Uniform stabilization of the damped Cauchy–Ventcel problem with variable coefficients and dynamic boundary conditions

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Abstract

In this paper, the uniform stabilization of the Cauchy–Ventcel problem with variable coefficients is considered, and the uniform energy decay rate for the problem is established by Riemannian geometry methods. © 2006 Elsevier Inc. All rights reserved.

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1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) \((n \geq 2)\) having a boundary \( \Gamma = \partial \Omega \) of class \( C^2 \). Let \( \Gamma_0 \) and \( \Gamma_1 \) be closed and disjoint subsets of \( \Gamma \) with \( \Gamma = \Gamma_0 \cup \Gamma_1 \). We denote by \( \nabla_0 \) the gradient (respectively by \( \nabla_T \) the tangential-gradient on \( \Gamma \)) and by \( \text{div}_0 \) the divergence (respectively by \( \text{div}_0T \) the tangential-divergence on \( \Gamma \)) in the Euclidean metric. This paper is devoted to the study of the uniform stabilization of solutions of the following damped Cauchy–Ventcel problem:

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\[
\begin{align*}
\left\{ \begin{array}{ll}
utt + Au + a(x)g(u_t) = 0 & \text{in } \Omega \times \mathbb{R}^+,
\vtt + \frac{\partial u}{\partial \nu_A} + A_T v + g_2(v_t) = 0 & \text{on } \Gamma_1 \times \mathbb{R}^+,
u = 0 & \text{on } \Gamma_0 \times \mathbb{R}^+,
(u(0), v(0)) = (u_0, v_0) & \text{in } \Omega \times \Gamma,
(u_t(0), v_t(0)) = (u_1, v_1) & \text{in } \Omega \times \Gamma,
\end{array} \right.
\end{align*}
\]  

(1.1)

where
\[
a \in L^\infty(\Omega), \quad a(x) \geq a_0 > 0, \quad \text{a.e. in } \omega,
\]

and \( \omega \subset \Omega \) is an open, nonempty subset of \( \Omega \); \( a_0 \) is a constant.

\[
A = (a_{ij}) \text{ is a matrix function, } a_{ij} = a_{ji} \text{ are } C^\infty \text{ functions in } \mathbb{R}^n, \quad \frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \nu_i,
\]

\[
v = (v_1, v_2, \ldots, v_n)^T \text{ is the unit normal of } \Gamma \text{ pointing toward the exterior of } \Omega, \quad \text{and } v_A = Av.
\]

We suppose that the second-order differential operators \( A \) and \( A_T \) satisfy the uniform ellipticity condition:
\[
\sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j > \lambda \sum_{i=1}^n \xi_i^2, \quad x \in \tilde{\Omega}, \quad 0 \neq \xi = (\xi_1, \xi_2, \ldots, \xi_n)^T \in \mathbb{R}^n,
\]

(1.4)

for some constant \( \lambda > 0 \) and \( g_i : \mathbb{R} \to \mathbb{R} \) are continuous, nondecreasing functions such that \( g_i(0) = 0, \ i = 1, 2 \).

Assume further that
\[
\sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j > 0, \quad x \in \mathbb{R}^n, \quad 0 \neq \xi = (\xi_1, \xi_2, \ldots, \xi_n)^T \in \mathbb{R}^n.
\]

(1.5)

Stability for the wave equation
\[
utt - \Delta u + f(u) + a(x)g(u_t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,
\]

(1.6)

has been studied for long time by many authors. When the feedback term depends on the velocity in a linear way Zuazua [22] proved that the energy related to the above equation decays exponentially if the damping region contains a neighborhood of the boundary \( \Gamma \) or, at least, contains a neighborhood \( \omega \) of the particular part given by \( \{ x \in \Gamma : (x - x_0) \cdot v(x) \geq 0 \} \). In the same direction, but when \( f = 0 \), it is important to mention the result of Bardos et al. [2], based on microlocal analysis, that ensures a necessary and sufficient condition to obtain exponential decay, namely, the damping region satisfies the well-known geometric control condition. The classical example of an open subset \( \omega \) verifying this condition is when \( \omega \) is a neighborhood of the boundary. Later, Nakao [16,17] extended the results of Zuazua [22], again considering \( f = 0 \), treating first the case of a linear degenerate equation, and then the case of a nonlinear dissipation \( g(x, u_t) \) assuming, as usually, that the function \( g \) has a polynomial growth near the origin. Martinez [15] improved the previous results mentioned above in what concerns the linear wave equation subject to a nonlinear dissipation \( g(x, u_t) \) (here, again, \( f = 0 \) was considered), avoiding the polynomial growth of the function \( g(x, s) \) in zero. His proof is based on the piecewise multiplier technique developed by Liu [14] combined with nonlinear integral inequalities to show that the energy of the system decays to zero with a precise decay rate estimate if the damping region satisfies some...
geometrical conditions. More recently, and still considering $f = 0$, Alabeau-Boussouira [1] extended the results due to Martinez [15] by showing optimal decay rates of energy. In addition, we would like to mention the work of Cavalcanti and Oquendo [3], who showed exponential and polynomial decay rates for the partially viscoelastic nonlinear wave equation subject to a nonlinear and localized frictional damping given by

$$u_{tt} - \kappa_0 \Delta u + \int_0^t \text{div} \left[ a(x)g(t - s)\nabla u(s) \right] + f(u) + b(x)h(u_t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

where $a, b$ are nonnegative functions, $a \in C^1(\bar{\Omega})$, $b \in L^\infty(\Omega)$, under the assumption

$$a(x) + b(x) \geq \delta > 0, \quad \forall x \in \Omega. \quad (1.7)$$

We observe that assumption (1.7) gives us a wide assortment of possibilities from which to choose the functions $a(x)$ and $b(x)$, and the most interesting case occurs when one has simultaneous and complementary damping mechanisms. Taking this point of view into account, a distinctive feature of the above mentioned paper is exactly to consider different and localized damping mechanisms acting in the domain but not necessarily ‘strategically localized dissipations’ as considered in the prior literature.

In the present paper, we establish the uniform stabilization for system (1.1) with variable coefficients, by considering a nonlinear feedback $g_2(v_t)$ and a localized frictional dissipation $a(x)g_1(u_t)$ acting on the system. The strategy to solve it is to combine the method which was firstly introduced into the boundary-control problem by Yao [21] for the exactly controllability of wave equations (and subsequently by Lasiecka et al. [11] for hyperbolic equations in a general setting), the technique developed in Lions [13] and finally the method developed by Lasiecka and Tataru [10].

In our main result (Theorem 4.1), we prove that the $(V \times H)$-energy at time $t = T$, or at time $t = 0$, is dominated by the $L^2(\omega \times [0, T])$-norms of $g_1(u_t)$ and $u_t$, and the $L^2(\Sigma)$-norms of the boundary traces $g_2(v_t)$, $v_t$, modulo an interior and boundary lower-order terms. Our result yield-under a uniqueness theorem, needed to absorb the lower-order terms.

It is important to be mentioned that the localized dissipation $a(x)g_1(u_t)$ is strong enough to assure the asymptotic stability, which will be clarified during the proof. However, from the Physical point of view, frictional dissipations can occur in both situations: inside or on the boundary. Reciprocally, to consider $g_1(s) = 0$ and $g_2(s) \neq 0$ is a very hard open problem because of the dynamic boundary condition. In fact, to prove that the boundary feedback $g_2(v_t)$ is strong enough to assure the asymptotic stability remains an open problem in the literature.

The approach adopted in this paper was inspired in [19], an approach which in turn originated in [21] adapted to our case. Namely, we shall generate appropriate estimates for the energy functional $\int_0^T E(t) \, dt$, as opposed to the classical method of constructing a particular Lyapunov function for a general nonlinear equation, and subsequently proving differential inequalities with respect to this Lyapunov function.

Our paper is organized as follows. Section 2 is devoted to the Riemannian metric generated by the principal part $\mathcal{A}$. In Section 3 we introduce some notations and the statement of the problem. Our main result is stated in Section 4. Section 5 is devoted to the proof of the main result.

Before dealing with the coupled system (1.1), we will need in the following section, the background material which is due to Yao [21].
2. Riemannian metric generated by the principal part $A$

This section is devoted to the introduction of Riemannian geometric tools which will play an essential role in our computations. The results present here are verbatim the same as those introduced in Lasiecka et al. [12, Section 2], which we are repeating just for the reader’s convenience.

Recalling the coefficients $a_{ij} = a_{ji}$ of $A$, let $A(x)$ and $G(x)$ be, respectively, the coefficient matrix and its inverse

$$A(x) = (a_{ij}(x)), \quad G(x) = [A(x)]^{-1} = (g_{ij}(x)), \quad i, j = 1, \ldots, n, \ x \in \mathbb{R}^n. \quad (2.1)$$

Both $A(x)$ and $G(x)$ are $n \times n$ matrices, $A(x)$ is positive definite for any $x \in \mathbb{R}^n$ by assumption (1.5).

2.1. Riemannian metric

Let $\mathbb{R}^n$ have the usual topology and $x = [x_1, \ldots, x_n]$ be the natural coordinate system. For each $x \in \mathbb{R}^n$, define the inner product and the norm on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$g(X, Y) = (X, Y)_g = \sum_{i, j = 1}^n g_{ij}(x) \alpha_i \beta_j, \quad (2.2)$$

$$|X|_g = (X, X)_g^{1/2}, \quad \forall X = \sum_{i = 1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i = 1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \quad (2.3)$$

It is easily checked from (1.5) that $(\mathbb{R}^n, g)$ is a Riemannian manifold with the Riemannian metric $g$. We shall denote $g = \sum_{i, j = 1}^n g_{ij}(x) \, dx_i \, dx_j$. (If, $A(x) \equiv I$, i.e., $A = -\Delta$, then $G(x) \equiv I$, and $g$ is the Euclidean $\mathbb{R}^n$-metric.)

2.2. Euclidean metric

For each $x \in \mathbb{R}^n$, denote by

$$X \cdot Y = \sum_{i = 1}^n \alpha_i \beta_i, \quad |X|_0 = (X, X)_0^{1/2},$$

$$\forall X = \sum_{i = 1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i = 1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \quad (2.4)$$

the Euclidean metric on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, and with reference to (2.1), set

$$A(x)X = \sum_{i = 1}^n \left( \sum_{j = 1}^n a_{ij}(x) \alpha_j \right) \frac{\partial}{\partial x_i}, \quad \forall X = \sum_{i = 1}^n \alpha_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \quad (2.5)$$

Thus, recalling the co-normal derivative, we have

$$\frac{\partial u}{\partial \nu_A} = \sum_{i = 1}^n \left( \sum_{j = 1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \right) v_i = (A(x) \nabla_0 u) \cdot v. \quad (2.6)$$
In (2.6), and hereafter, we denote by a sub “0” entities in the Euclidean metric. Thus, for \( f \in C^1(\Omega) \) and \( X = \sum_{i=1}^{n} \alpha_i(x) \frac{\partial}{\partial x_i} \) a vector field on \( \mathbb{R}^n \),

\[
\nabla_0 f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \quad \text{and} \quad \text{div}_0(X) = \sum_{i=1}^{n} \frac{\partial \alpha_i(x)}{\partial x_i},
\]

(2.7)
denote gradient of \( f \) and divergence of \( X \) in the Euclidean metric.

