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The determinacy strength of Π^1_2-comprehension

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ABSTRACT

Determinacy axioms state the existence of winning strategies for infinite games played by two players on natural numbers. We show that a base theory enriched by a certain scheme of determinacy axioms is proof-theoretically equivalent to Π^1_2-comprehension.

1. Introduction

The aim of the program of reverse mathematics, founded by Harvey Friedman, is the exact determination of the axioms which are needed for proving theorems of ordinary mathematics. Exact determination here means not only showing that the theorem is provable from the axioms but also proving that the axioms can be regained from the theorem. In this article, we examine determinacy theorems which can be described in the following way. A game between two players is given by a set A of sequences of natural numbers. The two players alternately choose natural numbers, and thereby produce an infinite sequence of natural numbers. Player I wins iff this sequence is an element of the given set; otherwise player II wins. Axioms of determinacy say that for special complexity of the set A either player I or player II has a winning strategy.

Determinacy strength has been examined for many axiom systems. For example, ZFC proves Δ^1_1-determinacy but not Σ^1_1-determinacy or Π^1_1-determinacy (see for example [3]). Full second-order arithmetic does not prove $\mathbf{\Sigma^0_4}$-determinacy (see [4]). For the determinacy strength of some subsystems of second-order arithmetic see [9].

We will examine Π^1_2-CA_0, that is second-order arithmetic with comprehension restricted to Π^1_2-formulas [9]. We will use a previous result that Π^1_2-CA_0 is strongly connected to a system of iterated monotone inductive definitions called μ-calculus.

Definition 1.1 (*Language of the μ-calculus $\mathcal{L}_\mu$*). We start with the language of second-order arithmetic and add a set-constructor μ. For each X-positive formula $\varphi(x, X)$ which contains no second-order quantifiers we add a set term $\mu x X \varphi(x, X)$, which is intended to denote the least fixed point of the monotone operator $\Gamma_\varphi(X) := \{x \mid \varphi(x, X)\}$. We say that a free variable Y occurs positively in $t \in \mu x X \varphi(x, X)$ ($t \notin \mu x X \varphi(x, X)$) if and only if φ is Y-positive (Y-negative). φ may contain further μ-terms such that nestings of fixed points are possible.

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Definition 1.2. For each X-positive formula $\varphi(x, X)$, let LFP(φ, I) be an abbreviation for the conjunction of the formulas

- $\forall x[x \in I \leftrightarrow \varphi(x, I)]$, which means that I is a fixed point of the operator given by φ;
- $\forall Y[\forall x(\varphi(x, Y) \rightarrow x \in Y) \rightarrow I \subset Y]$, which means that I is a subset of each fixed point of the operator given by φ.

LFP means "**l**east **f**ixed **p**oint".

Definition 1.3 (*μ-calculus*)**.** The μ-calculus is formulated in $\mathcal{L}_\mu$ and contains the following axioms:

- the axioms of ACA_0 (see [9]) with comprehension for all $\mathcal{L}_\mu$-formulas without second-order quantifiers;
- LFP$(\varphi(x, X), \mu x X \varphi(x, X))$ for each X-positive first-order formula φ of $\mathcal{L}_\mu$.

Definition 1.4. $(S, \preceq, \prec)$ is called a φ-compatible prewellordering iff

- $(S, \preceq, \prec)$ is a prewellordering, i.e.,
  - $\forall x(x \in S \rightarrow x \preceq x)$,
  - $\forall x, y(x \preceq y \rightarrow x \in S)$,
  - $\forall x, y(x \preceq x \wedge y \npreceq x \rightarrow x \prec y)$,
  - $\forall x, y(x \prec y \rightarrow x \preceq y)$,
  - $\forall x, y, z[x \prec y \preceq z \rightarrow x \prec z]$,
  - $\prec$ is wellfounded,
- $\forall x, y[x \preceq y \leftrightarrow x \prec y \vee \varphi(x, \{z \mid z \prec y\})]$.

We abbreviate that by IGF$(\varphi, S, \preceq, \prec)$; IGF means "**i**nductively **g**enerated **f**ixed point".

The following lemma is proved in [5](lemma 3.17).

