K)$ onto that of $p$ and the convex hulls of the faces of $\sigma(K)$ into the boundary complex of $p$. Since $\phi$ is an affine map, the result follows.

In the remainder of the section, the link between the space of affine dependences and the possibility to lift a simplicial complex to the boundary of a polytope of given dimension will be stated. In particular, it will be shown how the construction of $D(\pi)$ for some projection $\pi$ in $P(\mathbb{B})$ helps finding polytope projections that induce given simplicial complexes on $\pi(B)$. Let $\pi$ be a projection in $P(\mathbb{B})$. For any vector $x$ in $D(\pi)$ consider the following two subsets of $\pi(B)$:

$$s_\pi^-(x) = \{y \in \mathbb{B} : x \cdot y < 0\} \quad \text{and} \quad s_\pi^+(x) = \{y \in \mathbb{B} : x \cdot y > 0\}.$$ 

Observe that for all $x \in D(\pi) \setminus \{0\}$, the sets $s_\pi^-(x)$ and $s_\pi^+(x)$ are disjoint subsets of $\pi(B)$ whose convex hulls have non-disjoint relative interiors. If $x$ is an affine dependence that corresponds to a circuit $z$, then $s_\pi^-(x), s_\pi^+(x)$ is the Radon partition of $z$ [19]. Now consider a subset $A$ of $\pi(B)$. The following lemma characterizes the faces of $\text{conv}(A)$ using conditions on $s_\pi^-(x)$ and $s_\pi^+(x)$. Intuitively, it states that the convex hull of some set $s \subset A$ is a face of $\text{conv}(A)$ if and only if any set $t \subset A \setminus s$ so that $\text{conv}(s \cup t)$ have non-disjoint relative interiors is necessarily a subset of $\text{conv}(s)$. The proof relies on Stiemke's theorem of the alternative [24], which is similar to Farkas' lemma.

**Lemma 2.** Let $\pi$ be a projection in $P(\mathbb{B})$ and $A$ a subset of $\pi(B)$. For any subset $s$ of $A$, the convex hull of $s$ is a face of $\text{conv}(A)$ if and only if for all $x \in D(\pi)$, at least one of the following three statements holds:

i. $s_\pi^+(x)$ is not a subset of $s$,
ii. $s_\pi^-(x)$ is a subset of $\text{conv}(s)$,
iii. $s_\pi^-(x)$ is not a subset of $A$. 

Proof. Denote by $\mathcal{V}$ the subset of $\mathcal{B}$ that projects onto $\mathcal{A}$ under $\pi$. First assume that the convex hull of $s$ is a face of $\text{conv}(\mathcal{A})$. Consider a vector $x$ in $D(\pi)$ and assume that $s^+_{\mathcal{A}}(x) \subset s$ and that $s^-_{\mathcal{A}}(x) \subset \mathcal{A}$. It follows that all the terms in the affine dependence stated by $x$ correspond to points of $\mathcal{A}$:

$$\sum_{y \in \mathcal{V}} (x \cdot y)\pi(y) = 0. \quad (1)$$

Denote by $t$ the set of all vectors in $\mathcal{B}$ whose image under $\pi$ belongs to $\text{conv}(s)$. Since the convex hull of $s$ is a face of $\text{conv}(\mathcal{A})$, there exists an affine map $\psi : \pi(\mathbb{R}^n) \to \mathbb{R}$ that projects the points of $\pi(t)$ to zero and the rest of $\mathcal{A}$ to positive numbers. As $x$ is a vector of $H$, the dot products in the left-hand side of (1) sum to 0. Hence, applying $\psi$ to this equation yields:

$$\sum_{y \in \mathcal{V} \setminus t} (x \cdot y)\psi \circ \pi(y) = 0. \quad (2)$$

As $s^+_{\mathcal{A}}(x) \subset s$, and as $s \subset \pi(t)$, then for all $y \in \mathcal{V} \setminus t$, $x \cdot y \leq 0$. Moreover, $\psi$ maps the points of $\pi(\mathcal{V} \setminus t)$ to positive numbers. Hence, the left-hand side of Eq. (2) is a sum of non-positive terms, and all these terms are therefore equal to zero. It then follows from the strict positivity of $\psi$ on $\pi(\mathcal{V} \setminus t)$ that for all $y \in \mathcal{V} \setminus t$, $x \cdot y = 0$. As a consequence, $s^-_{\mathcal{A}}(x)$ is a subset of $\pi(t)$. Hence, statement ii. holds and the first implication is proven.

The second implication will be obtained using Stiemke’s theorem [24]. Assume that every vector $x$ in $D(\pi)$ satisfies at least one of the three conditions in the statement of the lemma. Denote by $d$ and by $d'$ the respective dimensions of $\mathcal{A}$ and $s$. One can find a set $t \subset \mathcal{V}$ of $d' + 1$ vectors whose image under $\pi$ is affinely independent and contains exactly $d' + 1$ points of $s$. Denote by $u$ the set of the $d + 1$ vectors of $t$ so that $\pi(u) \subset s$. According to this construction, the affine hulls of $\pi(t)$ and $\pi(u)$ are respectively aff$(\mathcal{A})$ and aff$(s)$. Denote by $\mathcal{K}$ the set obtained by removing from $\mathcal{V}$ the vectors whose image under $\pi$ belongs to $\text{conv}(s) \setminus \pi(u)$:

$$\mathcal{K} = \{ y \in \mathcal{V} : \pi(y) \notin \text{conv}(s) \setminus \pi(u) \}.$$ 

Observe that $t$ is a subset of $\mathcal{K}$ and that $\pi(\mathcal{K})$ is a subset of $\mathcal{A}$. Consider a vector $a \in \mathcal{K} \setminus t$. As $\pi(t)$ affinely spans aff$(\mathcal{A})$, then point $a$ can be written as an affine combination of $\pi(t)$. In other words, there exists a vector $x_a \in D(\pi)$ so that $x_a \cdot a = 1$ and $x_a \cdot b = 0$ for all $b \in \mathcal{B} \setminus \{a\}$. In order to apply Stiemke’s theorem, it will now be established that the following system of inequalities in the real unknowns $(\lambda_a)_{a \in \mathcal{K} \setminus t}$ does not admit non-zero solutions:

$$\forall b \in \mathcal{K} \setminus u, \quad \sum_{a \in \mathcal{K} \setminus t} \lambda_a (x_a \cdot b) \leq 0. \quad (3)$$

Assume that all inequalities in this system are satisfied and consider the following vector of $D(\pi)$:

$$x = \sum_{a \in \mathcal{K} \setminus t} \lambda_a x_a. \quad (4)$$

Let $b$ be a vector in $\mathcal{B} \setminus \mathcal{K}$. As $x_a \cdot b = 0$ for all $a \in \mathcal{K} \setminus t$, it follows from (4) that $x \cdot b = 0$. In other words, both $s^-_{\mathcal{A}}(x)$ and $s^+_{\mathcal{A}}(x)$ are subsets of $\pi(\mathcal{K})$. Since $\pi(\mathcal{K}) \subset \mathcal{A}$, this shows that statement iii. does not hold. Now observe that (3) states that $s^+_{\mathcal{A}}(x)$ and $\pi(\mathcal{K} \setminus u)$ are disjoint. Since $s^+_{\mathcal{A}}(x)$ is a subset of $\pi(\mathcal{K})$, one obtains that $s^+_\mathcal{A}(x) \subset \pi(u)$. As $s$ admits $\pi(u)$ as a subset, it follows that statement i. does not hold either. As a consequence, $x$ satisfies statement ii., that is $s^-_{\mathcal{A}}(x)$ is a subset of $\text{conv}(s)$. In addition, as $s^-_{\mathcal{A}}(x)$ is a subset of $\pi(\mathcal{K})$ and as the only points of $\pi(\mathcal{K})$ that belong to $\text{conv}(s)$ are the points of $\pi(u)$, one obtains that $s^-_{\mathcal{A}}(x) \subset \pi(u)$.

