The distance matrix eigensystem of an equally spaced row of points

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Abstract

The distance matrix of an equally spaced row of points can be taken to be the matrix with \( ij \)th entry \(|i - j|\). Denote the characteristic polynomial of this matrix by \( \text{ch}(n, x) \). Because this matrix is symmetric and centrosymmetric, it is similar to a 2 by 2 block diagonal matrix. This corresponds to a factorization of \( \text{ch}(n, x) \) into two characteristic polynomials, \( \text{sym}(n, x) \) and \( \text{ant}(n, x) \). The eigenvalues from \( \text{sym} \) have symmetric eigenvectors and those from \( \text{ant} \) have antisymmetric eigenvectors. Expansion by minors gives recursions for \( \text{sym} \) and \( \text{ant} \). The system of relations between \( \text{sym}(n, x) \), \( \text{ant}(n, x) \) and a third set \( \text{din}(n, x) \) is derived. These relations allow the simple recursive calculation of the polynomials. They are used to show that the eigenvalues are simple and to determine when eigenvalues for different \( n \) can be the same. Eigenvectors built by repeating smaller eigenvectors cause eigenvalues for one \( n \) to be repeated for multiples of \( n \). An unexpected result following from this is that the \( \text{ant} \) and \( \text{din} \) polynomials have factorizations that are parallel to the factorization of \( x^n - 1 \) into cyclotomic polynomials. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

The distance matrix of a set of points, \( S = \{p_1, p_2, p_3, \ldots, p_n\} \), has as its \( ij \)th entry the distance, \( \|p_i - p_j\| \), between the \( i \)th and \( j \)th points. The eigenvalues of
S’s distance matrix are invariants (under congruence) of the point set S and hence describe geometric properties of it. Since the rows of the distance matrix are associated with the points of S, the entries of an eigenvector can be associated with the points of S. Thus they are also invariants of S and hence geometric properties of S. Although each eigenvalue and eigenvector is the measure of a property of S, these properties are not simply related to properties of point sets that are traditionally studied in geometry. In this paper, we will study how one aspect of the geometry of the point set, the presence of a symmetry, interacts with the properties of the distance matrix eigensystem. An involutory symmetry of the point set produces invariant subspaces for the distance matrix, and this means that the distance matrix is similar to a 2 by 2 block diagonal matrix. The characteristic polynomial of a block diagonal matrix factors; and in this case, the factors collect the eigenvalues who’s eigenvectors have a fixed symmetry—either symmetric or antisymmetric. By examining how the eigensystem of a symmetric point set relates to the eigensystems of subsets that have the same symmetry, we derive three-term recursions for the factors of the distance matrix. Common eigenvalues of the point set and a symmetric subset are associated with repetitive eigenvectors.

In this paper, we will not develop the program sketched above in full generality; that would take too much space. We will simplify to a representative special case: a one-dimensional point set with a reflection symmetry. Even this is too general, so we further specialize to the case of a set consisting of equally spaced points on a line. This special case allows a few tricks that simplify (and shorten) the exposition. The most drastic simplification is that in one dimension the triangle inequality is actually an equality. This causes our recursions to have only three terms. Another simplification is that a reflection symmetry has order 2 and we need only real roots of unity. For a rotation of order n, the antisymmetric part is repeated n − 1 times and is most conveniently written using nth roots of unity. Thus when n > 2, we must use complex numbers, and we have repetition of eigenvalues arising for a second reason (in addition to repetitive eigenvectors). Finally, we define the din polynomials using an identity relating them to the ant polynomials. In general, the din polynomials should be defined using the determinant of a certain bordered distance matrix. This (more lengthy) approach shows that din polynomials are the characteristic polynomials for a type of relative eigenvalue and thus why they satisfy recursions like those of the other characteristic polynomials. The associated relative eigenvectors can be repeated and that explains why din polynomials satisfy multiplicative identities and have cyclotomic-like factorizations. This approach will be developed in a sequel.

There is one trick possible for a line of equally spaced points that we do not use. For a line of equally spaced points, the distance matrix is Toeplitz. But the Toeplitz property does not apply to more irregular point sets to which the methods of this paper do apply. The results and methods of this paper can be easily generalized to the symmetric sections of a periodic one-dimensional point set with reflection symmetries. But the distance matrices of these sections are not Toeplitz and need not be minors of the Toeplitz bi-infinite distance matrix of the integers. For another
approach using Fourier methods on Toeplitz distance matrices, see Baxter’s papers [3,4]. At an even more general level, the basic three-term recursions for the din polynomials do not require the overlaying infinite set to have a translation symmetry or reflection symmetries; they apply to the finite sections of any one-dimensional point set.

Distance matrices occur in radial basis function interpolation, which uses functions of the form

\[ g(x) = \sum a_i f(\|x - p_i\|) \]

to interpolate scattered point data. When the basis function, \( f \), is the identity and \( g(x) \) is required to interpolate data given at the points of \( S \), the values of the \( a_i \) are found by solving a linear system whose matrix is the distance matrix of \( S \). Most of the early work on eigenvalues of distance matrices was from the viewpoint of interpolation theory. Indeed, that is how I came to the subject, using multiquadrics to interpolate geophysical survey data from Airborne Electromagnetic Induction Profilers, see [7,10]. In 1937, Schoenberg [12] proved that distance matrices (of distinct points) are always nonsingular and that they have one positive eigenvalue and the rest negative. An important paper by Micchelli [9] extended the nonsingularity result to a broad class of basis functions that included the multiquadrics. In papers in 1992, Ball [2] and Sun [13] gave estimates for the smallest size eigenvalue of a distance matrix. For one-dimensional point sets, they both derived the best possible bound of \( 1/2 \), but computer work in dimensions 2 through 6 seems to indicate that both of their bounds are too low for higher dimensions.

Since the distance matrix is invariant under Euclidean motions of the point set, and scales with a similarity; we may take our equally spaced row of points to be the points 1 through \( n \) on a number line. See [8, Section 2] for more details on this and a simple derivation of the determinant of the distance matrix of any finite one-dimensional set.

**Definition 1.1.** The matrix \( M_n \) has as its \( ij \)th entry \( M_n[i,j] = |i - j| \).

\( M_n \) is a symmetric \( n \) by \( n \) matrix of integers and, because the point set defining \( M_n \) has a reflection symmetry, \( M_n \) is also centrosymmetric. This reflection symmetry is reflected in symmetries of the eigenvectors; they are either symmetric or antisymmetric. We will show that all the eigenvalues are simple so that they too may be classified as symmetric or antisymmetric. Centrosymmetry makes \( M_n \) similar to a block diagonal matrix because the symmetric and antisymmetric vectors of \( \mathbb{R}^n \) are invariant subspaces.

**Definition 1.2.** Let \( \text{ch}(0, x) = 1 \). The characteristic polynomial of the distance matrix \( M_n \) is

\[ \text{ch}(n, x) = \det(x \cdot \text{IdentityMatrix}(n) - M_n). \]
With this definition, $\text{ch}(n, x)$ is a monic, integer coefficients polynomial of degree $n$. Note that $\text{ch}(1, x) = x$. Since the main diagonal of $M_n$ is all zeroes, the trace of $M_n$ is zero and hence the degree $n - 1$ coefficient of $\text{ch}(n, x)$ is zero. The similarity of $M_n$ to a block diagonal matrix shows that $\text{ch}(n, x)$ factors into the product of the characteristic polynomials of the two blocks. The roots of one of these factors, $\text{ant}(n, x)$, are the eigenvalues of the antisymmetric eigenvectors, and the roots of the other, $\text{sym}(n, x)$, are the eigenvalues of the symmetric eigenvectors. Although the polynomials $\text{ch}(n, x)$ do not seem to have many simple recursions, the symmetric and especially the antisymmetric polynomials do satisfy a wealth of simple identities. These identities allow the recursive calculation of $\text{ant}(n, x)$ and $\text{sym}(n, x)$ and thus of $\text{ch}(n, x)$. More importantly, they allow us to do what might be called algebraic set theory. By embodying the set of eigenvalues in a polynomial, we can use algebraic operations on polynomials to do set theoretic manipulations. Greatest common divisors (GCDs) correspond to intersections. A divisibility relation corresponds to a containment relation. Relatively prime polynomials correspond to disjoint sets. Multiplication is related to unions. At a more complicated level, taking a linear combination of two polynomials produces a polynomial whose roots are a kind of average of the roots of the summands. The interlacing properties of our polynomials are produced by this kind of averaging resulting from the three-term recursions.

An unexpected result is that $\text{ant}(n, x)$ and $\text{din}(n, x)$, a series of polynomials derived from $\text{ant}(n, x)$, show analogies to the cyclotomic polynomials. If we write $\text{cyl}(n, x) = x^n - 1$, then a characteristic formula is

$$ \text{GCD}(\text{cyl}(m, x), \text{cyl}(n, x)) = \text{cyl}(\text{GCD}(m, n), x). $$

This formula can be used to show that $\text{cyl}(n, x)$ is the product over the divisors of $n$ of factors called the cyclotomic polynomials $\Phi_d(x)$. The $\text{din}(n, x)$ polynomials satisfy the same identity and the $\text{ant}(n, x)$ polynomials satisfy a closely related identity. In both cases the identities can be used to give factorizations of these polynomials with the factors corresponding to divisors of the index of the polynomial. The cyclotomic polynomials are irreducible and we conjecture that the two analogous series here also consist of irreducibles. These polynomial divisibility results have the algebraic set theoretic consequences that if $m$ is an odd multiple of $n$, then the antisymmetric eigenvalues of $M_n$ are also antisymmetric eigenvalues of $M_m$. In fact we find that an eigenvector of size $m$ can be obtained by concatenating alternatingly positive and negative copies of an eigenvector of size $n$. This result hints at the fact that the eigenvectors themselves satisfy some interesting identities, a fact that we will elaborate in a sequel.

Since this paper developing basic properties is already quite long; we will omit most applications of the results, saving them for later papers. In the last section we will use the theory developed to prove a few facts about the $\text{ch}(n, x)$ polynomials. An indication of another direction of applications is the following. The eigenvalues derived from nested sequences of sections of a periodic set satisfy uniform inter-
lacing theorems. These theorems lead to eigenvalue density results of the following type. If $D(n, [a, b])$ is the proportion of eigenvalues of the distance matrix of size $n$ that fall in the interval $[a, b]$, then

$$
\lim_{n \to \infty} D(n, [a, b])
$$

exists and, in effect, defines a probability density on the appropriate semi-infinite interval of the negative reals. For example, for $\text{ch}(n, x)$ half of the eigenvalues are in $[-1, -1/2]$, half are in $[-\infty, -1]$, one-fourth are in $[-2/3, -1/2]$, one-fourth in $[-2, -1]$, and so on. We do not yet know how general the conditions on a set can be and still permit results of this type.

2. Antisymmetric and symmetric eigenvectors

Symmetries of a point set are directly manifested in symmetries of the eigenvectors of its distance matrix. The manifestation of symmetries for the eigenvalues is more subtle. At the risk of being vague, we can say that there is a link between the eigenvalues whose eigenvectors share the same symmetry properties. For one-dimensional sets, we have the luxury of an integer distance matrix and the association among similar type eigenvalues is reflected in a factorization of the characteristic polynomial over the integers. In this section we take up the basic example of this phenomenon: a point set with a reflection symmetry will have eigenvectors that are symmetric or antisymmetric. At the start of the following section we will mention another example of this phenomenon. Later we will see another example of an association being exhibited by a polynomial factorization over the integers, in a context that also illustrates the analogy with cyclotomic polynomials. Antisymmetric eigenvectors can be repeated, with alternating signs, to make antisymmetric eigenvectors with the same eigenvalue over a larger set. Just as each root of unity is primitive for its smallest power that equals 1; each antisymmetric eigenvalue will be primitive for some smallest size set. This smallest set gives an eigenvector that is the fundamental domain into which the eigenvectors sharing that eigenvalue can be broken. The roots of unity primitive for a certain $d$ are exactly the roots of the cyclotomic polynomial of size $d$, and this cyclotomic polynomial is a factor of $x^n - 1$ over the integers for any $n$ that is a multiple of $d$. Similarly, the antisymmetric eigenvalues primitive for a certain size, $d$, are exactly the roots of $\text{ann}(d, x)$ (see Section 8) and this polynomial is a factor over the integers of $\text{ant}(n, x)$ for any $n$ that is an odd multiple of $d$.

The set $\{1, 2, 3, \ldots, n\}$ has the property that the permutation given by $\pi(i) = n + 1 - i$ is the restriction of a Euclidean motion of the number line and we have

$$
M_n[i, j] = |i - j| = |\pi(i) - \pi(j)| = M_n[n + 1 - i, n + 1 - j].
$$

A matrix that satisfies this condition is said to be centrosymmetric. See [1,6,15], for expositions of most of the following theory of centrosymmetric matrices. Since a distance matrix is always a symmetric matrix, centrosymmetry is an additional
symmetry of the distance matrix of a symmetric point set. To facilitate the discussion of centrosymmetric matrices we define a matrix operation, the spin $S$, somewhat analogous to the transpose $T$. Weaver called this operation the reflection $R$, but we prefer $S$ since the operation is not really a reflection. $R$ would be a more appropriate name for the operation of reflection around the off-diagonal of a matrix.

**Definition 2.1.** The spin $A^S$ of an $m$ by $n$ matrix $A$ has $ij$th entry $A^S[i, j] = A[m + 1 - i, n + 1 - j]$. $A^S$ is also $m$ by $n$. The $S$ operation is a half turn about the center of $A$. A centrosymmetric matrix is defined as a matrix that satisfies $A^S = A$.

Except that the product rule is not twisted, $S$ satisfies the usual set of identities:

\[
A^{SS} = A, \quad (A + B)^S = A^S + B^S, \quad (aA)^S = aA^S, \quad (AB)^S = A^S B^S, \quad A^{TS} = A^{ST}, \quad (A^{-1})^S = (A^S)^{-1}, \quad \text{Det}(A^S) = \text{Det}(A), \quad \chi_{A^S}(x) = \chi_A(x).
\]

There is an $n$ by $n$ permutation matrix, $E_n$, that is convenient here. Good, Andrew and Weaver all call this matrix $J$, but we prefer to let $J$ denote the matrix with all entries equal to 1. We follow Golub and Van Loan [5, p. 125] who refer to this matrix as $E$, the exchange matrix.

**Definition 2.2.** The $n$ by $n$ matrix $E_n$ is defined by

\[
E_n[i, j] = \begin{cases} 
1 & \text{if } i + j = n + 1, \\
0 & \text{otherwise}. 
\end{cases}
\]

We will omit the size subscript $n$ of $E_n$ when convenient. Note that $E$ is a symmetric, centrosymmetric involution. For an $m$ by $n$ matrix $A$, the definition of $E_n$ quickly gives $(E_n A)[i, j] = A[m + 1 - i, n + 1 - j]$ so that $E_n A$ has exchanged the rows of $A$ in reverse order. $(AE_n)[i, j] = A[i, n + 1 - j]$ so that $AE_n$ has exchanged the columns of $A$ in reverse order. Putting these two calculations together shows $A^S = E_n A E_n$.