Lemma 1.5. *Let $\varphi(x, X)$ be an X-positive first-order formula of $\mathcal{L}_2$. Then* ACA_0 *proves the following. Let S be a set and $\preceq, \prec$ binary relations with* IGF$(\varphi, S, \preceq, \prec)$*. Then* LFP$(\varphi, S)$.

Theorem 1.6 (*Möllerfeld*)**.** *The μ-calculus and Π^1_2-CA_0 prove the same Π^1_1-sentences; therefore they are proof-theoretically equivalent.*

For the proof see [5].

We therefore can examine the μ-calculus instead of Π^1_2-CA_0. As pointed out in [5], the μ-calculus is strongly connected to generalized quantifiers: a generalized quantifier Q on ω is a subset of $\mathcal{P}(\omega)$ satisfying $\emptyset \notin \mathsf{Q}$, $\mathsf{Q} \neq \emptyset$ and $X \subset Y \wedge X \in \mathsf{Q} \Rightarrow Y \in \mathsf{Q}$. $(\mathsf{Q}x)\varphi(x)$ is an abbreviation for $\{x \mid \varphi(x)\} \in \mathsf{Q}$. In this notation, the familiar quantifier $\forall$ is $\{\mathbb{N}\}$ and $\exists$ is $\{X \subset \mathbb{N} \mid X \neq \emptyset\}$. For each quantifier Q, the inverse $\overline{\mathsf{Q}} := \{X^c \mid X \notin \mathsf{Q}\}$ is again a quantifier, and it holds that $\overline{\overline{\mathsf{Q}}} = \mathsf{Q}$ and $(\overline{\mathsf{Q}}x)\varphi(x) \leftrightarrow \neg \mathsf{Q}x \neg\varphi(x)$.

For each quantifier Q, we define the next quantifier $\mathsf{Q}^\vee$:

$$(\mathsf{Q}^\vee x)\varphi(x) :\Leftrightarrow (\overline{\mathsf{Q}}x_0)(\overline{\mathsf{Q}}x_1)(\overline{\mathsf{Q}}x_2)\cdots \bigvee_{n\in\mathbb{N}} \varphi(\langle x_0, \ldots, x_n\rangle).$$

Note that $\exists^\vee$ is the Souslin quantifier. By iteration we define generalized quantifiers $\exists^n$ and $\forall^n$ for each $n \in \omega$ putting $\exists^0 := \exists$, $\exists^{n+1} := (\exists^n)^\vee$ and $\forall^n := \overline{\exists^n}$.

We now introduce a language and a theory where we can talk about $\exists^n$ and $\forall^n$.

Definition 1.7 (*Language $\mathcal{L}_\partial$ of the theory* ∂ame)**.** We enlarge the language $\mathcal{L}_2$ by quantifiers $\forall^n$ and $\exists^n$ for each $n \in \omega$. The rules in the definition of formulas for these quantifiers are analogous to the rules for the ordinary first-order quantifiers $\forall$ and $\exists$ with the restriction that $\forall^n x\varphi(x)$ and $\exists^n\varphi(x)$ can only be built if φ contains no second-order quantifiers (but φ may contain further quantifiers $\exists^m$ or $\forall^m$). We do not count formulas containing $\forall^n$ or $\exists^n$ as first-order formulas.

Definition 1.8 (*Theory* ∂ame)**.** The theory ∂ame is formulated in $\mathcal{L}_\partial$ and contains the following axioms:

- the axioms of ACA_0, with comprehension for all $\mathcal{L}_\partial$-formulas without second-order quantifiers;
- $\exists^0 x\varphi(x) \leftrightarrow \exists x\varphi(x)$
- $\exists^{n+1}x\varphi(x, \vec{y}, \vec{Y}) \leftrightarrow \forall X\big(\forall x(\varphi^{\exists^n}(x, \vec{y}, X, \vec{Y}) \rightarrow x \in X) \rightarrow \langle\rangle \in X\big)$;
- $\forall^n x\varphi(x) \leftrightarrow \neg\exists^n x \neg\varphi(x)$;

where φ varies over $\mathcal{L}_\partial$-formulas without second-order quantifiers. $\varphi^{\mathsf{Q}}(s, \vec{y}, X, \vec{Y})$ is an abbreviation for

$$\varphi(s, \vec{y}, \vec{Y}) \vee (\overline{\mathsf{Q}}x)s^\frown\langle x\rangle \in X.$$

Since φ^{Q} is X-positive, the formula

$$\forall X\big(\forall x(\varphi^{\exists^n}(x, \vec{y}, X, \vec{Y}) \rightarrow x \in X) \rightarrow \langle\rangle \in X\big)$$

expresses " the least fixed point of the operator given by $\varphi^{\exists^n}(x, X)$ contains $\langle\rangle$." Therefore $\exists^{n+1}$ is the next quantifier of $\exists^n$.