It has been shown in the preceding paragraph that $s^+_{\mathcal{A}}(x)$ and $s^+_{\mathcal{A}}(x)$ are two subsets of $\pi(u)$. As $\pi(u)$ is affinely independent, this shows that $x$ is equal to 0. Now recall that for all $a \in \mathcal{K} \setminus t$, $x_a \cdot a = 1$ and $x_a \cdot b = 0$ for all $b \in \mathcal{B} \setminus \{a\}$. Hence, (4) yields $\lambda_a = 0$ for all $a \in \mathcal{K} \setminus t$, proving that (3) does not admit non-zero solutions. According to Stiemke’s theorem [24], there exists a family $(\mu_b)_{b \in \mathcal{K} \setminus u}$ of positive numbers so that for all $a \in \mathcal{K} \setminus t$:

$$\sum_{b \in \mathcal{K} \setminus u} \mu_b (x_a \cdot b) = 0. \quad (5)$$

Since $\pi(t)$ is affinely independent, there exists an affine map $\psi : \text{aff}(\mathcal{A}) \to \mathbb{R}$ so that for all $b \in u$, $\psi \circ \pi$ maps $b$ to 0 and for all $b \in \mathcal{V} \setminus u$, $\psi \circ \pi$ maps $b$ to $\mu_b$. Recall that $\pi(u)$ and $s$ have the same affine span. Hence, $\psi$ projects every point of aff$(s)$ to 0. Now let $a$ be a vector in $\mathcal{K} \setminus t$. Since $x_a$ belongs to $D(\pi)$, and since $x_a \cdot b = 0$ for all $b \in \mathcal{B} \setminus \mathcal{V}$, the following equation holds:

$$\sum_{b \in \mathcal{V}} (x_a \cdot b)\pi(b) = 0. \quad (6)$$

As the dot products in the left-hand side of (6) sum to zero and as $\psi$ projects the points of $\text{conv}(s)$ to 0, applying $\psi$ to Eq. (6) yields:

$$\sum_{b \in \mathcal{K} \setminus u} \psi \circ \pi(b) (x_a \cdot b) = 0. \quad (7)$$
Now subtract Eq. (7) from Eq. (5). Since for all $b \in \mathcal{B} \setminus (t \cup \{a\})$, the dot product $x_a \cdot b$ is equal to zero and since for all $b \in t \setminus u$, $\psi \circ \pi (b) = \mu_b$, only one term remains in the left-hand side of this difference:

$$(\mu_a - \psi \circ \pi (a)) (x_a \cdot a) = 0.$$ 

As $x_a \cdot a = 1$, one obtains $\psi \circ \pi (a) = \mu_a$, which proves that $\psi$ projects the points of $\text{conv}(s)$ to 0 and the points of $\mathcal{A} \setminus \text{conv}(s)$ to positive values. Hence, the convex hull of $s$ is a face of $\text{conv}(\mathcal{A})$ and the lemma is proven. □

**Lemma 2** can be used to identify the projections that map the faces of a simplicial complex on $\mathcal{B}$ into the boundary complex of a lower-dimensional polytope. Let $\pi$ be a projection in $P(\mathcal{B})$ and $C$ a simplicial complex on $\pi(\mathcal{B})$. Consider the three conditions given in the statement of this lemma. One can see that if $\mathcal{A}$ is the vertex set of $C$ and if $s$ is a face of $C$, then the second condition implies the third one. This suggests that the following set should be considered:

$$I_\pi(C) = \{ x \in D(\pi) \setminus \{0\} : s^+_x (x) \in C, \ s^-_x (x) \subset \pi(\text{conv}(C)) \}.$$ 

One can see that $I_\pi(C)$ is the set of the vectors $x$ in $D(\pi)$ that do not satisfy any of the three conditions in the statement of **Lemma 2** when $\mathcal{A} = \pi(C)$ and $s = s^+_x (x)$. In other words, every vector $x \in I_\pi(C)$ defines a face of $C$ whose convex hull does not belong to the boundary complex of $\text{conv}(\pi(C))$. Inversely, if $C$ contains a face whose convex hull is not in the boundary complex of $\text{conv}(\pi(C))$, then this face causes a such vector $x$ to belong to $I_\pi(C)$. The following proposition is an immediate consequence of these observations:

**Proposition 4.** Let $\pi$ be a projection in $P(\mathcal{B})$ and $C$ a simplicial complex on $\pi(\mathcal{B})$. The convex hull of every face of $C$ is a face of $\text{conv}(\pi(\text{conv}(C)))$ if and only if $I_\pi(C)$ is empty.

In order to provide some intuition on the placement of $I_\pi(C)$ within the space of affine dependences, Example 4.1.41. is now borrowed from [6]. Let $a, b, c, d,$ and $e$ denote the vertices of a regular pentagon, disposed as shown in the left of Fig. 1. Call $\mathcal{A}$ the set of these five vertices and consider the triangulation $T$ of $\mathcal{A}$ depicted in the left of Fig. 1. According to **Proposition 1**, $\mathcal{A}$ can be identified with the image by a projection $\pi$ of the canonical basis of $\mathbb{R}^5$. Now, consider the two following affine dependences of $\mathcal{A}$:

$$-a a - a b + b c + b e - 2 d = 0 \ \text{and} \ 2 a - 2 b + 2 a c - 2 a e = 0,$$

where $\alpha = 2 \cos(2\pi/5)$ and $\beta = 1 + \alpha$. The first of these affine dependences states that the convex hull of $\{a, b, d\}$ contains the centroid of $\{c, e\}$, and the second one states that edges $\{a, c\}$ and $\{b, e\}$ are crossing. Call $x$ and $y$ the vectors of $\mathbb{R}^3$ whose coordinates in the canonical basis are the coefficients in the left-hand side of these respective affine dependences. These two vectors are orthogonal and, by definition, they belong to $D(\pi)$.

Fig. 1. A regular pentagon (left) and its Gale transform (right) built in space $D(\pi)$ using basis $(x, y)$. Each dashed line is indexed by a vertex of the pentagon and contains the affine dependences whose coefficient for this vertex is zero. The hatched surface is set $I_\pi(T)$, where $T$ is the triangulation shown in the left of the figure. The face of $T$ that causes a given cone to be in this surface is explicitly indicated.
and represented using the same orthogonal basis of $D(\pi)$. In particular, $\mathcal{H}$ can be generally defined as the arrangement of the linear hyperplanes of $D(\pi)$ orthogonal to the vectors of $\mathcal{A}^*$. According to this construction, the pair $(s_{\pi}(x), s_{\pi}^+(x))$ is the same for all the vectors $x$ in the relative interior of any given cone in the fan defined by $\mathcal{H}$. In particular, set $I_\pi(T)$ is obtained by removing 0 from the union of a subset $S$ of these cones. Set $I_\pi(T)$ is hatched in Fig. 1, and the faces of $T$ that cause cones in $S$ to contribute to $I_\pi(T)$ are indicated.

As can be seen on Fig. 1, the line in $\mathcal{H}$ associated to point $e$ is disjoint from $I_\pi(T)$. The link between the existence of such linear subspaces in $D(\pi)$ and the possible values for $\delta$ is now explained. Consider two projections $\pi$ and $\sigma$ in $P(\mathcal{B})$ and a simplicial complex $K$ on $\mathcal{B}$. Under given conditions, $I_\pi(\sigma(K))$ is a subset of $I_\pi(\tau(K))$:

**Lemma 3.** Let $\pi$ and $\sigma$ be two projections in $P(\mathcal{B})$ and $K$ a simplicial complex on $\mathcal{B}$. If $\pi(K)$ is a simplicial complex and if $\pi(\mathbb{R}^n)$ is a subset of $\sigma(\mathbb{R}^n)$, then $I_\pi(\sigma(K))$ is equal to the intersection of $I_\pi(\tau(K))$ with $D(\sigma)$.

