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Triple-point defective ruled surfaces

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Abstract

In [L. Chiantini, T. Markwig, Triple-point defective regular surfaces. arXiv:0705.3912, 2007] we studied triple-point defective very ample linear systems on regular surfaces, and we showed that they can only exist if the surface is ruled. In the present paper we show that we can drop the regularity assumption, and we classify the triple-point defective very ample linear systems on ruled surfaces.

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Let $S \subset \mathbb{P}^n$ be a smooth projective surface, $K = K_S$ the canonical class and L the hyperplane divisor class on S.

We study a classical interpolation problem for the pair (S, L), namely whether for a general point $p \in S$ the linear system $|L - 3p|$ has the expected dimension

$$\operatorname{expdim} |L - 3p| = \max\{-1, \dim |L| - 6\}.$$

If this is not the case we call the pair (S, L) *triple-point defective*.

The terminology is a little bit different from the classical one, of differential flavour, where such surfaces are referred to as *satisfying one Laplace equation*. Some classical results on these surfaces are contained, e.g., in the papers of Terracini [9] and Togliatti [10]. However, no general classification of these surfaces has been achieved in classical projective geometry.

This paper is a continuation of [3], where indeed some classification of triple-point defective pairs is achieved, under the following assumptions:

$$L - K \textit{ very ample, and } (L - K)^2 > 16,$$

conditions that we will take all over the paper.

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With these assumptions, the main result of [3] says that all triple-point defective *regular* surfaces are rationally ruled.

In the present paper, we show that we can drop the regularity assumption. Our main results is:

Theorem 1. *Suppose that the pair (S, L) is triple-point defective where L and $L - K$ are very ample with $(L - K)^2 > 16$. Then S admits a ruling $\pi : S \to C$, i.e. a morphism whose generic fibre is isomorphic to $\mathbb{P}^1$.*

In Theorem 3 we also show a classification of line bundles L on a ruled surface S, such that the pair (S, L) is triple-point defective.

Notice that the previous classification is far from being complete if we drop the assumption that S is *smooth*. Namely, as explained in [3], a classical result of [2] shows that if Y is a (necessarily singular) *developable* scroll in planes of dimension 3 then *any* surface contained in Y is triple-point defective.

Our method follows Reider's analysis of rank 2 bundles arising from points which do not impose independent conditions, adapted to the case of fat points, as explained in the paper [1].

Let us observe that our assumption $(L - K)^2 > 16$ is essential, if one wants to apply Reider's construction.

On the other hand, the assumption $L - K$ *very ample* might be more restrictive than necessary for the purpose of getting a classification. However, our investigations in [3] heavily rely on the embedding of S provided by $L - K$, and even with an extensive use of the known vanishing theorems (following a procedure well explained in [1]), we were not able to exclude a wide range of possibilities without this assumption. We note that the assumption becomes sometimes automatic (e.g. when K is trivial), and it could be replaced by merely numerical (but a little stronger) hypotheses, in the spirit e.g. of [5].

Finally, let us point out that our initial aim was not a detailed, deep and heavy chase for minimal conditions yielding a classification, but we wanted to show that, in some non-trivial environment, the study of rank 2 bundles could lead to an effective classification of surfaces with unexpected interpolation behaviour, missing in classical geometry.

Let us describe the method in more details. In [3] we tackled the problem by considering $|L - 3p|$ as fibres of the map α in the following diagram,

$$|L| = \mathbb{P}(H^0(L)^*) \xleftarrow{\beta} \mathcal{L}_3 \xrightarrow{\alpha} S\ , \tag{1}$$

where $\mathcal{L}_3$ denotes the incidence variety

$$\mathcal{L}_3 = \{(C, p) \in |L| \times S \mid \mathrm{mult}_p(C) \geq 3\}$$

and α and β are the obvious projections.