As shown in [5], the theories ⅁ame and the μ-calculus are equivalent in the following way.

Theorem 1.9 (*Möllerfeld*). *The μ-calculus and* ⅁ame *prove the same $\mathcal{L}_2$-sentences; they especially prove the same Π^1_1-sentences and are therefore proof-theoretically equivalent.*

Let $<\omega$-$\mathbf{\Sigma^0_2}$ denote the union of the finite levels of the difference hierarchy over $\mathbf{\Sigma^0_2}$. It is a tedious exercise to prove that this is the Boolean closure (i.e., the closure under finite union and complement) of the $\mathbf{\Sigma^0_2}$-sets. A proof is given in [2]. Let $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 be ACA_0 with determinacy for $<\omega$-$\mathbf{\Sigma^0_2}$-formulas. For lightface versions of $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0, see [7,8]. In the first paper, they show that Π^0_2-comprehension proves $<\omega$-Σ^0_2-determinacy. In the second one, they show a kind of the reversal.

Now we can state the main theorem of this article.

Theorem 1.10. *The μ-calculus and $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 prove the same $\mathcal{L}_2$-sentences.*

Together with Theorem 1.6, we obtain the following.

Corollary 1.11. *Π^1_2-CA_0 and $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 prove the same Π^1_1-sentences and are therefore proof-theoretically equivalent.*

2. The μ-calculus proves $<\omega$-Σ^0_2-determinacy

To show that $<\omega$-$\mathbf{\Sigma^0_2}$-determinacy is provable within the μ-calculus we need a finer characterization of $<\omega$-$\mathbf{\Sigma^0_2}$.

Definition 2.1. We define the hierarchy $\mathbf{\Sigma^\delta_n}$ as follows:

- $\mathbf{\Sigma^\delta_0} = \mathbf{\Sigma^0_1}$,
- $\mathbf{\Pi^\delta_n} = \neg\mathbf{\Sigma^\delta_n}$ for $n \in \omega$,
- $\mathbf{\Sigma^\delta_{n+1}} = \mathbf{\Sigma^0_2} \wedge \mathbf{\Pi^\delta_n}$ for $n \in \omega$,

where the elements of $\mathbf{\Sigma^0_2} \wedge \mathbf{\Pi^\delta_n}$ are the intersections between a $\mathbf{\Sigma^0_2}$-set and a $\mathbf{\Pi^\delta_n}$-set.

Further, let $\mathbf{\Sigma^\delta} := \bigcup_{n\in\mathbb{N}} \mathbf{\Sigma^\delta_n}$.

It follows directly from the definitions that $\mathbf{\Sigma^\delta} = \mathrm{Diff}(<\omega; \mathbf{\Sigma^0_2})$. For the first direction of Theorem 1.10, it therefore suffices to prove the following lemma.

We use lower-case Greek letters as syntactical variables for infinite sequences of natural numbers. By $\alpha[x]$ we denote the initial segment of length x of α. By $\varphi(\vec{X}^+, \vec{Y}^-, \vec{Z}^{+-})$ we denote a formula where the variables $\vec{X}$ occur positively (i.e., inside an even number of negations), $\vec{Y}$ occur negatively (i.e., inside an odd number of negations) and $\vec{Z}$ occur both positively and negatively.

We say that a set term depends uniformly on a formula φ if it can be defined by comprehension over a formula which may contain φ as a subformula.

