In the quoted paper J. de Groot introduced a generalization of compactness with the purpose of establishing the Baire category theorem also in a higher cardinal version (differing from the previous such versions which are not valid in compact spaces). Unfortunately the proof of the theorem as given at the end of the paper is incorrect for higher cardinals (the argument following the third displayed equation on p. 767 cannot be justified for limit ordinals beyond the first since the sequence chosen inductively need not be decreasing, as appears to be tacitly assumed). Nevertheless the theorem as stated is correct for the first uncountable cardinal, and with a slightly strengthened hypothesis for the others as well, as the following lines will show.

A regular Hausdorff space is \textit{subcompact} relative to an open base \{U\} if every regular filter base drawn from \{U\} has non-void intersection: a \textit{regular filter base} is a non-empty collection of non-void sets every finite intersection of whose elements contains the closure of one of its elements. Of course the union of an increasing sequence of regular filter bases is again one.

The usual Baire theorem concludes that every countable intersection of dense open sets is dense. de Groot proposed replacing dense sets— which meet every non-void open subset—with \textit{puffed sets}, defined as those which meet every non-void \(G_δ\). His \(\aleph_1\) Baire theorem reads: \textit{In a subcompact space every intersection of \(\aleph_1\) puffed open subsets is non-void.} His proof
starts out to show the intersection dense: it is little more trouble to see that it is even puffed.

PROOF: It must be shown that every non-void $G_\delta$ subset $S$ has a point common to all the puffed open sets $P_\alpha$, which may be supposed indexed by the countable ordinals. This will be achieved by choosing basic $U_\alpha \subset P_\alpha$ such that those chosen up to each limit ordinal constitute a regular filter base, refining a sequence with intersection $S$, hence have non-void intersection contained in $S$.

Let $\lambda$ be a limit ordinal and $\bigcap O_\alpha$ a non-void $G_\delta$ subset of $S \cap \bigcap_{\alpha<\lambda} P_\alpha$ (for $\lambda=0$ take $\bigcap O_\alpha=S$) thus with a point in the puffed open $P_\lambda$, hence also with a basic open neighborhood of the point, say $U_\lambda \subset P_\lambda \cap O_0$. The non-void $U_\lambda \cap \bigcap O_\alpha$ has a point in $P_{\lambda+1}$ which will have a basic $U_{\lambda+1} \subset \bigcap P_{\lambda+1} \cap O_1$ and by regularity of the topology such that $U_{\lambda+1} \subset U_\lambda$. Continuing by integer induction one obtains a regular filter base for which the non-void $\bigcap U_{\lambda+1} \subset S \cap \bigcap_{\alpha<\lambda+\omega} P_\alpha$. If $\{O_\alpha\}$ had been drawn from $\{U\}$, $\{U_{\lambda+1}, O_\alpha\}$ would be a regular filter base: hence at the limit of an increasing sequence of limit ordinals one can take the union of the increasing sequence of constructed regular filter bases which, being regular, will again have non-void intersection. Finally, for the transfinite sequence of all countable limit ordinals, this yields $S \cap \bigcap P_\alpha \neq \emptyset$.

This argument cannot be carried beyond an uncountable limit ordinal. However if $\{U\}$ is closed for finite intersection, the pairwise intersections of sets from two regular filter bases, if always non-void, form a regular filter base (as follows from $U \cap U' \subset U \cap U'$); and so one could enlarge a regular filter base whose non-void intersection is in $S \cap \bigcap_{\alpha<\lambda} P_\alpha$ to a similar one for $\lambda+\omega$ even for uncountable $\lambda$ (the requirement on the $O_\alpha$ no longer being needed). It is then only necessary to take the $P_\alpha$, indexed by the ordinals $<\alpha$ cardinal $\aleph$, as open and $\aleph$-puffed in the sense that they meet every non-void intersection of fewer than $\aleph$ open sets (equivalent now to meeting just these intersections $S$ of regular filter bases) to obtain the generalized Baire theorem: In a space subcompact relative to a base closed for finite intersection, every intersection of $\aleph$ open $\aleph$-puffed subsets is $\aleph$-puffed.

It may be observed that the proof uses only the non-voidness of intersections of regular filter bases of at most $\aleph$ $U$'s. This is the analogue of de Groot's observation that countable subcompactness suffices for the ordinary Baire theorem, and should counter Zakon's reservations about the usefulness of the notion. Indeed, the above Baire theorem immediately yields the other higher cardinal generalization for the spaces in which it has been established: e.g. for the $\aleph_1$-version these are spaces in which non-void $G_\delta$ have non-void interiors (christened "almost-$P$-spaces" by Levy). Now in such a space denseness is equivalent to (rather than just being implied by) puffedness, hence in a subcompact such space intersections of $\aleph_1$ dense open sets are dense—under the stronger hypothesis
of local compactness this is proved as 6.15 in Comfort and Negrepontis (the $\aleph_1$-version already appears as Prop. 3.2 in Plank). The other setting in which this conclusion is established there may also be reduced to the above by noting that subcompactness relative to an intersection closed \{U\} is not destroyed by strengthening the topology to have $G_\delta$'s open: the intersections of countable, or in general fewer than $\aleph$, regular filter bases form a base for the strengthened topology such that every filter base (they are now all regular) drawn from it has non-void intersection – 15.8 (due to Sikorski) follows from this and the compactness of a product of two-element sets. More generally a product, topologized by the base of those products of open subsets in which fewer than $\aleph$ factors differ from the whole space, of spaces subcompact relative to bases closed for fewer than $\aleph$ intersections, enjoys the property that intersections of $\aleph$ dense open subsets are dense. Finally, that this property holds in spaces, uniformized by descending type $\aleph$ well-ordered entourage bases, in which decreasing chains of the basic neighborhoods given by the uniformity have non-void intersection (Cohen-Goffman, Zakon) again follows from the $\aleph$-subcompactness of such spaces relative to these bases coupled with the openness of intersections of fewer than $\aleph$ (basic) open sets.

The essence of the proof above could be carried out in a more abstract setting: in a partially ordered set, representing the non-void open subsets of the space in their "regular" order (i.e. the closure of the smaller contained in the larger). A puffed (open) element relative to a subset \{U\}, should now be defined as one below which there exist decreasing sequences from \{U\} refining each countable filter base from \{U\}: the conclusion is that any $\aleph_1$ puffed elements are simultaneously refined by a single filter base from \{U\}.

REFERENCES