2.3. Further relationships

If \( f \in C^1(\bar{\Omega}) \), we define the gradient \( \nabla g f \) of \( f \) in the Riemannian metric \( g \), via the Riesz representation theorem, by

\[
X(f) = \langle \nabla g f, X \rangle_g,
\]

(2.8)
where \( X \) is any vector field on the manifold \((\mathbb{R}^n, g)\). The following lemma provides further relationships [21, Lemma 2.1].

**Lemma 2.1.** Let \( x = [x_1, \ldots, x_n] \) be the natural coordinate system in \( \mathbb{R}^n \). Let \( f, h \in C^1(\bar{\Omega}) \). Finally, let \( H, X \) be vector fields. Then, with reference to the above notation, we have

(a) \( \{ H(x), A(x)X(x) \}_g = H(x) \cdot X(x), \ x \in \mathbb{R}^n \),

(2.9)
(b) \( \nabla g f(x) = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i} = A(x) \nabla_0 f, \ x \in \mathbb{R}^n \),

(2.10)
(c) if \( X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} \), then by (2.8) and (2.10),

\[
X(f) = \langle \nabla g f, X \rangle_g = \langle A(x) \nabla_0 f, X \rangle_g = \nabla_0 f \cdot X = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i},
\]

(2.11)
(d) by (2.6) and (2.10),

\[
\frac{\partial u}{\partial \nu_A} = (A(x) \nabla_0 u) \cdot \nu = \nabla g u \cdot \nu,
\]

(2.12)
(e) by (2.8)–(2.10),

\[
\langle \nabla g f, \nabla g h \rangle_g = \nabla g f(h) = \langle A(x) \nabla_0 f, \nabla g h \rangle_g = \nabla_0 f \cdot \nabla g h = \nabla_0 f A(x) \nabla_0 h, \ x \in \mathbb{R}^n,
\]

(2.13)
(f) if \( H \) is a vector field in \((\mathbb{R}^n, g)\) (see, e.g., (2.16)),

\[
\langle \nabla g f, \nabla g (H(f)) \rangle_g = DH \langle \nabla g f, \nabla g f \rangle_g + \frac{1}{2} \text{div}_0(|\nabla g f|^2_g H)(x) - \frac{1}{2} (|\nabla g f|_g^2(x))(\text{div}_0 H)(x), \ x \in \mathbb{R}^n,
\]

(2.14)
where \( DH \) is the covariant differential discussed below.
(g) by (1.1), (2.7), (2.10),
\[ Au = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = -\text{div}_0 (A(x)\nabla_0 u) \]
\[ = -\text{div}_0 (\nabla_g u), \quad u \in C^2(\Omega). \quad (2.15) \]

2.4. Covariant differential

Denote the Levi-Civita connection in the Riemannian metric \( g \) by \( D \). Let
\[ H = \sum_{k=1}^{n} h_k \frac{\partial}{\partial x_k}, \quad X = \sum_{k=1}^{n} \xi_k \frac{\partial}{\partial x_k} \]
be vector fields on \((\mathbb{R}^n, g)\). The covariant differential \( DH \) of \( H \) determines a bilinear form on \( \mathbb{R}_x^n \times \mathbb{R}_x^n \), for each \( x \in \mathbb{R}^n \), defined by
\[ DH(Y, X) = \langle DXH, Y \rangle_g, \quad \forall X, Y \in \mathbb{R}_x^n, \quad (2.17) \]
where \( DXH \) is the covariant derivative of \( H \) with respect to \( X \).

Let \( H \) be a vector field on \( \mathbb{R}^n \) and \( f \in C^1(\overline{\Omega}) \). We have the formulae for divergence in the Euclidean metric
\[ \text{div}_0(fH) = f \text{div}_0(H) + H(f) \quad (2.18) \]
and
\[ \int_{\Omega} \text{div}_0(H) \, dx = \int_{\Gamma} H \cdot \nu \, d\gamma. \quad (2.19) \]

3. Statement of problem

Let \( \Omega \) be a bounded open, connected subset in \( \mathbb{R}^n \) \((n \geq 2)\), with \( C^2 \) boundary \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), and with both \( \Gamma_i \), \( i = 0, 1 \), being closed and disjoint. In this paper, we investigate the stability properties of functions \([u(x, t), u_t(x, t)]\) and \([v(x, t), v_t(x, t)]\) which solve the following coupled system consisting of a damped Cauchy–Ventcel problem:
\[ \begin{cases} 
    u_{tt} + Au + a(x)g_1(u_t) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\
    v_{tt} + \frac{\partial u_t}{\partial A} + A_T v + g_2(v_t) = 0 & \text{on } \Gamma_1 \times \mathbb{R}_+, \\
    u = v & \text{on } \Gamma \times \mathbb{R}_+, \\
    u = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+, \\
    (u(0), v(0)) = (u^0, v^0) & \text{in } \Omega \times \Gamma, \\
    (u_t(0), v_t(0)) = (u^1, v^1) & \text{in } \Omega \times \Gamma, 
\end{cases} \quad (3.1) \]
where the \( g_i \in C^1(\mathbb{R}) \) are functions which satisfy the following assumptions for \( i = 1, 2 \):

(H.1) (i) \( g_i(s) \) are continuous and monotone increasing,
(ii) \( g_i(s)s > 0 \) for \( s \neq 0 \),
(iii) \( m_i s \leq g_i(s) \leq M_i s \) for \( |s| > 1 \),
where \( m_i \) and \( M_i \) are positive constants.
We assume
\[ a \in L^\infty(\Omega), \quad a(x) \geq a_0 > 0, \quad \text{a.e. in } \omega, \]
where \( \omega \subset \Omega \) is an open, nonempty subset of \( \Omega \); \( a_0 \) is a constant.

The coupled system is a version (with variable coefficients and nonlinear feedback) of the Cauchy–Ventcel model derived by Lemrabet [7] to describe the asymptotic vibrations of an elastic body with a thin of high rigidity on its boundary. See also [5,8,9].

In addition, to obtain the boundary stabilization of problem (3.1), we shall need the following geometrical assumptions:

(H.2) There exists a vector field \( H \) on the Riemannian manifold \((\mathbb{R}^n, g)\) such that
\[ DH(X, X) \geq b|X|^2_g, \quad \forall X \in \mathbb{R}^n, \quad x \in \tilde{\Omega}, \]
for some constant \( b > 0 \).

(H.3) We assume that \( \omega \) is a neighborhood of \( \Gamma_1 \), where
\[ \Gamma_1 := \{ x \in \Gamma; \ H(x) \cdot \nu(x) > 0 \}. \]

Remark 3.1. (1) The existence of vector field \( H \) in (H.2) has been proved in [21], where some examples are given, too. In particular, if \( a_{ij} = \delta_{ij} \), we have \( H = x - x_0 \).

(2) The growth condition in (H.1) is imposed on \( g_i \) for large value of \( |s| \), but is not necessary near the origin.

As an example of a domain \( \Omega \) satisfying the above assumptions let us consider Fig. 1.

Remark 3.2. The assumption of a not simply connected region, that is, \( \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset \), as in Fig. 1, is dedicated in view of necessity of regularity when dealing with regular solutions.

In the sequel we define by \( Q = \Omega \times ]0, T[ \), \( \omega_T = \omega \times ]0, T[ \), \( \Sigma = \Gamma \times ]0, T[ \), \( \Sigma_i = \Gamma_i \times ]0, T[ \), \( i = 0, 1 \).

We set
\[ H^{1}_{\Gamma_0}(\Omega) = \{ u \in H^1(\Omega)/u|_{\Gamma_0} = 0 \}, \]
\[ V = \{ z = (u, v) \in H^{1}_{\Gamma_0}(\Omega) \times H^1(\Gamma)/u|_\Gamma = v \}, \]
\[ H = L^2(\Omega) \times L^2(\Gamma). \]

![Fig. 1.](image-url)
Equipped with the canonical norms
\[ |z|^2_H = |u|^2_{L^2(\Omega)} + |v|^2_{L^2(\Gamma)}, \]
\[ \|z\|^2_V = \|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Gamma)}, \]
\( V \) and \( H \) are two Hilbert spaces and \( V \) is dense in \( H \) with continuous injection.
We define the operator \( A \) by
\[ A = \begin{pmatrix} -A_0 & 0 \\ -\frac{\partial}{\partial \nu} & -A_T \end{pmatrix}, \]
and the domain of \( A \) by
\[ D_A = \{(u,v) \in V, A(u,v) \in H\}. \]

Before dealing with the coupled system (3.1), let us consider the linear version of problem (3.1) which is written as
\[ \Phi_{tt} - A \Phi = F, \quad (3.3) \]
\[ \Phi(0) = \Phi^0, \quad (3.4) \]
\[ \Phi_t(0) = \Phi^1, \quad (3.5) \]
where \( F(x,t) = (f_1(x,t), f_2(x,t))^T \), and for which we will need the following background material.

For \( \Phi = (u, v) \), where \( u : \Omega \to \mathbb{R} \) and \( v : \Gamma \to \mathbb{R} \), we define the gradient of \( \Phi \) in the Riemannian metric \( g \) by \( \text{Grad} \Phi = (\nabla g u, (\nabla_T g)v) \) and for \( \Psi = (\psi_1, \psi_2) \), where \( \psi_1 : \Omega \to \mathbb{R}^n \) and \( \psi_2 : \Gamma \to \mathbb{R}^n \), we define the divergence of \( \Psi \) by \( \text{Div} \Psi = (\text{div}_0(\psi_1), \text{div}_0 T(\psi_2)) \).