Assuming that for a general point $p \in S$ there is a curve in L_p with a triple point in p — and hence α surjective, we considered then the *equimultiplicity scheme* Z_p of a curve $L_p \in |L - 3p|$ defined by

$$\mathcal{J}_{Z_p,p} = \left\langle \frac{\partial f_p}{\partial x_p}, \frac{\partial f_p}{\partial y_p} \right\rangle + \langle x_p, y_p\rangle^3.$$

One easily sees that (S, L) triple-point defective necessarily implies that

$$h^1\left(S, \mathcal{J}_{Z_p}(L)\right) \neq 0.$$

Non-zero elements in $H^1\left(S, \mathcal{J}_{Z_p}(L)\right)$ determine by Serre duality a non-trivial extension $\mathcal{E}_p$ of $\mathcal{J}_{Z_p}(L - K)$ by $\mathcal{O}_S$, which turns out to be a rank 2 bundle on the surface. Due to the assumption $(L - K)^2 > 16$, $\mathcal{E}_p$ is Bogomolov unstable. We then exploited the destabilising divisor A_p of $\mathcal{E}_p$ in order to obtain the above mentioned result.

In this analysis, we prove that, when S is regular, then $L - K - A_p$ is embedded by L as a line, so S is ruled. The analysis requires a careful examination of many cases. In some of them, we could simply assume $L - K$ big and nef, but there are cases in which we cannot proceed without assuming $L - K$ *very ample*.

For non-regular surfaces, the argument of Chiantini and Markwig [3] shows the following lemma (see [3], Propositions 17 and 18), where we denote by $|D|_a = \{C \mid C \sim_a D, C \text{ a curve in } S\}$ the family of curves in S which are algebraically equivalent to the divisor D:

Proposition 2. *Suppose that, with the notation in (1), α is surjective, and suppose as usual that L and $L-K$ are very ample with $(L-K)^2 > 16$.*

For p general in S and for $L_p \in |L-3p|$ general, call Z'_p the minimal subscheme of the equimultiplicity scheme Z_p of L_p such that

$$h^1\left(S, \mathcal{J}_{Z'_p}(L)\right) \neq 0.$$

Then either:

(1) $\mathrm{length}(Z'_p) = 3$ *and S is ruled; or*

(2) $\mathrm{length}(Z'_p) = 4$ *and, for $p \in S$ general, there are smooth, elliptic curves E_p and F_p in S through p such that $E_p^2 = F_p^2 = 0$, $E_p.F_p = 1$ and $L.E_p = L.F_p = 3$. In particular, both $|E|_a$ and $|F|_a$ are pencils inducing an elliptic fibration with section on S over an elliptic curve.*

This is our starting point. We will in this paper show that the latter case actually cannot occur, and we will classify the triple-point defective linear systems L as above on ruled surfaces. It will in particular follow that the fibre of the ruling is contained exactly twice, and thus that the map β above is generically finite.

We call a ruled surface $\pi : S \to C$ *geometrically ruled* if π is minimal, i.e. if all fibres are isomorphic to $\mathbb{P}^1$. For the classification result, call C_0 a section of the ruling C, $\mathfrak{e}$ the line bundle on the base curve given by the determinant of the defining bundle, and call E_i the exceptional divisors (see Section 2 for a more precise setting of the notation):

Theorem 3. *Assume that $\pi : S \to C$ is a ruled surface and that the pair (S, L) is triple-point defective, where L and $L-K$ are very ample with $(L-K)^2 > 16$.*

Then π is minimal, i.e. S is geometrically ruled, and for a general point $p \in S$ the linear system $|L-3p|$ contains a fibre of the ruling as fixed component with multiplicity two.

Moreover, in the previous notation, the line bundle L is of type $C_0 + \pi^*\mathfrak{b}$ for some divisor $\mathfrak{b}$ on C such that $\mathfrak{b}+\mathfrak{e}$ is very ample.

In Section 1 we will first show that a surface S admitting two elliptic fibrations as required by Case (2) of Proposition 2 would necessarily be a product of two elliptic curves and the triple-point defective linear system would be of type $(3, 3)$. We then show that such a system is never triple-point defective, setting the first part of the main theorem.

In Section 2 we classify the triple-point defective linear systems on ruled surfaces, thus completing our main results.

1. Products of elliptic curves

In the above setting, consider a triple-point defective pair (S, L) where the equimultiplicity scheme Z_p (see [3]) of a general element $L_p \in |L-3p|$ admits a complete intersection subscheme Z'_p of length *four* with

$$h^1\left(S, \mathcal{J}_{Z'_p}(L)\right) \neq 0.$$

As explained in the introduction, Proposition 2, after [3] we know that, for $p \in S$ general, there are smooth, elliptic curves E_p and F_p in S through p such that $E_p^2 = F_p^2 = 0$, $E_p.F_p = 1$ and $L.E_p = L.F_p = 3$.