**Proof.** Assume that $\pi(\mathbb{R}^n) \subset \sigma(\mathbb{R}^n)$ and that $\pi(K)$ is a simplicial complex. First observe that since $\pi(\mathbb{R}^n)$ is a subset of $\sigma(\mathbb{R}^n)$, an immediate consequence of Proposition 2 is that $D(\sigma)$ is a subset of $D(\pi)$. As in addition, $\pi$ and $\sigma$ are orthogonal projections, then there exists a third orthogonal projection $\pi'$ from $\sigma(\mathbb{R}^n)$ onto $\pi(\mathbb{R}^n)$ so that $\pi = \pi' \circ \sigma$. Now consider a vector $x \in D(\sigma)$. As $\pi' = \pi' \circ \pi$, then following the definitions of these sets, $s_{\pi'}^-(x)$ and $s_{\pi'}^+(x)$ are the images under $\pi'$ of respectively $s_{\pi}^- (x)$ and $s_{\pi}^+ (x)$. Hence, $s_{\pi'}^+(x) \in \pi(K)$ if and only if $s_{\pi}^+(x) \in \pi(\sigma(K))$. For the same reason, $s_{\pi'}^- (x) \subset \pi(\sigma(K))$ if and only if $s_{\pi}^- (x) \subset \pi(\sigma(K))$. In other words, the two conditions on $x$ in the definition of $I_\pi(\tau(K))$ are equivalent to the corresponding conditions in the definition of $I_\pi(\sigma(K))$. As $x$ is a vector of $D(\sigma)$, this proves that $I_\pi(\sigma(K))$ is equal to the intersection of $I_\pi(\tau(K))$ with $D(\sigma)$. $\square$

If $\pi$ is a projection in $P(\mathcal{B})$ and $C$ a simplicial complex on $\pi(\mathcal{B})$, next theorem characterizes the possible values of $\delta(C)$ using the linear subspaces of $D(\pi)$ disjoint from $I_\pi(C)$. This theorem will be the main tool used in the next sections to bound $\delta(C)$. It provides a way to tell whether a simplicial complex is $k$-regular or not. Hence, it is likely to produce further results in addition to those obtained in this paper.

**Theorem 1.** Let $\pi$ be a projection in $P(\mathcal{B})$ and $C$ a simplicial complex on $\pi(\mathcal{B})$. There exists a $k$-dimensional linear subspace of $D(\pi)$ disjoint from $I_\pi(C)$ if and only if $\delta(C) \leq n - \dim(\pi(\mathcal{B})) - k - 1$.

**Proof.** Since $\pi$ does not project distinct vectors of $\mathcal{B}$ to a same point, there exists a unique simplicial complex $K$ on $\mathcal{B}$ whose image under $\pi$ is precisely $C$.

Assume first that $M$ is a $k$-dimensional linear subspace of $D(\pi)$ that does not intersect $I_\pi(C)$. Denote by $\sigma$ the orthogonal projection from $\mathbb{R}^n$ onto the orthogonal complement of $M$ in $H$. According to this construction, $\pi(\mathbb{R}^n)$ is a subset of $\sigma(\mathbb{R}^n)$. Since $\pi$ and $\sigma$ are two orthogonal projections so that $\pi(\mathbb{R}^n) \subset \sigma(\mathbb{R}^n)$, the orthogonal projection $\pi'$ from $\sigma(\mathbb{R}^n)$ onto $\pi(\mathbb{R}^n)$ satisfies $\pi' \circ \sigma = \pi$. This proves in particular that the restriction of $\sigma$ to $B$ is injective and that $\sigma$ belongs to $P(\mathcal{B})$. It then follows from Proposition 2 that $D(\sigma)$ is equal to $M$. By hypothesis, $D(\sigma)$ and $I_\pi(C)$ are therefore disjoint, and according to Lemma 3, $I_\pi(\sigma(K))$ is necessarily empty. Hence, according to Proposition 4, the convex hulls of the faces of $\sigma(K)$ are faces of $\conv(\psi(\sigma(K)))$. Further observe that $\pi'$ projects $\sigma(K)$ precisely onto $C$. As a consequence, $C$ is induced by $\pi'$ from the convex hull of $\psi(\sigma(K))$. Since $D(\sigma)$ is a $k$-dimensional, $\sigma(\mathbb{R}^n)$ has dimension $n - k - 1$, which yields the desired inequality.

Now assume that $\delta(C) \leq n - \dim(\pi(\mathcal{B})) - k - 1$. According to Lemma 1, there exists a projection $\sigma$ in $P(\mathcal{B})$ of rank $\dim(\pi(\mathcal{B})) + \delta(\pi(\mathcal{B}))$ so that $\sigma(\mathbb{R}^n) \subset \pi(\mathbb{R}^n)$ and the convex hulls of the faces of $\sigma(K)$ are faces of $\conv(\psi(\sigma(K)))$. As a first consequence, $\sigma(\mathbb{R}^n)$ has dimension at most $n - k - 1$. In addition, it follows from Proposition 4 that $I_\pi(\sigma(K))$ is empty. Moreover, as $\pi(\mathbb{R}^n)$ is a subset of $\sigma(\mathbb{R}^n)$, then one obtains from Lemma 3 that $I_\pi(\sigma(K))$ is the intersection of $I_\pi(C)$ with $D(\sigma)$, proving that this intersection is empty. Finally, a consequence of Proposition 2 and of the inclusion of $\pi'(\mathbb{R}^n)$ into $\sigma(\mathbb{R}^n)$ is that $D(\sigma)$ is a subset of $D(\pi)$. This shows that $D(\sigma)$ is a linear subspace of $D(\pi)$ disjoint from $I_\pi(C)$. Since $\sigma(\mathbb{R}^n)$ has dimension at most $n - k - 1$, then $D(\sigma)$ is at least $k$-dimensional and the result follows. $\square$

Now recall that, in the example discussed above and depicted in Fig. 1, the line in $\mathcal{H}$ corresponding to point $e$ is disjoint from $I_\pi(T)$. It immediately follows from Theorem 1 that $\delta(T) \leq 1$. In other words, $T$ is regular. This is not a surprise since the vertex set of any convex polygon only admits regular triangulations. This bound on $\delta(T)$ leaves another possibility open, though: could $T$ be 0-regular? The answer is obviously negative. Indeed, a simplicial complex is 0-regular when the convex hulls of its faces belong to the boundary complex of some polytope, which is not the case for triangulation $T$. This conclusion can be alternatively obtained from Theorem 1. In the case of $T$, this theorem states that 0-regularity is equivalent to the existence of a 2-dimensional linear subspace of $D(\pi)$ disjoint from $I_\pi(T)$. One can see on Fig. 1 that if a linear subspace of $D(\pi)$ is disjoint from $I_\pi(T)$ then its dimension is at most 1, which shows that $T$ is not 0-regular. The tools developed in this section, and especially Theorem 1, are used in the remainder of the paper to obtain bounds on $\delta$.

3. Bounds on $\delta$

Let $C$ be a simplicial complex on a $d$-dimensional configuration $\mathcal{A}$ of $n$ points. This section provides general upper bounds on $\delta(C)$. Theorem 2 and Corollary 1 give bounds that depend on $n$ and $d$. Theorem 3 gives sharper bounds that have a greater dependence on the geometry of $C$. Finally, the vertex set of the $d$-dimensional cube is studied and it is shown that all its triangulations are $(2^{d-1} - d + 1)$-regular. According to Proposition 1, one can find a projection $\pi$ in $P(\mathcal{B})$ so that $\mathcal{A}$ can be
affinely identified with \( \pi (B) \). The following lemma shows that if \( n \geq d + 3 \), then \( D(\pi) \) contains at least one non-zero vector that does not belong to \( I_n (C) \).

**Lemma 4.** Let \( A \) be a \( d \)-dimensional configuration of \( n \) points and \( C \) a simplicial complex on \( A \). If \( n \geq d + 3 \), then \( A \) admits two disjoint subsets that do not belong to \( C \) and whose convex hulls have non-disjoint relative interiors.

**Proof.** Assume that \( n \geq d + 3 \). One can assume without loss of generality that \( A \) is the image of \( B \) under a projection \( \pi \) in \( P (B) \). Moreover, it follows from Proposition 3 that \( D(\pi) \) has dimension at least 2. Let \( y \) and \( z \) be two linearly independent vectors of \( D(\pi) \). Since the convex hulls of \( s_{\pi}^- (y) \) and \( s_{\pi}^+ (y) \) have non-disjoint relative interiors, these sets cannot both belong to \( C \), and one can assume without loss of generality that \( s_{\pi}^- (y) \) is not a face of \( C \), by exchanging \( y \) for its opposite if necessary. Invoking the same argument, one can assume that \( s_{\pi}^+ (z) \) is not a face of \( C \) either. Call \( N \) the affine hull of \( \{ y, z \} \) and consider the following two sets:

\[
I^- = \{ x \in N : s_{\pi}^- (x) \not\subseteq C \} \quad \text{and} \quad I^+ = \{ x \in N : s_{\pi}^+ (x) \not\subseteq C \}.
\]

These are open subsets of \( N \); indeed, let \( x^* \) be an element of \( I^- \). For all \( a \in A \), let \( \phi (a) \) denote the vector in \( B \) whose image by \( \pi \) is \( a \) and consider the map \( f_a : N \rightarrow \mathbb{R} \) defined by \( f_a (x) = x \cdot \phi (a) \). By continuity of \( x \mapsto f_a (x) \) and because \( s_{\pi}^- (x^*) \) is finite, the set

\[
U = \bigcap_{a \in I^* (x^*)} f_a^{-1} (1 - \infty, 0],
\]

is an open subset of \( N \). Observe that \( x^* \) belongs to \( U \). Moreover, by construction, \( f_a (x) \) is negative for all \( x \in U \) and all \( a \in s_{\pi}^- (x^*) \). As a consequence, \( s_{\pi}^- (x^*) \) is a subset of \( s_{\pi}^- (x) \) for all \( x \in U \). As \( C \) is an abstract simplicial complex and as \( s_{\pi}^- (x^*) \) does not belong to \( C \), then \( U \subset I^- \), which proves that \( I^- \) is open. Using similar arguments, one shows that \( I^+ \) is open too.