**Proposition 3.1.** (i) Under the conditions above, problem (3.3)–(3.5) is well posed in the space \( V \times H \), i.e., for any initial data \( \{\Phi^0, \Phi^1\} \in V \times H \), and \( F \in L^2(0,T; H) \), there exists a unique weak solution of (3.3)–(3.5) in the class
\[ (u,v) \in C^0(0,T; V) \cap C^1(0,T; H). \]

(ii) There exists a \( C > 0 \), such that
\[ \|\Phi_t\|_{L^\infty(0,T; H)} + \|\Phi\|_{L^\infty(0,T; V)} \leq C \left[ \|\Phi^0\|_V + \|\Phi^1\|_H + \|F\|_{L^1(0,T; H)} \right]. \]

**Proof.** For the existence of solutions we use the Galerkin or Semigroup method and the following equality of energy
\[ \frac{d}{dt} E(t) = (F(t), \Phi_t(t))_H, \]
where
\[ E(t) = \frac{1}{2} \left[ \|\Phi_t\|^2_H + \|\text{Grad} \Phi\|^2_H \right] \]
\[ = \frac{1}{2} \left\{ \int_\Omega [u_t]^2 + |\nabla g u|^2 \, dx + \int_{\Gamma} [v_t]^2 + |(\nabla_T g)v|^2 \, d\gamma \right\}. \]
Proposition 3.2. For any initial data \( \{ \Phi^0, \Phi^1 \} \in D_A \times \mathbb{V} \), and \( F \in L^2(0, T; \mathbb{V}) \), there exists a unique weak solution of (3.3) in the class

\[
(u, v) \in C^0(0, T; D_A) \cap C^1(0, T; \mathbb{V}).
\]

Furthermore, there exist a \( C > 0 \), such that

\[
\| \phi \|_{L^\infty(0,T;\mathbb{V})} + \| \phi \|_{L^\infty(0,T;D_A)} \leq C [\| \phi^0 \|_{D_A} + \| \phi^1 \|_{\mathbb{V}} + \| F \|_{L^1(0,T;\mathbb{V})}].
\]

In this case, the equality of energy is given by

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \text{Grad} \phi_t \|_{\mathbb{H}}^2 + \| A \phi_t \|_{\mathbb{H}}^2 \right] = \langle \text{Grad} F(t), \text{Grad} \phi_t(t) \rangle_{\mathbb{H}}.
\]

Proof. See Lemrabet [8] and references therein. □

Remark 3.3. When \( \Omega \) is of class \( C^2 \), \( D_A = H^2(\Omega) \times H^2(\Gamma) \) see Lemrabet [8].

We observe that the problem (3.1) can be written in the following form:

\[
\frac{dU}{dt} + AU = G(U),
\]

where

\[
A = \begin{pmatrix} 0 & -I \\ -A & 0 \end{pmatrix}
\]

is a maximal monotone operator and \( G(\cdot) \) represents a locally Lipschitz perturbation. So, making use of standard semigroup arguments we have the following result:

Theorem 3.1. (i) Under the conditions above, problem (3.1) is well posed in the space \( \mathbb{V} \times \mathbb{H} \), i.e., for any initial data \( \{ u^0, v^0, u^1, v^1 \} \in \mathbb{V} \times \mathbb{H} \), there exists a unique weak solution of (3.1) in the class

\[
(u, v) \in C(\mathbb{R}^+; \mathbb{V}) \cap C^1(\mathbb{R}^+; \mathbb{H}).
\]

(ii) In addition, the velocity terms of the solution have the following regularity:

\[
(u_t, v_t) \in L^2_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)) \times L^2_{\text{loc}}(\mathbb{R}^+; L^2(\Gamma))
\]

(Consequently, \( (g_1(u_t), g_2(v_t)) \) \( L^2_{\text{loc}}(\mathbb{R}^+; L^2(\Omega)) \times L^2_{\text{loc}}(\mathbb{R}^+; L^2(\Gamma)) \) by (H.1)(iii)). Furthermore, if \( \{ u^0, u^1, v^0, v^1 \} \in \{ H^2(\Omega) \cap H^1_{0} (\Omega) \times H^1_{0} (\Omega) \} \times \{ H^2(\Gamma) \times H^1(\Gamma) \} \) and \( g_i, i = 1, 2, \) are globally Lipschitz continuous, then the solution has the following regularity:

\[
(u, v) \in L^\infty(\mathbb{R}^+; H^2(\Omega)) \cap W^{1,\infty}(\mathbb{R}^+; H^1_{0} (\Omega)) \cap W^{2,\infty}(\mathbb{R}^+; L^2(\Omega))
\]

\[
\times L^\infty(\mathbb{R}^+; H^2(\Gamma)) \cap W^{1,\infty}(\mathbb{R}^+; H^1(\Gamma)) \cap W^{2,\infty}(\mathbb{R}^+; L^2(\Gamma)).
\]

Suppose that \( (u, v) \) is the unique global weak solution of problem (3.1), we define the corresponding energy functional by

\[
E(t) = \frac{1}{2} \left\{ \int_{\Omega} |u_t|^2 + |\nabla g u|^2 \, dx + \int_{\Gamma} |v_t|^2 + |(\nabla T)_v|^2 \, d\gamma \right\}.
\]

(3.12)
For every solution of (3.1), in the class (3.10), the following identity holds:

\[
E(t_2) - E(t_1) = - \left[ \int_{t_1}^{t_2} \int_{\omega} a(x) g_1(u_t(x,t)) \, dx \, dt + \int_{t_1}^{t_2} \int_{\Gamma} g_2(v_t) v_t(x,t) \, d\gamma \, dt \right],
\]

\(\forall t_2 > t_1 \geq 0,\)  \(\text{(3.13)}\)

and therefore the energy is a nonincreasing function of the time variable \(t.\)

4. Main result

Before stating our stability result, we will define some needed functions. For this purpose we are following the ideas firstly introduced in Lasiecka and Tataru \([10]\). For the reader’s comprehension, we will repeat them briefly. Let \(h\) be defined by

\[
h(x) = h_1(x) + h_2(x),
\]

where the \(h_i\) are concave, strictly increasing functions, with \(h_i(0) = 0, i = 1, 2,\) and such that

\[
h_i (s g_i(s)) \geq s^2 + g_i^2(s), \quad \text{for } |s| \leq 1.
\]  \(\text{(4.1)}\)

Note that such function can be straightforwardly constructed, given the hypotheses on the \(g_i\) in (H.1). With those functions, we define

\[
r(x) = h \left( \frac{x}{N_1} \right),
\]  \(\text{(4.2)}\)

where \(N_1 = \max\{\text{meas}(\Sigma_1), \text{meas}(\omega_T)\}. As r is monotone increasing, then \(cI + r\) is invertible for all \(c \geq 0.\) For \(K\) a positive constant, we then set

\[
p(x) = (cI + r)^{-1}(Kx),
\]  \(\text{(4.3)}\)

the function \(p\) is easily seen to be positive, continuous and strictly increasing with \(p(0) = 0.\) Finally, let

\[
q(x) = x - (I + p)^{-1}(x).
\]  \(\text{(4.4)}\)

We can now proceed to state our stability result.

**Theorem 4.1.** Assume that the hypotheses (H.1)–(H.3) are in place. Let \((u, v)\) be the weak solution of the coupled system (3.1). With the energy \(E(t)\) as defined in (3.12), there then exists a \(T_0 > 0\) such that

\[
E(t) \leq S \left( \frac{t}{T_0} - 1 \right), \quad \forall t > T_0,
\]  \(\text{(4.5)}\)

with \(\lim_{t \to \infty} S(t) = 0,\) where the contraction semigroup \(S(t)\) is the solution of the differential equation

\[
\frac{d}{dt} S(t) + q(S(t)) = 0, \quad S(0) = E(0)
\]  \(\text{(4.6)}\)

(where \(q\) is as given in (4.4)). Here, the constant \(K\) (from definition (4.3)) will depend on \(E(0)\) and time \(T_0,\) and the constant \(c\) (from definition (4.3)) is taken here to be \(c = ((m_1^{-1} + M_1)/a_0 + m_2^{-1} + M_2)/N_1.\)
Remark 4.1. If the feedbacks are linear, e.g., $g_1(u_t) = u_t$ and $g_2(v_t) = v_t$, then, under the same assumptions as in Theorem 4.1, we have that the energy of problem (3.1) decays exponentially with respect to the initial energy, e.g., there exist two positive constants $C > 0$ and $k > 0$ such that

$$E(t) \leq Ce^{-kt} E(0), \quad t > 0. \tag{4.7}$$

5. Proof of main result

5.1. Preliminaries

We collect, below, some few formulas to be invoked in the sequel.

5.1.1. Some notations and results

Let $x \in \Gamma$; we denote by $T_x(\Gamma)$ the tangent plane at $x$ on $\Gamma$, $\pi(x)$ the orthogonal projection on $T_x(\Gamma)$ and, for a given vector field $v$, we will write

$$\forall x \in \Gamma, \quad v(x) = v_T(x) + v_v(x)v(x),$$

with

$$v_T(x) = \pi(x)v(x), \quad v_v(x) = v(x) \cdot v(x).$$

We denote by $\partial_T$ (respectively $\partial_v$) the tangential (respectively normal) derivative. If $v$ is some regular function, the transposed vector of $\partial_T v$, denoted by $\partial_T^T v$, is the tangential gradient of $v$ and is denoted by $\nabla_T v$. So, we have

$$\nabla v = \nabla_T v + \partial_v v v \quad \text{on} \; \Gamma.$$ 

Lemma 5.1.1. Let $f$ be a function of class $C^2(\overline{\Omega})$ defined on $\Gamma$; then

$$\pi \partial_T (\partial_T f) = \pi \partial_T (\partial_T f), \tag{5.1}$$

where the barre denotes the transposed of a vector.

Proof. See Lemrabet [7]. \qed

Lemma 5.1.2. If $u_T$ and $v_T$ are two tangent vector fields of class $C^1$ defined on $\Gamma$, we have:

$$\partial_T (u_T \cdot v_T) = u_T \partial_T (v_T) + v_T \pi (\partial_T u_T). \tag{5.2}$$

Lemma 5.1.3. Let $f$ be a function of class $C^2(\overline{\Omega})$ and $q_T$ a tangent field of class $C^1$ defined on $\Gamma$; then

$$\nabla_T f \nabla_T (\nabla_T f \cdot q_T) = \nabla_T f (\pi(\partial_T q_T)) (\nabla_T f) + \frac{1}{2} \partial_T (|\nabla_T f|^2) q_T, \tag{5.3}$$

where $\nabla_T$ is the tangential gradient.

Proof. We have

$$\nabla_T f \nabla_T (\nabla_T f \cdot q_T) = (\partial_T f) (\partial_T (\partial_T f) q_T) = \partial_T ((\partial_T f) \cdot q_T) \partial_T f.$$
From Lemma 5.1.1, we have
\[ \partial T f (\partial T f \cdot q_T) (\partial T f) = (\partial T f) (\pi \partial T q_T) + \frac{1}{2} \partial T (|\nabla_T f|^2) q_T. \]

So,
\[ \nabla_T f \nabla_T (\nabla_T f \cdot q_T) = (\partial T f) (\pi \partial T q_T) (\partial T f) + \frac{1}{2} \partial T (|\nabla_T f|^2) q_T. \]

From Lemma 5.1.1, we have
\[ \pi \partial T (\partial T f) = \pi \partial T (\partial T f) \]
thus
\[ \nabla_T f \nabla_T (\nabla_T f \cdot q_T) = (\partial T f) (\pi \partial T q_T) (\partial T f) + \frac{1}{2} \partial T (|\nabla_T f|^2) q_T. \]

Finally from Lemma 5.1.1, we obtain
\[ \partial T (|\nabla_T f|^2) = \partial T (\partial T f \cdot \overline{\partial T f}) = 2(\partial T f) (\pi \partial T (\overline{\partial T f})). \]

Then
\[ \nabla_T f \nabla_T (\nabla_T f \cdot q_T) = (\partial T f) (\pi \partial T q_T) (\partial T f) + \frac{1}{2} \partial T (|\nabla_T f|^2) q_T. \]

Lemma 5.1.4. If \( u \in C^1(\Gamma) \) and \( q_T \in [C^2(\overline{\Omega})]^n \) are a function and a tangent vector field to \( \Gamma \), then we have the following Stokes formulae (see Nedelec [18]):
\[ \int_{\Gamma} (\nabla_T u \cdot q_T) \, d\gamma + \int_{\Gamma} u \text{div}_T q_T \, d\gamma = 0. \] (5.4)

The proof of Theorem 4.1 proceeds through several steps.