Square centrosymmetric matrices can be block diagonalized. This follows from the equivalence of the centrosymmetry of $M$ to the identity $ME = EM$. $E$ is similar to the matrix $D$ defined below. Thus $E$ has invariant subspaces that are the eigenspaces of the eigenvalues 1 and $-1$. The matrix $X$ below arises from an obvious choice of bases for these two subspaces. Since centrosymmetric matrices commute with $E$, they share these invariant subspaces. Vectors belonging to these subspaces themselves have symmetry, so we make the following definitions. Andrew and Weaver use the term skew symmetric but we prefer antisymmetric.

**Definition 2.3.** A vector, $v$, is symmetric iff it satisfies $Ev = v$ considered as a column vector of length $n$. This means that $v[i] = v[n + 1 - i]$. A vector, $v$, is antisymmetric iff it satisfies $Ev = -v$ considered as a column vector of length $n$. This means that $v[i] = -v[n + 1 - i]$ and if $n = 2k + 1$, then $v[k + 1] = 0$. Since $v^S = Ev$ for column vectors, the definition could have been equally well stated using $S$. 
The following theorems give the structure of a symmetric, centrosymmetric matrix using a block partition. If the size of the matrix is even, say $2n$, then the blocks of the partition are $n$ by $n$. In the following theorem, $A$ and $B$ are square $n$ by $n$ blocks.

**Theorem 2.1.** If $M$ is symmetric, centrosymmetric and of even size $2n$, then $M$ can be written as

\[ M = \begin{pmatrix} A & BE \\ EB & A^S \end{pmatrix} \] \[ \text{and } M \text{ is orthogonally similar to } \begin{pmatrix} A + B & 0 \\ 0 & (A - B)^S \end{pmatrix}. \]

where $A^T = A$ and $B^T = B$. If $v$ is an eigenvector of $A + B$, then the block partitioned vector $[v, Ev]^T$ is a symmetric eigenvector of $M$ with the same eigenvalue. If $v$ is an eigenvector of $(A - B)^S$, then the block partitioned vector $[Ev, -v]^T$ is an antisymmetric eigenvector of $M$ with the same eigenvalue.

**Proof.** These facts are straightforward calculations, and the references give proofs. If $D$ is the diagonal matrix with $n$ 1’s followed by $n$ ($-1$)’s, and $X = (E_{2n} + D)/\sqrt{2}$, then $XMX$ is the block diagonal matrix required. □

In the case of $M_{2n}$, the matrix $A$ is just $M_n$ so $A[i, j] = |i - j|$. The matrix $B$ satisfies $B[i, n + 1 - j] = BE[i, j] = n + j - i$, so that

\[ B[i, j] = 2n + 1 - i - j. \]

Adding these two matrices gives the matrix of the symmetric eigenvectors

\[ (A + B)[i, j] = 2n + 1 - 2 \text{Min}(i, j). \]

Subtracting gives $(A - B)[i, j] = -(2n + 1 - 2 \text{Max}(i, j))$. Thus the matrix for the antisymmetric eigenvectors is

\[ (A - B)^S[i, j] = (A - B)[n + 1 - i, n + 1 - j] = -2 \text{Min}(i, j) + 1. \]

**Definition 2.4.** Let $\text{sym}(0, x) = 1$ and $\text{ant}(0, x) = 1$. For $n > 0$, we use the matrices calculated above to define

\[ \text{sym}(2n, x) = \text{Det}(x \cdot \text{IdentityMatrix}(n) - (A + B)), \]

\[ \text{ant}(2n, x) = \text{Det}(x \cdot \text{IdentityMatrix}(n) - (A - B)^S). \]

Note that $\text{sym}(2n, x)$ and $\text{ant}(2n, x)$ are monic integer coefficients polynomials of degree $n$ all of whose roots are real (since $A + B$ and $(A - B)^S$ are symmetric). The first few values are: $\text{sym}(2, x) = x - 1$, $\text{sym}(4, x) = x^2 - 4x - 6$, $\text{sym}(6, x) = x^3 - 9x^2 - 36x - 20$, and $\text{ant}(2, x) = x + 1$, $\text{ant}(4, x) = x^2 + 4x + 2$, $\text{ant}(6, x) = x^3 + 9x^2 + 12x + 4$.

**Corollary 2.2.** For $n \geq 0$, $\text{ch}(2n, x) = \text{sym}(2n, x) \cdot \text{ant}(2n, x)$. 
When the size of the matrix, $M$, is odd, say $2n + 1$, $M$ can be written as a $3 \times 3$ block matrix using four $n \times n$ blocks in the corners, column and row vectors of length $n$ between the square blocks and a one by one block in the center. In the following theorems, $A$ and $B$ are square $n \times n$ blocks, $u$ is a column vector and $x$ is a scalar. We use the shorthand $UCB(A + B, u, x)$ to denote the $2 \times 2$ block matrix in the upper left corner of the matrix to the right below. Note that, unlike Theorem 2.1, $UCB$ is not symmetric. A symmetric version of this theorem, such as Theorem 9 of Weaver [15], would not be suitable for our purposes because it involves $\sqrt{2}$ multiplying $u$ and $u^T$. Thus we modify the similarity to put a 2 in one place and a 1 in the other.

**Theorem 2.3.** If $M$ is symmetric and centrosymmetric and of odd size $2n + 1$, then $M$ can be written as

$$
M = \begin{pmatrix}
A & u & BE \\
u^T & x & u^{TS} \\
EB & u^S & A^S
\end{pmatrix}
$$

and $M$ is similar to

$$
\begin{pmatrix}
A + B & 2u & 0 \\
u^T & x & 0 \\
0 & 0 & (A - B)^S
\end{pmatrix},
$$

where $A^T = A$ and $B^T = B$. If an eigenvector of $UCB(A + B, u, x)$ is partitioned as $[w, y]^T$, where $w$ is a length $n$ column vector and $y$ is a scalar, then the block partitioned vector $[w, 2y, Ew]^T$ is a symmetric eigenvector of $M$ with the same eigenvalue. If $w$ is an eigenvector of $(A - B)^S$, then the block partitioned vector $[Ew, 0, -w]^T$ is an antisymmetric eigenvector of $M$ with the same eigenvalue.

**Proof.** Let $D$ be the diagonal matrix with $n + 1$ 1’s followed by $n (-1)$’s. $E_{2n+1} + D$ has a 2 as its central entry; let $X$ be the same matrix except for a 1 in the central entry. It is straightforward to show that the similarity $(X/2)M(E_{2n+1} + D)$ gives the required block diagonal matrix. □

In the case of $M_{2n+1}$, the matrix $A$ is again $M_n$ so $A[i, j] = |i - j|$. The vector $u$ is $\{n, n - 1, \ldots, 1\}^T$, and the scalar $x$ is 0. The matrix $B$ satisfies $B[i, n + 1 - j] = BE[i, j] = n + 1 + j - i$, so that $B[i, j] = 2n + 2 - i - j$. Adding these two matrices gives $(A + B)[i, j] = 2n + 2 - 2 \text{Min}(i, j)$. Note that $2u$ is the same as the last column of $A + B$. Subtracting $A$ and $B$ gives $(A - B)[i, j] = -(2n + 2 - 2 \text{Max}(i, j))$. Thus


**Definition 2.5.** Let $\text{sym}(1, x) = x$ and $\text{ant}(1, x) = 1$. For $n > 0$, use the matrices calculated above to define

$$
\text{sym}(2n + 1, x) = \text{Det}(x \cdot \text{IdentityMatrix}(n) - UCB(A + B, u, 0)),
$$
$$
\text{ant}(2n + 1, x) = \text{Det}(x \cdot \text{IdentityMatrix}(n) - (A - B)^S).
$$

Note that $\text{sym}(2n + 1, x)$ and $\text{ant}(2n + 1, x)$ are monic integer coefficients polynomials all of whose roots are real. As in the even case, $(A - B)^S$ is symmetric.
$UCB(A + B, u, 0)$ is not symmetric but it is similar to a symmetric matrix, and in any case its roots must all be real because those of $ch(n, x)$ are. The degree of $\text{sym}(2n+1, x)$ is $n + 1$ and the degree of $\text{ant}(2n+1, x)$ is $n$. The first few values are: $\text{sym}(3, x) = x^2 - 2x - 2$, $\text{sym}(5, x) = x^3 - 6x^2 - 18x - 8$, $\text{sym}(7, x) = x^4 - 12x^3 - 72x^2 - 80x - 24$ and $\text{ant}(3, x) = x + 2$, $\text{ant}(5, x) = x^2 + 6x + 4$, $\text{ant}(7, x) = x^3 + 12x^2 + 20x + 8$.

**Corollary 2.4.** For $n \geq 0$,

$$ch(2n+1, x) = \text{sym}(2n+1, x) \text{ant}(2n+1, x).$$

Although these are eigenvectors which are symmetric or antisymmetric, we will show that every eigenvalue of $M_n$ is simple; so its eigenvalues can be described as symmetric or antisymmetric based on their eigenvectors. For this reason we will refer to $\text{ant}(n, x)$ as the characteristic polynomial of the antisymmetric eigenvalues and $\text{sym}(n, x)$ as the characteristic polynomial of the symmetric eigenvalues.

### 3. Antisymmetric characteristic polynomials

We now begin the development of the theory of three-term identities for $\text{ant}(n, x)$, $\text{sym}(n, x)$ and $\text{din}(n, x)$. $\text{din}(n, x)$ will be defined in the following section. In the following three sections we will derive a basic set of identities that are sufficient tools for the rest of this paper, but they are just a beginning. Let us introduce some notation to discuss the general situation. Although it is important that $\text{ant}(n, x)$, $\text{sym}(n, x)$ and $\text{din}(n, x)$ are each a single series of polynomials, in many ways they act like two interlaced series, the odd index series and the even index series. This is another example of the symmetry principle, since the odd size point sets have their reflection axis on a point of the set and the even size point sets have it at a midpoint. Although certain key identities are the same for both series, the coefficients for most types of identity differ between the odd and the even series. So, in the following, we separate the evens and odds by using the variable $2n$ rather than $n$. Let $f_1$, $f_2$ and $f_3$ be selected from $\text{ant}$, $\text{sym}$ and $\text{din}$ and let $d_1$, $d_2$ and $d_3$ be fixed integers. There are nonzero integer coefficient polynomials, $p_1$, $p_2$ and $p_3$, in the variables $n$ and $x$ such that

$$p_1(n, x)f_1(2n + d_1, x) + p_2(n, x)f_2(2n + d_2, x) = p_3(n, x)f_3(2n + d_3, x)$$

holds for all positive $n$ and all $x$. For example: in Lemma 3.4 all three $f_i$ are $\text{ant}$, and we could take $d_1 = 0$, $d_2 = -1$ and $d_3 = 1$. Then $p_1 = 2$, $p_2 = -x$ and $p_3 = 1$. In this paper we will give many examples of such identities but we will not prove any general theorems about the whole class of identities. We do make the following observations. The $p_i$ seem to need the $n$ only when at least one of the $f_i$ is $\text{sym}$. Three-term identities involving only $\text{ant}$ and $\text{din}$ have coefficients that are independent of
A frequent technique in the following three sections is combining two three-term identities by elimination to produce a third three-term identity. The idea is this: suppose we have two three-term identities and they share two $f-d$ pairs. Then we can multiply by suitable polynomials and add the identities so that one of the repeated $f-d$ pairs is cancelled out and the coefficients of the other repeated $f-d$ pair can be combined. This leaves us with only three terms and we have a new three-term identity involving the nonshared $f-d$’s from the original two identities and the $f-d$ that was not eliminated. One goal of the next three sections is to derive a set of three-term identities with $|di| \leq 2$ that is sufficient to derive any three-term identity by repeated elimination. In Section 6, we will give some quadratic three-term identities that summarize an infinite number of linear three-term identities.

The basic formulae for both ant$(n, x)$ and sym$(n, x)$ are in terms of ant$(odd, x)$. The following theorem and Theorem 5.1 use expansion by minors to make the transition from matrices and determinants to polynomial identities.

**Theorem 3.1.** For $n \geq 0$

\[
ant(2n + 1, x) = x^n + \sum_{i=1}^{n} 2i x^{i-1} \text{ant}(2(n - i) + 1, x),
\]

\[
ant(2n, x) = x^n + \sum_{i=1}^{n} (2i - 1)x^{i-1} \text{ant}(2(n - i) + 1, x).
\]

**Proof.** We will do the odd case in more detail. The determinant can be written as a sum of two determinants by writing the first row of $\text{Det}(x \cdot \text{IdentityMatrix}(n) - (A - B)^5)$ as the sum of $[x, 0, \ldots, 0]$ and $[2, 2, \ldots, 2]$. In the second determinant, subtract the first row from all the other rows and then expand by minors on the first column. The 2 at the top of the column is the only nonzero entry and its minor is the matrix for ant$(2(n - 1) + 1, x)$. This gives the $i = 1$ term in the sum. For the first determinant, pull out the factor of $x$ and expand by minors on the first row. We can now apply the same procedure to the first row of the resulting minor by writing its first row as the sum of $[x, 0, \ldots, 0]$ and $[4, 4, \ldots, 4]$. The second determinant gives $2 \cdot 2x^{2-1} \text{ant}(2(n - 2) + 1, x)$, the $i = 2$ term in the sum, and the first determinant gives another factor of $x$ and, after expansion by minors on the first row, the determinant for the next stage of the process. After $n - 1$ stages, we have obtained the terms $i = 1$ to $i = n - 1$ of the sum and the residual

\[
x^{n-2} \text{Det}\begin{pmatrix} x & 0 \\ 2n - 2 & x + 2n \end{pmatrix} = x^n + 2nx^{n-1} \text{ant}(2(n - n) + 1, x),
\]

since ant$(1, x) = 1$. The even case proceeds similarly; since in each stage we subtract odds from odds, we again get the matrices of even numbers that give rise to the ant’s of odd order. However the coefficients are from the matrix before subtraction and hence are odd. □
Corollary 3.2. For \( n \geq 1 \), all of the coefficients of \( \text{ant}(n, x) \) are positive integers. All the roots of \( \text{ant}(n, x) \) are negative.