Now observe that the union of \( I^- \) and \( I^+ \) is equal to \( N \). Indeed, for any \( x \in N \setminus (I^- \cup I^+) \), \( s_{\pi}^- (x) \) and \( s_{\pi}^+ (x) \) are two faces of \( C \) whose convex hulls have non-disjoint interiors. Moreover, as neither \( s_{\pi}^- (y) \) nor \( s_{\pi}^+ (z) \) is a face of \( C \), then \( y \in I^- \) and \( z \in I^+ \). It follows that \( I^- \) and \( I^+ \) are non-empty open subsets of \( N \) whose union is \( N \); thus they are non-disjoint. Let \( x \in I^- \setminus I^+ \). The sets \( s_{\pi}^- (x) \) and \( s_{\pi}^+ (x) \) are disjoint subsets of \( A \) that do not belong to \( C \) and whose convex hulls have non-disjoint interiors, which finishes the proof. \( \square \)

It can be deduced from Lemma 4 that \( D(\pi) \) admits a 1-dimensional linear subspace disjoint from \( I_n (C) \). The following result thus comes as a consequence of Theorem 1:

**Theorem 2.** Let \( A \) be a \( d \)-dimensional configuration of \( n \) points. If \( n \geq d + 3 \) then every simplicial complex on \( A \) is \((n - d - 2)\)-regular.

**Proof.** Assume that \( n \geq d + 3 \). According to Proposition 1, one can assume without loss of generality that there exists \( \pi \in P (B) \) so that \( A = \pi (B) \). Let \( C \) be a simplicial complex on \( A \). According to Lemma 4, \( A \) admits two disjoint subsets that do not belong to \( C \) and whose convex hulls have non-disjoint relative interiors. In other words, there exists a vector \( y \) in \( D(\pi) \setminus \{ 0 \} \) so that \( \pi_{\pi}^- (y) \not\subseteq C \) and \( \pi_{\pi}^+ (y) \not\subseteq C \).

Call \( M \) the linear subspace of \( D(\pi) \) spanned by \( y \). It is now shown that \( M \) and \( I_n (C) \) are disjoint. Let \( x \in M \). There exists a real number \( \lambda \) so that \( x \lambda y \). If \( \lambda = 0 \) then \( x = 0 \) and therefore \( x \not\in I_n (C) \). If \( \lambda > 0 \), then \( s_{\pi}^- (x) \) and \( s_{\pi}^+ (x) \) are identical. As \( s_{\pi}^+ (y) \) does not belong to \( C \), then \( x \not\in I_n (C) \). Finally, if \( \lambda < 0 \) then \( s_{\pi}^- (x) = s_{\pi}^+ (y) \). As \( s_{\pi}^- (y) \) does not belong to \( C \), then \( x \) does not belong to \( I_n (C) \). This shows that \( M \) and \( I_n (C) \) are disjoint. As \( M \) is a 1-dimensional linear subspace of \( D(\pi) \) the result follows from Theorem 1. \( \square \)

A well-known result by Lee [13] is a consequence of Theorem 2: triangulations of \( d \)-dimensional configurations of \( d + 3 \) points are regular. In the case of triangulations of such point configurations, it follows from the equivalence between 1-regularity and regularity that Theorem 2 is identical to the result by Lee. Using Theorem 2 together with an example taken from [4], it is now shown how simplicial complex \( k \)-regularity strictly generalizes triangulation regularity, even when \( k = 1 \). Let \( \varepsilon \) a positive real number. Consider the following points, defined as the vectors of their coordinates in the canonical basis of \( \mathbb{R}^3 \):

\[
a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad a_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}; \quad a_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad a_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}; \quad a_5 = \begin{pmatrix} -\varepsilon \\ \varepsilon \\ 1 \end{pmatrix}; \quad a_6 = \begin{pmatrix} \varepsilon \\ -\varepsilon \\ -1 \end{pmatrix}.
\]

Call \( A \) the set of these six points and \( C \) the simplicial complex on \( A \) obtained by adding to \( A \) the segments \( \{ a_{2i-1}, a_{2i} \} \) for all \( i \in \{ 1, 2, 3 \} \). Simplicial complex \( C \) is depicted in Fig. 2. According to Theorem 2, \( C \) is 1-regular: there exists a polytope \( p \subset \mathbb{R}^3 \) and a projection \( \pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) so that \( (p, \pi) \) is a polytope projection and \( C \) is induced by \( \pi \) from \( p \). One may require without loss of generality that \( p \) admits exactly 6 vertices. It is shown in [4] that if \( \varepsilon \) is small enough, then \( C \) cannot be found as a subset in any triangulation of \( A \). Hence, for these values of \( \varepsilon \), the faces of \( p \) whose images under \( \pi \) are convex hulls of faces of \( C \) cannot all belong to the lower boundary of \( p \). This situation never occurs with regular triangulations, whose faces are either all projected from the lower or from the upper faces of \( p \). Such a general setting where a simplicial complex does not completely triangulate the convex hull of its vertex set will be needed hereafter.
Let $C$ be a simplicial complex on a point configuration $\mathcal{A}$. The bound on $\delta(C)$ provided by Theorem 2 is obtained by considering a pair of subsets of $\mathcal{A}$ that do not belong to $C$. One can improve this bound by considering a larger collection of subsets of $\mathcal{A}$ that do not belong to $C$ instead of just a pair. The following lemma sets the ground for this generalization.

**Lemma 5.** Let $C$ be a simplicial complex on a point configuration $\mathcal{A}$. If $s$ and $t$ are disjoint subsets of $\mathcal{A}$ that do not belong to $C$ and whose convex hulls have non-disjoint relative interiors, then:

$$\delta(C) \leq \delta(C \cap \mathcal{P}(\mathcal{A} \setminus s)) + |s| + |t \setminus v(C)| - 1.$$  

**Proof.** Assume that $s$ and $t$ are disjoint subsets of $\mathcal{A}$ that do not belong to $C$ and whose convex hulls have non-disjoint relative interiors. One can assume without loss of generality that the cardinality of $\mathcal{A}$ is that of basis $\mathcal{B}$ and, following Proposition 1, that $\mathcal{A}$ is the image of $\mathcal{B}$ under a projection $\pi$ in $P(\mathcal{B})$. Denote by $\mathcal{V}$ the union of $v(C)$ with $s \cup t$. As $s$ and $t$ are disjoint, one obtains the following equality:

$$|\mathcal{V}| = |v(C)| + |s \setminus v(C)| + |t \setminus v(C)|.$$  

Consider the simplicial complex $C' = C \cap \mathcal{P}(\mathcal{A} \setminus s)$ and observe that $v(C') = v(C) \setminus \lambda$. Denote by $n'$ the cardinality of $v(C')$ and consider the subset $\mathcal{B}'$ of $\mathcal{B}$ that projects onto $v(C')$ under $\pi$. One can identify $\mathbb{R}^{n'}$ with the linear subspace of $\mathbb{R}^n$ spanned by $\mathcal{B}'$, and its canonical basis with $\mathcal{B}'$. According to this construction, there exists a projection $\pi'$ in $P(\mathcal{B}')$ that coincides with the restriction of $\pi$ to $\mathbb{R}^{n'}$. It follows that $D(\pi')$ and $I_{\pi'}(C')$ are precisely the intersections of $\mathbb{R}^{n'}$ with respectively $D(\pi)$ and $I_{\pi}(C)$. In addition, for all $x \in D(\pi')$, sets $s_{\pi'}(x)$ and $s_{\pi'}^+(x)$ are respectively equal to $s_\pi(x)$ and $s_\pi^+(x)$. Call $k = \delta(C)$ and let $d$ and $d'$ denote the respective dimensions of $v(C)$ and $v(C')$. It follows from Theorem 1 that $D(\pi')$ admits a linear subspace $M'$ of dimension $n' - d' - k - 1$ disjoint from $I_{\pi'}(C')$. As $I_{\pi'}(C')$ is the intersection of $\mathbb{R}^{n'}$ with $I_{\pi'}(C)$, and as $M'$ is a subset of $\mathbb{R}^{n'}$, then $M'$ is also disjoint from $I_{\pi'}(C)$.

