On trees with perfect matchings

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Received 28 November 2001; accepted 1 June 2002
Submitted by S. Friedland

Abstract

A tree is said to have a perfect matching if it has a spanning forest whose components are paths on two vertices only. In this paper we develop upper bounds on the algebraic connectivity of such trees and we consider other eigenvalue properties of its Laplacian matrix. Furthermore, for trees with perfect matchings, we refine a result, due to Kirkland, Neumann, and Shader, concerning the connection between the maximal diagonal entry of the group inverse of the Laplacian matrix of a (general) tree and the pendant vertices of the tree, and use this refinement to narrow down the set of the pendant vertices of a tree with a perfect matching which can correspond to the maximal diagonal entry in the group inverse of its Laplacian matrix.

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1. Introduction

An undirected graph \( G = (V, E) \) on \( n \) vertices is a finite set \( V \) of cardinality \( n \), whose elements are called vertices, together with a set \( E \) of two-element subsets of \( V \) called edges. It will be convenient to label the vertices 1, 2, \ldots, \( n \).

With \( G \) we can associate the so-called Laplacian matrix which is the \( n \times n \) matrix \( L = (\ell_{i,j}) \) whose entries are determined as follows:

\[
\ell_{i,j} = \begin{cases} 
-1, & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\
0, & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\
-\sum_{k \neq i} \ell_{i,k}, & \text{if } i = j 
\end{cases}
\]

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1 The work of this author was supported in part by NSF grant no. DMS9973247.
It is known that the Laplacian matrix is a symmetric positive semidefinite M-matrix. We shall always consider its eigenvalues to have been arranged in a non-descending order: \(0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\). For an extensive survey on the Laplacian matrix see [10].

A graph \(G\) is called connected if there is a path linking any two of its vertices. Fiedler [4, p. 298] has shown that \(\lambda_2 > 0\) if and only if \(G\) is connected and, partly for this reason, he has called \(\mu(G) := \lambda_2\) the algebraic connectivity of \(G\).

A tree \(T\) is a graph in which any pair of vertices is linked by a unique path. For any two vertices \(u, v \in T\) we define the distance between \(u\) and \(v\), \(d(u, v)\), as the number of edges which lie on the path connecting the two vertices.

In this paper, we investigate a type of tree that is said to have a perfect matching:

**Definition 1.1.** A tree is said to have a perfect matching if there exists a spanning forest whose components are solely paths on two vertices.

In Section 2 of the paper we develop upper bounds on the algebraic connectivity of trees with perfect matchings. Such trees must, of necessity, have \(n = 2k\) vertices. The converse of this is not true as shown by taking the star graph on 4 vertices. We shall show that for \(k \geq 2\), the algebraic connectivity of a tree with perfect matching is bounded above by \(2 - \sqrt{2}\). In particular, for \(k \geq 3\), all such trees have an algebraic connectivity which is bounded above by \((3 - \sqrt{5})/2\). Furthermore, \(\lambda_2 = 2\) is an eigenvalue of the Laplacian of any tree with a perfect matching.

Recall, now, from [1] or [3], that an \(n \times n\) square matrix \(A\) is said to have a group (generalized) inverse if there exists an \(n \times n\) matrix \(X\) such that \(AXA = A\), \(XAX = X\), and \(AX = XA\). Recall that if \(A\) has a group inverse, then it is unique and it is customary to denote it by \(A^g\). In [7–10], the authors studied the relationship between the vertices of a graph and the corresponding diagonal entries of the group inverse of its Laplacian matrix \(L\). In particular in [8, Theorem 3.5] it was shown that the maximal diagonal entry of \(L^g\) must occur at an index corresponding to a pendant vertex of the tree. Here we shall show that, for a tree with a perfect matching, the maximal diagonal entry in the group inverse of its Laplacian must occur at an index which corresponds to a pendant vertex whose adjacent vertex has degree 2.

2. Trees with a perfect matching

We begin with an important result from [6] about trees with a perfect matching:

**Claim 2.1** [6, p. 2]. If a tree has a perfect matching, then the matching is unique.
In other words, if a tree \( T \) has a perfect matching, then there is a unique spanning forest for \( T \) whose components are solely paths on two vertices. A further important result from \([6]\) which we will need here is the following:

**Claim 2.2** \([6, \text{Lemma 2.4}]\). If a tree \( T \) has a perfect matching, then \( T \) has at least two pendant vertices such that each are adjacent to vertices of degree 2.