Proof. The coefficients being positive integers follow by induction using Theorem 3.1. (Note that the \( i = 1 \) term of each sum gives a positive contribution to all but the lead coefficient.) The roots are negative since the coefficients are all positive. \( \square \)

In the odd and even index series for \( \text{ant} \), \( \text{sym} \) and \( \text{din} \), the degree increases by 1 from each polynomial to the next. To discuss this situation with a simple notation, let \( f(n, x) \) be a series of polynomials with \( \text{Degree}(f(n, x)) = n \). What we will call differencing the series gives the series with \( n \)th term \( f(n + 1, x) - xf(n, x) \). A (centered) second differencing yields the series with \( n \)th term \( f(n + 1, x) - 2xf(n, x) + x^2f(n - 1, x) \). Differencing gives connections between \( \text{sym} \), \( \text{ant} \) and \( \text{din} \). Theorem 5.1 shows that the difference of \( \text{sym} \) is \( \text{ant} \) up to a scalar. The definition of \( \text{din} \) is that it is the difference of \( \text{ant} \) (even and odd done separately), and Proposition 4.2 shows that the difference of \( \text{din} \) is 2 \( \text{ant} \). Thus \( \text{ant} \) and \( \text{din} \) form a differencing pair like \( \text{sinh} \) and \( \text{cosh} \) with derivative. This means that \( \text{ant} \) and \( \text{din} \) both have their second difference equal to twice themselves. Because all of the gaps in our point set are the same, this fundamental identity below has coefficients independent of \( n \) and it is the same for even and odd indexes even though it is working separately in the two series. A somewhat more involved proof gives a similar identity for any one-dimensional point set with a reflection symmetry. In the general identity, the term \( x + 1 \) is replaced by \( x + g_n \), where \( g_n \) is the \( n \)th gap from the expansion center.

Theorem 3.3. For \( n \geq 0 \),

\[
\text{ant}(n + 4, x) = 2(x + 1)\text{ant}(n + 2, x) - x^2\text{ant}(n, x).
\]

Proof. We use the formulae from Theorem 3.1, the odd \( n \) case first.

\[
\text{ant}(2n + 3, x) - 2x \text{ant}(2n + 1, x) + x^2\text{ant}(2n - 1, x)
= x^{n+1} + \sum_{i=1}^{n+1} 2ix^{i-1}\text{ant}(2(n + 1 - i) + 1, x) - 2x^{n+1}
- \sum_{j=1}^{n} 4jx^j\text{ant}(2(n - j) + 1, x) + x^{n+1}
+ \sum_{k=1}^{n-1} 2kx^{k+1}\text{ant}(2(n - 1 - i) + 1, x)
= 2\text{ant}(2n + 1, x) + \sum_{j=1}^{n} 2(j + 1)x^j\text{ant}(2(n - j) + 1, x)
\]
\[- \sum_{j=1}^{n} 4j x^j \operatorname{ant}(2(n - j) + 1, x) \]
\[+ \sum_{j=2}^{n} 2(j - 1)x^j \operatorname{ant}(2(n - j) + 1, x) \]
\[= 2 \operatorname{ant}(2n + 1, x). \]

For the even \(n\) case we proceed as above, using the other formula from Theorem 3.1. This time the three sums from \(j = 2\) to \(j = n\) again cancel but the two \(j = 1\) terms do not completely cancel and we are left with

\[\operatorname{ant}(2n + 2, x) - 2x \operatorname{ant}(2n, x) + x^2 \operatorname{ant}(2n - 2, x)\]
\[= \operatorname{ant}(2n + 1, x) + x \operatorname{ant}(2n - 1, x). \]

Thus to complete the proof, we need the following lemma. □

**Lemma 3.4.** For \(n \geq 1\),
\[\operatorname{ant}(2n + 1, x) = 2 \operatorname{ant}(2n, x) - x \operatorname{ant}(2n - 1, x). \]

**Proof.** First use the odd case of Theorem 3.1, and then, after combining sums, use the even case.
\[\operatorname{ant}(2n + 1, x) + x \operatorname{ant}(2n - 1, x)\]
\[= x^n + \sum_{i=1}^{n} 2ix^{i-1} \operatorname{ant}(2(n - i) + 1, x) + x^n\]
\[+ \sum_{j=1}^{n-1} 2jx^j \operatorname{ant}(2(n - 1 - j) + 1, x)\]
\[= 2 \left( x^n + \sum_{i=1}^{n} (2i - 1)x^{i-1} \operatorname{ant}(2(n - i) + 1, x) \right)\]
\[= 2 \operatorname{ant}(2n, x). \quad \square \]

The formulae for the derivatives of the various polynomials are surprisingly simple. This and the odd case formula below could be used to derive a second-order differential equation for \(\operatorname{ant}(n, x)\). We will not do this since we do not need it here. Unfortunately, these simple derivative formulae do not seem to generalize to other point sets. We include them for their intrinsic interest and because they give an easy proof that the roots of \(\operatorname{ant}(n, x)\) are all simple.

**Theorem 3.5.** For \(n \geq 1\),
\[\operatorname{ant}'(2n, x) = n \operatorname{ant}(2n - 1, x). \]
Proof. The proof will be by induction on \( n \). We need two basis cases

\[
\begin{align*}
\text{ant}'(2, x) &= (x + 1)' = 1 = 1 \cdot \text{ant}(1, x), \\
\text{ant}'(4, x) &= (x^2 + 4x + 2)' = 2x + 4 = 2 \cdot (x + 2) = 2 \cdot \text{ant}(3, x).
\end{align*}
\]

Differentiating the formula of Theorem 3.3 with \( n \) replaced by \( 2n - 2 \), we get

\[
\begin{align*}
\text{ant}'(2n + 2, x) &= 2(x + 1)\text{ant}'(2n, x) + 2 \text{ant}(2n, x) \\
&\quad - x^2 \text{ant}'(2n - 2, x) - 2x \text{ant}(2n - 2, x).
\end{align*}
\]

Now use the induction hypothesis to replace the two derivatives of lower-order \( \text{ant} \)'s. Recombining gives

\[
\begin{align*}
&\quad = n \big(2(x + 1)\text{ant}(2n - 1, x) - x^2 \text{ant}(2n - 3, x)\big) \\
&\quad + x^2 \text{ant}(2n - 3, x) + 2 \text{ant}(2n, x) - 2x \text{ant}(2n - 2, x).
\end{align*}
\]

Replace the first two terms using Theorem 3.3, and the third using Lemma 3.4:

\[
\begin{align*}
&\quad = n \text{ant}(2n + 1, x) + x \big(2 \text{ant}(2n - 2, x) - \text{ant}(2n - 1, x)\big) \\
&\quad + 2 \text{ant}(2n, x) - 2x \text{ant}(2n - 2, x) \\
&\quad = (n + 1)\text{ant}(2n + 1, x).
\end{align*}
\]

The last equation is by Lemma 3.4.

Proposition 3.6. For \( n \geq 1 \),

\[
\text{ant}(2n, x) = (2x + 1)\text{ant}(2n - 1, x) - x \text{ant}(2n - 2, x).
\]

Proof. Differentiating the formula of Theorem 3.3 with \( n \) replaced by \( 2n - 2 \) and replacing the derivatives using Theorem 3.5, we get

\[
\begin{align*}
(n + 1)\text{ant}(2n + 1, x) &= 2 \text{ant}(2n, x) + 2(x + 1)n \text{ant}(2n - 1, x) \\
&\quad - 2x \text{ant}(2n - 2, x) - x^2(n - 1)\text{ant}(2n - 3, x).
\end{align*}
\]

By Theorem 3.3 we also have

\[
\begin{align*}
(n - 1)\text{ant}(2n + 1, x) &= 2(x + 1)(n - 1)\text{ant}(2n - 1, x) \\
&\quad - x^2(n - 1)\text{ant}(2n - 3, x).
\end{align*}
\]

Subtracting these two equations gives

\[
\begin{align*}
2 \text{ant}(2n + 1, x) &= 2 \text{ant}(2n, x) + 2(x + 1)\text{ant}(2n - 1, x) \\
&\quad - 2x \text{ant}(2n - 2, x).
\end{align*}
\]

Now divide by 2, replace the LHS using Lemma 3.4, and combine the like terms.
Corollary 3.7. For \( n \geq 0 \),
\[
(2x + 1)\text{ant}'(2n + 1, x) = (2n + 1)\text{ant}(2n, x) - \text{ant}(2n + 1, x).
\]

Proof. Differentiate the formula of Proposition 3.6, use Theorem 3.5 for the even order derivatives and simplify using Lemma 3.4. □

We now catalog some basic facts about the ant polynomials for reference and later use.

Corollary 3.8.
\[
\begin{align*}
\text{ant}(n, 0) &= 2^{\text{Floor}((n-1)/2)}, & \text{ant}(4n, -1) &= (-1)^n, \\
\text{ant}(2n + 1, -1) &= (-1)^{\text{Floor}(n/2)}, & \text{ant}(4n + 2, -1) &= 0, \\
\text{ant}(2n, -1/2) &= 2^{-n}, & \text{ant}(2n + 1, -1/2) &= (2n + 1)2^{-n}.
\end{align*}
\]
The next to leading coefficient of \( \text{ant}(2n, x) \) is \( n^2 \) and of \( \text{ant}(2n + 1, x) \) is \( n(n + 1) \).

Proof. For the values at 0, set \( x = 0 \) in Theorem 3.3 and use induction from the initial values. For the values at \( -1 \), set \( x = -1 \) in Theorem 3.3. For the values at \( -1/2 \) use Proposition 3.6 for even order and then Corollary 3.7 for odd. The formulæ for the next to leading coefficient can be proved by summing the contributions to it in the formulæ of Theorem 3.1 or by induction using Theorem 3.3. □

With the tools now available, we can formulate some algebraic set theory. These results will be the tools used in proving more such results later.

Proposition 3.9. For \( n \geq 0 \), \( \text{GCD}(\text{ant}(n, x), \text{ant}(n + 2, x)) = 1 \).

Proof. A common factor of \( \text{ant}(n + 2, x) \) and \( \text{ant}(n, x) \) would give these polynomials a common root, \( \alpha \). By Corollary 3.8 \( \alpha \) is neither 0 nor \(-1\), so substitution of \( \alpha \) into Theorem 3.3 shows that \( \alpha \) is also a root of \( \text{ant}(n - 2, x) \). We now reach a contradiction by descending to \( \alpha \) being a root of either \( \text{ant}(1, x) = 1 \) or \( \text{ant}(0, x) = 1 \). □

Proposition 3.10. For \( n \geq 0 \), \( \text{GCD}(\text{ant}(n, x), \text{ant}(n + 1, x)) = 1 \).

Proof. Substituting a common root of \( \text{ant}(n, x) \) and \( \text{ant}(n + 1, x) \) into Lemma 3.4 would show that this root (again it is not 0) is also a root of \( \text{ant}(n - 1, x) \) if \( n \) is even or \( \text{ant}(n + 1, x) \) if \( n \) is odd. In either case this contradicts Proposition 3.9. □

We now have a simple proof of the key fact that the roots of \( \text{ant}(n, x) \) are all simple.

Theorem 3.11. For \( n \geq 0 \), \( \text{ant}(n, x) \) does not have repeated roots.
Proof. By Theorem 3.5, a repeated root of ant(2n, x) would also be a root of ant(2n − 1, x), and by Corollary 3.7, a repeated root of ant(2n + 1, x) would also be a root of ant(2n, x); in both cases contradicting Proposition 3.10. □

4. A family of polynomials related to ant(n, x)

We now repeat the work of the last section for the din family of polynomials. The din polynomials seem to have little obvious relevance to the distance matrix of {1, 2, 3, . . . , n}, although it turns out they are the values of the determinant of a bordered distance matrix. The din polynomials are in fact a sort of characteristic polynomial. Their roots are values that are eigenvalues too within a multiple of the all ones vector-relative eigenvalues. The din polynomials give the symmetric relative eigenvalues corresponding to the antisymmetric true eigenvalues given by the ant polynomials. In a sense, the din polynomials are what the sym polynomials should have been but are not. We need to study the din polynomials because they keep popping up and because they prove to be so useful. The din polynomials are the key to the generalization of the results of this paper to more irregular point sets. They have properties similar to those of the ant polynomials, but are in some ways simpler. This will become clear in Section 7 when we do the divisibility results. As mentioned before Theorem 3.3, the din polynomials are obtained by differencing the ant polynomials.

Definition 4.1. Let din(0, x) = 0, and for n ≥ 1, let
\[ \text{din}(n, x) = \text{ant}(n + 1, x) - x \text{ant}(n - 1, x). \]

din(n, x) is not a monic polynomial, so at least some of its roots are not algebraic integers, as all of the roots of ch(n, x) are. The first few values are: din(1, x) = 1, din(2, x) = 2, din(3, x) = 3x + 2, din(4, x) = 4x + 4, din(5, x) = 5x^2 + 10x + 4, din(6, x) = 6x^2 + 16x + 8, and din(7, x) = 7x^3 + 28x^2 + 28x + 8. Some obvious patterns are confirmed by the following Proposition.

Proposition 4.1. The degree of din(2n, x) is n − 1. The degree of din(2n + 1, x) is n. The leading coefficient of din(n, x) is n.

Proof. Using the definition and Corollary 3.8, we have
\[ \text{din}(2n, x) = (x^n + n(n + 1)x^{n-1} + \cdots) - x(x^{n-1} + n(n - 1)x^{n-2} + \cdots) \]
\[ = 2nx^{n-1} + \cdots, \]
\[ \text{din}(2n + 1, x) = (x^{n+1} + (n + 1)^2x^n + \cdots) - x(x^n + n^2x^{n-1} + \cdots) \]
\[ = (2n + 1)x^n + \cdots. \] □
The next two results show the effects of differencing. Differencing din gives back 2 ant, so second differencing will come back to 2 din. Both results are the same for even and odd indexes.

**Proposition 4.2.** For \( n \geq 1 \),
\[
2 \text{ant}(n, x) = \text{din}(n + 1, x) - x \text{din}(n - 1, x).
\]

**Proof.** Just difference the definition of din and use Theorem 3.3. \( \square \)

As in the case of Theorem 3.3, a more complicated proof starting from the matrix definition of the din polynomials shows that the following recursion generalizes to any one-dimensional point set.

**Proposition 4.3.** For \( n \geq 0 \),
\[
\text{din}(n + 4, x) - 2(x + 1) \text{din}(n + 2, x) + x^2 \text{din}(n, x) = 0.
\]

**Proof.** Just difference Proposition 4.2 and use the definition of din. \( \square \)

The next two results relate the even and odd series of ant and din. As explained in the beginning of the section on ant polynomials, these results (along with the results for ant in that section) enable us to derive a three-term identity for any combination of ant and din.