Since the convex hulls of $s$ and $t$ have non-disjoint relative interiors, there exists a vector $y \in D(\pi)$ so that $s_{\pi}(y)$ and $s_{\pi}^+(y)$ are equal to $s$ and $t$ respectively. Denote by $N$ the 1-dimensional subspace of $D(\pi)$ spanned by $y$ and consider the space $M = M' \oplus N$. Let $x \in M$. There exist $\lambda \in \mathbb{R}$ and $x' \in M'$ so that $x = x' + \lambda y$. It will now be shown that $x$ does not belong to $I_{\pi'}(C)$. If $\lambda < 0$, then $x$ is a subset of $s_{\pi}(x)$ and as $s$ is not a face of $C$, it follows that $x$ does not belong to $I_{\pi'}(C)$. If $\lambda = 0$, then $x \in M'$ and therefore $x \not\in I_{\pi'}(C)$ because $M'$ and $I_{\pi'}(C)$ are disjoint. If $\lambda > 0$ and $x' = 0$, then $s_{\pi}^+(x) = t$ and as $t$ is not a face of $C$, then $x$ does not belong to $I_{\pi'}(C)$.

Now assume that $\lambda > 0$ and $x' \neq 0$. As $x' \in M'$, and as $M'$ is disjoint from $I_{\pi'}(C')$, then at least one of the two conditions that define set $I_{\pi'}(C')$ is not satisfied by $x'$. As $v(C') = \pi'(\mathcal{B}')$, then necessarily $s_{\pi'}^+(x')$ is a subset of $v(C')$ and therefore, $s_{\pi'}^+(x')$ does not belong to $C$. In addition, $s_{\pi'}^+(x')$ is also a subset of $v(C')$. Since $C'$ is precisely made up of the faces of $C$ whose vertices belong to $v(C')$, and since $s_{\pi'}^+(x') = s_{\pi'}^+(x)$, one obtains that $s_{\pi'}^+(x') \not\in C$. Moreover, as $\lambda > 0$ and $x' \neq 0$ then $s_{\pi}^+(x')$ is a subset of $s_{\pi}^+(x)$. As a consequence, $s_{\pi}^+(x) \not\in C$, and it follows from the definition of $I_{\pi'}(C)$ that $x$ does not belong to $I_{\pi'}(C)$.

This shows that $M$ is a linear subspace of $D(\pi)$ disjoint from $I_{\pi'}(C)$. Since $y$ does not belong to $M'$, the dimension of $M$ is $n' - d' - k$. As $n' = |v(C)| - |s \cap v(C)|$, the following inequality is a consequence of Theorem 1:

$$\delta(C) \leq |\mathcal{V}| - (|v(C)| - |s \cap v(C)| - d' - k) - d - 1.$$  

Since $d' \leq d$, combining (8) with (9) produces the desired inequality. \hfill \square

Next theorem is proven using Lemma 5. It gives bounds on $\delta(C)$ that depend on the particular structure of simplicial complex $C$. This theorem will be invoked in Section 5 to show that the subgraph induced by 3-regular triangulations in the flip-graph of a point configuration is not always connected.

**Theorem 3.** Let $C$ be a simplicial complex on a $d$-dimensional configuration of $n$ points, and $\{s_1, \ldots, s_k\}$ a partition of $v(C)$ disjoint with $C$. If for all $i \in \{2, \ldots, k\}$, there exists $t_i \subset \bigcup_{1 \leq j < i} s_j$ so that $t_i \not\in C$ and the convex hulls of $s_i$ and $t_i$ have non-disjoint relative interiors, then $\delta(C) \leq n - d - k$. 

---

**Fig. 2.** A simplicial complex on a 3-dimensional configuration of 6 points that is not found as a subset in any triangulation of its vertex set (see [4]).
Proof. The result is proven by induction on $k$. If $k = 1$ then the result is immediate. Indeed, as mentioned earlier, a simplicial complex on a $d$-dimensional configuration of $n$ points is always $(n - d - 1)$-regular. Now assume that $k \geq 2$. By hypothesis, there exist $t_k \subset v(C) \setminus s_k$ so that $t_k \not\subseteq C$ and the convex hulls of $s_k$ and $t_k$ have non-disjoint relative interiors. As $t_k \subset v(C)$, the following inequality is obtained from Lemma 5:

$$\delta(C) \leq \delta(C \cap \mathcal{P}(v(C) \setminus s_k)) + |s_k| - 1.$$ 

As $|v(C) \setminus s_k| = n - |s_k|$, one also obtains by induction that:

$$\delta(C \cap \mathcal{P}(v(C) \setminus s_k)) \leq n - |s_k| - d - (k - 1).$$

Combining these two inequalities yields $\delta(C) \leq n - d - k$. \hfill \Box

Theorem 3 and Lemma 4 can be invoked together to obtain an improved upper bound on $\delta$:

**Corollary 1.** If $\mathcal{A}$ is a $d$-dimensional configuration of $n$ points and $C$ a simplicial complex on $\mathcal{A}$, then:

$$\delta(C) \leq n - d - 2 - \left\lceil \frac{n - d - 3}{[d/2 + 1]} \right\rceil. \tag{10}$$

Proof. First observe that inequality (10) is obviously satisfied when $1 \leq n - d \leq 2$ and follows from Theorem 2 when $n - d = 3$. It will be shown by induction on $n - d$ that for all $n - d \geq 3$, there exists a partition $\{s_1, \ldots, s_k\}$ of $\mathcal{A}$ disjoint from $C$ so that for all $i \in \{2, \ldots, k\}$, the convex hulls of $s_i$ and $\bigcup_{j<i} s_j$ are non-disjoint, and:

$$k \geq 2 + \left\lceil \frac{n - d - 3}{[d/2 + 1]} \right\rceil. \tag{11}$$

If $n - d = 3$, then the right-hand side of (11) is equal to 2, and the desired assertion immediately follows from Lemma 4. Assume that $n - d > 3$. Consider a subset $\mathcal{A}'$ of $\mathcal{A}$ of cardinality $d + 3$. According to Lemma 4, $\mathcal{A}'$ admits two disjoint subsets $s$ and $t$ that do not belong to $C$ and whose convex hulls have non-disjoint relative interiors. As a first consequence, the convex hulls of $s$ and $\mathcal{A} \setminus s$ are non-disjoint. Moreover, $s$ and $t$ cannot both contain more than $[d/2 + 1]$ points. Hence, it can be assumed without loss of generality that $s$ has cardinality at most $[d/2 + 1]$. Denote by $n'$ the cardinality of $\mathcal{A} \setminus s$ and by $d'$ the dimension of its affine hull. Since $d' \leq d$, one obtains:

$$n' - d' \geq n - d - [d/2 + 1]. \tag{12}$$

If $n' - d' < 3$, then (12) states that $[d/2 + 1]$ is greater than $n - d - 3$. As a consequence, the right-hand side of inequality (11) is precisely equal to 2, and $\{s, \mathcal{A} \setminus s\}$ is a partition of $\mathcal{A}$ with the desired properties.

It is now assumed that $n' - d' \geq 3$. Let $C'$ denote the intersection of $C$ with $\mathcal{P}(\mathcal{A} \setminus s)$. If $n' - d' < n - d$, then by induction, there exists an integer $k$ satisfying:

$$k - 1 \geq 2 + \left\lceil \frac{n' - d' - 3}{[d'/2 + 1]} \right\rceil. \tag{13}$$

and a partition $\{s_1, \ldots, s_{k-1}\}$ of $\mathcal{A} \setminus s$ so that $\{s_1, \ldots, s_{k-1}\}$ is disjoint with $C'$ and for all $i \in \{2, \ldots, k - 1\}$, the convex hulls of $s_i$ and $\bigcup_{j<i} s_j$ are non-disjoint. As $d' \leq d$, then $[d'/2 + 1] \leq [d/2 + 1]$. Using this inequality along with (12) provides a lower bound for the right-hand side of (13) according to which inequality (11) holds. Calling $s_k = s$, the set $\{s_1, \ldots, s_k\}$ then is a partition of $\mathcal{A}$ with the desired properties.