Lemma 2.2. *For each $\mathbf{\Sigma^\delta_n}$-game G given by the formula*

- $\exists x\varphi(\alpha[x], \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-})$ *for* $n = 0$,
- $\exists x\forall y\varphi(x, \alpha[y], \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-})$ *for* $n = 1$,
- $\exists x\forall y\varphi(x, \alpha[y], \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-}) \wedge R(\alpha, \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-})$ *with* $R(\alpha, \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-}) \in \mathbf{\Pi^\delta_{n-1}}$ *for* $n > 1$,

there exist set terms of μ-calculus W^G, S^G_I and S^G_{II} which depend uniformly on φ and R, and W^G depends only positively on $\vec{X}$ and only negatively on $\vec{Y}$, such that the μ-calculus proves that

- *if $\langle\rangle \in W^G$, then S^G_I is a winning strategy for player I in G;*
- *if $\langle\rangle \notin W^G$, then S^G_{II} is a winning strategy for player II in G.*

This statement holds also for a $\mathbf{\Pi^\delta_n}$-game G of the form

$$\forall x\exists y\varphi(x, \alpha[y], \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-}) \vee R(\alpha, \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-}),$$

with $R(\alpha, \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-}) \in \mathbf{\Sigma^\delta_{n-1}}$.

Proof. The proof is by meta induction on n (that means the induction from outside and not inside the μ-calculus).

For $n = 0$, the claim is the theorem of Gale and Stewart that all open sets are determined [6]. This proof is formalizable in the μ-calculus.

For $n = 1$, the proof is a formalization of Wolfe's proof of $\mathbf{\Sigma^0_2}$-determinacy (see [6]). This is in fact Tanaka's formalization of Wolfe's theorem in $\mathbf{\Sigma^1_1}$-MI_0 (see [10]). Because this is implicit in the case $n > 1$ we skip it.

The proof for $n > 1$ is a generalization of Wolfe's theorem to $<\omega$-$\mathbf{\Sigma^0_2}$-games. This idea was first published by Bradfield in [1]. We will formalize it in the μ-calculus.

Because x does not occur freely in R, the game G is of the form

$$\exists x\big(\forall y\varphi(x, \alpha[y], \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-}) \wedge R(\alpha, \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-})\big).$$

Let Q_i be the following $\mathbf{\Pi^0_1}$-game:

$$Q_i(\alpha, \vec{X}, \vec{Y}, \vec{Z}) :\Leftrightarrow \forall y\varphi(i, \alpha[y], \vec{X}, \vec{Y}, \vec{Z}).$$

For each Q_i, we consider the tree T_i given by

$$s \in T_i :\Leftrightarrow (\forall t \subset s)\forall y\varphi(i, t, \vec{X}, \vec{Y}, \vec{Z}).$$

We then obtain

$$\gamma \in Q_i \leftrightarrow \gamma \in [T_i]$$

with $[T] := \{\alpha \mid \forall s(s \subset \alpha \to s \in T)\}$; $s \subset \alpha$ means that s is an initial segment of α.

We will now define a set of secure winning positions for player I as follows. We start with all positions s (a finite sequence of natural numbers) such that there exists a natural number i such that player I can win the game $[T_i] \cap R$. If P is the set of the already secured winning positions, we add all positions in which player I can stay in R and in $T_i \cup P$. All positions obtained in this way are winning positions for player I in G, because player I can follow his strategy that holds him inside R and $T_i \cup P$. At the moment the position leaves T_i it is in P and player I wins by induction hypothesis. If T_i is never left, the played path is in $[T_i]$ and in R, so player I wins, too. To formalize this in the μ-calculus we define a $\mathbf{\Pi^\delta_{n-1}}$-game $A_{w,i}(P)$ by

$$[A_{w,i}(P)](\gamma) :\Leftrightarrow (w^\frown a \in T_i \cup P \text{ for all } a \subset \gamma) \wedge R(w^\frown\gamma).$$

The sets $W^{A_{w,i}(P)}$, $S_I^{A_{w,i}(P)}$ and $S_{II}^{A_{w,i}(P)}$ exist by induction hypothesis. Let

$$\psi(\langle s, i\rangle, V) :\Leftrightarrow \langle\rangle \in W^{A_{s,i}((V)^1)}$$

with $(V)^1 := \{x \mid \exists i\langle x, i\rangle \in V\}$. There is such a formula ψ because W^A is uniform in A for each game A. V occurs positively in $A_{s,i}((V)^1)$, so V is positive in $W^{A_{s,i}((V)^1)}$ by the induction hypothesis for $A_{s,i}((V)^1)$. Therefore we can build the least fixed point of ψ:

$$I_\psi := \mu x X\psi(x, X).$$

Let $W^G := (I_\psi)^1$. Notice that the variables $\vec{X}$ and $\vec{Y}$ occur positively (respectively negatively) in W^G.