**Proposition 4.4.** For \( n \geq 0 \),
\[
\text{ant}(2n + 1, x) = \text{din}(2n + 1, x) - x \text{din}(2n, x).
\]

**Proof.** By induction on \( n \). Subtract the induction hypothesis from Proposition 4.2 getting
\[
\text{ant}(2n - 1, x) = \text{din}(2n, x) - \text{din}(2n - 1, x).
\]
Multiply by \( x \) and subtract again from Proposition 4.2,
\[
\text{din}(2n + 1, x) - x \text{din}(2n, x) = 2 \text{ant}(2n, x) - x \text{ant}(2n - 1, x)
= \text{ant}(2n + 1, x).
\]
The last step is by Lemma 3.4. \( \square \)

**Corollary 4.5.** For \( n \geq 1 \),
\[
\begin{align*}
\text{din}(2n, x) &= \text{din}(2n - 1, x) + \text{ant}(2n - 1, x), \\
2 \text{ant}(2n + 1, x) &= 2 \text{ant}(2n, x) + \text{din}(2n, x), \\
2 \text{ant}(2n, x) &= (2x + 1)\text{ant}(2n - 1, x) + \text{din}(2n - 1, x), \\
2 \text{ant}(2n, x) &= \text{din}(2n, x) + 2x \text{ant}(2n - 1, x), \\
2 \text{ant}(2n, x) &= (2x + 1)\text{din}(2n, x) - 2x \text{din}(2n - 1, x).
\end{align*}
\]
Proof. For the first formula, eliminate \( x \) \( \text{din}(2n - 2, x) \) from Proposition 4.2 and Proposition 4.4. For the second formula, eliminate \( \text{ant}(2n - 1, x) \) from the definition of \( \text{din} \) and Lemma 3.4. For the third formula, eliminate \( x \) \( \text{ant}(2n - 2, x) \) from the definition of \( \text{din} \) and Proposition 3.6. For the fourth formula, eliminate \( \text{ant}(2n + 1, x) \) from formula 2 and Lemma 3.4. For the fifth formula, eliminate \( \text{ant}(2n - 1, x) \) from formulas 3 and 4. □

Corollary 4.6. For \( n \geq 1 \), all of the coefficients of \( \text{din}(n, x) \) are positive integers.

Proof. The proof is by induction on \( n \); \( \text{din}(1, x) = 1 \) and \( \text{din}(2, x) = 2 \). We use Corollary 3.2 and Proposition 4.4 for the step from even \( n \) to \( n + 1 \), and the first formula of Corollary 4.5 for the step up from odd \( n \). □

Here we collect for reference some miscellaneous facts about the values of \( \text{din} \).

Corollary 4.7.

\[
\begin{align*}
\text{din}(2n, 0) &= 2^n, \\
\text{din}(2n + 1, 0) &= 2^n, \\
\text{din}(4n, -1) &= 0, \\
\text{din}(4n + 1, -1) &= (-1)^n, \\
\text{din}(4n + 2, -1) &= 2(-1)^n, \\
\text{din}(4n + 3, -1) &= (-1)^{n+1}, \\
\text{din}(2n, -1/2) &= n \ 2^{-n+2}, \\
\text{din}(2n + 1, -1/2) &= 2^{-n}.
\end{align*}
\]

Proof. The proofs are straightforward applications of the above identities and Corollary 3.8. □

The next series of results are more of algebraic set theory. These are the starter results which we need for the much more complete analysis in Section 7.

Corollary 4.8. For \( n \geq 1 \), \( \text{GCD}(\text{din}(n, x), \text{din}(n + 1, x)) = 1 \).

Proof. There can be no integer common factor because the leading coefficient of \( \text{din}(n, x) \) is \( n \) and of \( \text{din}(n + 1, x) \) is \( n + 1 \). For \( n \) even, a common root of \( \text{din}(2n, x) \) and \( \text{din}(2n + 1, x) \) would, by Proposition 4.4, also be a root of \( \text{ant}(2n + 1, x) \), and then a common root of \( \text{din}(2n, x) \) and \( \text{ant}(2n + 1, x) \) would, by the definition of \( \text{din}(2n, x) \), also be a root of \( \text{ant}(2n - 1, x) \), a contradiction to Proposition 3.9. For \( n \) odd, a common root of \( \text{din}(2n + 1, x) \) and \( \text{din}(2n + 2, x) \) would, by the first formula of Corollary 4.5, also be a root of \( \text{ant}(2n + 1, x) \), and then a common root of \( \text{din}(2n + 2, x) \) and \( \text{ant}(2n + 1, x) \) would, by Proposition 4.2, also be a root of \( \text{din}(2n, x) \), a contradiction to the even \( n \) case of this proof. □

Corollary 4.9. For \( n \geq 1 \), \( \text{GCD}(\text{ant}(n, x), \text{din}(n, x)) = 1 \).
Proof. There is no integer common factor because the leading coefficient of ant\((n, x)\) is 1. For \(n\) even, a common root of ant\((2n, x)\) and din\((2n, x)\) would, by the second formula of Corollary 4.5, also be a root of ant\((2n + 1, x)\). For \(n\) odd, a common root of ant\((2n + 1, x)\) and din\((2n + 1, x)\) would, by Proposition 4.4, also be a root of din\((2n, x)\). □

Corollary 4.10. For \(n \geq 1\), GCD\((\text{ant}(n, x), \text{din}(n + 1, x)) = 1\) and GCD\((\text{ant}(n + 1, x), \text{din}(n, x)) = 1\).

Proof. In both cases, a common root would, by the definition of din, propagate to another ant. □

Corollary 4.11. For \(n \geq 1\),
\[
\text{GCD}(\text{din}(n, x), \text{din}(n + 2, x)) = \begin{cases} 
1 & \text{if } n \text{ odd}, \\
2 & \text{if } n \text{ even}.
\end{cases}
\]

Proof. A common root could be run back to din\((2, x) = 2\) or to din\((1, x) = 1\) using Proposition 4.3. For \(n\) odd there will be no common integer factor because the leading coefficients are relatively prime. For \(n\) even, the GCD of the leading coefficients is 2, and we can show 2\(\text{din}(2n, x)\) by induction starting from din\((2, x)\) and din\((4, x) = 4x + 4\) using Proposition 4.3. □

The formulae for the derivative of even and odd index din polynomials are again simple and useful for proving that din does not have repeated roots. These formula do not seem to generalize.

Theorem 4.12. For \(n \geq 1\),
\[
\begin{align*}
\text{din}'(2n + 1, x) &= (n + 1/2)\text{din}(2n, x), \\
(2x + 1)\text{din}'(2n, x) &= 2n \text{ din}(2n - 1, x) - \text{din}(2n, x).
\end{align*}
\]

Proof. In the first formula, note that din\((2n, x)\) has a factor of 2 in it, so the equation does indeed involve integer coefficients polynomials. To prove the first formula, we differentiate the definition of din\((2n + 1, x)\) and replace the derivatives of the ant terms using Theorem 3.5 to get
\[
(n + 1) \text{ ant}(2n + 1, x) - nx \text{ ant}(2n - 1, x) - \text{ant}(2n, x),
\]
\[
= n (\text{ant}(2n + 1, x) - x \text{ ant}(2n - 1, x)) + \text{ant}(2n + 1, x) - \text{ant}(2n, x)
\]
\[
= n \text{ din}(2n, x) + (1/2)\text{din}(2n, x).
\]

The last line is by the definition of din\((2n, x)\) and the second formula of Corollary 4.5. To prove the second formula, again differentiate the definition, and then this time multiply by \(2x + 1\) so that Corollary 3.7 can be used to give
\[(2x + 1) \text{din}'(2n, x)\]
\[= (2n + 1) \text{ant}(2n, x) - \text{ant}(2n + 1, x) - x(2n - 1) \text{ant}(2n - 2, x)\]
\[+ x \text{ant}(2n - 1, x) - (2x + 1) \text{ant}(2n - 1, x)\]
\[= (2n - 1) \text{din}(2n - 1, x) - \text{din}(2n, x) + 2 \text{ant}(2n, x)\]
\[- (2x + 1) \text{ant}(2n - 1, x)\]
\[= 2n \text{din}(2n - 1, x) - \text{din}(2n, x).\]

The second line is by the definition of din twice and the third line is by the third formula of Corollary 4.5. □

**Corollary 4.13.** For \(n \geq 2\), \(\text{din}(n, x)\) has no repeated roots.

**Proof.** For odd \(n\) the GCD of \(\text{din}(2n + 1, x)\) and its derivative is 1 by Theorem 4.12 and Corollary 4.8. For even \(n\) we note that, by Corollary 4.7, \(-1/2\) is never a root of a din so we can multiply by \(2x + 1\) and proceed as for odd \(n\):

\[
\text{GCD}(\text{din}(2n, x), \text{din}'(2n, x))
\]
\[= \text{GCD}(\text{din}(2n, x), (2x + 1)\text{din}'(2n, x))\]
\[= \text{GCD}(\text{din}(2n, x), 2n \text{din}(2n - 1, x) - \text{din}(2n, x))\]
\[= \text{GCD}(\text{din}(2n, x), \text{din}(2n - 1, x))\]
\[= 1.\]

(The GCD’s are over \(\mathbb{Q}[x]\) since we are only concerned with common roots.) □

**5. Symmetric characteristic polynomials**

In this section we continue the development of three-term identities, now involving the sym polynomials. The sym polynomials have a different flavor than the ant and din polynomials. All sym polynomials are relatively prime to each other; there are no divisibility patterns and symmetric eigenvalues do not re-occur. Symmetric eigenvectors are not repeated to build larger symmetric eigenvectors. The symmetric relative eigenvalues given by the din polynomials are the ones that build bigger eigenvectors by repeating a smaller one. Three-term identities involving sym often have a degree dependent coefficient. For example, in Theorem 5.1 we see that ant is essentially the difference of sym but there is a degree dependent coefficient. This means that the second difference of sym is not simply din, the difference of ant; another term appears. Nevertheless, the sym polynomials can be related to the ant and din polynomials by three-term identities and we will provide the basic tools to do so in this section. The first theorem again makes the transition from matrices and determinants to polynomials using expansion by minors.
Theorem 5.1. For \( n \geq 2 \),
\[
sym(n, x) = x \sym(n - 2, x) - (n - 1) \ant(n - 1, x).
\]

Proof. The even case is simpler. The determinant for \( \sym(2n, x) \) can be written as a sum of two determinants by writing the first row of
\[
\text{Det}(x \cdot \text{IdentityMatrix}(n) - (A + B))
\]
as the sum of \([x, 0, \ldots, 0]\) and \([-2n + 1, -2n + 1, \ldots, -2n + 1]\). In the second determinant, subtract the first row from all the other rows and then expand by minors on the first column. The \(-2n + 1\) at the top of the column is the only nonzero entry and its minor is the matrix for \( \ant(2n - 1, x) \). For the first determinant, pull out the factor of \( x \) and expand by minors on the first row, with the minor of \( x \) being the matrix for \( \sym(2n - 2, x) \).

The odd case is somewhat complicated by the extra row and column in \( \text{UCB}(A + B, u, 0) \). We begin as above, writing the first row of
\[
\text{Det}(x \cdot \text{IdentityMatrix}(n) - \text{UCB}(A + B, u, 0))
\]
as the sum of \([x, 0, \ldots, 0]\) and \([-2n, -2n, \ldots, -2n]\). The first determinant can again be expanded on the first row to give \( x \sym(2n - 1, x) \). In the second determinant, we again subtract the first row from rows 2 through \( n \), subtract half the first row from the last row and then prepare to expand along the first column. If we double the last row, the minor of \(-2n\) would be the determinant for \( \ant(2n + 1, x) \) except the lower right corner entry which is \( 2x - 2n \) instead of \( x - 2n \). So we evaluate this minor by breaking it into two determinants writing the last row as \([0, \ldots, 0, x]\) plus \([2, 4, \ldots, x - 2n]\). The second determinant here is \( \ant(2n + 1, x) \) and the first can be expanded along the last row; the minor of \( x \) being just \( \ant(2n - 1, x) \). Putting the pieces together we get
\[
sym(2n + 1, x) = x \sym(2n - 1, x) - 2n \left( \frac{1}{2} \right) (\ant(2n + 1, x) + x \ant(2n - 1, x))
\]
\[
= x \sym(2n - 1, x) - 2n \ant(2n, x),
\]
where the last equation follows by Lemma 3.4. ☐

We can use the theorem above to produce sum of ant formulae for sym that are analogous to Theorem 3.1.

Corollary 5.2. For \( n \geq 2 \), all coefficients of \( \sym(n, x) \) are negative integers except the lead coefficient (which is 1).
Proof. If we iterate the formula of Theorem 5.1, first for even orders
\[ \text{sym}(2n, x) = x^n - \sum_{i=1}^{n} (2i - 1)x^{n-i}\text{ant}(2i - 1, x) \]
and then for odd orders
\[ \text{sym}(2n + 1, x) = x^{n+1} - \sum_{i=1}^{n} 2ix^{n-i}\text{ant}(2i, x), \]
the result follows from Corollary 3.2. □

The second difference of sym is not as elegant as the formulae for the second differences of ant and din.

Proposition 5.3. For \( n \geq 2 \),
\[ \text{sym}(n + 2, x) - 2x \text{sym}(n, x) + x^2 \text{sym}(n - 2, x) = -n \text{din}(n, x) - \begin{cases} 2\text{ant}(n, x), & n \text{ even,} \\ (2x + 1)\text{ant}(n, x), & n \text{ odd.} \end{cases} \]

Proof. Writing Theorem 5.1 as a difference, differencing again, and using the definition of din show that the LHS is equal to \(-n \text{din}(n, x) - (\text{ant}(n + 1, x) + x \text{ ant}(n - 1, x))\). We then replace the ant terms using Lemma 3.4 for \( n \) even and Proposition 3.6 for \( n \) odd. □

As was the case for ant, some of the first identities for sym must be proved by induction. After we have a few we can grind out more using elimination.