Now assume that $n' - d' \geq n - d$. Let $t' \subset \mathcal{A} \setminus s$ be an affinely independent set of $d' + 1$ points. Recall that $n \geq d + 2$. Subtracting $|s| - d'$ from both sides of this inequality yields:

$$n - |s| \geq (d - d' - |s| + 1) + (d' + 1).$$

Following this, $\mathcal{A} \setminus s$ contains at least $d - d' - |s| + 1$ distinct points that do not belong to $t'$. Let $s' \subset \mathcal{A} \setminus (s \cup t)$ be a set of cardinality $d - d' - |s| + 1$. According to this construction, $\mathcal{A} \setminus (s \cup s')$ is a $d'$-dimensional configuration of $n - d + d' - 1$ points. Hence, $|\mathcal{A} \setminus (s \cup s')| - d' = n - d - 1$. By induction, there exists an integer $k$ satisfying:

$$k - 1 \geq 2 + \left\lceil \frac{|\mathcal{A} \setminus (s \cup s')| - d' - 3}{[d'/2 + 1]} \right\rceil. \tag{14}$$

and a partition $\{s_1, \ldots, s_{k-1}\}$ of $\mathcal{A} \setminus (s \cup s')$ so that $\{s_1, \ldots, s_{k-1}\}$ is disjoint with $C'$ and for all $i \in \{2, \ldots, k - 1\}$, the convex hulls of $s_i$ and $\bigcup_{j<i} s_j$ are non-disjoint. Denote by $s_k$ the union of $s$ with $s'$. Since $[d'/2 + 1] \leq [d/2 + 1]$ and since $|\mathcal{A} \setminus s_k| - d' = n - d - 1$, inequality (11) follows from (14). As a consequence, $\{s_1, \ldots, s_k\}$ is a partition of $\mathcal{A}$ with the desired properties.

It has been proven that there exists an integer $k$ satisfying (11) and a partition $\{s_1, \ldots, s_k\}$ of $\mathcal{A}$ disjoint with $C$ so that for all $i \in \{2, \ldots, k\}$, the convex hulls of $s_i$ and $\bigcup_{j<i} s_j$ are non-disjoint. Hence, the result follows from Theorem 3 and inequality (11). \hfill \Box
The bounds on $\delta$ given by (10) are reported in Table 1 for $1 \leq d \leq 28$ and $1 \leq n - d \leq 20$. Note that each column in this table corresponds to two consecutive dimensions. The bounds given in the first column (i.e. for $d \in \{1, 2\}$) are not sharp. Indeed, all simplicial complexes of a 1-dimensional point configuration are obviously regular. Furthermore, it is shown in [16] that any triangulation of a 2-dimensional point configuration is 2-regular. As a simplicial complex on a 2-dimensional point configuration $\mathcal{A}$ is always found as a subset in some triangulation of $\mathcal{A}$, then any simplicial complex on a 2-dimensional point configuration is 2-regular.

Denote by $\mathcal{A}$ the vertex set of the $d$-dimensional cube. This point configuration and its triangulations are now studied. The set of these triangulations has an exponential complexity: while the 2-dimensional cube only admits two triangulations, the 3-dimensional cube has seventy-four of them [5], and there are more than eighty-seven million ways to triangulate the 4-dimensional cube and this number only takes into account regular triangulations (see [6,11]). If $2 \leq d \leq 3$, all the triangulations of $\mathcal{A}$ are regular and if $d = 4$, $\mathcal{A}$ admits non-regular triangulations [5]. It is proven in the following that all triangulations of $\mathcal{A}$ are $(2d^2 - d + 1)$-regular for arbitrary $d$. This is obviously not a sharp bound (for example when $d = 3$), but this bound holds in any dimension and it is much sharper than those given by (10) in the general case. Recall that $\mathcal{A}$ contains $2^d$ points. Therefore, according to Theorem 1, the following result corresponds to finding a linear subspace of dimension $2d^2 - 2$ in the space of affine dependencies of $\mathcal{A}$.

**Theorem 4.** Let $\mathcal{A}$ be the vertex set of the $d$-dimensional cube. If $T$ is a triangulation of $\mathcal{A}$ then $\delta(T) \leq 2d^2 - 2d - 1$.

**Proof.** Observe that the $d$-dimensional cube admits $2d^2 - 2d$ diagonals. Moreover, any two of these diagonals intersect at the center of the cube. In other words, there exists a partition $\{s_i : 1 \leq i \leq 2d^2 - 2d\}$ of $\mathcal{A}$ so that the convex hulls of $s_i$ and $s_j$ have non-disjoint relative interiors for all $i \neq j$. It follows that any triangulation of $\mathcal{A}$ contains at most one of these diagonals. Let $T$ be a triangulation of $\mathcal{A}$. One can assume that $s_i \notin T$ for all $2 \leq i \leq 2d^2 - 2d$. As all the points of $\mathcal{A}$ are vertices of $T$, the set:

$$\{s_1 \cup s_2 \cup \{s_i : 3 \leq i \leq 2d^2 - 1\}\}$$

is a partition of $v(T)$ into $2d^2 - 2d - 1$ subsets that do not belong to $T$. Let $i$ be an integer so that $3 \leq i \leq 2d^2 - 1$. As any two diagonals of the cube intersect at their center, then the convex hull of $s_i$ and $\cup_{1 \leq j < i} s_j$ have non-disjoint relative interiors. As in addition $\cup_{1 \leq j < i} s_j \notin T$, the desired inequality follows from Theorem 3.

4. Results on flip-graph connectivity

Consider a point configuration $\mathcal{A}$. Flips are local operations that transform a triangulation of $\mathcal{A}$ into another triangulation of $\mathcal{A}$ (see [21,23] for several equivalent definitions of flips). The flip-graph of $\mathcal{A}$ is the graph $\gamma(\mathcal{A})$ whose vertices are the triangulations of $\mathcal{A}$ and whose edges connect two triangulations that can be obtained from one another by performing a flip. When $\gamma(\mathcal{A})$ is connected, flips provide an efficient way to enumerate the triangulations of $\mathcal{A}$ or to obtain triangulations with desired properties for numerous applications [14,17,18]. This naturally raises the question of flip-graph connectedness. If $\mathcal{A}$ is 2-dimensional, $\gamma(\mathcal{A})$ is connected [12]. Point configurations of dimensions 5 and 6 with disconnected flip-graphs have been found by Santos [6,21–23]. So far, the question remains open in general for point configurations of dimensions 3 and 4. As already mentioned in the introduction, some subgraphs of the flip-graph are known to be connected for point configurations of any dimension, though. Denote by $\gamma_k(\mathcal{A})$ the subgraph induced by $k$-regular triangulations in $\gamma(\mathcal{A})$. It is
known that $\gamma_2(A)$ is connected \cite{8,9,17} and it has been shown recently that $\gamma_3(A)$ is connected as well (see Theorem 2 in \cite{15}). The fourth row of Table 1 indicates that if $n - d = 4$, then $\gamma(A)$ and $\gamma_2(A)$ are identical and therefore $\gamma(A)$ is connected in this case. This result has been obtained by Azaola and Santos a decade ago \cite{1}. They actually proved a stronger statement: if $n - d = 4$, then $\gamma(A)$ is 3-connected. The connectivity status of the flip-graph remains open when $n - d = 5$ (see comments in \cite{1} on the difficulty of this problem). Partial results are obtained in this section.

Assume that $n - d = 5$. Looking at the fifth row of Table 1, one can see that all the triangulations of $A$ are 3-regular. While this does not imply the connectedness of $\gamma(A)$ (it will be shown in Section 5 that $\gamma_3(A)$ is not always connected), one can still find conditions on a triangulation $T$ of $A$ under which 2-regularity holds. If it does, there exists a path in $\gamma(A)$ between $T$ and any regular triangulation of $A$. Such conditions are given in this section. For triangulations, some of the requirements of Theorem 3 can be weakened as stated by the following lemma:

**Lemma 6.** Let $T$ be a triangulation of a point configuration $A$, and $s$ a subset of $A$ so that the convex hulls of $s$ and $A$ have non-disjoint relative interiors. If $s$ does not belong to $T$ and all the proper subsets of $s$ belong to $T$, then the convex hulls of $s$ and $A \setminus s$ have non-disjoint relative interiors.