Claim 2.2 in \([6]\) gives rise to the following definition:

**Definition 2.3.** A \( p_2 \)-set is a set of two adjacent vertices \( v \) and \( w \), where \( v \) is pendant and \( \deg(w) = 2 \) together with their incident edges.

We see from Claim 2.4 that if a tree has a perfect matching, then it must contain at least two \( p_2 \)-sets. The following claim gives us an insight into how we can inductively construct trees with a perfect matching.

**Claim 2.4.** If a tree \( T \) on \( n \) vertices has a perfect matching, then there exists a tree \( T' \) on \( n - 2 \) vertices with a perfect matching such that \( T \) can be constructed from \( T' \) by attaching a path on 2 vertices.

**Proof.** Let \( v, w \in T \) come from a \( p_2 \)-set (in \( T \)). (We know that such vertices exist by Claim 2.2.) Removing \( v \) and \( w \) along with the incident edges to \( w \), yields the desired tree \( T' \) as \( T' \) remains a tree spanned by a forest whose components are paths on two vertices only. \( \square \)

The following result and its immediate corollary show the relationship between the algebraic connectivities of the trees \( T \) and \( T' \) in Claim 2.4. A similar proof can be found in \([5]\), but the proof is included here for the sake of completeness.

**Claim 2.5.** Let \( G \) be a graph on \( n \) vertices. Suppose \( \hat{G} \) is a graph on \( n + 1 \) vertices created from \( G \) by affixing to \( G \) an isolated vertex \( u \) together with an edge joining \( u \) to some vertex in \( G \). Then \( \mu(\hat{G}) \leq \mu(G) \).

**Proof.** Let \( \hat{G} \) be the graph on \( n + 1 \) vertices obtained by joining \( u \), without loss of generality, to vertex 1 in \( G \) and let \( L = (\ell_{i,j}) \) and \( \hat{L} = (\hat{\ell}_{i,j}) \) be the Laplacian matrices for \( G \) and \( \hat{G} \), respectively. Let \( L' = (\ell'_{i,j}) \) be the \((n + 1) \times (n + 1)\) matrix whose entries are given by:

\[
\ell'_{i,j} = \begin{cases} 
\ell_{i,j}, & \text{if } i \neq n + 1 \text{ and } j \neq n + 1, \\
0, & \text{if } i = n + 1 \text{ or } j = n + 1.
\end{cases}
\]
Next, let \( \lambda_1' = \lambda_2' \leq \lambda_3' \leq \cdots \leq \lambda_{n+1}' \) and \( 0 = \hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \hat{\lambda}_3 \leq \cdots \leq \hat{\lambda}_{n+1} \) be the eigenvalues of \( L' \) and \( \hat{L} \), respectively. By the construction of \( \hat{G} \) we have that 
\[
\hat{L} = L' + z z^T,
\]
where 
\[
z = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \end{bmatrix}^T.
\]
It now follows from the interlacing property of eigenvalues of symmetric matrices that 
\[
\mu(\hat{G}) = \hat{\lambda}_2 \leq \lambda_3' = \mu(G).
\]
□

An immediate outcome of the above result is the following:

**Corollary 2.6.** Let \( T \) and \( T' \) be the trees with a perfect matching as in Claim 2.4. Then \( \mu(T) \leq \mu(T') \).

In the remainder of this section it will be convenient to let \( T_k \) be the set of all trees on \( 2k \) vertices which have a perfect matching. Clearly \( k \) represents here the number of connected components in a spanning forest whose components are paths on two vertices only. We easily see that the only tree in \( T_2 \) is the path on four vertices and one is easily convinced that there are only two trees in \( T_3 \): the path on six vertices and the tree \( B_6 \) which we define now:

**Definition 2.7.** Let \( S_k \) denote the star on \( k \) vertices, where \( k \geq 3 \) is an integer. Then \( B_{2k} \) is the tree created from \( S_{k+1} \) by adding a pendant edge to \( k - 1 \) of the pendant vertices of \( S_{k+1} \). For example:

- \( B_6 \)
- \( B_8 \)
- \( B_{10} \)

We are now ready to obtain our first upper bound on the algebraic connectivity of trees with a perfect matching:

**Theorem 2.8.** Let \( T \) be a tree in \( T_k \). If \( k = 2 \), then \( \mu(T) = 2 - \sqrt{2} \). If \( k \geq 3 \), then \( \mu(T) \leq (3 - \sqrt{5})/2 \).
Proof. For \( k = 2 \), the path on 4 vertices is the unique tree in \( \mathcal{T}_2 \) and it is easy to check that \( \mu(T) = 2 - \sqrt{2} \). For \( k = 3 \), we compute the algebraic connectivity of both trees in \( \mathcal{T}_3 \) and observe that for both,

\[
\mu(T) \leq \frac{3 - \sqrt{5}}{2}.
\]

Suppose now that \( k > 3 \) and \( T \in \mathcal{T}_k \). From Claim 2.4 we know that the removal of a \( p_2 \)-set from \( T \) will result in a tree \( T' \in \mathcal{T}_{k-1} \). We now repeat the process until we have obtained a tree with a perfect matching in \( \mathcal{T}_3 \). The result is now a consequence of repeated application of Claim 2.5. □

We see from Theorem 2.8 that if \( T \in \mathcal{T}_k \), \( k \geq 3 \), then \( \mu(T) \leq (3 - \sqrt{5})/2 \). It is natural to ask for which trees in \( \mathcal{T}_k \) is the bound sharp? To answer this question we need the following lemma:

**Lemma 2.9.** For all integers \( k \geq 3 \),

\[
\mu(B_{2k}) = \frac{3 - \sqrt{5}}{2}.
\]

**Proof.** Observe that the Laplacian matrix of \( B_{2k} \) can be written as

\[
L(B_{2k}) = \begin{bmatrix}
I & -I \\
-I & L(S_k) + I
\end{bmatrix},
\]

where \( L(S_k) \) is the Laplacian matrix of \( S_k \). Therefore, for each eigenvalue \( \lambda \) of \( L(S_k) \), both eigenvalues of

\[
\begin{bmatrix}
1 & -1 \\
-1 & \lambda + 1
\end{bmatrix}
\]

are eigenvalues of \( L(B_{2k}) \). Since the eigenvalues of \( L(S_k) \) are known to be 0, 1, and \( k \), the result immediately follows. □

The following theorem is a sharpening of the results in Theorem 2.8:

**Theorem 2.10.** If a tree \( T \) on at least 6 vertices has a perfect matching, then \( \mu(T) \leq (3 - \sqrt{5})/2 \). Moreover, equality holds if and only if \( T = B_{2k} \), for some positive integer \( k \geq 3 \).

**Proof.** Suppose \( T \) is any tree whose diameter is at least 5. Then \( P_6 \), the path on 6 vertices, is a subgraph of \( T \) and it follows from repeated applications of Claim 2.5 that

\[
\mu(T) \leq \mu(P_6) = 2 - \sqrt{3} < \frac{3 - \sqrt{5}}{2}.
\]
To complete the proof, we need to consider all trees that have a perfect matching with at least 6 vertices whose diameter is less than 5. If $T$ has diameter 1 then $T = K_2$ and there is nothing to prove since $K_2$ has less than 6 vertices. There are no trees with a perfect matching that have a diameter of 2. It is readily established that each tree on at least 6 vertices with diameter 3 fails to have a perfect matching. Finally, the only trees of diameter 4 that have a perfect matching are trees of the form $B_{2k}$, where $k \geq 3$. Since it was established in Lemma 2.9 that $\mu(B_{2k}) = (3 - \sqrt{5})/2$, the theorem is proved. □

Having investigated the algebraic connectivity of trees with a perfect matching, we now consider another eigenvalue of the Laplacian matrices of such trees. To this end we need the following result:

**Theorem 2.11.** Let $L$ be the Laplacian matrix for a tree $T \in \mathcal{T}_k$. Then $L$ admits the representation as

$$
\begin{bmatrix}
L_1 + I & -I \\
-I & L_2 + I
\end{bmatrix},
$$

(2.1)

where $L_1$ and $L_2$ are $k \times k$ Laplacian matrices for two subgraphs $\mathcal{V}_1(T)$ and $\mathcal{V}_2(T)$ of $T$.