Theorem 5.4. For \( n \geq 1 \),
\[ \begin{align*}
\text{sym}(2n, x) &= \text{sym}(2n - 1, x) - n \text{ant}(2n - 1, x), \\
\text{sym}(2n + 1, x) &= x \text{sym}(2n, x) - n \text{ant}(2n + 1, x).
\end{align*} \]

Proof. The proof of the first equation is by induction on \( n \). Multiplying the induction hypothesis by \( x \)
\[ \begin{align*}
0 &= x \text{sym}(2n - 2, x) - x \text{sym}(2n - 3, x) + (n - 1)x \text{ant}(2n - 3, x), \\
0 &= x \text{sym}(2n - 2, x) - x \text{sym}(2n - 3, x) \\
& \quad + (n - 1)(2\text{ant}(2n - 2, x) - \text{ant}(2n - 1, x)), \\
0 &= (x \text{sym}(2n - 2, x) - (2n - 1)\text{ant}(2n - 1, x)) + n \text{ant}(2n - 1, x) \\
& \quad - (x \text{sym}(2n - 3, x) - 2(n - 1)\text{ant}(2n - 2, x)), \\
0 &= \text{sym}(2n, x) + n \text{ant}(2n - 1, x) - \text{sym}(2n - 1, x).
\end{align*} \]
The second line is by Lemma 3.4 and the last by Theorem 5.1 twice. For the second equation, we start with the first equation
\[
sym(2n + 1, x) = sym(2n + 2, x) + (n + 1)ant(2n + 1, x),
\]
and Theorem 5.1
\[
x sym(2n, x) = sym(2n + 2, x) + (2n + 1)ant(2n + 1, x),
\]
and subtract. □

Corollary 5.5. For \( n \geq 1 \),
\[
n sym(2n + 2, x) = (2n + 1)sym(2n + 1, x) - x(n + 1)sym(2n, x).
\]

Proof. Eliminate the ant(odd, x) from the two formulae of Theorem 5.4. □

Theorem 5.6. For \( n \geq 1 \),
\[
2n ant(2n + 1, x) = (2n + 1)ant(2n, x) - sym(2n, x),
\]
\[
(2n - 1)ant(2n, x) = 2nx ant(2n - 1, x) - sym(2n, x),
\]
\[
(2x + 1)sym(2n, x) = 2x sym(2n - 1, x) - (2n - 1)ant(2n, x),
\]
\[
2 sym(2n + 1, x) = (2x + 1)sym(2n, x) - (2n + 1)ant(2n, x),
\]
\[
(2n - 1)sym(2n + 1, x) = 2n(2x + 1)sym(2n, x)
\]
\[
- (2n + 1)x sym(2n - 1, x).
\]

Proof. The first formula is proved by induction. Multiplying the induction hypothesis by \( x \) and then using Theorem 5.1 give
\[
2x(n - 1)ant(2n - 1, x)
\]
\[
= x(2n - 1)ant(2n - 2, x) - sym(2n, x) - (2n - 1)ant(2n - 1, x),
\]
\[
(2n - 1 + 2x(n - 1))ant(2n - 1, x)
\]
\[
= (2n - 1)((2x + 1)ant(2n - 1, x) - ant(2n, x)) - sym(2n, x),
\]
\[
4n ant(2n, x) - 2xn ant(2n - 1, x) = (2n + 1)ant(2n, x) - sym(2n, x),
\]
\[
2n ant(2n + 1, x) = (2n + 1)ant(2n, x) - sym(2n, x).
\]

Line 2 is by Proposition 3.6, and line 4 is by Lemma 3.4. The second formula follows by elimination from the first formula and Lemma 3.4. The third formula follows by elimination from the second formula of this theorem and the first formula of Theorem 5.4. The fourth formula follows by elimination from the first formula of this theorem and the second formula of Theorem 5.4. The fifth formula follows by elimination from the third and fourth formulae of this theorem. □

Combining results from the last three sections using elimination, we can derive three-term identities connecting all three kinds of polynomials.
Theorem 5.7. For \( n \geq 0 \),

\[
\begin{align*}
n \ \operatorname{din}(2n, x) &= \operatorname{ant}(2n, x) - \operatorname{sym}(2n, x), \\
(2n + 1)\operatorname{din}(2n + 1, x) &= (2x + 1)\operatorname{ant}(2n + 1, x) - 2\operatorname{sym}(2n + 1, x).
\end{align*}
\]

**Proof.** For the first formula, eliminate \( \operatorname{ant}(2n + 1, x) \) from the first formula of Theorem 5.6 and the second formula of Corollary 4.5. For the second formula, eliminate \( \operatorname{sym}(2n, x) \) from the second formula of Theorem 5.4 and the first formula of Theorem 5.6 to get an identity with \( \operatorname{sym}(2n + 1, x), \operatorname{ant}(2n + 1, x) \), and \( \operatorname{ant}(2n, x) \). Then eliminate \( \operatorname{ant}(2n + 2, x) \) from the third formula of Corollary 4.5 and the definition of \( \operatorname{din} \) to get an identity with \( \operatorname{din}(2n + 1, x), \operatorname{ant}(2n + 1, x) \), and \( \operatorname{ant}(2n, x) \). Then eliminate \( \operatorname{ant}(2n, x) \) from these two derived identities. \( \square \)

The formulae for the derivatives of sym polynomials are much like those for the ant polynomials and they can be used to show that sym polynomials do not have repeated roots.

Theorem 5.8. For \( n \geq 1 \),

\[
(2x + 1)\operatorname{sym}'(2n, x) = 2n\operatorname{sym}(2n, x) + n(2n + 1)\operatorname{ant}(2n - 1, x).
\]

**Proof.** The proof is by induction on \( n \); starting with Theorem 5.1 differentiate

\[
\begin{align*}
\operatorname{sym}'(2n + 1, x) &= \operatorname{sym}(2n - 1, x) + nx\operatorname{sym}(2n - 2, x) - 2n^2\operatorname{ant}(2n - 1, x) \\
&= \operatorname{sym}(2n - 1, x) + n\operatorname{sym}(2n, x) - n\operatorname{ant}(2n - 1, x) \\
&= n\operatorname{sym}(2n, x) + \operatorname{sym}(2n, x) = (n + 1)\operatorname{sym}(2n, x).
\end{align*}
\]

The first line is by the induction hypothesis and Theorem 3.5, the second line by Theorem 5.1, and the last line by the first formula of Theorem 5.4. \( \square \)

Proposition 5.9. For \( n \geq 1 \),

\[
(2x + 1)\operatorname{sym}'(2n, x) = 2n\operatorname{sym}(2n, x) + n(2n + 1)\operatorname{ant}(2n - 1, x).
\]

**Proof.** Differentiate the fourth formula of Theorem 5.6 using Theorem 3.5 and Theorem 5.8. \( \square \)

As in the previous two sections, we collect for reference some miscellaneous facts about the values of sym.

**Corollary 5.10.**

\[
\begin{align*}
\operatorname{sym}(2n, 0) &= -(2n - 1)2^{n-1}, \\
\operatorname{sym}(2n + 1, 0) &= -n \ 2^n, \\
\operatorname{sym}(4n, -1) &= (-1)^n, \\
\operatorname{sym}(4n + 1, -1) &= (2n + 1)(-1)^{n+1},
\end{align*}
\]
\[ \text{sym}(4n + 2, -1) = 2(2n + 1)(-1)^{n+1}, \quad \text{sym}(4n + 3, -1) = (2n + 1)(-1)^n, \]
\[ \text{sym}(2n + 1, -1/2) = -(2n + 1)2^{-n-1}, \quad \text{sym}(2n, -1/2) = -(4n^2 - 1)2^{-n}. \]

**Proof.** For the values at 0, set \( x = 0 \) in Theorem 5.1 and use the values for \( \text{ant}(k, 0) \).

For the values at \(-1\), use the first two formulae of Theorem 5.6 and the values of \( \text{ant}(k, -1) \) for even \( n \) and then use Theorem 5.4 to get odd \( n \). For the values at \(-1/2\), use the values of \( \text{ant}(k, -1/2) \) and Proposition 5.9 for even \( n \) and the fourth formula of Theorem 5.6 for odd \( n \). \( \Box \)

The divisibility pattern of the sym polynomials differs markedly from that of ant and din. The typical ant or din polynomial breaks up into several factors over the integers and these factors reappear in infinitely many other ant and din polynomials. The situation for the sym polynomials is summarized in the following conjecture.

**Conjecture 5.11.** The sym polynomials for different indexes are relatively prime, so the roots of the various \( \text{sym}(n, x) \) are all different; and furthermore, for \( n \geq 1 \), \( \text{sym}(n, x) \) is irreducible over the integers.

We will only prove a small portion of this conjecture in this paper. The following algebraic set theory results are needed in later sections, but they also can be seen as fragments of the above conjecture.

**Proposition 5.12.** For \( n \geq 0 \), \( \text{GCD}(\text{sym}(n + 1, x), \text{sym}(n, x)) = 1 \).

**Proof.** For \( n \) odd, a common root of \( \text{sym}(2k + 2, x) \) and \( \text{sym}(2k + 1, x) \) will also be a root of \( \text{sym}(2k, x) \) by Corollary 5.5. For \( n \) even, a common root \( \text{sym}(2k + 1, x) \) and \( \text{sym}(2k, x) \) will also be a root of \( \text{sym}(2k - 1, x) \) by the fifth formula of Theorem 5.6. With these two methods to perform the descent, we can run a common root down to \( \text{sym}(0, x) = 1 \) and a contradiction. \( \Box \)

**Proposition 5.13.** For \( n \geq 0 \), \( \text{GCD}(\text{sym}(n + 2, x), \text{sym}(n, x)) = 1 \).

**Proof.** For \( n \) even, a common root of \( \text{sym}(2k + 2, x) \) and \( \text{sym}(2k, x) \) will also be a root of \( \text{sym}(2k + 1, x) \) by Corollary 5.5. For \( n \) odd, a common root \( \text{sym}(2k + 1, x) \) and \( \text{sym}(2k - 1, x) \) will also be a root of \( \text{sym}(2k, x) \) by the fifth formula of Theorem 5.6. Either way we contradict Proposition 5.12. \( \Box \)

**Theorem 5.14.** For \( n \geq 0 \), \( \text{sym}(n, x) \) has no repeated roots.

**Proof.** For \( n \) odd, a repeated root of \( \text{sym}(2k + 1, x) \) would give a common root of it and its derivative \((k + 1)\text{sym}(2k, x)\) contradicting Proposition 5.12. For \( n \) even, a repeated root of \( \text{sym}(2k, x) \) would also be a root of its derivative and by
Proposition 5.9 it would also be a root of \( \text{ant}(2k - 1, x) \). Then by the first formula of Theorem 5.4 it would also be a root of \( \text{sym}(2k - 1, x) \), and now we have contradicted Proposition 5.12 again. □

6. Multiplicative identities

The three-term identities of the last three sections involved polynomials whose indices had (small) fixed differences. We now move on to some identities involving polynomials whose indices are related multiplicatively. We will use these identities to explore the rich divisibility structure of the ant and din polynomials in the following two sections. The first theorem below could be proved by induction using Theorem 6.4 but that would not reveal the connection with the eigenvectors. The real import of the following theorem is not just that the eigenvalues repeat, but that the eigenvectors also repeat.

**Theorem 6.1.** For \( n \geq 1 \) and \( k \geq 0 \), \( \text{ant}(n, x) \) divides \( \text{ant}(n(2k + 1), x) \).

**Proof.** Since ant polynomials have no repeated roots, the theorem claims that every root of \( \text{ant}(n, x) \) is also a root of \( \text{ant}(n(2k + 1), x) \). We will use the fact that the roots of \( \text{ant}(n(2k + 1), x) \) are the antisymmetric eigenvalues of \( M_{n(2k + 1)} \) and produce an antisymmetric eigenvector of \( M_{n(2k + 1)} \) whose eigenvalue is a given root, \( \alpha \), of \( \text{ant}(n, x) \). Let \( v = [a_1, a_2, \ldots, a_n]^T \) be an antisymmetric eigenvector of \( M_n \) for eigenvalue \( \alpha \). We will show that the block vector \([v, -v, v, \ldots, v]^T\) with \( k + 1 \) copies of \( v \) alternating with \( k \) copies of \(-v\) is an antisymmetric (this is clear) eigenvector of \( M_{n(2k + 1)} \) for eigenvalue \( \alpha \). \( M_{n(2k + 1)} \) can be written as a \((2k + 1)\) by \((2k + 1)\) block matrix whose blocks are all \( n \) by \( n \) matrices. Let \( J[i, j] = 1 \) and \( A[i, j] = n + j - i \). Then the diagonal blocks are copies of \( M_n \) and the off diagonal \( ij \)th block is \( A + (j - i - 1)nJ \) for \( j > i \) and \( A^T + (i - j - 1)nJ \) for \( i > j \). Note that \( M_nv = \alpha v \), \( Jv = 0 \) (by antisymmetry), and \((A + A^T)v = 2nJv = 0\). The \( j \)th row of the product of the block matrix and the block vector is

\[
(A^T + (j - 2)nJ)v - (A^T + (j - 3)nJ)v + \cdots + (-1)^j A^Tv \\
+(-1)^{j+1}Mv + (-1)^{j+2}Av + \cdots + (A + (2k - j)nJ)v.
\]

In this sum there are no \( A^T \) terms if \( j = 1 \) and no \( A \) terms if \( j = 2k + 1 \). After dropping products \( Jv \), we can cancel pairs \( A^Tv \) and \(-A^Tv \) and pairs \( Av \) and \(-Av \). If \( j \) is odd, there is nothing left of the terms from off-diagonal. If \( j \) is even, there is an \( A^Tv \) and an \( Av \) left from the off-diagonal terms, but they then cancel each other. The diagonal term gives \((-1)^{j+1}Mv = (-1)^{j+1}\alpha v\) as required. □

We now develop explicit identities relating the ant and din of index \( n \) to the ant and din with index a multiple of \( n \). The key formula is the second formula of the
following lemma, a special case of Theorem 6.5 and of the factorization of \( \text{din} \) to be developed in the following section. This formula holds for any one-dimensional point set with a reflection symmetry. It can be written as

\[
\text{din}(2n, x) = 2 \text{din}(n, x) \text{ant}(n, x),
\]

and it shows that the product of \( \text{ant} \) and \( \text{din} \) can be interpreted as a \( \text{din} \) function. Thus the product can be expected to have the sort of properties a \( \text{din} \) function has. Lemma 6.3 is a good example of this re-interpretation of a \( \text{din} \) identity as an identity for the product of \( \text{ant} \) and \( \text{din} \).