**Proof.** Assume that $s \not\subset T$ and that all the proper subsets of $s$ belong to $T$. Observe that $s$ is affinely independent. Otherwise, two of its proper subsets would have convex hulls that intersect within their relative interiors, and they could not both belong to $T$.

By hypothesis, there exists a point $a$ that lies in the relative interiors of both $\text{conv}(s)$ and $\text{conv}(A)$. As $a$ belongs to $\text{conv}(A)$, $T$ admits a face $t$ whose convex hull contains $a$ in its relative interior. As $a$ lies in the relative interiors of both $\text{conv}(s)$ and $\text{conv}(t)$, it can be written as convex combinations of $s$ and $t$ with positive coefficients:

$$a = \sum_{x \in s} \alpha_x x \quad \text{and} \quad a = \sum_{x \in t} \beta_x x. \quad (15)$$

Assume that $y$ is a point in $s \cap t$. Observe that as $s$ is not a face of $T$ then it cannot be a subset of $t$ and as a consequence, $s \setminus \{y\}$ is non-empty. Moreover, $t$ is not a subset of $s$ because the convex hulls of $s$ and $t$ have non-disjoint relative interiors while $s$ is affinely independent. It follows that $t \setminus \{y\}$ is non-empty. Eq. (15) yield:

$$\sum_{x \in s \setminus \{y\}} \alpha_x x = (\beta_y - \alpha_y) y + \sum_{x \in t \setminus \{y\}} \beta_x x. \quad (16)$$

If $\alpha_y \leq \beta_y$, then Eq. (16) states that the convex hulls of a proper face of $s$ and of a face of $t$ have non-disjoint relative interiors. Since all the faces of $t$ and all the proper faces of $s$ belong to $T$, this cannot happen, which proves that $\alpha_y > \beta_y$ for all $y \in s \cap t$.

Now, recall that $a$ belongs to the relative interior of $\text{conv}(A)$. As a consequence, $a$ can be written as a convex combination of $A$ with positive coefficients:

$$a = \sum_{x \in A} \gamma_x x. \quad (17)$$

For all $x \in A \setminus t$, denote $\beta_x = 0$, and for all $x \in A \setminus s$, denote $\alpha_x = 0$. It follows that $\alpha_x$ is strictly greater than $\beta_x$ for all $x \in s$ and as a consequence, there exists a real number $\lambda \in ]0, 1[$ so that for all $x \in s$, the following inequality holds:

$$\lambda(\gamma_x - \beta_x) < \alpha_x - \beta_x. \quad (18)$$

Combining Eqs. (15) and (17) yields:

$$\sum_{x \in A} (\lambda \gamma_x - \alpha_x + (1 - \lambda) \beta_x) x = 0. \quad (19)$$

Since inequality (18) is satisfied for all $x \in s$, the points of $s$ have negative coefficients in the left-hand side of (19). Moreover, recall that $\alpha_x = 0$ for all $x \in A \setminus s$. As $\lambda \in ]0, 1[$, it follows that the points of $A \setminus s$ have positive coefficients in the left-hand side of (19).

Therefore, Eq. (19) states that the convex hulls of $s$ and $A \setminus s$ have non-disjoint relative interiors, which completes the proof. \hfill $\Box$

The following theorem is the most general result in this section. It provides additional bounds on $\delta$. Its corollaries state results on the structure of the flip-graph.

**Theorem 5.** Let $A$ be a $d$-dimensional configuration of $n$ points, $T$ a triangulation of $A$, and $s$ a subset of $A$ so that $s \not\subset T$ and $n - |s| \geq d + 3$. If the convex hulls of $s$ and $A$ have non-disjoint relative interiors and all the proper subsets of $s$ belong to $T$, then $T$ is $(n - d - 3)$-regular.

**Proof.** Assume that the convex hulls of $s$ and $A$ have non-disjoint relative interiors and that all the proper subsets of $s$ belong to $T$. Consider the simplicial complex $C = T \cap P(A \setminus s)$. As $n - |s| \geq d + 3$, then $A \setminus s$ contains at least $d + 3$ points. As in
addition $A \setminus s$ is at most $d$-dimensional, it follows from Lemma 4 that $A \setminus s$ admits two disjoint subsets $t$ and $s_2$ that do not belong to $c$ and whose convex hulls have non-disjoint relative interiors. Consider the set:

$$s_1 = A \setminus (s_2 \cup s).$$

As $t \subset s_1$ and as $t \not\subseteq T$, then $s_1 \not\subseteq T$. According to Lemma 6, the convex hulls of $s$ and $A \setminus s$ have non-disjoint relative interiors. Moreover, $A \setminus s$ is not a face of $T$ because it contains more than $d + 1$ points. Therefore, denoting $s_3 = s$, the set $\{s_1, s_2, s_3\}$ is a partition of $A$ that satisfies the hypotheses of Theorem 3 (for instance, one can use $t_2 = t$ and $t_3 = A \setminus s$), and as a consequence, $T$ is $(n - d - 3)$-regular. □

Recall that a $d$-dimensional point configuration is in general position if all its subsets of $d + 1$ points are affinely independent. The following two corollaries of Theorem 5 give results for such point configurations:

**Corollary 2.** Let $A$ be a $d$-dimensional configuration of $n$ points in general position, $T$ a triangulation of $A$, and $s$ a subset of $A$ that does not belong to $T$. If $n - |s| \geq d + 3$, then $T$ is $(n - d - 3)$-regular.

**Proof.** Assume that $n - |s| \geq d + 3$. Since the empty set belongs to $T$, $s$ admits a subset $t$ so that $t$ does not belong to $T$ and all the proper subsets of $t$ belong to $T$. As $A$ is in general position, then $T$ contains all the subsets of $A$ whose convex hulls do not intersect the relative interior of $\text{conv}(A)$. Therefore the convex hulls of $t$ and $A$ have non-disjoint relative interiors. As $t$ is a subset of $s$, then $n - |t| \geq d + 3$ and the result follows from Theorem 5. □

**Corollary 3.** Let $T$ be a triangulation of a $d$-dimensional configuration $A$ of $d + 5$ points in general position. If the $1$-skeleton of $T$ is not a complete graph, then there exists a path in $\gamma(A)$ from $T$ to any regular triangulation of $A$.

**Proof.** As the $1$-skeleton of $T$ is not a complete graph, there exists a set $s \subset A$ of cardinality 2 that does not belong to $T$. In this case, $n - |s| = d + 3$ and according to Corollary 2, $T$ is 2-regular. Since the subgraph of $\gamma(A)$ induced by 2-regular triangulations is connected, there exists a path in $\gamma(A)$ from $T$ to any regular triangulation of $A$. □

Observe that Corollary 3 cannot be used to investigate the vertex set of a 2-neighborly polytope: in this case, the convex hull of any pair of points of $A$ is a face of $\text{conv}(A)$. Hence, the $1$-skeleton of every triangulation of $A$ is necessarily a complete graph. Note however, that among such point configurations, one finds the vertex sets of cyclic polytopes of dimension at least 4, whose flip-graphs were shown to be connected in [20].

**Corollary 4.** Let $A$ be a $d$-dimensional configuration of $d + 5$ points. If $A$ contains a subset of three collinear points whose middle point is in the relative interior of $\text{conv}(A)$, then $\gamma(A)$ is connected.

**Proof.** Let $a$, $b$, and $c$ be mutually distinct points of $A$ so that $c$ belongs to the relative interiors of both $\text{conv}(\{a, b\})$ and $\text{conv}(A)$. It will be proven that all the triangulations of $A$ are 2-regular which, according to [15] implies the connectedness of $\gamma(A)$. Let $T$ be a triangulation of $A$. If $\{a, b, c\} \not\subseteq v(T)$, then according to Theorem 2, $T$ is 2-regular. It is therefore assumed that points $a$, $b$, and $c$ are vertices of $T$. Since $T$ is a simplicial complex, it does not both admit $c$ as a vertex and $\{a, b\}$ as a face. Hence, $\{a, b\}$ does not belong to $T$. Moreover, the relative interiors of $\text{conv}(\{a, b\})$ and $\text{conv}(A)$ are non-disjoint because they both contain $c$. Finally, the proper subsets of $\{a, b\}$ are $\{a\}$, $\{b\}$, and $\emptyset$. Since they all belong to $T$, it follows from Theorem 5 that $T$ is 2-regular. □

Let $T$ be a triangulation of a 3-dimensional point configuration $A$. A geometric bistellar neighbor of $T$ is a triangulation obtained from $T$ by performing a flip. If the 1-skeleton of $T$ is a complete graph, then any flip removes at least one edge from $T$ (see the definition of flips in [7] or [21]):

**Proposition 5.** Let $A$ be a 3-dimensional point configuration and $T$ a triangulation of $A$. If the 1-skeleton of $T$ is a complete graph, then the 1-skeleton of any geometric bistellar neighbor of $T$ is not the complete graph on $v(T)$.