In particular, both $|E|_a$ and $|F|_a$ induce an elliptic fibration with section on S over an elliptic curve.

We will now show that this situation indeed cannot occur. Namely, for general p and L_p there cannot exist such a scheme Z'_p.

Lemma 4. *Suppose that the surface S has two elliptic fibrations $\pi : S \longrightarrow E_0$ and $\pi' : S \longrightarrow F_0$ with general fibre E respectively F satisfying $E.F = 1$.*

Then E_0 and F_0 are elliptic curves, and S is the blow-up of a product of two elliptic curves $S' = E \times E_0 \cong E \times F$.

Proof. Since $E.F = 1$ we have that F is a section of π, and thus $F \cong E_0$ via π. In particular, E_0 and, similarly, F_0 are elliptic curves.

It is well known that there are no non-constant maps from a rational curve to a curve of positive genus ([6], IV.2.5.4). Thus any exceptional curve of S sits in some fibre. Thus we can reach relatively minimal models of π

and π' by successively blowing down exceptional curves which belong to fibres of both π and π', i.e. we have the following commutative diagram

$$\begin{array}{ccccc} S & & & & \\ & \overset{\phi}{\searrow} & & \overset{\pi}{\searrow} & \\ & & S' & \xrightarrow{\widetilde{\pi}} & E_0 \\ & \underset{\pi'}{\searrow} & \downarrow{\scriptstyle \widetilde{\pi}'} & & \\ & & F_0 & & \end{array}$$

where S' is actually a minimal surface. Since a general fibre of π or π' is not touched by the blowing-down ϕ we may denote the general fibres of $\widetilde{\pi}$ and $\widetilde{\pi}'$ again by E respectively F, and we still have $E.F = 1$.

We will now try to identify the minimal surface S' in the classification of minimal surfaces.

There are $r, s \in \mathbb{Q}$ such that

$$sE \sim_n K_{S'} \sim_n rF,$$

where $\sim_n$ means numerically equivalent. Suppose that $s \neq 0$ then $0 = E^2 = \frac{r}{x} \cdot E.F = \frac{r}{s}$ leads to the contradiction $0 \sim_n K_{S'} \sim_n sE \not\sim_n 0$. Thus $s = 0$ and $K_{S'}$ is numerically trivial. Moreover, no multiple of $K_{S'}$ can be effective, since it otherwise would intersect at least one of E or F positively, and hence the Kodaira dimension $\kappa(S')$ of S' is zero.

Taking into account that by the Nakai–Moishezon Criterion $E + F$ is ample, $(S', E + F)$ is a polarised surface of sectional genus

$$p_a(E + F) = 1 + \frac{(K_{S'} + E + F).(E + F)}{2} = 2.$$

Note that $\widetilde{\pi}$ and $\widetilde{\pi}'$ induce injective morphisms $\Omega^1(E_0) \to \Omega^1(S')$ and $\Omega^1(F_0) \to \Omega^1(S')$ with distinct images, so that $h^0(S', \Omega^1_{S'}) \geq 2$. Since by Hodge Theory $q(S') = h^1(S', \mathcal{O}_{S'}) = h^0(S', \Omega^1_{S'})$ the irregularity $q(S')$ is at least two, and S' must be an abelian surface. But then necessarily $S' \cong E_0 \times F_0$. □

Lemma 4 implies that in order to show that the situation of Proposition 2 cannot occur, we have to understand products of elliptic curves.

Let us, therefore, consider a surface $S = C_1 \times C_2$ which is the product of two smooth elliptic curves.

Let us set some notation. We will use some results of Keilen [7] Appendices G.b and G.c in the sequel.