**Proof.** We will consider $\mathcal{T}_k$ and prove the theorem by induction on $k$. By letting $\mathcal{V}_1(P_4) = K_2$ and $\mathcal{V}_2(P_4) = E_2$, the empty graph on 2 vertices, we see that $L(P_4)$ is permutationally similar to the matrix in (2.1). Since $P_4$ is the only tree in $\mathcal{T}_2$, the theorem is true for $k = 2$. Fix an integer $k \geq 2$ and assume that the theorem is true for all trees in $\mathcal{T}_{k+1}$. Let $T \in \mathcal{T}_{k+1}$. Since $T$ must have a $p_2$-set, let $T'$ be a tree in $\mathcal{T}_k$ by removing a $p_2$-set, $Q$, from $T$. By the inductive hypothesis, $L(T')$ can be written as in (2.1). Let $x \in Q$ be pendant in $T$ and let $y \in Q$ be adjacent to a vertex $z \in T'$. Without loss of generality, let $z$ be a vertex in $\mathcal{V}_1(T')$. Letting

$$\mathcal{V}_1(T) = y \cup \mathcal{V}_1(T')$$

and

$$\mathcal{V}_2(T) = x \cup \mathcal{V}_2(T')$$

we see that in $T$, each vertex in $\mathcal{V}_1(T)$ is adjacent to exactly one vertex in $\mathcal{V}_2(T)$, and each vertex in $\mathcal{V}_2(T)$ is adjacent to exactly one vertex in $\mathcal{V}_1(T)$. Therefore, $L(T)$ is permutationally similar to the matrix in (2.1) and the theorem is proved. □

The following corollary gives us further information about the spectrum of Laplacian matrices of trees with a perfect matching:
Corollary 2.12. Let $T \in \mathcal{T}_k$ and $L(T)$ be its Laplacian matrix. Then 2 is an eigenvalue of $L(T)$. Moreover, the corresponding eigenvector can be normalized so that exactly $k$ of its entries are 1 and $k$ of its entries are $-1$.

Proof. From Theorem 2.11 we see that we can partition the Laplacian matrix of a tree with a perfect matching as in (2.1). Letting $e_k$ be the all ones vector of order $k$, it is easy to check that

$$\begin{bmatrix} e_k^T \\ -e_k^T \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue 2. □

3. The group inverse of the Laplacian matrix

Recall from the introduction that group generalized inverses of singular matrices have been studied extensively in the books [1–3]. Applications of group generalized inverses of to Laplacian matrices have been considered, for example, in [7–9]. The group inverse of the Laplacian matrix for a tree gives insight into the structure of the tree.

As an example for the above, we refer to the use of the group inverse to determine the location of the centroid (see [9]) of a tree. The smaller a diagonal entry in the group inverse is, the closer that corresponding vertex is to the centroid of the tree. It was proven in [7] that the maximal diagonal entry in the group inverse of the Laplacian matrix of a tree must occur at an index which corresponds to a pendant vertex in the tree. The proof of this result makes heavy use of the inverse status number of a vertex $v \in T$. Specifically, let $u$ and $v$ be vertices in $T$ and let $P_{v,u}$ denote the path from $v$ to $u$. Then the inverse status number of a vertex $v$ is defined \(^3\) in [7] to be:

$$\hat{d}_v := \sum_{u \in T} d(v,u). \quad (3.1)$$

Letting $L^#$ denote the group inverse of the Laplacian matrix $L$ for a tree $T$ on $n$ vertices, it was shown in [7] that if $L^#_{ij}$ is the diagonal entry in $L^#$ corresponding to vertex $i$, then

$$L^#_{ij} = \hat{d}_i - \frac{1}{2n^2}.$$

Therefore, according to [7, Corollary 3.4], the maximal diagonal entry in the group inverse of the Laplacian matrix of $T$ has an index which corresponds to a pendant

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3 The definition of inverse status number can be extended to the more general setting of weighted trees (see [7]).
vertex with a maximal inverse status number. In this section we shall strengthen [7, Corollary 3.4] for trees with a perfect matching. We shall show that for such trees, we can narrow down the set of pendant vertices whose index corresponds to an index of a diagonal entry in the group inverse, which is maximal in value. We begin by stating two definitions:

**Definition 3.1.** Let $T$ be a tree. A $p_2$-vertex is a vertex belonging to a $p_2$-set in $T$. A $p_2$-path is a path in $T$ in which both of its terminal vertices are pendant $p_2$-vertices in $T$.