**Lemma 6.2.** For \( n \geq 0 \),

\[
\begin{align*}
\text{din}(2n + 2, x) &= \text{ant}(n + 2, x) \text{din}(n, x) + \text{ant}(n, x) \text{din}(n + 2, x), \\
\text{din}(2n + 2, x) &= 2 \text{din}(n + 1, x) \text{ant}(n + 1, x).
\end{align*}
\]

**Proof.** The proof is by induction; we will use \( n \) and \( n + 1 \) to show \( n + 2 \). Thus the induction hypotheses are the two above equations for \( \text{din}(2n + 2, x) \) and for \( \text{din}(2n + 4, x) \). We then use Proposition 4.3

\[
\text{din}(2n + 6, x) = 2(x + 1) \text{din}(2n + 4, x) - x^2 \text{din}(2n + 2, x).
\]

Next we use the induction hypothesis to get this equal to

\[
\begin{align*}
4(x + 1) &\text{din}(n + 2, x) \text{ant}(n + 2, x) \\
&- x^2 \left( \text{ant}(n + 2, x) \text{din}(n, x) + \text{ant}(n, x) \text{din}(n + 2, x) \right) \\
&= 2(x + 1) \text{din}(n + 2, x) \text{ant}(n + 2, x) - x^2 \text{ant}(n + 2, x) \text{din}(n, x) \\
&+ 2(x + 1) \text{din}(n + 2, x) \text{ant}(n + 2, x) - x^2 \text{ant}(n, x) \text{din}(n + 2, x) \\
&= \text{ant}(n + 2, x) \text{din}(n + 4, x) + \text{ant}(n + 4, x) \text{din}(n + 2, x)
\end{align*}
\]

using Proposition 4.3 and Theorem 3.3. To show the other equation we again start with expression for \( \text{din}(2n + 6, x) \) and substitute using the induction hypothesis to get

\[
4(x + 1) \text{din}(n + 2, x) \text{ant}(n + 2, x) - 2x^2 \text{din}(n + 1, x) \text{ant}(n + 1, x).
\]

We now use the definition of \( \text{din} \) and Proposition 4.2 to get

\[
\begin{align*}
4x &\text{din}(n + 2, x) \text{ant}(n + 2, x) - 2x^2 \text{din}(n + 1, x) \text{ant}(n + 1, x) \\
&+ 2(\text{ant}(n + 3, x) - x \text{ant}(n + 1, x)) \left( \text{din}(n + 3, x) - x \text{din}(n + 1, x) \right) \\
&= 4x \text{din}(n + 2, x) \text{ant}(n + 2, x) + 2 \text{ant}(n + 3, x) \text{din}(n + 3, x) \\
&- 2x(\text{ant}(n + 3, x) \text{din}(n + 1, x) + \text{ant}(n + 1, x) \text{din}(n + 3, x)) \\
&= 2 \text{ant}(n + 3, x) \text{din}(n + 3, x)
\end{align*}
\]

using the induction hypothesis one more time for the last equation. □
Lemma 6.3. For $n \geq 1$,

$$\text{ant}(n + 1, x) \text{din}(n + 1, x) = 2(x + 1)\text{ant}(n, x) \text{din}(n, x) - x^2\text{ant}(n - 1, x) \text{din}(n - 1, x).$$

**Proof.** Apply the second formula of Lemma 6.2 to Proposition 4.3. □

The following two theorems can be regarded as providing an infinite list of three-term identities, one for each value of $k$. If the ant()din() products are replaced using the second formula of Lemma 6.2, we could take $2k$ as the index and get another infinite series of three-term identities, one for each $n$. In the following section we will use these theorems in a multiplicative role, replacing $k$ with $nk$, or replacing $n$ by 0, or both.

**Theorem 6.4.** For $n \geq 0$ and $k \geq 1$

$$\text{ant}(n + 2k, x) = \text{ant}(k, x) \text{din}(k, x)\text{ant}(n + 2, x) - x^2\text{ant}(k - 1, x) \text{din}(k - 1, x)\text{ant}(n, x).$$

**Proof.** The proof is by induction on $k$. $k = 1$ is trivial and $k = 2$ is Theorem 3.3. Using Theorem 3.3 and then the induction hypothesis we get

$$\begin{align*}
\text{ant}(n + 2(k + 1), x) &= 2(x + 1)\text{ant}(n + 2k, x) - x^2\text{ant}(n + 2(k - 1), x) \\
&= 2(x + 1)(\text{ant}(k, x) \text{din}(k, x)\text{ant}(n + 2, x) \\
&\quad - x^2\text{ant}(k - 1, x) \text{din}(k - 1, x)\text{ant}(n, x)) \\
&\quad - x^2(\text{ant}(k - 1, x) \text{din}(k - 1, x)\text{ant}(n + 2, x) \\
&\quad - x^2\text{ant}(k - 2, x) \text{din}(k - 2, x)\text{ant}(n, x)) \\
&= (2(x + 1)\text{ant}(k, x) \text{din}(k, x) - x^2\text{ant}(k - 1, x) \text{din}(k - 1, x))\text{ant}(n + 2, x) \\
&\quad - x^2(2(x + 1)\text{ant}(k - 1, x) \text{din}(k - 1, x) \\
&\quad - x^2\text{ant}(k - 2, x) \text{din}(k - 2, x))\text{ant}(n, x) \\
&= \text{ant}(k + 1, x) \text{din}(k + 1, x)\text{ant}(n + 2, x) - x^2\text{ant}(k, x) \text{din}(k, x)\text{ant}(n, x).
\end{align*}$$

The last line is by two applications of Lemma 6.3. □

**Theorem 6.5.** For $n \geq 0$ and $k \geq 1$,

$$\text{din}(n + 2k, x) = \text{din}(k, x)\text{ant}(k, x) \text{din}(n + 2, x) - x^2\text{din}(k - 1, x)\text{ant}(k - 1, x) \text{din}(n, x).$$
Proof. The proof follows the same pattern as the proof of Theorem 6.4. The induction on $k$ again devolves to Lemma 6.3 (which is symmetric in ant and din). \endproof

These last two formulae are generalizations of the results in this section and can be proved from them using induction. Since we will not make use of these results in this paper, the proofs are omitted.

Theorem 6.6. If $a, b, c$ are nonnegative integers and at least one of $a$ and $b$ is even

$$x^a \text{ant}(b, x) \text{din}(2c, x) = \text{din}(a + 2c, x) \text{ant}(a + b, x)$$

$$- \text{din}(a, x) \text{ant}(a + b + 2c, x).$$

Theorem 6.7. If $a, b, c$ are nonnegative integers and both $a$ and $b$ are odd

$$(2x + 1)x^a \text{ant}(b, x) \text{din}(2c, x)$$

$$= 2(\text{din}(a + 2c, x) \text{ant}(a + b, x) - \text{din}(a, x) \text{ant}(a + b + 2c, x)).$$

7. The factors of din

Examining the factorizations over $\mathbb{Z}[x]$ of the first few dozen ant and din polynomials discloses a striking pattern. The polynomials’ irreducible factors correspond to the divisors of their indices. For the din polynomials the correspondence is complete; for the ant polynomials only certain divisors appear. The pattern is simpler for the din polynomials so we do them first. In this section we will just establish the basic facts about the factors of din. In another paper, we will show that as the index gets bigger and eigenvalues from smaller indices reappear, the new eigenvalues distribute themselves uniformly into the gaps between the old ones. The following result is analogous to Theorem 6.1 and can be given as analogous proof using relative eigenvectors.

Theorem 7.1. For $n \geq 1$ and $k \geq 1$, \text{din}(n, x) \text{divides} \text{din}(kn, x) in \mathbb{Z}[x].$

Proof. \text{din}(1, x) = 1 so the result is trivial for $n = 1$. Since \text{din}(2, x) = 2 and we do not wish to treat 2 as a unit, the theorem refers to divisibility over $\mathbb{Z}[x]$. The proof will be by a doubling induction on $k$. The case $k = 1$ is trivial. We will show that the truth of the result for $k$ implies its truth for $2k$ and for $2k + 1$. Thus starting from a basis of $k = 1$, successive doublings inductively prove it for all positive integers. To show $k \Rightarrow 2k$ we use the second formula of Lemma 6.2 with $n + 1$ replaced by $kn$:

$$\text{din}(2kn, x) = 2 \text{din}(kn, x) \text{ant}(kn, x).$$
To show \( k \Rightarrow 2k + 1 \) we use Theorem 6.5 with \( k \) replaced by \( kn \):

\[
\text{d}_n(n(2k + 1), x) = \text{d}_n(kn, x) \text{ant}(kn, x) \text{d}_n(n + 2)
- x^2 \text{d}_n(kn - 1, x) \text{ant}(kn - 1, x) \text{d}_n(n, x).
\]

In both cases \( \text{d}_n(n, x) \) is multiplied by an integer coefficients polynomial. □

Theorem 7.1 shows that \( \text{d}_n \) has divisibility properties like the cyclotomic polynomials. See [14] for a discussion of cyclotomic polynomial somewhat like that given below. We recursively define a new series of functions \( \text{d}_m(n, x) \) by \( \text{d}_m(1, x) = 1 \) and

\[
\text{d}_n(n, x) = \prod_{d|n} \text{d}_m(d, x).
\]

Mobius inversion gives an explicit formula:

\[
\text{d}_m(n, x) = \prod_{d|n} \text{d}_m(d, x)^\mu(n/d).
\]

For example, the positive divisors of 36 are \{1, 2, 3, 4, 6, 9, 12, 18, 36\} and we get

\[
\text{d}_m(36, x) = \frac{\text{d}_m(36, x) \text{d}_m(6, x)}{\text{d}_m(18, x) \text{d}_m(12, x)}.
\]

The first few values of \( \text{d}_m \) are: \( \text{d}_m(2, x) = 2, \text{d}_m(3, x) = 3x + 2, \text{d}_m(4, x) = 2x + 2, \text{d}_m(5, x) = 5x^2 + 10x + 4, \text{d}_m(6, x) = x + 2, \text{d}_m(7, x) = 7x^3 + 28x^2 + 28x + 8, \text{d}_m(8, x) = 2x^2 + 8x + 4 \). From the definition we see that \( \text{d}_m(n, x) \) is a rational function with rational coefficients, but as the first few values illustrate, \( \text{d}_m(n, x) \) is in fact a polynomial with integer coefficients. To prove this we need to further clarify the divisibility structure of the \( \text{d}_n \) polynomials.

**Theorem 7.2.** For \( m, n \geq 1 \),

\[
\text{GCD}(\text{d}_m(m, x), \text{d}_n(n, x)) = \text{d}_n(\text{GCD}(m, n), x).
\]

*The polynomial GCD is taken in \( \mathbb{Z}[x] \).*

**Proof.** Let \( d = \text{GCD}(m, n) \). Theorem 7.1 shows that \( \text{d}_n(d, x) \) divides both \( \text{d}_n(m, x) \) and \( \text{d}_n(n, x) \) and hence their GCD. What is left is to show that common divisors of \( \text{d}_n(m, x) \) and \( \text{d}_n(n, x) \) also divide \( \text{d}_n(d, x) \). Since the \( \text{d}_n \) polynomials have no repeated roots, we must show that a common root of \( \text{d}_n(m, x) \) and \( \text{d}_n(n, x) \) must also be a root of \( \text{d}_n(d, x) \). First suppose that \( d = 2k \) is even and write \( d = am - bn \), where \( a \) and \( b \) are positive integers. Theorem 6.5 and Lemma 6.2 give

\[
2 \text{d}_n(am, x) = 2 \text{d}_n(bn + 2k, x)
= \text{d}_n(d, x) \text{d}_n(bn + 2, x) - x^2 \text{d}_n(d - 2, x) \text{d}_n(bn, x).
\]
A common root of \( \text{din}(m, x) \) (and hence \( \text{din}(am, x) \)) and \( \text{din}(n, x) \) (and hence \( \text{din}(bn, x) \)) will thus be a root of \( \text{din}(d, x) \text{din}(bn + 2, x) \). But a root of \( \text{din}(bn, x) \) cannot also be a root of \( \text{din}(bn + 2, x) \), so \( \text{din}(d, x) \) gets the root. Next suppose that at least one of \( m \) and \( n \) is even, say \( n = 2k \). Again write \( d = am - bn \), where \( a \) and \( b \) are positive integers. We have

\[
2 \text{din}(am, x) = 2 \text{din}(d + 2kb, x) = \text{din}(nb, x) \text{din}(d + 2, x) - x^2 \text{din}(nb - 2, x) \text{din}(d, x).
\]

A common root of \( \text{din}(m, x) \) and \( \text{din}(n, x) \) will thus be a root of \( \text{din}(d, x) \).

**Theorem 7.3.** For all \( n \), \( \text{dmn}(n, x) \) is an integer coefficients polynomial.

**Proof.** Note that if \( d|n \), then since \( \text{din}(n, x) \) divided by \( \text{din}(d, x) \) is an integer coefficients polynomial, if \( \text{din}(d, x) \) divided by \( \text{dmn}(d, x) \) is an integer coefficients polynomial, then so is \( \text{din}(n, x) \) divided by \( \text{dmn}(d, x) \). We prove by induction on \( n \) the statement that \( \text{dmn}(n, x) \) is an integer coefficients polynomial and that for all \( m \leq n \) we have

\[
\text{GCD}(\text{dmn}(m, x), \text{dmn}(n, x)) = 1,
\]

where the GCD is over \( \mathbb{Q}[x] \). The basis \( n = 1 \) is trivial. Let us denote the positive divisors of \( n \), taken in increasing order, by \( d_i \) for \( i = 1 \) to \( i = k \). Since \( n \) is at least 2, \( k \) is at least 2. We have \( d_1 = 1 \) and \( d_k = n \). By the definition of \( \text{dmn} \)

\[
\text{din}(n, x) = \prod_{i=1}^{k} \text{dmn}(d_i, x). \tag{1}
\]

We will now inductively proceed through the divisors of \( n \). Define \( q_0(x) \) to be \( \text{din}(n, x) \) and suppose we have established that the quotient

\[
\text{din}(n, x) \prod_{i=1}^{h} \text{dmn}(d_i, x) = q_h(x)
\]

is an integer coefficients polynomial. By induction \( \text{dmn}(d_{h+1}, x) \) integrally divides \( \text{din}(d_{h+1}, x) \) and hence integrally divides \( \text{din}(n, x) \). Since, by induction, \( \text{dmn}(d_{h+1}, x) \) is relatively prime to \( \text{dmn}(d_g, x) \) for \( g \leq h \), it must be that \( \text{dmn}(d_{h+1}, x) \) integrally divides \( q_h(x) \). Thus we may define \( q_{h+1}(x) \) as \( q_h(x) \) divided by \( \text{dmn}(d_{h+1}, x) \). When we get to \( q_{k-1}(x) \), the factorization (1) of \( \text{din}(n, x) \) shows that \( q_{k-1}(x) = \text{dmn}(n, x) \) and so it is an integer coefficients polynomial. To complete the proof we must show that \( \text{dmn}(n, x) \) is relatively prime to \( \text{dmn}(m, x) \) for all \( m \leq n \). If \( m|n \), then the facts that the product (1) is a product of polynomials and that \( \text{din}(n, x) \) has no repeated...
roots show that $d_{mn}(n, x)$ and $d_{mn}(m, x)$ cannot have a common root. Now suppose that $m \nmid n$ and that $\text{GCD}(m, n) = d$. By the definition of $d_{mn}$ we have

$$d_{in}(n, x) = d_{in}(d, x) \prod_{f \mid d} d_{mn}(f, x),$$

where the product is over the divisors of $n$ that do not divide $d$. There is a similar expression for $d_{in}(m, x)$. Since the GCD of $d_{in}(n, x)$ and $d_{in}(m, x)$ is $d_{in}(d, x)$ we have

$$1 = \text{GCD} \left( \prod_{f \mid d} d_{mn}(f, x), \prod_{f \mid d} d_{mn}(f, x) \right)$$

and we see that $d_{mn}(n, x)$ is relatively prime to $d_{mn}(m, x)$. □

**Proposition 7.4.** For $n \geq 1$, $d_{mn}(2^n, x) = 2a_{n \times t}(2^n - 1, x)$.

**Proof.** The definition of $d_{mn}$ and setting $n = 0$ and $k = 2^n - 1$ in Theorem 6.5 gives

$$d_{in}(2^n - 1, x)d_{mn}(2^n, x) = d_{in}(2^n, x) = 2d_{in}(2^n - 1, x)a_{n \times t}(2^n - 1, x).$$

In the next results, we give explicit formulae for $d_{mn}(n, 0)$ and the degree of $d_{mn}(n, x)$ in terms of the Euler $\phi$ function and note a relation between the value at 0 and the degree. There are analogous results for $a_{mn}$ in the following section.