5.2. An identity

We began by proving the following proposition.

Proposition 5.2.1. Let \( \Omega \) be a bounded, open, connected set in \( \mathbb{R}^n \) (\( n \geq 2 \)) having a boundary \( \Gamma = \partial \Omega \) of class \( C^2 \) and \( H \in (C^1(\overline{\Omega}))^n \), with \( H = H_T + (H \cdot \nu) \nu \). Then, for every weak solution \((u, v)\) of (3.1) we have the following identity:
\[ \frac{1}{2} \int_0^T \int_{\Gamma_0} \frac{|\nabla_T u|^2}{|\nu_\mathcal{A}|^2} \, d\gamma + \frac{1}{2} \int_0^T \int_{\Gamma_1} (H \cdot \nu) \left[ |v_t|^2 - |(\nabla_T g)_{\nu}\nu|^2 + \frac{1}{|\nu_\mathcal{A}|^2} \frac{\partial u}{\partial \nu_\mathcal{A}} \right] \, d\gamma + \frac{1}{2} \int_0^T \int_{\Omega} DH (\nabla g u, \nabla g u) \, dx \, dt \]
\[ + \int_0^T \int_{\Gamma_1} [ (\nabla_T g)v] (\pi \partial_T HT) \left[ (\nabla_T g)v \right] d\gamma dt + \frac{1}{2} \int_0^T \int_{\Omega} (\text{div}_0 H) \left( u_t^2 - |\nabla g u|^2_g \right) dx dt \\
+ \frac{1}{2} \int_0^T \int_{\Gamma_1} (\text{div}_0 T HT) \left\{ |v_t|^2 - |(\nabla_T g)v|^2_g \right\} d\gamma dt + \int_0^T \int_{\omega} a(x) g_1(u_t) H(u) dx dt \\
+ \int_0^T \int_{\Gamma_1} g_2(v_t) H_T(v) d\gamma dt, \quad (5.5) \]

where \( H_T \) is the tangential component of \( H \) and \( (\nabla_T g)v \) are the components of tangential gradient of \( v \).

**Proof.** Multiplying the first equation of (3.1) by the multiplier \( H(u) \) and integrating on \( \Omega \times ]0, T[ \), we obtain the following lemma that linear version is due to Yao [21, Proposition 2.1, Part 1]. \( \square \)

**Lemma 5.2.1.** Let \( u \) be a solution of the following problem:

\[ u_{tt} + Au + a(x) g_1(u_t) = 0, \quad \text{in} \; \Omega \times ]0, T[. \]

Let \( H \) be a vector field on \( \overline{\Omega} \). Then

\[ \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \nu} H(u) d\gamma dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} (u_t^2 - |\nabla g u|^2_g) H \cdot \nu d\gamma dt = \int_\Omega u_t H(u) dx \bigg|_0^T + \int_0^T \int_{\Omega} DH(\nabla g u, \nabla g u) dx dt \\
+ \frac{1}{2} \int_0^T \int_{\Omega} (u_t^2 - |\nabla g u|^2_g) \text{div}_0 H dx dt + \int_0^T \int_{\omega} a(x) g_1(u_t) H(u) dx dt. \quad (5.6) \]

We have also for the boundary condition the following lemma.

**Lemma 5.2.2.** Assume that \( (u, v) \) is a solution of the problem (3.1). Then for the boundary condition we have

\[ - \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \nu} H_T(v) d\gamma dt = \left[ \int_{\Gamma_1} v_t H_T(v) d\gamma \right]_0^T \\
+ \int_0^T \int_{\Gamma_1} [(\nabla_T g)v] (\pi \partial_T HT) \left[ (\nabla_T g)v \right] d\gamma dt. \]
\[
\frac{1}{2} \int_0^T \int_{\Gamma_1} (\text{div}_{0T} H_T)[|v_t|^2 - |(\nabla_T)_g v|^2_g] \, d\gamma \, dt \\
+ \int_0^T \int_{\Gamma_1} g_2(v_t) H_T(v) \, d\gamma \, dt.
\] (5.7)

**Proof.** We work with regular solutions and by density arguments our result follows for weak solutions. Multiplying the second equation of (3.1) by the multiplier \( H_T(v) \), integrating on \( \Gamma_1 \times [0, T] \), we obtain

\[
\int_0^T \int_{\Gamma_1} v_{tt} H_T(v) \, d\gamma \, dt + \int_0^T \int_{\Gamma_1} \left( \frac{\partial u}{\partial v_A} + A_T v + g_2(v_t) \right) H_T(v) \, d\gamma \, dt = 0.
\] (5.8)

For the first integral of the left-hand side of (5.8) we obtain,

\[
\int_0^T \int_{\Gamma_1} v_{tt} H_T(v) \, d\gamma \, dt = \left[ \int_{\Gamma_1} H_T(v) v_t \, d\gamma \right]^T_0 + \frac{1}{2} \int_0^T \int_{\Gamma_1} v_t^2 \text{div}_{0T} H_T \, d\gamma \, dt,
\] (5.9)

where \( H_T(v) = \langle (\nabla_T)_g v, H_T \rangle_g = H_T \cdot \nabla_T v \). Indeed, integrating by parts in \( t \), and recalling that \( H_T \) is time-independent, we compute

\[
\int_0^T \int_{\Gamma_1} v_{tt} H_T(v) \, d\gamma \, dt = \left[ \int_{\Gamma_1} v_t H_T(v) \, d\gamma \right]^T_0 - \int_0^T \int_{\Gamma_1} v_t H_T(v_t) \, d\gamma \, dt.
\] (5.10)

Now the last term in (5.10), where \( H_T(v_t) = H_T \cdot \nabla_T v_t \), is rewritten, by the usual formula for divergence, as

\[
\int_0^T \int_{\Gamma_1} v_t H_T(v_t) \, d\gamma \, dt = \frac{1}{2} \int_0^T \int_{\Gamma_1} H_T(v_t^2) \, d\gamma \, dt = \frac{1}{2} \int_0^T \int_{\Gamma_1} H_T \cdot \nabla_T (v_t^2) \, d\gamma \, dt
\]

\[
= \frac{1}{2} \int_0^T \int_{\Gamma_1} v_t^2 H_T \cdot v \, d\gamma \, dt - \frac{1}{2} \int_0^T \int_{\Gamma_1} v_t^2 \text{div}_{0T} H_T \, d\gamma \, dt.
\] (5.11)

Using (5.11) in (5.10) and the fact that \( H_T \cdot v = 0 \) yields (5.9), as desired.

For the second integral of the left-hand side of (5.8), taking (5.3) and Lemma 5.1.4 into account, we deduce

\[
\int_0^T \int_{\Gamma_1} \left( \frac{\partial u}{\partial v_A} + A_T v + g_2(v_t) \right) H_T(v) \, d\gamma \, dt
\]

\[
= \int_0^T \int_{\Gamma_1} (\nabla_T)_g v (\nabla_T)_g [H_T(v)] \, d\gamma \, dt + \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial v_A} H_T(v) \, d\gamma \, dt
\]
\[ + \int_0^T \int_{\Gamma_1} g_2(v_t) H_T(v) \, d\gamma \, dt \]

\[ = \int_0^T \int_{\Gamma_1} \left( \nabla_{T} g \right)v \left( \pi \partial_T H_T \right) \left( \nabla_{T} g \right)v \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} \partial_T \left( \left( \nabla_{T} g \right)v \right)^2 \, H_T \, d\gamma \, dt \]

\[ + \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial v_A} H_T(v) \, d\gamma \, dt + \int_0^T \int_{\Gamma_1} g_2(v_t) H_T(v) \, d\gamma \, dt \]

\[ = \int_0^T \int_{\Gamma_1} \left( \nabla_{T} g \right)v \left( \pi \partial_T H_T \right) \left( \nabla_{T} g \right)v \, d\gamma \, dt - \frac{1}{2} \int_0^T \int_{\Gamma_1} \text{div}_0 T |(\nabla_{T} g)v|^2 g \, d\gamma \, dt \]

\[ + \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial v_A} H_T(v) \, d\gamma \, dt + \int_0^T \int_{\Gamma_1} g_2(v_t) H_T(v) \, d\gamma \, dt. \quad (5.12) \]

Combining (5.9) and (5.12), we deduce (5.7). So, Lemma 5.2.2 is proved. \( \Box \)

**Proof of Proposition 5.2.1.** Now adding (5.6) and (5.7), we obtain

\[ \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial v_A} H(u) \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma} \left( u_t^2 - |\nabla g u|^2 g \right) H \cdot v \, d\gamma \, dt - \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial v_A} H_T(v) \, d\gamma \, dt \]

\[ = \int_{\Omega} u_t H(u) \, dx \bigg|_0^T + \int_0^T \int_{\Gamma_1} v_t H_T(v) \, d\gamma \bigg|_0^T + \int_0^T \int_{\Omega} D H(\nabla g u, \nabla g u) \, dx \, dt \]

\[ + \int_0^T \int_{\Gamma_1} \left( \nabla_{T} g \right)v \left( \pi \partial_T H_T \right) \left( \nabla_{T} g \right)v \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Omega} \left( \text{div}_0 T \right) (u_t^2 - |\nabla g u|^2 g) \, dx \, dt \]

\[ + \frac{1}{2} \int_0^T \int_{\Gamma_1} \left( \text{div}_0 T \right) |v_t|^2 - |(\nabla_{T} g)v|^2 g \, d\gamma \, dt + \int_0^T \int_{\omega} a(x) g_1(u_t) H(u) \, dx \, dt \]

\[ + \int_0^T \int_{\Gamma_1} g_2(v_t) H_T(v) \, d\gamma \, dt. \quad (5.13) \]

We decompose the left-hand side of (5.13) in two integrals on \( \Gamma_0 \) and \( \Gamma_1 \), respectively,
\[
E = \int_0^T \int_{\Gamma_0} \frac{\partial u}{\partial \nu} H(u) \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} \left( u_t^2 - |\nabla_g u|_g^2 \right) H \cdot v \, d\gamma \, dt
\]
\[
+ \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \nu} H(u) \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} \left( u_t^2 - |\nabla_g u|_g^2 \right) H \cdot v \, d\gamma \, dt
\]
\[
- \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \nu} H_T(v) \, d\gamma \, dt.
\] (5.14)