The surface S is naturally equipped with two projections $\pi_i : S \longrightarrow C_i$. If $\mathfrak{a}$ is a divisor on C_2 of degree a and $\mathfrak{b}$ is a divisor on C_1 of degree b then the divisor $\pi_2^*\mathfrak{a} + \pi_1^*\mathfrak{b} \sim_a aC_1 + bC_2$, where by abuse of notation we denote by C_1 a fixed fibre of π_2 and by C_2 a fixed fibre of π_1. Moreover, K_S is trivial, and given two divisors $D \sim_a aC_1 + bC_2$ and $D' \sim_a a'C_1 + b'C_2$ then the intersection product is

$$D.D' = (aC_1 + bC_2).(a'C_1 + b'C_2) = a \cdot b' + a' \cdot b.$$

We will consider first the case

$$L = \pi_2^*\mathfrak{a} + \pi_1^*\mathfrak{b},$$

where both $\mathfrak{b}$ on C_1 and $\mathfrak{a}$ on C_2 are divisors of degree 3. The dimension of the linear system $|L|$ is $\dim |L| = 8$, and thus for a point $p \in S$ the expected dimension is $\operatorname{expdim} |L - 3p| = \dim |L| - 6 = 2$.

Notice that a divisor of degree three on an elliptic curve is always very ample and embeds the curve as a smooth cubic in $\mathbb{P}^2$. Since the smooth plane cubics are classified by their normal forms $xz^2 - y \cdot (y - x) \cdot (y - \lambda \cdot x)$ with $\lambda \neq 0$ the following example reflects the behaviour of any product of elliptic curves embedded via a linear system of bidegree $(3, 3)$.

Example 5. Consider two smooth plane cubics

$$C_1 = V\left(xz^2 - y \cdot (y - z) \cdot (y - az)\right)$$

and

$$C_2 = V\left(xz^2 - y\cdot(y-z)\cdot(y-bz)\right).$$

The surface $S = C_1 \times C_2$ is embedded into $\mathbb{P}^8$ via the Segre embedding

$$\phi : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^8 : ((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \mapsto (x_0y_0 : \ldots : x_2y_2).$$

We may assume that both curves contain the point $p = (1 : 0 : 0)$ as a general non-inflexion point, and the point (p, p) is mapped by the Segre embedding to $\phi(p, p) = (1 : 0 : \ldots : 0)$. If we denote by $z_{i,j}$, $i, j \in \{0, 1, 2\}$, the coordinates on $\mathbb{P}^8$ as usual, then the maximal ideal locally at $\phi(p, p)$ is generated by $z_{0,2}$ and $z_{2,0}$, i.e. these are local coordinates of $\phi(S)$ at $\phi(p, p)$. A standard basis computation shows that locally at $\phi(p, p)$ the coordinates $z_{i,j}$ satisfy modulo the ideal of S and up to multiplication by a unit the following congruences (note, $z_{0,0} = 1$)

$$z_{0,1} \equiv \frac{1}{b}\cdot z_{0,2}^2, \qquad z_{1,0} \equiv \frac{1}{a}\cdot z_{2,0}^2, \qquad z_{1,1} \equiv \frac{1}{ab}\cdot z_{0,2}^2\cdot z_{2,0}^2,$$

$$z_{1,2} \equiv \frac{1}{a}\cdot z_{0,2}\cdot z_{2,0}^2, \qquad z_{2,1} \equiv \frac{1}{b}\cdot z_{0,2}^2\cdot z_{2,0}, \qquad z_{2,2} \equiv z_{0,2}\cdot z_{2,0}.$$

Thus a hyperplane section $H = a_{0,0}z_{0,0} + \cdots + a_{2,2}z_{2,2}$ of $\phi(S)$ is locally in $\phi(p, p)$ modulo $\mathfrak{m}^3 = \langle z_{0,2}, z_{2,0}\rangle^3$ given by

$$H \equiv a_{0,0} + a_{0,2}z_{0,2} + a_{2,0}z_{2,0} + \frac{a_{0,1}}{b}\cdot z_{0,2}^2 + \frac{a_{1,0}}{a}\cdot z_{2,0}^2 + a_{2,2}z_{0,2}z_{2,0},$$

and hence the family of hyperplane sections having multiplicity at least three in $\phi(p, p)$ is given by

$$a_{0,0} = a_{0,1} = a_{1,0} = a_{0,2} = a_{2,0} = a_{2,2} = 0.$$

But then the family has parameters $a_{1,1}, a_{1,2}, a_{2,1}$, and its dimension coincides with the expected dimension 2. Moreover, the 3-jet of a hyperplane section H through $\phi(p, p)$ with multiplicity at least three is

$$\mathrm{jet}_3(H) \equiv z_{0,2}\cdot z_{2,0}\cdot\left(\frac{a_{1,2}}{a}\cdot z_{2,0} + \frac{a_{2,1}}{b}\cdot z_{0,2}\right),$$

which shows that for a general choice of $a_{2,1}$ and $a_{1,2}$ the point $\phi(p, p)$ is an ordinary triple point.