There are uniformly in ψ stage comparison relations $\preceq_\psi$ and $\prec_\psi$ such that the μ-calculus proves that $\mathrm{IGF}(\psi, I_\psi, \preceq_\psi, \prec_\psi)$. We now define the strategy S_I^G according to our description above and show by induction on $\prec_\psi$ that for each position in $(I_\psi)^1$ our strategy is winning the game. This is possible because if we are in the position a with $\langle a, i\rangle \in I_\psi$ we can describe the set of the winning positions that are secured up to this moment as $\{b \mid \exists j\langle b, j\rangle \prec_\psi \langle a, i\rangle\}$.

Finally, we have to give a winning strategy for player II if $\langle\rangle \notin W^G$. But then there exists no i such that $\langle\langle\rangle, i\rangle \in I_\psi$. According to the second point of Definition 1.3 of the μ-calculus, we have

$$\langle\langle\rangle, i\rangle \notin I_\psi \to \neg\psi(\langle\langle\rangle, i\rangle, I_\psi),$$

and together with the definition of ψ

$$\langle\rangle \notin W^{A_{\langle\rangle,i}((I_\psi)^1)} \text{ for each } i.$$

According to our induction hypothesis, player II has a winning strategy in $A_{\langle\rangle,i}((I_\psi)^1)$ for each i. So player II can play as follows. He takes the least i_0 such that $\langle\rangle$ is in T_{i_0}, and then he plays according to his winning strategy in $A_{\langle\rangle,i_0}((I_\psi)^1)$. If the position does not leave $(I_\psi)^1 \cup T_{i_0}$ after finitely many moves, the played sequence is not in R, so player II wins the game G. Otherwise, let x be the first position that is not in $(I_\psi)^1 \cup T_{i_0}$. Then player II takes the least i_1 such that x is not in T_{i_1}, and he goes on in the manner described above (with x instead of $\langle\rangle$). So he forces the played sequence to be in none of the T_i, and he wins the game.

We now prove the statement for a $\mathbf{\Pi^\delta_n}$-game G of the form

$$\forall x\exists y\varphi(x, \alpha[y], \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-}) \vee R(\alpha, \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-})$$

with $R(\alpha, \vec{X}^+, \vec{Y}^-, \vec{Z}^{+-}) \in \mathbf{\Sigma^\delta_{n-1}}$. Let $\overline{G}$ be the game "G^c and the other player begins"; more formally,

$$\overline{G} := \{\langle n\rangle^\frown\gamma \mid \gamma \notin G \wedge n \in \omega\}.$$

Then $\overline{G}$ is $\mathbf{\Sigma^\delta_n}$, and by the first part of the lemma we obtain sets $W^{\overline{G}}$, $S_I^{\overline{G}}$ and $S_{II}^{\overline{G}}$. Now the claim follows for $W^G := \{s \mid \exists n(\langle n\rangle^\frown s \notin W^{\overline{G}})\}$, $S_I^G := \{s \mid \exists n(\langle n\rangle^\frown s \in S_{II}^{\overline{G}})\}$ and $S_{II}^G := \{s \mid \exists n(\langle n\rangle^\frown s \in S_I^{\overline{G}})\}$. The variables which occur positively in G occur negatively in $\overline{G}$, and hence negatively in $W^{\overline{G}}$ and positively in W^G.

The complete strength of the μ-calculus is used in this proof. In each step of the induction we build the least fixed point of

$$\psi(\langle s, i\rangle, V) :\Leftrightarrow \langle\rangle \in W^{A_{s,i}((V)^1)},$$

where $W^{A_{s,i}((V)^1)}$ is the set of winning positions in a $\mathbf{\Pi^{\delta}_{n-1}}$-game. But as we have seen in the last paragraph, this set is the complement of the set of winning positions in a $\mathbf{\Sigma^{\delta}_{n-1}}$-game, and therefore the complement of an inductive set built in the previous induction step. This occurrence of the complement of the previous fixed point ensures that the complexity of the fixed points is rising in the course of the induction. $\square$