On \( \Gamma \), we have
\[
|\nabla_g u|_g^2 = |(\nabla_T)_g u|_g^2 + \frac{1}{|\nu_A|_g^2} \left| \frac{\partial u}{\partial \nu_A} \right|_g^2
\]
and
\[
\frac{\partial u}{\partial \nu_A} H(u) = \frac{\partial u}{\partial \nu_A} H_T(u) + \frac{1}{|\nu_A|_g^2} \left| \frac{\partial u}{\partial \nu_A} \right|_g^2 H \cdot v.
\]

Since \( u = 0 \) on \( \Gamma_0 \), by Yao \[21\], it holds that \( \frac{\partial u}{\partial \nu_A} H(u) = |\nabla_g u|_g^2 H \cdot v \). Having this in mind and since \( u_t = 0 \) on \( \Gamma_0 \), we deduce that on \( \Gamma_0 \)
\[
\int_0^T \int_{\Gamma_0} \frac{\partial u}{\partial \nu_A} H(u) \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} \left( u_t^2 - |\nabla_g u|_g^2 \right) H \cdot v \, d\gamma \, dt
\]
\[
= \frac{1}{2} \int_0^T \int_{\Gamma_0} |\nabla_g u|_g^2 H \cdot v \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_0} \frac{H \cdot v}{|\nu_A|_g^2} \left| \frac{\partial u}{\partial \nu_A} \right|_g^2 \, d\gamma \, dt.
\] (5.15)

On \( \Gamma_1 \), we obtain
\[
\int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} H(u) \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} \left( u_t^2 - |\nabla_g u|_g^2 \right) H \cdot v \, d\gamma \, dt - \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \nu_A} H_T(v) \, d\gamma \, dt
\]
\[
= \frac{1}{2} \int_0^T \int_{\Gamma_1} (H \cdot v) \left[ |v_t|^2 - |(\nabla_T)_g v|_g^2 + \frac{1}{|\nu_A|_g^2} \left| \frac{\partial u}{\partial \nu_A} \right|_g^2 \right] \, d\gamma \, dt.
\] (5.16)

Inserting (5.15) and (5.16) into (5.13), we obtain (5.5). \( \square \)

We have the following identity

**Lemma 5.2.3.** Let \((u, v)\) be a solution of the problem (3.1) and \( P \in C^2(\overline{\Omega}) \). Then
\[
\int_0^T \int_\Omega P \left[ u_t^2 - |\nabla_g u|_g^2 \right] \, dx \, dt + \int_0^T \int_{\Gamma_1} P \left[ v_t^2 - |(\nabla_T)_g v|_g^2 \right] \, d\gamma \, dt
\]
\begin{align*}
&= (u_t, u P)_{\Omega} + (v_t, v P)_{\Gamma_1} + \frac{1}{2} \int_0^T \int_\Omega u^2 \nabla g P \cdot v \, dx \, dt + \frac{1}{2} \int_0^T \int_\Omega u^2 AP \, dx \, dt \\
&\quad + \frac{1}{2} \int_0^T \int_{\Gamma_1} v^2 (\nabla_T g) P \cdot v \, d\gamma \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} v^2 A_T P \, d\gamma \, dt \\
&\quad + \int_0^T \int_{\Omega} a(x) g_1 (u_t) u P \, dx \, dt + \int_0^T \int_{\Gamma_1} g_2 (v_t) v P \, d\gamma \, dt.
\end{align*}

**Proof.** We use here the same technique as in Yao [21, Proposition 2.1, part 2] applied to our context. From Lemma 2.1, we have

\begin{align*}
A_P &= - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n a_{ij}(x) \frac{\partial P}{\partial x_j} \right) = - \text{div}_0 \left( A(x) \nabla_0 P \right), \\
A_T P &= - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n a_{ij}(x) \frac{\partial P}{\partial x_j} \right) = - \text{div}_T \left( A(x) \nabla_T P \right), \quad P \in C^2(\bar{\Omega}).
\end{align*}

(5.18)

From (5.18) and formula (2.18), we deduce

\begin{align*}
\langle \nabla g u, \nabla g (P u) \rangle _g (x) &= P \langle \nabla g u, \nabla g (P u) \rangle _g (x) + u (\nabla g u, \nabla g P) _g (x) \\
&= P \langle \nabla g u, \nabla g (P u) \rangle _g (x) + \frac{1}{2} \nabla g P (u^2) \\
&= P \langle \nabla g u, \nabla g (P u) \rangle _g (x) + \frac{1}{2} \text{div}_0 (u^2 \nabla g P) + \frac{1}{2} u^2 AP, \\
\langle (\nabla_T g) v, (\nabla_T g) (P v) \rangle _g (x) &= P \langle (\nabla_T g) v, (\nabla_T g) (P v) \rangle _g (x) + v (\nabla_T g v, (\nabla_T g) P) _g (x) \\
&= P \langle (\nabla_T g) v, (\nabla_T g) (P v) \rangle _g (x) + \frac{1}{2} (\nabla_T g) P (v^2) \\
&= P \langle (\nabla_T g) v, (\nabla_T g) (P v) \rangle _g (x) + \frac{1}{2} \text{div}_T (v^2 (\nabla_T g) P) + \frac{1}{2} v^2 A_T P.
\end{align*}

(5.19)

It follows from (3.1), (5.19), (2.19), and Green’s formula that

\begin{align*}
(u_t, u P)_{\Omega} + (v_t, v P)_{\Gamma_1}
&= \int_0^T \left\{ (u_{tt}, u P)_{\Omega} + (u_t, u P)_{\Omega} + (v_{tt}, v P)_{\Gamma_1} + (v_t, v P)_{\Gamma_1} \right\} \, dt \\
&= \int_0^T \int_{\Omega} \left[ -\langle \nabla g u, \nabla g (P u) \rangle _g (x) + u_{tt}^2 P \right] \, dx \, dt \\
&\quad + \int_0^T \int_{\Gamma_1} \left[ -\langle (\nabla_T g) v, (\nabla_T g) (P v) \rangle _g (x) + v_{tt}^2 P \right] \, d\gamma \, dt.
\end{align*}
\[ + \int_0^T \int_\omega a(x)g_1(u_t)u \, dP \, dx \, dt + \int_0^T g_2(v_t) v \, P \, d\gamma \, dt \]
\[ = \int_0^T \int_\Omega P \left[ u_t^2 - |\nabla g u|_g^2 \right] \, dx \, dt + \int_0^T \int_{\Gamma_1} P \left[ v_t^2 - |(\nabla_T)_g v|_g^2 \right] \, d\gamma \, dt \]
\[ - \frac{1}{2} \int_0^T \int_\Omega u^2 \nabla g P \cdot v \, dx \, dt - \frac{1}{2} \int_0^T \int_\Omega u^2 A P \, dx \, dt \]
\[ - \frac{1}{2} \int_0^T \int_{\Gamma_1} v^2 (\nabla_T)_g P \cdot v \, d\gamma \, dt - \frac{1}{2} \int_0^T \int_{\Gamma_1} v^2 A_T P \, d\gamma \, dt \]
\[ + \int_0^T \int_\omega a(x)g_1(u_t)u \, P \, dx \, dt + \int_0^T \int_{\Gamma_1} g_2(v_t) v \, P \, d\gamma \, dt. \]
(5.20)

Equation (5.17) follows from (5.20).  

Now substituting \( P = \frac{1}{2} \text{div}_0 H \) in \( \Omega \) and \( P = \frac{1}{2} \text{div}_{0T} H_T \) on \( \Gamma \), in Lemma 5.2.3 (Eq. (5.17)) and combining the obtained result with the identity (5.5) of Lemma 5.2.1, we infer

\[
\left[ \int_\Omega u_t H(u) \, dx \right]_0^T + \frac{1}{2} (u_t, u \text{div}_0 H \big|_\Omega)_0^T + \left[ \int_{\Gamma_1} v_t H_T(v) \, d\gamma \right]_0^T + \frac{1}{2} (v_t, v \text{div}_{0T} H_T \big|_{\Gamma_1})_0^T
\]
\[ + \int_0^T \int_\Omega D\!H(\nabla g u, \nabla g u) \, dx \, dt + \int_0^T \int_{\Gamma_1} \left[ (\nabla_T)_g v \left( \pi \partial_T H_T \right) \left( (\nabla_T)_g v \right) \right] \, d\gamma \, dt \]
\[ + \frac{1}{4} \int_0^T \int_\Omega u^2 \nabla g (\text{div}_0 H) \cdot v \, dx \, dt + \frac{1}{4} \int_0^T \int_\Omega u^2 A (\text{div}_0 H) \, dx \, dt \]
\[ + \frac{1}{4} \int_0^T \int_{\Gamma_1} v^2 (\nabla_T)_g (\text{div}_{0T} H_T) \cdot v \, d\gamma \, dt + \frac{1}{4} \int_0^T \int_{\Gamma_1} v^2 A_T (\text{div}_{0T} H_T) \, d\gamma \, dt \]
\[ + \int_0^T \int_\omega a(x)g_1(u_t)H(u) \, dx \, dt + \int_0^T \int_{\Gamma_1} g_2(v_t) H_T(v) \, d\gamma \, dt \]
\[ + \frac{1}{2} \int_0^T \int_\omega a(x)g_1(u_t)u \, \text{div}_0 H \, dx \, dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} g_2(v_t) v \, \text{div}_{0T} H_T \, d\gamma \, dt \]
\[ = \frac{1}{2} \int_0^T \int_{\Gamma_0} \frac{H \cdot v}{|v_A|^2} \left( \frac{\partial u}{\partial v_A} \right)_g^2 \, d\gamma \, dt \]
\begin{equation}
\frac{1}{2} \int_0^T \int_{\Gamma_1} (H \cdot \nu) \left[ |v_t|^2 - |(\nabla T)_g v|^2_g + \frac{1}{|v_A|^2_g} \left\| \frac{\partial u}{\partial v_A} \right\|_g^2 \right] d\gamma dt.
\tag{5.21}
\end{equation}

On the other hand, by Cauchy–Schwarz inequality, we obtain that
\begin{align*}
\int_\Omega u_t H(u) dx &+ \frac{1}{2} \int_\Omega u_t \text{div}_0 H dx \\
+ \left[ \int_{\Gamma_1} v_t H_T(v) d\gamma \right] &\leq C[E(0) + E(T)]
\tag{5.22}
\end{align*}

and
\begin{align*}
\frac{1}{4} \int_0^T \int_\Omega u^2 \nabla_g (\text{div}_0 H) \cdot v \ dx \ dt &+ \frac{1}{4} \int_0^T \int_\Omega u^2 A(\text{div}_0 H) \ dx \ dt \\
+ \frac{1}{4} \int_0^T \int_{\Gamma_1} v^2 (\nabla T)_g (\text{div}_0 T H_T) \cdot v d\gamma dt &+ \frac{1}{4} \int_0^T \int_{\Gamma_1} v^2 A_T(\text{div}_0 T H_T) d\gamma dt \\
\leq C \{ l.o.t(u) + l.o.t(v) \},
\tag{5.23}
\end{align*}

where \( l.o.t(u) = \int_0^T \int_\omega u^2 \ dx \ dt \), \( l.o.t(v) = \int_0^T \int_{\Gamma_1} v^2 \ d\gamma dt \) and \( C \) will denote various positive constants which may be different at different occurrences.