Remark 6. We actually can say very precisely what it means that p is general in the product, namely that neither $\pi_1(p)$ is a inflexion point of C_1, nor $\pi_2(p)$ is a inflexion point of C_2.

Indeed, since $\mathfrak{a}$ is very ample of degree three, for each point $p \in S$ there is a unique point $q_a \in C_2$ such that $q_a + 2\cdot\pi_2(p) \sim_l \mathfrak{a}$. When $\pi_2(p)$ is a inflexion point of C_2, then $q_a = \pi_2(p)$ and thus the two-dimensional family

$$3C_{1,\pi_2(p)} + |\pi^*\mathfrak{b}| \subset |L - 3p|$$

gives a superabundance of the dimension of $|L - 3p|$ by one.

Similarly one can argue when $\pi_1(p)$ is a inflexion point of C_1.

Now we are ready for the proof of Theorem 1.

Proof (*Proof of* Theorem 1). By Proposition 2, it is enough to prove that when S has two elliptic fibrations as in the proposition, then S is not triple-point defective.

By Lemma 4, S is the blow-up $\pi : S \longrightarrow S'$ of a product $S' = C_1 \times C_2$ of two elliptic curves, and we may assume that the curves E_p and F_p in Proposition 2 are the fibres of π_1 respectively π_2.

Our first aim will be to show that actually $S = S'$. For this note that

$$\mathrm{Pic}(S) = \bigoplus_{i=1}^{k} E_i \oplus \pi^*\,\mathrm{Pic}(S'),$$

where the E_i are the total transforms of the exceptional curves arising throughout the blow-up, i.e. the E_i are (not necessarily irreducible) rational curves with self intersection $E_i^2 = -1$ and such that $E_i.E_j = 0$ for $i \neq j$ and

$E_i.\pi^*(C) = 0$ for any curve C on S'. In particular, since $K_{S'}$ is trivial we have that $K_S = \sum_{i=1}^k E_i$, and if $L = \pi^*L' - \sum_{i=1}^k e_i E_i$ then $L - K = \pi^*L' - \sum_{i=1}^k (e_i + 1)E_i$. We therefore have

$$16 < (L-K)^2 = (L')^2 - \sum_{i=1}^k (e_i+1)^2,$$

or equivalently

$$(L')^2 \geq 17 + \sum_{i=1}^k (e_i+1)^2 \geq 17 + 4k, \tag{2}$$

where the latter inequality is due to the fact that $e_i = L.E_i > 0$ since L is very ample. By the assumption of Proposition 2 we know that $L'.C_1 = L.E_p = 3$ and $L'.C_2 = L.F_p = 3$, and therefore by Hartshorne [6] Ex. V.1.9

$$(L')^2 \leq 2 \cdot (L'.C_1) \cdot (L'.C_2) = 18. \tag{3}$$

But (2) and (3) together imply that no exceptional curve exists, i.e. $S = S'$.

Since now S is a product of two elliptic curves, by Lange and Birkenhake [8] we know that the Picard number $\rho = \rho(S)$ satisfies $2 \leq \rho \leq 4$, and the Néron–Severi group can be generated by the two general fibres C_1 and C_2 together with certain graphs C_j, $3 \leq j \leq \rho$, of morphisms $\varphi_j : C_1 \longrightarrow C_2$. In particular, $C_j.C_2 = 1$ and $C_j.C_1 = \deg(\varphi_j) \geq 1$ for $3 \leq j \leq \rho$. Moreover, these graphs have self intersection zero. If we now assume that $L \sim_a \sum_{j=1}^{\rho} a_i C_i$ then

$$L^2 = 2 \cdot \sum_{i<j} a_i \cdot a_j \cdot (C_i.C_j)$$

is divisible by 2, and since $L = L - K$ with $(L-K)^2 > 16$ we deduce with [6] Ex. V.1.9 that

$$L^2 = (L-K)^2 = 18 = 2 \cdot (L.C_1) \cdot (L.C_2),$$

and thus that

$$L \sim_a 3C_1 + 3C_2,$$

or, equivalently, that

$$L = \pi_2^*\mathfrak{a} + \pi_1^*\mathfrak{b}$$

for some divisors $\mathfrak{a}$ on C_2 and $\mathfrak{b}$ on C_1, both of degree 3. That is, we are in the situation of Example 5, and we showed there that (S, L) then is not triple-point defective. □