Further assume that $A$ is the vertex set of a 3-dimensional simplicial polytope. It is shown in [7] that $T$ then necessarily admits a flip. In addition, since $\text{conv}(A)$ is simplicial, the convex hull of $A$ and that of any missing edge in the 1-skeleton of a triangulation of $A$ must have non-disjoint relative interiors. Hence, the following result is a consequence of Proposition 5 and Theorem 5:

**Theorem 6.** Let $A$ be a 3-dimensional configuration of 8 points. If $A$ is the vertex set of a simplicial polytope then $\gamma(A)$ is connected.

**Proof.** Assume that $A$ is the vertex set of a simplicial polytope. Let $T$ be a triangulation of $A$. If the 1-skeleton of $T$ is not a complete graph, then let $s$ be a subset of $A$ so that $|s| = 2$ and $s \not\subseteq T$. Since $A$ is the vertex set of a simplicial polytope, the relative interior of this polytope has a non-empty intersection with the relative interior of $\text{conv}(s)$. In addition, $T$ admits both elements of $s$ as vertices. According to Theorem 6, $T$ is therefore 2-regular and it can be flipped to any regular triangulation of $A$. Now assume that the 1-skeleton of $T$ is a complete graph. According to Theorem 2.2. in [7], $T$ admits a flip and it follows from Proposition 5 that the $T$ can be flipped to a triangulation of $A$ whose 1-skeleton is not a complete graph. Hence, $T$ can be flipped, also in this case, to any regular triangulation of $A$, which completes the proof. □
Observe that Corollaries 2 and 3 only apply to point configurations in general position. Moreover, Theorem 6 requires the point configuration to be the vertex set of a simplicial polytope. In contrast, Corollary 4 takes advantage of affinely dependent subsets with few points to provide results on flip-graph connectivity. The two following results also exploit the existence of such subsets:

**Theorem 7.** Let \( k > 0 \) be an integer and \( \mathcal{A} \) a \( d \)-dimensional configuration of \( n \) points so that \( n - \lceil k/2 \rceil = d + 4 \). If the convex hull of \( \mathcal{A} \) admits a proper \( k \)-dimensional face that contains at least \( k + 3 \) points of \( \mathcal{A} \) then all the triangulations of \( \mathcal{A} \) are \( \lceil k/2 + 1 \rceil \)-regular.

**Proof.** Assume that the convex hull of \( \mathcal{A} \) admits a proper \( k \)-dimensional face \( f \) that contains at least \( k + 3 \) points of \( \mathcal{A} \). Let \( T \) be a triangulation of \( \mathcal{A} \). Consider a \( k \)-dimensional set \( s \subset \mathcal{A} \cap f \) of cardinality \( k + 3 \). According to Lemma 4, \( s \) admits two disjoint subsets \( s_1 \) and \( s_3 \) that do not belong to \( T \) and whose convex hulls have non-disjoint relative interiors. As \( |s| = d + 3 \), one of those two subsets, say \( s_3 \), has cardinality at most \( \lceil k/2 + 1 \rceil \). Now as \( n - \lceil k/2 \rceil = d + 4 \), then \( \mathcal{A} \setminus s_3 \) has cardinality at least \( d + 3 \). As in addition, \( \mathcal{A} \setminus s_3 \) is at most \( d \)-dimensional, it follows from Lemma 4 that \( \mathcal{A} \setminus s_3 \) admits two disjoint subsets \( s_2 \) and \( s_3 \) that do not belong to \( T \) and whose convex hulls have non-disjoint relative interiors. Denoting \( s_1 = \mathcal{A} \setminus (s_2 \cup s_3) \), the set \( \{s_1, s_2, s_3\} \) then is a partition of \( \mathcal{A} \) that satisfies the conditions of Theorem 3, and since \( n - d - 3 = \lceil k/2 + 1 \rceil \), the proof is complete. \( \square \)

According to Theorem 2 in [15], the following result is a direct consequence of Theorem 7:

**Corollary 5.** Let \( k \in \{1, 2\} \) be an integer and \( \mathcal{A} \) a \( d \)-dimensional configuration of \( d + 5 \) points. If the convex hull of \( \mathcal{A} \) admits a \( k \)-dimensional face that contains at least \( k + 3 \) points of \( \mathcal{A} \) then \( \gamma(\mathcal{A}) \) is connected.

The results in this section exhibit several connected subgraphs in the flip-graph of a point configuration, which reduces the frontier between connected and non-connected flip-graphs. A related question was formulated in [15]: what is the largest integer \( k \) so that for every point configuration \( \mathcal{A} \), \( \gamma_k(\mathcal{A}) \) is connected? The answer is \( k = 2 \) as shown in the next section.

### 5. Graph \( \gamma_3 \) is not always connected

In the last decade, several point configurations with disconnected flip-graphs were found by Santos [6,21–23]. In this section, the point configuration he describes in [23] is further studied and it is shown that all its triangulations are 3-regular. Let \( \{u_1, u_2, u_3, u_4, u_5, u_6\} \) be a basis of \( \mathbb{R}^5 \). Consider the following vectors:

\[
\begin{align*}
  b_1 &= \sqrt{2}u_5, \\
  b_2 &= u_5 + u_6, \\
  b_3 &= \sqrt{2}u_6, \\
  b_4 &= -u_5 + u_6,
\end{align*}
\]


and for all \( i \in \{1, 2, 3, 4\} \), consider the following four points:

\[
\begin{align*}
  a_i^+ &= u_i + b_i, \\
  a_{2i+1}^+ &= u_i - b_i, \\
  a_i^- &= -u_i + b_i, \\
  a_{2i+1}^- &= -u_i - b_i. (20)
\end{align*}
\]

As shown in [23] (see also [6]), the point configuration:

\[
\mathcal{A} = \{0\} \cup \left[ \bigcup_{i=1}^{4} \{a_i^-, a_{2i+1}^+, a_i^+, a_{2i+1}^-\} \right],
\]

has a disconnected flip-graph. Let \( i \in \{1, 2, 3, 4\} \). Observe that according to (20), the following equalities hold:

\[
a_i^+ + a_{2i+1}^- = 0 \quad \text{and} \quad a_i^- + a_{2i+1}^+ = 0.
\]

These equalities state that \( \mathcal{A} \) admits 8 mutually disjoint subsets of 2 elements whose convex hulls contain 0 in their relative interiors. As a consequence, if one removes one or several of these subsets from \( \mathcal{A} \), the convex hull of the resulting point configuration always contains 0 in its relative interior. In addition, a triangulation of \( \mathcal{A} \) either admits 0 as a vertex and contains none of these subsets, or contains exactly one of these subsets and does not admit 0 as a vertex. It therefore follows from Theorem 3 that all the triangulations of \( \mathcal{A} \) are 3-regular. This provides a new bound for the problem of flip-graph connectivity. While the subgraph induced by 2-regular triangulations in the flip-graph of a point configuration is connected [15], the same observation does not hold for 3-regular triangulations.

Yet, \( \mathcal{A} \) is 6-dimensional and the other point configurations with disconnected flip-graph known so far are of dimensions 5 and 6. In dimensions 3 and 4, the flip-graph connectivity problem is still open. Other interesting related questions still remain unanswered. In dimension 1, all the triangulations are regular and in dimension 2, they are all 2-regular [16]. Does there exist any triangulation (and more generally, any polyhedral subdivision) of a 3-dimensional or of a 4-dimensional point configuration that is not 2-regular? Does there even exist a polyhedral subdivision of a \( d \)-dimensional point configuration that is not \( d \)-regular? As a guideline, it is conjectured in [15] that the latter question admits a negative answer.

### References