To lead us to the main results of this section, we need the following lemma:

**Lemma 3.2.** Suppose that $T$ is a tree with a perfect matching and suppose that $v \in T$ is a pendant vertex not belonging to a $p_2$-set. If $y_0 \in T$ is the vertex adjacent to $v$, then there exists a $p_2$-path in $T$ containing $y_0$.

**Proof.** Consider the forest $F$ formed from $T$ by deleting $v$, $y_0$, and all edges incident with those two vertices. Since $T$ has a perfect matching, so does $F$; let the connected components of $F$ be the trees $T_1, \ldots, T_k$, and note that $k = \deg(y_0) - 1 \geq 2$. Evidently each tree $T_i$ has a perfect matching. In particular, $T_1$ is either a single edge, or contains a $p_2$-set $\{u, w\}$ such that neither $u$ nor $w$ is adjacent to $y_0$ in $T$. It now follows that there is a $p_2$-set $\{v_1, w_1\}$ in $T_1$ which is also a $p_2$-set in $T$. A similar argument applies to $T_2$, giving us another $p_2$-set $\{v_2, w_2\}$ (in $T_2$). The path in $T$ from $v_1$ to $v_2$ is now the desired $p_2$-path containing $y_0$ and the lemma is proved. □

We now prove the following theorem.

**Theorem 3.3.** Let $\tilde{T}$ be a tree with a perfect matching and let $v \in \tilde{T}$ be a pendant vertex not belonging to a $p_2$-set. Then there exists a pendant $p_2$-vertex $u \in \tilde{T}$ such that $\hat{d}_u > \hat{d}_v$.

**Proof.** We will prove this by induction on the number of vertices. The result is certainly true for trees with a perfect matching that contain at most 4 vertices. Suppose then that $T$ is a tree with a perfect matching that has $n \geq 6$ vertices. Let $y_0$ be the vertex adjacent to $v$. We know from Lemma 3.2 that there exists a $p_2$-path $P$ in $T$ containing $y_0$. Let $q$ and $r$ be the terminal $p_2$-vertices of $P$. Let $\tilde{T}$ be the subtree of $T$ consisting of $P$ along with $v$ and the edge joining $v$ to $y_0$. Label the vertices of $\tilde{T}$ as follows:
If \( d(r, y_0) = 2 \), consider the tree \( T_1 \) on \( n - 3 \) vertices formed from \( T \) by deleting \( r, y_1, v \), and all incident edges. Let \( \delta \) be the inverse status number of \( y_0 \) in \( T_1 \), and note that we have

\[
\hat{d}_r = \delta + 2(n - 3) + 3 + 1 > \delta + n - 3 + 3 + 2 = \hat{d}_v.
\]

A similar argument applies if \( d(q, y_0) = 2 \).

So now suppose that \( d(r, y_0), d(q, y_0) \geq 3 \). Let \( T_2 \) be the tree on \( n - 4 \) vertices formed from \( T \) by deleting \( r, q, y_0, y_{d(r,y_0)+1}, y_{d(q,y_0)-1}, u \), and all incident edges, and note necessarily that \( T_2 \) has a perfect matching. For a vertex \( a \) of \( T_2 \), denote its inverse status number (considered as a vertex of \( T_2 \)) by \( d^*_a \) and as before let \( \hat{d}_a \) be the inverse status number considered as a vertex of \( T \). Note that the degree of \( y_0 \) as a vertex of \( T_2 \) is at least \( 3 \) since it coincides with the degree of \( y_0 \) as a vertex of \( T \). Thus \( v \) does not belong to a \( p_2 \)-set of \( T_2 \). Applying the induction hypothesis to \( T_2 \), we see that there is a pendant \( p_2 \)-vertex \( u \) of \( T_2 \) such that \( d^*_u > d^*_v \).

If \( u \) is on the path from \( r \) to \( q \) in \( T \), say with \( d(r, u) = 2 \) (without loss of generality) then as above we have

\[
\hat{d}_r = d^*_r + 2(n - 4) + 1 + d(r, q) + d(r, y_{d(q,y_0)-1})
\]

\[
= d^*_r + 2n - 7 + d(r, q) + d(r, y_{d(q,y_0)-1})
\]

\[
\geq d^*_r + 5 + d(r, q) + d(r, y_{d(q,y_0)-1})
\]

\[
= d^*_r + 4 + d(r, q) + d(r, q)
\]

\[
= d^*_r + 4 + 2d(r, q),
\]

where inequality (3.2) follows from the fact \( n \geq 6 \).