Remark 7. Notice that, in practise, since

$$h^1(S, L) = h^0(C_1, \mathfrak{b}) \cdot h^1(C_2, \mathfrak{a}) + h^0(C_2, \mathfrak{a}) \cdot h^1(C_1, \mathfrak{b}) = 0,$$

the non-triple-point defectiveness shows that for general $p \in S$ and $L_p \in |L - 3p|$ no Z'_p as in the assumptions of Proposition 2 can have length 4.

2. Geometrically ruled surfaces

Let $S = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ be a geometrically ruled surface with normalised bundle $\mathcal{E}$ (in the sense of [6] V.2.8.1). The Néron–Severi group of S is

$$\mathrm{NS}(S) = C_0\mathbb{Z} \oplus f\mathbb{Z},$$

with intersection matrix

$$\begin{pmatrix} -e & 1 \\ 1 & 0 \end{pmatrix},$$

where $f \cong \mathbb{P}^1$ is a fixed fibre of π, C_0 a fixed section of π with $\mathcal{O}_S(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and $e = -\deg(\mathfrak{e}) \geq -g$ where $\mathfrak{e} = \bigwedge^2 \mathcal{E}$. If $\mathfrak{b}$ is a divisor on C we will write $\mathfrak{b} f$ for the divisor $\pi^*\mathfrak{b}$ on S, and so for the canonical divisor we have

$$K_S \;\sim_l\; -2C_0 + (K_C + \mathfrak{e}) f \;\sim_a\; -2C_0 + (2g - 2 - e) f,$$

where $g = g(C)$ is the genus of the base curve C.

Example 8. Let $\mathfrak{b}$ be a divisor on C such that $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are both very ample and such that $\mathfrak{b}$ is non-special. If C is rational we should in addition assume that $\deg(\mathfrak{b}) + \deg(\mathfrak{b} + \mathfrak{e}) \geq 6$. Then the divisor $L = C_0 + \mathfrak{b} f$ is very ample (see e.g. [4] Prop. 24) of dimension

$$\dim |L| = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 1.$$

Moreover, for any point $p \in S$ we then have (see [4] Cor. 22)

$$\dim |C_0 + (\mathfrak{b} - 2\pi(p)) f| = \dim |C_0 + \mathfrak{b} f| - 4 = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 5,$$

and we have for p general

$$\dim |C_0 + (\mathfrak{b} - 2\pi(p)) f - p| = h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 6.$$

For this note that $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ very ample implies that this number is non-negative — in the rational case we need the above degree bound.

If we denote by $f_p = \pi^*(\pi(p))$ the fibre of π over $\pi(p)$, then by Bézout and since $L.f_p = (L - f_p).f_p = 1$ we see that $2 f_p$ is a fixed component of $|L - 3p|$ and we have

$$|L - 3p| = 2 f_p + |C_0 + (\mathfrak{b} - 2\pi(p)) f - p|,$$

so that

$$\begin{aligned} \dim |L - 3p| &= h^0(C, \mathfrak{b}) + h^0(C, \mathfrak{b} + \mathfrak{e}) - 6 = \dim |L| - 5 \\ &> \dim |L| - 6 = \operatorname{expdim} |L - 3p|. \end{aligned}$$

This shows that (S, L) is triple-point defective and $|L - 3p|$ contains a fibre of the ruling as double component. Moreover, for a general p the linear series $|L - 3p|$ cannot contain a fibre of the ruling more than twice due to the above dimension count for $|C_0 + (\mathfrak{b} - 2\pi(p)) f - p|$.

Next we are showing that a geometrically ruled surface is indeed triple-point defective with respect to a line bundle L which fulfills our assumptions, and in Corollary 13 we will see that this is not the case for non-geometrically ruled surfaces.

Proposition 9. *On every geometrically ruled surface $S = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ there exists some very ample line bundle L such that the pair (S, L) is triple-point defective, and moreover also $L - K$ is very ample with $(L - K)^2 > 16$.*

Proof. It is enough to take $L = C_0 + \mathfrak{b} f$, with $b = \deg(\mathfrak{b}) = 3a$ such that $a, a - e, a + e, a - 2g + 2 + e, a - 2g + 2 - e$ are all bigger or equal than $2g + 1$.