We also have
\begin{align*}
\left| \int_0^T \int_\omega a(x) g_1(u_t) H(u) \ dx \ dt \right| &\leq \varepsilon \int_0^T \int_\omega |\nabla_g u|_g^2 \ dx \ dt + \frac{C}{2\varepsilon} \int_0^T \int_\omega a(x) \left| g_1(u_t) \right|^2 \ dx \ dt,
\tag{5.24}
\end{align*}

and
\begin{align*}
\left| \int_0^T \int_\omega a(x) g_1(u_t) u \text{div}_0 H \ dx \ dt \right| &\leq \varepsilon \int_0^T \int_\omega |\nabla_g u|_g^2 \ dx \ dt + \frac{C}{2\varepsilon} \int_0^T \int_\omega a(x) \left| g_1(u_t) \right|^2 \ dx \ dt,
\tag{5.25}
\end{align*}
for any $\varepsilon$. Inserting (5.22)–(5.24) and (5.25) into (5.21) and using the fact that $H \cdot v \leq 0$ on $\Gamma_0$, yields

$$
\begin{aligned}
b \left\{ \int_0^T \int_\Omega |\nabla_g u|^2_g \, dx \, dt + \int_0^T \int_{\Gamma_1} |(\nabla_T)_g v|^2_g \, d\gamma \, dt \right\} \\
\leq C \left[ \int_0^T \int_\Omega \frac{1}{|v_A|^2_g} \left| \frac{\partial u}{\partial v_A} \right|^2_g \, dx \, dt + \int_0^T \int_\Omega \left\{ u_t^2 + 2\varepsilon |\nabla_g u|^2_g \right\} \, dx \, dt \\
+ \int_0^T \int_{\Gamma_1} \left\{ v_t^2 + 2\varepsilon |(\nabla_T)_g v|^2_g \right\} \, d\gamma \, dt + \int_0^T \int_\omega a(x) |g_1(u_t)|^2 \, dx \, dt \\
+ \int_0^T \int_{\Gamma_1} |g_2(v_t)|^2 \, d\gamma \, dt + E(0) + E(T) + l.o.t(u) + l.o.t(v) \right].
\end{aligned}
$$

(5.26)

Choose $P = \frac{1}{2} b$, where $b$ is a positive constant given in (3.2). By Lemma 5.2.3, we have

$$
\begin{aligned}
\frac{1}{2} b \left\{ \int_0^T \int_\Omega \left[ u_t^2 - |\nabla_g u|^2_g \right] \, dx \, dt + \int_0^T \int_{\Gamma_1} \left[ v_t^2 - |(\nabla_T)_g v|^2_g \right] \, d\gamma \, dt \right\} \\
= \frac{1}{2} b (u_t, u)_\Omega \big|_0^T + \frac{1}{2} b (v_t, v)_{\Gamma_1} \big|_0^T + \frac{1}{2} b \int_0^T \int_\omega a(x) g_1(u_t) u \, dx \, dt \\
+ \frac{1}{2} b \int_0^T \int_{\Gamma_1} g_2(v_t) v \, d\gamma \, dt \\
\leq C \left\{ \int_0^T \int_\Omega a(x) |g_1(u_t)|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} |g_2(v_t)|^2 \, d\gamma \, dt \\
+ E(0) + E(T) + l.o.t(u) + l.o.t(v) \right\}. 
\end{aligned}
$$

(5.27)

Combining (5.21)–(5.27) and choosing $\varepsilon$ small enough, we obtain

$$
\int_0^T E(t) \leq C \left[ \int_0^T \int_\Omega \frac{1}{|v_A|^2_g} \left| \frac{\partial u}{\partial v_A} \right|^2_g \, dx \, dt \\
+ \int_0^T \int_\omega a(x) |g_1(u_t)|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} |g_2(v_t)|^2 \, d\gamma \, dt \\
+ E(0) + E(T) + l.o.t(u) + l.o.t(v) \right].
$$

(5.28)
We now estimate the quantity
\[
\int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2_g d\gamma dt
\]
in terms of
\[
\int_0^T \int_\omega a(x) \left[ |u_t|^2 + |g_1(u_t)|^2 \right] dx dt + \int_0^T \int_{\Gamma_1} \left[ |v_t|^2 + |g_2(v_t)|^2 \right] d\gamma dt
\]
\[+ E(0) + E(T) + l.o.t(u) + l.o.t(v). \tag{5.29}\]

**Proposition 5.2.2.** Let \( \Omega \) be a bounded, open, connected set in \( \mathbb{R}^n \) (\( n \geq 2 \)) having a boundary \( \Gamma = \partial \Omega \) of class \( C^2 \). Let \( T > 0 \) large enough. Then, for every solution \((u, v)\) of (3.1), there exist some constant \( C > 0 \) such that
\[
\int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2_g d\gamma dt \leq \int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2_g d\gamma dt
\]
\[\leq C \left\{ \int_0^T \int_\omega a(x) \left[ |u_t|^2 + |g_1(u_t)|^2 \right] dx dt
\]
\[+ \int_0^T \int_{\Gamma_1} \left[ |v_t|^2 + |g_2(v_t)|^2 \right] d\gamma dt
\]
\[+ E(0) + E(T) + l.o.t(u) + l.o.t(v) \right\}. \tag{5.29}\]

**Proof.** See Appendix A. \( \square \)

Substituting now (5.29) into (5.28), we obtain
\[
\int_0^T E(t) \leq C \left[ \int_0^T \int_\omega a(x) \left[ |u_t|^2 + |g_1(u_t)|^2 \right] dx dt + \int_0^T \int_{\Gamma_1} \left[ |v_t|^2 + |g_2(v_t)|^2 \right] d\gamma dt
\]
\[+ E(0) + E(T) + l.o.t(u) + l.o.t(v) \right]. \tag{5.30}\]

Applying the dissipativity property inherent in the relation (3.13), i.e., \( \forall T \geq 0 \)
\[
E(0) = E(T) + \int_0^T \int_\omega a(x)g_1(u_t)u_t dx dt + \int_0^T \int_{\Gamma_1} g_2(v_t)v_t d\gamma dt, \tag{5.31}\]
we obtain

**Proposition 5.2.3.** For time $T$ large enough, the following estimate holds for the solution $(u, v)$ of (3.1):

$$E(T) \leq C_T(E(0)) \left\{ \int_0^T \int_\omega a(x)[|u_t|^2 + |g_1(u_t)|^2] \, dx \, dt 
+ \int_0^T \int_{\Gamma_1} [|v_t|^2 + |g_2(v_t)|^2] \, d\gamma \, dt + \text{l.o.t}(u) + \text{l.o.t}(v) \right\},$$

(5.32)

where the constant $C_T(E(0))$ remains bounded for bounded values of $E(0)$.

5.3. Absorption of the lower order terms

Via a “nonlinear” compactness/uniqueness argument as in Lasiecka et al. [10] we now proceed to eliminate the lower order terms $\text{l.o.t}(u)$ and $\text{l.o.t}(v)$ in (5.32).

**Lemma 5.3.1.** With $T$ sufficiently large, the inequality (5.32) implies that there exists a nonnegative constant $C(E(0))$ such that the solution $(u, v)$ of (3.1) obeys the following inequality:

$$\text{l.o.t}(u) + \text{l.o.t}(v) \leq C(E(0)) \left\{ \int_0^T \int_\omega a(x)[|u_t|^2 + |g_1(u_t)|^2] \, dx \, dt 
+ \int_0^T \int_{\Gamma_1} [|v_t|^2 + |g_2(v_t)|^2] \, d\gamma \, dt \right\},$$

(5.33)

where the constant $C(E(0))$ remains bounded for bounded values of $E(0)$.

**Proof.** If Lemma 5.3.1 is false, there then exists a sequence $\{(u^{(n)}, u^{(n)}_t, v^{(n)}, v^{(n)}_t), (t = 0)\}_{n=1}^{\infty}$ and a corresponding sequence $\{(u^{(n)}, u^{(n)}_t, v^{(n)}, v^{(n)}_t)\}_{n=1}^{\infty}$ which satisfies for all $n$, 

$$\begin{cases}
    u^{(n)}_{tt} + Au^{(n)} + a(x)g_1(u^{(n)}_t) = 0 & \text{on } \Omega \times \mathbb{R}_+, \\
    v^{(n)}_{tt} + \frac{\partial u^{(n)}}{\partial x} + A_T v^{(n)} + g_2(v^{(n)}_t) = 0 & \text{on } \Gamma_1 \times \mathbb{R}_+, \\
    u^{(n)} = v^{(n)} & \text{on } \Gamma \times \mathbb{R}_+, \\
    u^{(n)} = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+,
\end{cases}$$

(5.34)

with

$$\lim_{n \to \infty} \frac{\text{l.o.t}(u^{(n)}) + \text{l.o.t}(v^{(n)})}{\int_0^T \int_\omega a(x)[|u_t^{(n)}|^2 + |g_1(u^{(n)}_t)|^2] \, dx \, dt + \int_0^T \int_{\Gamma_1} [|v_t^{(n)}|^2 + |g_2(v^{(n)}_t)|^2] \, d\gamma \, dt} = \infty$$

(5.35)

while the sequence of initial energy $\{E((u^{(n)}, u^{(n)}_t, v^{(n)}, v^{(n)}_t), 0)\}_{n=1}^{\infty}$ is uniformly bounded in $n$. By the energy relation (3.12), the sequence $\{E((u^{(n)}, u^{(n)}_t, v^{(n)}, v^{(n)}_t), t)\}_{n=1}^{\infty}$ is also
bounded uniformly for \(0 \leq t \leq T\), and consequently there exists a subsequence, still denoted by \(\{(u^{(n)}, u_t^{(n)}, v^{(n)}, v_t^{(n)})\}_{n=1}^{\infty}\), such that

\[
(u^{(n)}, v^{(n)}) \to (u, v) \quad \text{weakly in } H^1(Q) \times H^1(\Sigma),
\]

\[
(u^{(n)}, v^{(n)}) \to (u, v) \quad \text{in } L^2(Q) \times L^2(\Sigma) \text{ strongly,
}\]

\[
(u^{(n)}, v^{(n)}) \to (u, v) \quad \text{a.e. in } (Q) \times (\Sigma).
\]

We now consider two possibilities:

**Case I.** \((u, u_t, v, v_t) \neq 0\). Then with this assumption, the equality (5.35) implies that \(\{u_t^{(n)}\}\), and \(\{g_1(u_t^{(n)})\}\) each converge to 0 in \(L^2(\omega T)\), and \(\{v_t^{(n)}\}\), and \(\{g_2(v_t^{(n)})\}\) each converge to 0 in \(L^2(\Sigma_1)\). Upon passage to the limit in (5.34), we then have that \((u, u_t, v, v_t)\) satisfies the system

\[
\begin{aligned}
\rho_{tt} + Au &= 0 \quad \text{on } \Omega \times \mathbb{R}_+,
\frac{\partial u}{\partial \nu_A} &= 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+,
\rho &= 0 \quad \text{on } \Gamma \times \mathbb{R}_+.
\end{aligned}
\]

Moreover, if we make the change of variable, \((\tilde{u}, \tilde{v}) = (u_t, v_t)\), then \((\tilde{u}, \tilde{v})\) solves

\[
\begin{aligned}
\rho_{tt} + A\tilde{u} &= 0 \quad \text{on } \Omega \times \mathbb{R}_+,
\frac{\partial \tilde{u}}{\partial \nu_A} &= 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+,
\tilde{u} &= \tilde{v} \quad \text{on } \Gamma \times \mathbb{R}_+.
\end{aligned}
\]

By a Uniqueness Continuation theorem of Triggiani and Yao [20] adapted to our case (see also Theorem 7.4. of Gulliver et al. [4]), we have that \((\tilde{u}, \tilde{v}) = (u_t, v_t) = (0, 0)\) on \(\omega T \times \Sigma_1\), and consequently

\[
(u, v) = (0, 0),
\]

after using the ellipticity of \(A\). So \((u, u_t, v, v_t) = 0\), which contradicts our opening assumption.