**Lemma 7.5.** The function, $f$, defined by $f(1) = 0$ and for $n \geq 2$ by

$$\text{Floor} \left( \frac{n}{2} \right) = \sum_{d \mid n} f(d)$$

is $f(2) = 1$ and $f(n) = \frac{\phi(n)}{2}$ for $n \geq 3$.

**Proof.** The function $f$ is uniquely defined by the given conditions, so we need only show that the formula given for it does indeed satisfy the definition. First note that $\text{Floor}(2/2) = 1 = f(1) + f(2) = f(2)$. Next note that $\phi(n)$ is even for $n \geq 3$ and that

$$n = \sum_{d \mid n} \phi(d).$$

Thus for $n \geq 3$ and even we have

$$\text{Floor} \left( \frac{n}{2} \right) = \frac{n}{2} = \sum_{d \mid n} \frac{\phi(d)}{2} = \frac{\phi(1)}{2} + \frac{\phi(2)}{2} + \sum_{d \mid n} \frac{\phi(d)}{2} = 1 + \sum_{d \mid n} \frac{\phi(d)}{2}$$

and for $n \geq 3$ and odd we have

$$\text{Floor} \left( \frac{n}{2} \right) = \frac{n}{2} - \frac{1}{2} = \sum_{d \mid n} \frac{\phi(d)}{2} - \frac{1}{2} = \sum_{d \mid n} \frac{\phi(d)}{2}.$$ □
**Proposition 7.6.** $\text{d}m_0(1, 0) = 2^0$ and $\text{d}m_0(2, 0) = 2^1$ and for $n \geq 3$, $\text{d}m_0(n, 0) = 2^{\varphi(n)/2}$.

**Proof.** $\text{d}i_0(1, 0) = 2^0$ and by Corollary 4.7 $\text{d}i_0(n, 0) = 2^{\text{Floor}(n/2)}$, so if we take logs base 2, we can apply Lemma 7.5. □

**Lemma 7.7.** The function, $f$, defined by $f(1) = 0$ and for $n \geq 2$ by

$\text{Floor}\left(\frac{n - 1}{2}\right) = \sum_{d|n} f(d)$

is $f(2) = 0$ and $f(n) = \varphi(n)/2$ for $n \geq 3$.

**Proof.** The proof is parallel to that of Lemma 7.5. □

**Proposition 7.8.** Degree($\text{d}m_0(1, x)$) = Degree($\text{d}m_0(2, x)$) = 0, and for $n \geq 3$, Degree($\text{d}m_0(n, x)$) = $\varphi(n)/2$.

**Proof.** Degree($\text{d}i_0(1, x)$) = 0 and by Proposition 4.1,

Degree($\text{d}i_0(n, x)$) = $\text{Floor}((n - 1)/2)$

so we can use the above lemma. □

Note that except for $n = 2$, $\text{d}m_0(n, 0) = 2^{\text{Degree}($\text{d}m_0(n, x)$)}$.

**Theorem 7.9.** For $p$ an odd prime, $\text{d}m_0(p, x) = \text{d}i_0(p, x)$ is irreducible over $\mathbb{Q}[x]$.

**Proof.** We will apply the Eisenstein irreducibility criterion in its reverse form. See [14] for a discussion of the Eisenstein theorem. Let $p = 2k + 1$ and note that

$\text{d}i_0(p, x) = \text{d}m_0(1, x)\text{d}m_0(p, x) = \text{d}m_0(p, x)$.

Let us write

$\text{d}i_0(p, x) = \sum_{n=0}^{k} a_n x^n$.

Then by Theorem 4.12

$\text{d}i_0'(p, x) = p \text{d}i_0(p - 1, x)/2 = p \text{ant}(k, x) \text{d}i_0(k, x)$

so that for $k \geq n \geq 1$ we have $p|n a_n$ and since $p > n$ we conclude that $p|a_n$. Since $a_0 = 2^k$ we have $p|a_0$. Finally $a_k = p$ so that $p^2|a_k$. □

**Proposition 7.10.** If $n = p^a$, where $p$ is prime, then the leading coefficient of $\text{d}m_0(n, x)$ is $p$. Otherwise $\text{d}m_0(n, x)$ is monic.
Proof. If we let \( f(n) \) be the leading coefficient of \( dmn(n, x) \), then \( f(n) \) is completely determined by the condition that \( f(1) = 1 \) and, since the leading coefficient of \( din(n, x) \) is \( n \),

\[
n = \prod_{d|n} f(d).
\]

So the proposition will be proved if we show that the formula for \( f(n) \) satisfies these conditions. Let the prime factorization of \( n \) be

\[
n = \prod_{i=1}^{k} p_i^{a_i}.
\]

In the product over the divisors of \( n \), the \( a_i \) terms that are powers of \( p_i \) each contribute a factor of \( p_i \) and all the mixed divisors contribute factors of 1. Thus the product is exactly \( n \). \( \square \)

We conjecture that there are further analogies between the \( dmn \) polynomials and the cyclotomic polynomials, specifically:

**Conjecture 7.11.** \( dmn(n, x) \) is irreducible for all \( n \).

8. The factors of \( ant \)

The pattern of the factors of \( ant(n, x) \) is similar to the pattern of the \( dmn \) polynomial factors of the \( din \) polynomial and the cyclotomic polynomial factors of \( x^n - 1 \), except that the degrees must divide with an odd quotient. Therefore we make the following definition.

**Definition 8.1.** For \( a \) and \( b \) integers, \( a \parallel b \) (\( a \) odd-divides \( b \)) iff there is an odd integer \( c \) such that \( b = ac \).

It is easy to see that odd-divisibility has properties similar to ordinary divisibility and that the odd-divisors of \( n \) are exactly the ordinary divisors of \( n \) that have the same number of factors of 2 as \( n \) does. Thus the lattice of positive odd-divisors of \( n \) is isomorphic to the lattice of ordinary divisors of the number obtained from \( n \) by removing all its factors of 2. This means that the Moebius function of the partially ordered set of positive odd-divisors of \( n \) can be computed from the ordinary Moebius function by \( \mu_{\parallel}(n, d) = \mu(n/d) \). See [11] for more on Moebius functions of partially ordered sets. We now associate with \( n \) a factor, \( ann(n, x) \), of \( ant(n, x) \) that contains the new roots of \( ant(n, x) \); that is, the roots that have not come from \( ant(d, x) \) of a proper odd-divisor, \( d \), of \( n \). In other words, we recursively define \( ann(n, x) \) by \( ann(1, x) = 1 \) and
\[ \text{ant}(n, x) = \prod_{d \mid n} \text{amn}(d, x). \]

By Mobius inversion this is equivalent to

\[ \text{amn}(n, x) = \prod_{d \mid n} \text{ant}(d, x)^{\mu(n/d)}. \]

For example, the positive odd-divisors of 90 are \{2, 6, 10, 18, 30, 90\} and this gives

\[ \text{amn}(90, x) = \text{ant}(90, x) \text{ant}(6, x) \text{ant}(30, x) \text{ant}(18, x). \]

As was the case for \( \text{dmn} \), \( \text{amn} \) is at worst a rational function with rational coefficients, but as the following values illustrate, \( \text{amn} \) too is actually a polynomial with integer coefficients. \( \text{amn}(2, x) = x + 1 \), \( \text{amn}(3, x) = x + 2 \), \( \text{amn}(4, x) = x^2 + 4x + 2 \), \( \text{amn}(5, x) = x^2 + 6x + 4 \), \( \text{amn}(6, x) = x^2 + 8x + 4 \), and \( \text{amn}(7, x) = x^3 + 12x^2 + 20x + 8 \).

Before proceeding to the analogs of Theorem 7.2 we have a loose end to tie up. By theorem 6.1 the (polynomial) GCD of \( \text{ant}(n, x) \) and \( \text{ant}((2k+1)n, x) \) is just \( \text{ant}(n, x) \). We can now answer the question as to what is the GCD of \( \text{ant}(n, x) \) and \( \text{ant}(kn, x) \) for even \( k \).

**Proposition 8.1.** For \( n \geq 1 \) and \( k \geq 1 \),

\[ \text{GCD}\left(\text{ant}(n, x), \text{ant}(2kn, x)\right) = 1. \]

**Proof.** The proof will be by induction on \( k \). Setting \( n = 0 \) and relabelling \( k \) as \( n \) in Theorem 6.4

\[ \text{ant}(2n, x) = \text{ant}(n, x) \text{din}(n, x)(x + 1) − x^2 \text{ant}(n − 1, x) \text{din}(n − 1, x). \]

Using Proposition 3.10, we get

\[
\begin{align*}
\text{GCD}&(\text{ant}(n, x), \text{ant}(2n, x)) \\
&= \text{GCD}(\text{ant}(n, x), -x^2 \text{ant}(n − 1, x) \text{din}(n − 1, x)) \\
&= \text{GCD}(\text{ant}(n, x), \text{din}(n − 1, x)) \\
&= 1,
\end{align*}
\]

since \( \text{ant}(n, x) \) and \( \text{din}(n − 1, x) \) are relatively prime by Corollary 4.10. For the induction step we again use Theorem 6.4:

\[
\begin{align*}
\text{ant}(2kn + 2n, x) &= \text{ant}(n, x) \text{din}(n, x) \text{ant}(2kn + 2, x) \\
&\quad - x^2 \text{ant}(n − 1, x) \text{din}(n − 1, x) \text{ant}(2kn, x).
\end{align*}
\]
Using the induction hypothesis we reduce the above to

\[ \gcd(\text{ant}(n, x), \text{ant}(2kn + 2n, x)) = \gcd(\text{ant}(n, x), \text{ant}(n - 1, x) \text{din}(n - 1, x)) = 1. \]

\[ \blacksquare \]

The analog for \text{ant} of Theorem 7.2 requires two cases depending on whether the numbers have the same maximal power of 2 dividing them.

**Theorem 8.2.** Let \( m, n \) be positive integers with the same maximal power of 2 dividing them. Then

\[ \gcd(\text{ant}(m, x), \text{ant}(n, x)) = \text{ant}(\gcd(m, n), x). \]

**Proof.** Write \( d = \gcd(m, n) \) and \( d = d_02^k, m = m_02^k \) and \( n = n_02^k \), where \( k \) is the maximal power of 2 dividing \( n \) and \( n \) and hence \( d \). The assumption on \( m \) and \( n \) implies that \( m_0, n_0 \) and \( d_0 \) are all odd. We wish to write \( d = an - bm \) with certain conditions on \( a \) and \( b \). Note that given one solution \((a, b)\), the next solution is \((a + m_0, b + n_0)\), that for each solution exactly one of \( a \) and \( b \) is even and which one it is alternates from one solution to the next. The conditions on \( a \) and \( b \) that we want are that they both be positive, \( a \) is odd, and \( b \) is twice an odd. (If \( b \) were congruent to 0 mod 4, then \( b + 2n_0 \) will be congruent to 2 mod 4.) If we write \( b = 2c \), then Theorem 6.4 gives

\[ \text{ant}(an, x) = \text{ant}(d + 2cm) \]

\[ = \text{ant}(cm, x) \text{din}(cm, x) \text{ant}(d + 2) \]

\[ - x^2 \text{ant}(cm - 1, x) \text{din}(cm - 1, x) \text{ant}(d, x). \]

A common root of \( \text{ant}(n, x) \) and \( \text{ant}(m, x) \) will also be a root of \( \text{ant}(an, x) \) since \( a \) is odd and of \( \text{ant}(cm, x) \) since \( c \) is odd. This root cannot be 0 and it cannot be a root of either \( \text{ant}(cm - 1, x) \) or \( \text{din}(cm - 1, x) \). Thus it is a root of \( \text{ant}(d, x) \). Of course, all roots of \( \text{ant}(d, x) \) are also roots of \( \text{ant}(n, x) \) and \( \text{ant}(m, x) \) because \( m/d \) and \( n/d \) are odd. \( \blacksquare \)

**Theorem 8.3.** Let \( m, n \) be positive integers that do not have the same maximal power of 2 dividing them. Then \( \text{ant}(m, x) \) and \( \text{ant}(n, x) \) are relatively prime.

**Proof.** Write \( d = \gcd(m, n) \) and \( d = d_02^k, m = m_02^k \) and \( n = n_02^k \), where \( k \) is the maximal power of 2 dividing \( n \) and \( d \) but a higher power of 2 divides \( m \). This means that \( n_0 \) and \( d_0 \) are odd but \( m_0 \) is even. As in the proof of Theorem 8.2 above, we wish to write \( d = an - bm \) with certain conditions on \( a \) and \( b \). Note that from one solution to the next, \( a \) is always odd and \( b \) alternates between even and odd. The conditions on \( a \) and \( b \) that we want are that they both be positive and \( b \) is twice an odd. (If \( b \) were congruent to 0 mod 4, then \( b + 2n_0 \) will be congruent to 2 mod 4.) If we write \( b = 2c \), then Theorem 6.4 gives
\[ \text{ant}(an, x) = \text{ant}(d + 2cm) \]
\[ = \text{ant}(cm, x)\text{din}(cm, x)\text{ant}(d + 2) \]
\[ - x^2\text{ant}(cm - 1, x)\text{din}(cm - 1, x)\text{ant}(d, x). \]

A common root of \( \text{ant}(n, x) \) and \( \text{ant}(m, x) \) will also be a root of \( \text{ant}(an, x) \) since \( a \) is odd and of \( \text{ant}(cm, x) \) since \( c \) is odd. This root cannot be 0 and it cannot be a root of either \( \text{ant}(cm - 1, x) \) or \( \text{din}(cm - 1, x) \). It also cannot be a root of \( \text{ant}(d, x) \) because \( m \) is an even multiple of \( d \) and so by Proposition 8.1 \( \text{ant}(m, x) \) and \( \text{ant}(d, x) \) are relatively prime. Thus \( \text{ant}(n, x) \) and \( \text{ant}(m, x) \) can have no common roots. □

**Proposition 8.4.** For \( n \geq 1 \), \( \text{ant}(2^n, x) = amn(2^n, x) \).

**Proof.** The product for \( \text{ant}(2^n, x) \) has only one term, since \( 2^n \) has itself as its only positive odd-divisor. □