Indeed in this case $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are both very ample. For $p \in C$ general, we also have that both $\mathfrak{b} - p$ and $\mathfrak{b} + \mathfrak{e} - p$ are non-special. It follows that L is very ample (by [6] Ex. V.2.11.b) and (S, L) is triple-point defective, by the previous example. Moreover, in this situation we have:

$$L - K \sim_l 3C_0 + (\mathfrak{b} - K_C - \mathfrak{e}) f.$$

Hence

$$(L - K)^2 = (3C_0 + (\deg(\mathfrak{b}) - 2g + 2 + e) f)^2 \geq 18 > 16.$$

Finally, if we fix a divisor $\mathfrak{a}$ of degree a on C, then $L - K$ is the sum of the divisors $C_0 + (\mathfrak{a} - K_C) f$, $C_0 + (\mathfrak{a} - \mathfrak{e}) f$, $C_0 + \mathfrak{a} f$, which are very ample ([6] Ex. V.2.11). Thus $L - K$ is very ample. □

Next, let us describe which linear systems L on a ruled surface $\pi : S \to C$ determine a triple-point defective pair (S, L).

We will show that Example 8 describes, in most cases, the only possibilities. In order to do so we first have to consider the possible algebraic classes of irreducible curves with self intersection zero on a geometrically ruled surface.

Lemma 10. *Let $B \in |bC_0 + b'f|_a$ be an irreducible curve with $B^2 = 0$ and $\dim |B|_a \geq 0$, then we are in one of the following cases:*

(1) $B \sim_a f$,
(2) $e = 0$, $b \geq 1$, $B \sim_a bC_0$, *and* $|B|_a = |B|_l$, *or*
(3) $e < 0$, $b \geq 2$, $b' = \frac{be}{2} < 0$, $B \sim_a bC_0 + \frac{be}{2} f$ *and* $|B|_a = |B|_l$.

Moreover, if $b = 1$, then $S \cong C_0 \times \mathbb{P}^1$.

Proof. See [7] App. Lemma G.2. □

We can now classify the triple-point defective linear systems on a geometrically ruled surface. In order to do so we should recall the result of [3] Prop. 18.

Proposition 11. *Suppose that, with the notation in (1), α is surjective, and suppose that L and $L - K$ are very ample with $(L - K)^2 > 16$. Moreover, suppose that for $p \in S$ general and for $L_p \in |L - 3p|$ general the equimultiplicity scheme Z_p of L_p has a subscheme Z'_p of length 3 such that $h^1(S, \mathcal{J}_{Z'_p}(L)) \neq 0$.*

Then for $p \in S$ general there is an irreducible, smooth, rational curve B_p in a pencil $|B|_a$ with $B^2 = 0$, $(L - K).B = 3$ and $L - K - B$ big.

In particular, $S \to |B|_a$ is a ruled surface and $2B_p$ is a fixed component of $|L - 3p|$.

Theorem 12. *With the above notation let $\pi : S \to C$ be a geometrically ruled surface, and let L be a line bundle on S such that L and $L - K$ are very ample. Suppose that $(L - K)^2 > 16$ and that for a general $p \in S$ the linear system $|L - 3p|$ contains a curve L_p such that $h^1(S, \mathcal{J}_{Z_p}(L)) \neq 0$ where Z_p is the equimultiplicity scheme of L_p at p.*

Then $L = C_0 + \mathfrak{b} f$ for some divisor $\mathfrak{b}$ on C such that $\mathfrak{b} + \mathfrak{e}$ is very ample and $|L - 3p|$ contains a fibre of π as fixed component with multiplicity two. Moreover, if $e \geq -1$ then $\deg(\mathfrak{b}) \geq 2g + 1$ and we are in the situation of Example 8.

Proof. As in the proof of [3] Thm. 19, since the case in which the length of Z_p is 4 has been ruled out in Remark 7, we only have to consider the situations in Proposition 11.