**Case II.** \((u, u_t, v, v_t) = 0\). In this case, denoting

\[
\lambda_n \equiv \left[ \left\| v^{(n)} \right\|_{L^2(0,T;L^2(\Gamma_1))}^2 + \left\| u^{(n)} \right\|_{L^2(0,T;L^2(\Omega))}^2 \right]^{\frac{1}{2}},
\]

\[
(\tilde{u}^{(n)}, \tilde{u}_t^{(n)}, \tilde{v}^{(n)}, \tilde{v}_t^{(n)}) = \frac{1}{\lambda_n} (u^{(n)}, u_t^{(n)}, v^{(n)}, v_t^{(n)}),
\]

then

\[
inf(n) + l.o.t(\tilde{u}^{(n)}) = 1 \quad \text{for every } n,
\]

and as \((u, u_t, v, v_t) = 0\), we have from (5.36) that \(\lim_{n \to \infty} \lambda_n = 0\). Also, one has a fortiori that \((\tilde{u}^{(n)}, \tilde{u}_t^{(n)}, \tilde{v}^{(n)}, \tilde{v}_t^{(n)})\) satisfies

\[
\begin{aligned}
\tilde{u}_{tt} + A\tilde{u} &= 0 \quad \text{on } \Omega \times \mathbb{R}_+,
\frac{\partial \tilde{u}}{\partial \nu_A} + A_T \tilde{v} + \frac{g_2(v_t^{(n)})}{\lambda_n} &= 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+,
\tilde{u} &= \tilde{v} \quad \text{on } \Gamma \times \mathbb{R}_+,
\tilde{u} &= 0 \quad \text{on } \Gamma_0 \times \mathbb{R}_+.
\end{aligned}
\]

with \((\tilde{u}^{(n)}, \tilde{u}_t^{(n)}, \tilde{v}^{(n)}, \tilde{v}_t^{(n)}) = (0, 0) = \frac{1}{\lambda_n} (u^{(n)}, u_t^{(n)}, v^{(n)}, v_t^{(n)})(t = 0)\). In addition, (5.43) and (5.35) imply that
\[ (\tilde{u}_t^{(n)}, \tilde{v}_t^{(n)}) \to (0, 0) \quad \text{in} \ L^2(0, T; L^2(\omega)) \times L^2(0, T; L^2(\Gamma_1)) \quad \text{as} \ n \to \infty. \] (5.46)

Moreover, using the dissipative relation (5.31) (applied to \( (u^{(n)}, u_t^{(n)}, v^{(n)}, v_t^{(n)}) \), followed by the estimate (5.32), we have for all \( t \in (0, T] \),

\[
E_n(t) dt \leq C \left( E(0) \right) \left\{ \int_0^T \int_\omega a(x) \left[ |u_t^{(n)}|^2 + |g_1(u_t^{(n)})|^2 \right] dx \, dt \\
+ \int_0^T \int_{\Gamma_1} \left[ |v_t^{(n)}|^2 + |g_2(v_t^{(n)})|^2 \right] d\gamma \, dt + \text{l.o.t}(u^{(n)}) + \text{l.o.t}(v^{(n)}) \right\} (5.47)
\]

(where the constant \( C_T \) here is different from that in (5.32)).

Dividing both sides of this inequality by \( \lambda_n \), we then have that \( E((\tilde{u}^{(n)}, \tilde{u}_t^{(n)}, \tilde{v}^{(n)}, \tilde{v}_t^{(n)}), t) \) is uniformly bounded for \( 0 \leq t \leq T \), and thus there is a subsequence \( (\tilde{u}(n), \tilde{u}_t^{(n)}, \tilde{v}(n), \tilde{v}_t^{(n)}) \) and \( (\tilde{u}, \tilde{u}_t, \tilde{v}, \tilde{v}_t) \) such that

\[
(\tilde{u}^{(n)}, \tilde{v}^{(n)}) \to (\tilde{u}, \tilde{v}) \quad \text{weak in} \ L^2(0, T; V),
\]

\[
(\tilde{u}^{(n)}, \tilde{v}^{(n)}) \to (\tilde{u}, \tilde{v}) \quad \text{in} \ L^2(Q) \times L^2(\Sigma) \text{ strongly},
\]

\[
(\tilde{u}^{(n)}, \tilde{v}^{(n)}) \to (\tilde{u}, \tilde{v}) \quad \text{a.e. in} \ (Q) \times (\Sigma).
\]

The last two convergences above and (5.44) yield that

\[ \text{l.o.t}(\tilde{u}) + \text{l.o.t}(\tilde{v}) = 1. \] (5.48)

But as \( (\sqrt{a(x)} \frac{g_1(u_t^{(n)})}{\lambda_n}, \frac{g_2(v_t^{(n)})}{\lambda_n}) \to (0, 0) \) in \( L^2(0, T; L^2(\omega)) \times L^2(0, T; L^2(\Gamma_1)) \), by (5.35), we can then pass to the limit in (5.45), after recalling the convergences in (5.46), and subsequently invoking ellipticity and the Uniqueness Continuation theorem of Triggiani and Yao [20], as was done in the final part of Case I, we arrive at \( (\tilde{u}_t, \tilde{v}_t) = (0, 0) \) and \( (\tilde{u}, \tilde{v}) = (0, 0) \), a conclusion which contradicts (5.48). The proof of Lemma 5.3.1 is hence complete. \( \Box \)

5.4. Conclusion of Theorem 4.1

In what follows we will proceed exactly as in Lasiecka and Tataru’s work [10] (see Lemmas 3.2 and 3.3 of the referred paper) adapted to our context. Let

\[
\omega_\alpha = \{ (x, t) \in \omega_T / |u_t| > 1 \text{ a.e.} \}, \quad \omega_\beta = \omega_T \setminus \omega_\alpha,
\]

\[
\Sigma_\alpha = \{ (x, t) \in \Sigma / |v_t| > 1 \text{ a.e.} \}, \quad \Sigma_\beta = \Sigma \setminus \Sigma_\alpha.
\]

Then using hypothesis (H.1)(iii), we obtain

\[
\int_{\omega_\alpha} (g_1^2(u_t) + (u_t)^2) \, dx \, dt \leq \frac{(m_1^{-1} + M_1)}{a_0} \int_{\omega_\alpha} a(x) g_1(u_t) u_t \, dx \, dt. \] (5.49)

Moreover, from (4.1)

\[
\int_{\omega_\beta} (g_1^2(u_t) + (u_t)^2) \, dx \, dt \leq \int_{\omega_\beta} h_1(g_1(u_t) u_t) \, dx \, dt. \] (5.50)
Then by Jensen’s inequality
\[
\int_0^T \int_{\omega_T} h_1(g_1(u_t)u_t) \, dx \, dt \leq \text{meas}(\omega_T) h_1\left( \frac{1}{a_0 \text{meas}(\omega_T)} \int_0^T \int_{\omega_T} a(x) g_1(u_t) u_t \, dx \, dt \right)
\]
\[
= \text{meas}(\omega_T) r_1 \left( \int_0^T \int_{\omega_T} a(x) g_1(u_t) u_t \, dx \, dt \right),
\]
where \( r_1(s) = h_1\left( \frac{s}{N_0} \right) \), \( N_0 = \min\{\text{meas}(\Sigma), a_0 \text{meas}(\omega_T)\} \). Thus
\[
\int_{\omega_T} \left( g_2^2(u_t) + (u_t)^2 \right) \, dx \, dt \leq \frac{m_1^{-1} + M_1}{a_0} \int_{\omega_T} a(x) g_1(u_t) u_t \, dx \, dt
\]
\[
+ \text{meas}(\omega_T) r_1 \left( \int_0^T \int_{\omega_T} a(x) g_1(u_t) u_t \, dx \, dt \right),
\]
(5.51)
Analogously we have done above, we have
\[
\int_{\Sigma} \left( g_2^2(v_t) + (v_t)^2 \right) \, dx \leq (m_2^{-1} + M_2) \int_{\Sigma} g_2(v_t) v_t \, d\Sigma
\]
\[
+ \text{meas}(\Sigma) r_2 \left( \int_{\Sigma} g_2(v_t) v_t \, d\Sigma \right),
\]
(5.52)
where \( r_2(s) = h_2\left( \frac{s}{N_1} \right) \). Splicing together (5.32), (5.33), (5.52) and (5.53), and further recalling the definition (4.2), we have
\[
E(T) \leq C_T(E(0)) \left[ M_0 \left\{ \int_{\omega_T} a(x) g_1(u_t) u_t \, dx \, dt + \int_{\Sigma} g_2(v_t) v_t \, d\Sigma \right\}
\]
\[
+ N_1 r \left( \int_{\omega_T} a(x) g_1(u_t) u_t \, dx \, dt + \int_{\Sigma} g_2(v_t) v_t \, d\Sigma \right) \right],
\]
(5.54)
where \( M_0 = \frac{m_1^{-1} + M_1}{a_0} + m_2^{-1} + M_2, N_1 = \max\{\text{meas}(\Sigma), \text{meas}(\omega_T)\} \). Setting
\[
K = \frac{1}{C_T(E(0)) N_1}, \quad c = \frac{M_0}{N_1},
\]
we, then, obtain
\[
p[E(T)] \leq \int_{\omega_T} a(x) g_1(u_t) u_t \, dx \, dt + \int_{\Sigma} g_2(v_t) v_t \, d\Sigma = E(0) - E(T),
\]
(5.55)
where the function \( p \) is as defined in (4.3). To finish the proof of Theorem 4.1, we invoke the following result from Lasiecka and Tataru [10].
Lemma A. Let $p$ be a positive, increasing function such that $p(0) = 0$. Since $p$ is increasing we can define an increasing function $q$, $q(x) = x - (I + p)^{-1}(x)$. Consider a sequence $s_n$ of positive numbers which satisfies

$$s_{n+1} + p(s_{n+1}) \leq s_n.$$ 

Then $s_n \leq S(m)$, where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt} S(t) + q(S(t)) = 0, \quad S(0) = s_0.$$ 

Moreover, if $p(x) > 0$ for $x > 0$, then $\lim_{t \to \infty} S(t) = 0$.