We also have

\[
\hat{d}_v = d^*_v + d(r, v) + d(q, v) + d(v, y_{d(q,y_0)-1}) + d(v, y_{d(q,y_0)-1})
\]

\[
= d^*_v + d(r, q) + 2 + d(r, q)
\]

\[
= d^*_v + 2 + 2d(r, q).
\]

Since \( d^*_u > d^*_v \), it follows that \( \hat{d}_r > \hat{d}_v \), as desired.

Suppose that \( u \) is not on the path from \( r \) to \( q \) in \( T \); let \( y_c \) be the vertex on the path which is closest to \( u \). Then

\[
\hat{d}_u = d^*_u + d(r, u) + d(q, u) + d(y_{d(q,y_0)-1}, u) + d(y_{d(q,y_0)-1}, u)
\]

\[
= d^*_u + d(r, y_c) + d(y_c, u) + d(q, y_c) + d(y_c, u)
\]

\[
+ d(y_{d(q,y_0)+1}, y_c) + d(y_c, u) + d(y_{d(q,y_0)-1}, y_c) + d(y_c, u)
\]

\[
= d^*_u + d(r, q) + d(y_{d(q,y_0)+1}, y_{d(q,y_0)-1}) + 4d(y_c, u).
\]

Next note that

\[
\hat{d}_v = d^*_v + d(r, v) + d(q, v) + d(y_{d(q,y_0)+1}, v) + d(y_{d(q,y_0)-1}, v)
\]

\[
= d^*_v + d(r, q) + d(y_{d(q,y_0)+1}, y_{d(q,y_0)-1}) + 4.
\]

It now follows that \( \hat{d}_r > \hat{d}_u \), completing the induction step. \( \square \)
Remark. Observe that in the proof of Theorem 3.3, the only place where we essentially used the fact that $T$ has a perfect matching occurred where it was necessary to show that if $v$ is a pendant vertex not belonging to a $p_2$-set, then the vertex adjacent to $v$ must lie on a $p_2$-path. Also observe that the proof would still be correct if we assumed $n \geq 5$ in the inductive step of the proof since inequality (3.2) would still hold. Thus, the conditions of Theorem 3.3 can be weakened to requiring (only) that $T$ be a tree in which every pendant vertex $v$ not belonging to a $p_2$-set must be adjacent to a vertex which lies on a $p_2$-path. We cannot, however, weaken the conditions of the theorem any further. Consider the following example of the tree $U$:

Observe that $v$ is a pendant vertex not belonging to a $p_2$-set and the vertex adjacent to $v$ does not lie on a $p_2$-path. Also observe that $q$ is the only pendant $p_2$-vertex in $U$. However, upon calculating $\hat{d}_v$ and $\hat{d}_q$ we see that

$$\hat{d}_v = 33 > \hat{d}_q = 32.$$ 

It should be noted that if a tree has a perfect matching, then from Lemma 3.2 we know that it is impossible for a pendant vertex not belonging to a $p_2$-set to be adjacent to a vertex that is not on some $p_2$-path. Therefore, Theorem 3.3 yields an immediate corollary concerning trees with a perfect matching:

**Corollary 3.4.** Let $T$ be a tree with a perfect matching and let $L$ be its Laplacian matrix. Then the maximal diagonal entry in the group inverse $L^\#$ of $L$ occurs at a pendant vertex belonging to a $p_2$-set.

**Proof.** According to [7, Corollary 3.4], the diagonal entry $L_{i,i}^\#$ that is the largest corresponds to the vertex $i$ for which $\hat{d}_i$ of (3.1) is largest. Since Lemma 3.2 states that in a tree with a perfect matching, the vertex adjacent to a pendant vertex not belonging to a $p_2$-set must lie on a $p_2$-path, we can apply Theorem 3.3 to see that the diagonal entry $L_{i,i}^\#$ which is maximal must occur at a pendant vertex belonging to a $p_2$-set. □

Acknowledgement

The authors would like to thank the anonymous referee for his or her many valuable suggestions for improving this paper.
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