Using the notation there we have a divisor $A := L - K - B \sim_a aC_0 + a'f$ and a curve $B \sim_a bC_0 + b'f$ satisfying certain numerical properties, in particular $p_a(B) = 0$, $B^2 = 0$, and $a > 0$ since A is big. Moreover,

$$3 = A.B = -eab + ab' + a'b \tag{4}$$

and

$$a \cdot (2a' - ae) = A^2 = (L - K)^2 - 2 \cdot A.B - B^2 \geq 17 - 2 \cdot A.B - B^2 = 11. \tag{5}$$

By Lemma 10 there are three possibilities for B to consider. If $e < 0$ and $B \sim_a bC_0 + \frac{eb}{2} f$ with $b \geq 2$, then Riemann–Roch leads to the impossible equation

$$-2 = 2p_a(B) - 2 = B.K = (2g - 2) \cdot b.$$

If $e = 0$ and $B \sim_a bC_0$, then similarly Riemann–Roch shows

$$-2 = B.K = (2g - 2) \cdot b,$$

which now implies that $b = 1$ and $g = 0$. In particular, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $L \sim_a A + B + K \sim_a (a - 1)C_0 + f$, since $3 = A.B = a'$. But this is then one of the cases of Example 8.

Finally, if $B \sim_a f$ then (4) gives $a = 3$, and thus

$$L \sim_a A + B + K \sim_a C_0 + (\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e}) f,$$

where $A = 3C_0 + \mathfrak{a}' f$. Moreover, by the assumptions of Case (b) the linear system $|L - 3p|$ contains the fibre of the ruling over p as double fixed component, and since L is very ample it induces on C the very ample divisor $\mathfrak{e} + (\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e})$. Note also, that (5) implies that

$$a' - 2 - e \geq \frac{e}{2},$$

and thus for $e \geq -1$ we have

$$\deg(\mathfrak{a}' + \pi(p) + K_C + \mathfrak{e}) = 2g + 1 + (a' - 2 - e) \geq 2g + 1,$$

so that then the assumptions of Example 8 are fulfilled. This finishes the proof. □

If $\pi : S \longrightarrow C$ is a ruled surface, then there is a (not necessarily unique (if $g(C) = 0$)) minimal model

$$\begin{array}{ccc} S & \overset{\pi}{\longrightarrow} & C \\ {\scriptstyle\phi}\searrow & & \nearrow{\scriptstyle\widetilde{\pi}} \\ & S' & \end{array} \qquad S' \overset{\widetilde{\pi}}{\longrightarrow} C,$$

and the Néron–Severi group of S is

$$\mathrm{NS}(S) = C_0 \cdot \mathbb{Z} \oplus f \cdot \mathbb{Z} \oplus \bigoplus_{i=1}^{k} E_i \cdot \mathbb{Z},$$

where f is a general fibre of π, C_0 is the total transform of the section of $\widetilde{\pi}$, and the E_i are the total transforms of the exceptional divisors of the blow-up ϕ. Moreover, for the Picard group of S we just have to replace $f \cdot \mathbb{Z}$ by $\pi^* \operatorname{Pic}(C)$. We may, therefore, represent a divisor class A on S as

$$L = aC_0 + \pi^* \mathfrak{b} - \sum_{i=1}^{k} c_i E_i. \tag{6}$$

Corollary 13. *Suppose that (S, L) is a pair as in Proposition 2 with ruling $\pi : S \to C$, and suppose that the Néron–Severi group of S is as described before with general fibre $f = B_p$.*

Then S is minimal, $L = C_0 + \pi^* \mathfrak{b}$ for some divisor $\mathfrak{b}$ on C such that $\mathfrak{b} + \mathfrak{e}$ is very ample and $|L - 3p|$ contains a fibre of π as fixed component with multiplicity two.

Proof. Let $L = C_0 + \pi^* \mathfrak{b} - \sum_{i=1}^{k} c_i E_i$, as described in (6). Then

$$L - K = (a + 2)C_0 + \pi^*(\mathfrak{b} - K_C - \mathfrak{e}) - \sum_{i=1}^{k} (c_i + 1) E_i,$$

and thus considering Proposition 11

$$3 = (L - K).B = a + 2.$$

The very ampleness of L implies now that $c_i > 0$ for all i. Therefore, if S is not minimal and f' is the strict transform of a fibre of the minimal model meeting some E_i, then $L.f' \leq 0$, a contradiction. □

By [3] we get Theorem 3 as an immediate corollary.

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