With this result in mind, we replace $T$ (respectively, 0) in (5.55) with $(m + 1)T$ (respectively, $mT$) to obtain

$$E((m + 1)T) + p(E((m + 1)T)) \leq E(mT), \quad \text{for } m = 0, 1, \ldots. \quad (5.56)$$

Applying Lemma A with $s_m = E(mT)$ thus results in

$$E(mT) \leq S(m), \quad m = 0, 1, \ldots. \quad (5.57)$$

Finally, using the dissipativity of $E(t)$ inherent in the relation (3.13), we have for $t = mT + \tau$, $0 \leq \tau \leq T$,

$$E(t) \leq E(mT) \leq S(m) \leq S\left(\frac{t - \tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right) \quad \text{for } t > T,$$

where we have used above the fact that $S(\cdot)$ is dissipative. The proof of Theorem 4.1 is now completed.

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Appendix A. Proof of Proposition 5.2.2

We proceed as in Khemmoudj and Medjden [6] following the same procedure as in Lions [13].

Following the method of proof of Lemma 2.3, Chapter VII in Lions [13], we construct a tubular neighborhood (see Fig. A.1) $\tilde{\omega} = \omega_\epsilon$ of $\Gamma_1$ such that

$$\tilde{\omega} \cap \Omega \subset \omega$$

and a vector field $h \in (C^1(\tilde{\Omega}))^n$ such that

$$h = v \quad \text{on } \Gamma_1, \quad h \cdot v \geq 0 \quad \text{a.e. on } \Gamma, \quad (5.58)$$

and

$$h = 0 \quad \text{on } \Omega \setminus \tilde{\omega}. \quad (5.59)$$

(See Lemma 3.1 and Remarks 3.1 and 3.2 of Chapter I (pp. 29–31) and also, Lemma 2.3 of Chapter VII (pp. 411–412) of Lions [13]; for the construction of this vector field.)
Applying identity (5.5) with $H = h$, we easily deduce

\[
\frac{1}{2} \int_0^T \int_{\hat{\omega}} (h \cdot v) \left| \frac{\partial u}{\partial v_A} \right|_g^2 d\gamma dt
\]

\[
\leq \frac{1}{2} \int_0^T \int_{\hat{\omega}} (h \cdot v) \left| \frac{\partial u}{\partial v_A} \right|_g^2 d\gamma dt + \frac{1}{2} \int_0^T \int_{\Gamma_1} (h \cdot v) \left| \frac{\partial u}{\partial v_A} \right|_g^2 d\gamma dt
\]

\[
= \left[ \int_0^T u_t h(u) dx \right]_{\hat{\omega}} + \left[ \int_0^T v_t h_T(v) \right]_{\Gamma_1}
\]

\[
+ \int_0^T \int_{\hat{\omega}} Dh(\nabla_g u, \nabla_g u) dx dt + \int_0^T \int_{\Gamma_1} (\nabla_T g v)(\pi \partial_T h_T)[(\nabla_T g v)] d\gamma dt
\]

\[
+ \frac{1}{2} \int_0^T \int_{\hat{\omega}} (\text{div}_0 h)(u_t^2 - |\nabla_g u|^2_g) dx dt
\]

\[
+ \frac{1}{2} \int_0^T \int_{\hat{\omega}} (\text{div}_0 T h_T)(v_t^2 - |(\nabla_T g v)|^2_g) d\gamma dt
\]

\[
+ \frac{1}{2} \int_0^T \int_{\hat{\omega}} a(x) g_1(u_t) h(u) dx dt + \int_0^T \int_{\Gamma_1} g_2(v_t) h_T(v) d\gamma dt
\]

\[- \frac{1}{2} \int_0^T \int_{\hat{\omega}} (h \cdot v)(|v_t|^2 - |(\nabla_T g v)|^2_g) d\gamma dt
\]

which gives

\[
\int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial v_A} \right|_g^2 d\gamma dt
\]
\[ \leq C \left\{ \int_0^T \int_\omega |u_t|^2 + |\nabla_g u_t|^2_g \, dx \, dt + \int_0^T \int_{\Gamma_1} |v_t|^2 + |(\nabla_T)_g v_t|^2_g \, d\gamma \, dt \right\} \\
+ 2 \left( \int_0^T u_t h(u) \, dx \right|_0^T + 2 \left( \int_{\Gamma_1} v_t h_T(v) \, d\gamma \right|_0^T \\
+ 2 \int_0^T \int_\omega a(x) g_1(u_t) h(u) \, dx \, dt + 2 \int_0^T \int_{\Gamma_1} g_2(v_t) h_T(v) \, d\gamma \, dt, \right. \\
\]

where \( C \) is a positive constant.

We construct then a function \( \eta \in W^{1,\infty}(\bar{\Omega}) \) (see Fig. A.2) satisfying

\begin{align*}
0 \leq \eta \leq 1 & \quad \text{a.e. in } \bar{\Omega}, \quad \eta = 1 \quad \text{a.e. in } \hat{\omega}, \quad (5.61) \\
\eta = 0 & \quad \text{a.e. in } \Omega \setminus \omega, \quad (5.62)
\end{align*}

and

\[ \frac{|
abla \eta|^2}{\eta} \in L^{\infty}(\bar{\Omega}). \quad (5.63) \]

(See Lemma 2.4 Chapter VII in Lions [13, pp. 413–414] for the construction of this function.)

Applying identity (5.17) with \( P = \eta \) and noting that \( a(x) \geq a_0 > 0 \), we deduce

\[ \int_0^T \int_{\omega_2} \eta |\nabla_g u_t|^2_g \, dx \, dt - \int_0^T \int_{\omega_2} u \nabla_g \eta \cdot \nabla_g u \, dx \, dt \\
+ \int_0^T \int_{\Gamma_1} \eta |(\nabla_T)_g v_t|^2_g \, d\gamma \, dt - \int_0^T \int_{\Gamma_1} v(\nabla_T)_g \eta \cdot (\nabla_T)_g v \, d\gamma \, dt \\
\leq C \left\{ \int_0^T \int_\omega a(x) [ |u_t|^2 + |g_1(u_t)|^2 ] \, dx \, dt \\
+ \int_0^T \int_{\Gamma_1} [ |v_t|^2 + |g_2(v_t)|^2 ] \, d\gamma \, dt + l.o.t(v) + Y \right\}, \]
where
\[ Y = \left| \int_{\Gamma_1} \eta v_t v \bigg|_0^T + \int_{\omega_{2x}} \eta u_t u \bigg|_0^T \right|. \]  
\( (5.65) \)

On the other hand,
\[ \left| \int_0^T \left| \int_{\omega_{2x}} u \nabla g \eta \cdot \nabla g u \, dx \, dt \right| \leq \epsilon' \int_0^T \int_{\omega_{2x}} \eta |\nabla g u|^2 \, dx \, dt + \frac{1}{2\epsilon'} \int_0^T \int_{\omega_{2x}} \frac{|\nabla g \eta|^2}{\eta} |u|^2 \, dx \, dt, \]  
\[ (5.66) \]

\[ \left| \int_0^T \left| \int_{\Gamma_1} v (\nabla T)_g \eta \cdot (\nabla T)_g v \, d\gamma \, dt \right| \right| \leq \epsilon' \int_0^T \int_{\Gamma_1} \eta \left| (\nabla T)_g v \right|^2 \, d\gamma \, dt \]
\[ + \frac{1}{2\epsilon'} \int_0^T \int_{\Gamma_1} \frac{|(\nabla T)_g \eta|^2}{\eta} |v|^2 \, d\gamma \, dt. \]

Combining (5.64) with (5.66) for \( \epsilon' \in ]0, 1[ \) sufficiently small, noting that \( \eta = 1 \) on \( \tilde{\omega} \), we deduce
\[ \int_0^T \int_{\tilde{\omega}} \left| \nabla g u \right|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} \left| (\nabla T)_g v \right|^2 \, d\gamma \, dt \]
\[ = \int_0^T \int_{\tilde{\omega}} \eta |\nabla g u|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} \eta |(\nabla T)_g v|^2 \, d\gamma \, dt \]
\[ \leq \int_0^T \int_{\omega_{2x}} \eta |\nabla g u|^2 \, dx \, dt + \int_0^T \int_{\Gamma_1} \eta |(\nabla T)_g v|^2 \, d\gamma \, dt \]
\[ \leq C \left\{ \int_0^T \int_{\omega} a(x) \left[ |u_t|^2 + |g_1(u_t)|^2 \right] \, dx \, dt \right. 
\left. + \int_0^T \int_{\Gamma_1} \left[ |v_t|^2 + |g_2(v_t)|^2 \right] \, d\gamma \, dt + l.o.t(u) + l.o.t(v) + Y \right\}. \]  
\( (5.67) \)

From (5.60) and (5.67), we obtain
\[ \int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial v_A} \right|^2 \, d\gamma \, dt \leq C \left\{ \int_0^T \int_{\omega} a(x) \left[ |u_t|^2 + |g_1(u_t)|^2 \right] \, dx \, dt 
+ \int_0^T \int_{\Gamma_1} \left[ |v_t|^2 + |g_2(v_t)|^2 \right] \, d\gamma \, dt \right\}. \]
\[ + \int_{\omega} u_t h(u) \, dx|_0^T + \int_{\Gamma_1} v_t h_T(v) \, dx|_0^T + Y \right). \tag{5.68} \]

We now remark that
\[ \left| \int_{\omega} u_t h(u) \, dx|_0^T + \int_{\Gamma_1} v_t h_T(v) \, dx|_0^T + Y \right| \leq C \left\{ E(0) + E(T) \right\}. \tag{5.69} \]

Replacing (5.69) in (5.68) yields (5.29). The proof of Proposition 5.2.2 is now completed. \( \square \)

**Remark A.1.** It is important to be noted that from the proof of the main theorem, the localized dissipation \( a(x)g_1(u_t) \) is strong enough to assure the asymptotic stability. Indeed, this comes from the fact that
\[ \int_0^T \int_{\Gamma_1} |v_t|^2 \, d\gamma \, dt = \int_0^T \int_{\Gamma_1} |u_t|^2 \, d\gamma \, dt \leq a_0^{-1} \int_\omega a(x) |u_t|^2 \, dx \, dt. \]

So, it is easy to see (following the computations) that considering \( g_2 = 0 \) and taking the above inequality into account, we obtain the same decay rate state in the main theorem. However, the reciprocal procedure is not true, or in other words: To consider \( g_1 = 0 \) and \( g_2 \neq 0 \) is still a hard open problem because of the dynamic boundary conditions.

**References**