**Theorem 8.5.** For all \( n \), \( amn(n, x) \) is an integer coefficients polynomial.

**Proof.** Note that if \( d \parallel n \), then since \( \text{ant}(n, x) \) divided by \( \text{ant}(d, x) \) is an integer coefficients polynomial, if \( \text{ant}(d, x) \) divided by \( amn(d, x) \) is an integer coefficients polynomial, then so is \( \text{ant}(n, x) \) divided by \( amn(d, x) \). We prove by induction on \( n \) the statement that \( amn(n, x) \) is an integer coefficients polynomial and that for all \( m \leq n \) we have

\[ \text{GCD}(amn(m, x), amn(n, x)) = 1, \]

where the GCD is over \( \mathbb{Q}[x] \). The basis \( n = 1 \) is trivial. If \( n \) is a power of 2, then Proposition 8.4 shows that \( amn(n, x) \) is an integer coefficients polynomial and Theorem 8.3 shows that for \( m \leq n \) \( amn(m, x) \) and \( amn(n, x) \) are relatively prime. Let us denote the positive odd-divisors of \( n \) not a power of 2, taken in increasing order, by \( d_i \) for \( i = 1 \) to \( i = k \). Since \( n \) is not a power of 2, \( k \) is at least 2. We have \( d_1 = 2^a \) the maximal power of 2 dividing \( n \), and \( d_k = n \). By the definition of \( amn \) we have

\[ \text{ant}(n, x) = \prod_{i=1}^{k} amn(d_i, x). \]  \hspace{1cm} (2)

We will now inductively proceed through the odd-divisors of \( n \). Let \( q_0(x) = \text{ant}(n, x) \) and suppose we have established that the quotient

\[ \text{ant}(n, x) \div \prod_{i=1}^{h} amn(d_i, x) = q_h(x) \]

is an integer coefficients polynomial. By induction \( amn(d_{h+1}, x) \) integrally divides \( \text{ant}(d_{h+1}, x) \) and hence integrally divides \( \text{ant}(n, x) \). Since, by induction, \( amn(d_{h+1}, x) \) is relatively prime to \( amn(d_g, x) \) for \( g \leq h \), it must be that \( amn(d_{h+1}, x) \) integrally
divides \( q_h(x) \). Thus we may define \( q_h(x) + 1 \) as \( q_h(x) \) divided by \( \text{amn}(d_h + 1, x) \). When we get to \( q_{k-1}(x) \), the factorization (2) of \( \text{ant}(n, x) \) shows that \( q_{k-1}(x) = \text{amn}(n, x) \) and so is an integer coefficients polynomial. To complete the proof we must show that \( \text{amn}(n, x) \) is relatively prime to \( \text{amn}(m, x) \) for all \( m \leq n \). If \( m \parallel n \), then the facts that the product (2) is a product of polynomials and that \( \text{ant}(n, x) \) has no repeated roots show that \( \text{amn}(n, x) \) and \( \text{amn}(m, x) \) cannot have a common root. Similarly, if \( n \) and \( m \) do not have the same maximal power of 2 dividing them, then \( \text{ant}(n, x) \) and \( \text{ant}(m, x) \) are relatively prime and hence so are \( \text{amn}(n, x) \) and \( \text{amn}(m, x) \). Suppose now that \( m \) and \( n \) have the same maximal power of 2 dividing them but \( m \) does not divide (and hence odd-divide) \( n \). Write \( \text{GCD}(m, n) = d \) and note that \( d \) odd-divides both \( m \) and \( n \). By the definition of \( \text{amn} \) we have

\[
\text{ant}(n, x) = \text{ant}(d, x) \prod_{f \parallel d} \text{amn}(f, x),
\]

where the product is over the odd-divisors of \( n \) that do not divide \( d \). There is a similar expression for \( \text{ant}(m, x) \). Since the GCD of \( \text{ant}(n, x) \) and \( \text{ant}(m, x) \) is \( \text{ant}(d, x) \) we have

\[
1 = \text{GCD} \left( \prod_{f \parallel d} \text{amn}(f, x), \prod_{f \parallel d} \text{amn}(f, x) \right)
\]

and we see that \( \text{amn}(n, x) \) is relatively prime to \( \text{amn}(m, x) \). □

**Proposition 8.6.** For \( p \) an odd prime, \( \text{ant}(p, x) = \text{amn}(p, x) \).

**Proof.** Since \( \text{amn}(1, x) = 1 \), \( \text{amn}(p, x) \) is the only term in the product for \( \text{ant}(p, x) \).

□

**Lemma 8.7.** The function \( f \) defined by \( f(1) = 0 \) and for \( n \geq 2 \) by

\[
\text{Floor} \left( \frac{n}{2} \right) = \sum_{d \parallel n} f(n)
\]

is \( f(n) = \varphi(n) \) for \( n \) even and \( f(n) = \varphi(n)/2 \) for \( n \) odd and greater than 1.

**Proof.** The proof is parallel to that of Lemma 7.5. □

**Proposition 8.8.** Degree(\( \text{amn}(1, x) \)) = 0 and Degree(\( \text{amn}(n, x) \)) = \( \varphi(n) \) for \( n \) even and, Degree(\( \text{amn}(n, x) \)) = \( \varphi(n)/2 \) for \( n \) odd and greater than 1.

**Proof.** Degree(\( \text{ant}(1, x) \)) = 0 and by the definition of \( \text{ant} \)

\[
\text{Degree}(\text{ant}(n, x)) = \text{Floor}(n/2),
\]

so we just apply the above lemma. □
Lemma 8.9. The function $f$ defined by $f(1) = 0$ and for $n \geq 2$ by

$$\text{Floor} \left( \frac{n - 1}{2} \right) = \sum_{d \mid n} f(n)$$

is $f(n) = \varphi(n)/2$ for odd $n$, $f(n) = \varphi(n)$ for $n$ even but not a power of 2, and $f(2^n) = \varphi(2^n) - 1 = 2^{a-1} - 1$.

Proof. The proof is parallel to that of Lemma 7.5. □

Proposition 8.10. $a_{mn}(n, 0) = 2^{\varphi(n)/2}$ for $n$ odd, $a_{mn}(n, 0) = 2^{\varphi(n)}$ for $n$ even but not a power of 2, and $a_{mn}(2^a, 0) = 2^{2^{a-1} - 1}$.

Proof. $\text{ant}(1, 0) = 2^0$ and by Corollary 3.8 $\text{ant}(n, 0) = 2^{\text{Floor}(n-1)/2}$ so we can take logs base 2 and apply the above lemma. □

Note that $a_{mn}(n, 0) = 2^{\text{Degree}(a_{mn}(n, x))}$ for all $n$ except powers of 2 and that $a_{mn}(2^a, 0) = 2^{2^{a-1} - 1}$.

As was the conjecture for $d_{mn}$, we again conjecture that the $a_{mn}$ polynomials are all irreducible. The following theorem shows that there is a great deal of overlap between these two conjectures since most $a_{mn}$ polynomials are also $d_{mn}$ polynomials.

Conjecture 8.11. For $n \geq 1$, $a_{mn}(n, x)$ is a monic irreducible polynomial with all positive integer coefficients.

Theorem 8.12. For $n$ not a power of 2, $d_{mn}(2n, x) = a_{mn}(n, x)$.

Proof. We will write $n = 2^ak$, where $k$ is odd and proceed by induction on the number of prime factors of $k$. If $k$ is the prime $p$, the only factors of $2^{a+1}p$ that are not also factors of $2^ap$ are $2^{a+1}$ and $2^{a+1}p$. Therefore

$$\text{d}_{2^{a+1}p, x} = \text{d}_{2^ap, x} \text{d}_{2^{a+1}, x} \text{d}_{2^{a+1}p, x} = \text{d}_{2^ap, x} 2 \text{a}_{2^ap, x} \text{d}_{2^{a+1}p, x}.$$

Theorem 6.5 with $n = 0$ and $k = 2^ap$ gives

$$\text{d}_{2^{a+1}p, x} = 2 \text{d}_{2^ap, x} \text{a}_{2^ap, x} = 2 \text{d}_{2^ap, x} \text{a}_{2^ap, x} \text{a}_{2^ap, x}.$$

Assume now the result for all odds with fewer prime factors than $k$. As above, Theorem 6.5 gives
\[
din(2^{a+1}k, x) = 2 \din(2^a k, x) \ant(2^a k, x) \\
\quad = 2 \din(2^a k, x) \prod_{d|k} \amn(2^a d, x).
\]
The only factors of \(2^{a+1}k\) that are not also factors of \(2^a k\) are those with an \((a + 1)\)th power of 2. Therefore
\[
\din(2^{a+1}k, x) = \din(2^a k, x) \prod_{d|k} \dmn(2^{a+1} d, x).
\]
Setting these equal and cancelling the common \(\din\) factor give
\[
2 \prod_{d|k} \amn(2^a d, x) = \prod_{d|k} \dmn(2^{a+1} d, x).
\]
The \(d = 1\) term on the LHS gives \(\amn(2^a, x) = \ant(2^a, x)\). The \(d = 1\) term on the RHS gives \(\dmn(2^{a+1}, x) = 2 \ant(2^a, x)\). Thus we can cancel the 2 and the \(d = 1\) term from the LHS against the \(d = 1\) term on the RHS. The induction hypothesis applies to all of the other terms except the \(d = k\) terms so we can cancel them and get the \(d = k\) terms equal, which was what was required to be proved. \(\Box\)

9. Distance matrix characteristic polynomials

Due to the length of this paper, we will only do a few applications of the above theory, specifically to establish a few simple properties of the \(ch\) polynomials. There are several other results like the first proposition of this section, but we will not give them here. The technique in the proof of Proposition 9.1 of reducing the order of the identity can be applied to all of these quadratic identities. Small cases show what the linear identities must be and then the above theory can be used to prove them.

**Proposition 9.1.** For \(n \geq 1\),
\[
ch(n + 1, x) - 2x \ch(n, x) + x^2 \ch(n - 1, x) = -n \ant(n, x) \din(n, x).
\]

**Proof.** Replace \(ch\) by the product \(\ant \sym\), use Theorem 5.1 to replace \(\sym(n + 1, x)\), bring the terms with \(\ant(n, x)\) to the RHS, use the definition of \(\din\) to replace \(\ant(n + 1, x) - \din(n, x)\) and then divide by \(x\) to get
\[
\sym(n - 1, x) \left( \ant(n + 1, x) + x \ant(n - 1, x) \right) \\
\quad = \ant(n, x) \left( 2 \sym(n, x) + n \ant(n - 1, x) \right).
\]
Since \(\ant(n, x)\) and \(\sym(n - 1, x)\) are relatively prime, we can expect that this equation reduces to two linear identities: one relating \(\ant(n, x)\) to the second factor of the LHS, and the other relating \(\sym(n - 1, x)\) to the second factor of the RHS. This is the case, but the identities differ for \(n\) even and \(n\) odd. For \(n\) even, we can replace the
second LHS factor using Lemma 3.4 and reduce to the first formula of Theorem 5.4. For \(n\) odd, we can replace the second LHS factor using Proposition 3.6 and reduce to the fourth formula of Theorem 5.6. □

Next we establish some basic results about the distinctness of the roots of the \(ch\) polynomials. The conjecture here is that the only time a root of \(ch(n, x)\) is equal to a root of \(ch(m, x)\) is when \(n\) divides \(m\) and the root is a root of \(ant(n, x)\). Since we know about the common roots in this case, the conjecture is really that there are no other cases of equal roots.

**Proposition 9.2.** For \(n \geq 0\), \(\gcd(ant(n, x), \text{sym}(n, x)) = 1\).

**Proof.** For \(n\) even, a common root of \(ant(2k, x)\) and \(\text{sym}(2k, x)\) would, by the first formula of Theorem 5.6, also be a root of \(ant(2k + 1, x)\), contradicting Proposition 3.10. For \(n\) odd, a common root of \(ant(2k + 1, x)\) and \(\text{sym}(2k + 1, x)\) would, by the second formula of Theorem 5.4, also be a root of \(\text{sym}(2k, x)\), contradicting Proposition 5.12. □

**Corollary 9.3.** For \(n \geq 0\), \(ch(n, x)\) has no repeated roots.

**Proof.** By Theorems 3.11 and 5.14, \(ant(n, x)\) and \(\text{sym}(n, x)\) have no repeated roots, and by Proposition 9.2 they have no common roots. □

**Proposition 9.4.** For \(n \geq 0\), \(\gcd(ant(n, x), \text{sym}(n + 1, x)) = 1\) and

\[
\gcd(ant(n + 1, x), \text{sym}(n, x)) = 1.
\]

**Proof.** By the fourth formula of Theorem 5.6, a common root of \(ant(2n, x)\) and \(\text{sym}(2n + 1, x)\) would also be a root of \(\text{sym}(2n, x)\) contradicting Proposition 5.12. The other three cases are similar. □

**Corollary 9.5.** For \(n \geq 0\), \(\gcd(ch(n, x), ch(n + 1, x)) = 1\).

**Proposition 9.6.** For \(n \geq 0\), \(\gcd(ant(n, x), \text{sym}(n + 2, x)) = 1\) and

\[
\gcd(ant(n + 2, x), \text{sym}(n, x)) = 1.
\]

**Proof.** The four cases are similar; we construct a three-term identity relating the two functions so that a common root would also be a root of the third function. For example, eliminating \(ant(2n + 1, x)\) from the first formula of Theorems 5.6 and 5.1 gives

\[
2n \text{sym}(2n + 2, x) = (2nx + 2n + 1)\text{sym}(2n, x) - (2n + 1)^2\text{ant}(2n, x).
\]
Thus a common root of \( \text{sym}(2n + 2, x) \) and \( \text{ant}(2n, x) \) would also be a root of \( \text{sym}(2n, x) \) contradicting Proposition 5.13. Note that \( 2nx + 2n + 1 \neq 0 \) since rational roots of \( \text{sym} \) or \( \text{ant} \) must be integers. \( \square \)

**Corollary 9.7.** For \( n \geq 0 \), \( \text{GCD}(\text{ch}(n, x), \text{ch}(n + 2, x)) = 1 \).

Since \( 2 \text{din}(n, x)\text{ant}(n, x) = \text{din}(2n, x) \), Proposition 4.3 can be combined with Proposition 9.1 to give a (fairly ugly) recursion for the \( \text{ch} \) polynomials. The following is a much nicer recursion. It is probably related to the formula for the derivative of a determinant in terms of the principal minors but I have not been able to prove it that way.

**Theorem 9.8.** For \( n \geq 0 \),

\[
2 \text{ch}(n, x) = \frac{\text{ch}'(n + 1, x)}{n + 1} + \frac{x \text{ch}'(n, x)}{n}.
\]

Since we will not use this result for anything other than the corollary below, the proof is left to the reader. A proof can be obtained by replacing \( \text{ch} \) by \( \text{ant} \) \( \text{sym} \) and using the strategy of reduction to linear identities. Note that this result, when combined with the value of \( \text{ch}(n, 0) \), gives a simple recursive method for computing \( \text{ch}(n, x) \) directly (as opposed to computing it as the product \( \text{ant}(n, x)\text{sym}(n, x) \)). Using this result, we can establish by induction the following interesting fact.

**Corollary 9.9.** For \( n \geq 1 \), \( \text{ch}'(n, x)/n \) has integer coefficients.

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**References**