3. Embedding $\Game$*ame* in $<\omega$-$\mathbf{\Sigma^0_2}$-$\mathbf{Det_0}$

To prove the remaining direction of Theorem 1.10, it is sufficient to embed $\Game$ame in $<\omega$-$\mathbf{\Sigma^0_2}$-Det$_0$ because of Theorem 1.9. The idea of the proof is to describe the generalized quantifiers by games of complexities within $<\omega$-$\mathbf{\Sigma^0_2}$. For a formula $\exists^1 x\varphi(x)$, imagine a game where player I wants to prove that the formula is true, and player II that it is false. In this game, player II has to play natural numbers, and player I either passes or plays "break". Player I wins the game iff he has played "break" at some time and φ holds for the sequence of natural numbers played by player II before the break of player I. If player I wins the game, he wins it after finitely many moves; therefore it is an open game. If we change the roles of player I and player II, we get a game for the $\forall^1$-quantifier: player I plays natural numbers, and player II passes or plays break. This game is closed because player I loses in the finite. The game for $\exists^2$ is a kind of iteration of the $\forall^1$-game (that is no surprise because the $\exists^2$-quantifier is a kind of iteration of $\forall^1$). First, player I decides if he wants to play a $\forall^1$-game. If he decides to play, they do so until the first break of player II (if II never plays break, he loses). Then player I can decide if he wants to play another $\forall^1$-game, and so on. Suppose that player I decides after the nth $\forall^1$-game that he does not want a further $\forall^1$-game and plays break. Let s_n be the finite sequence played in the nth $\forall^1$-game. Then player I wins iff $\varphi(\langle s_1, \dots, s_n\rangle)$ holds. To find a convenient notation for this game, we imagine that player I plays functions from ω to ω, and player II answers with a natural number which chooses the initial sequence of length n of the function played by I. This is in fact the same game, and it can be denoted in the following way:

$$\begin{array}{llllllllll}
\mathrm{I} & f_1 & & f_2 & & f_3 & & f_4 & & \\
 & & & & & & & & & \cdots \\
\mathrm{II} & & n_1 & & n_2 & & n_3 & & n_4 &
\end{array}$$

Note that this is only a convenient notation for the game described above; our games remain games on ω and not on functions on ω!

Pursuing that we can describe all quantifiers $\exists^n$ and $\forall^n$ by games. We do that by meta induction (i.e., induction outside $<\omega$-$\mathbf{\Sigma^0_2}$-Det$_0$) on n. To be able to write down the sequences of natural numbers that are played during these games, we define the notations $rule_n$ and $\overline{rule_n}$ also by induction on n. The intended meaning is that $rule_n$ denotes the rules of the game for $\exists^n$, and $\overline{rule_n}$ the rules for $\forall^n$. $rule_1$ is "Player II plays natural numbers until player I plays a break; player I has to play his final break at some time"; $\overline{rule_1}$ is the same with player I and II swapped. $rule_n$ is "Play according to $\overline{rule_{n-1}}$. After this player I has the opportunity to play a final break. If he does this, the game is over, if he does not, another game according to $\overline{rule_{n-1}}$ is played. After this, player I again has the opportunity to play a final break, and so on, and this is repeated until player I plays his break. At some time, player I has to play his final break". So the game in the diagram above is played according to $rule_2$. $\overline{rule_n}$ is $rule_n$ with player I and II swapped.

During the following proofs we will often have to denote courses of games, i.e., sequences of natural numbers which were played according to these rules. Therefore we will denote by $rule_n$ also a sequence of natural numbers played according to $rule_n$. For example, we denote a game for $\exists^n$ like this:

$$\begin{array}{lllllll}
\mathrm{I} & & & & & & \\
 & \overline{rule_{n-1}} & \overline{rule_{n-1}} & \overline{rule_{n-1}} & \overline{rule_{n-1}} & \overline{rule_{n-1}} & \cdots \\
\mathrm{II} & & & & & &
\end{array}$$

Let $\alpha : \omega \to \omega$ be a sequence of natural numbers played in a $\exists^n$-game. Then $\alpha(m)$ is a natural number coding the sequence which was played in the mth part which was played according to $rule_{n-1}$. Let $(\alpha{\restriction}m) := \langle\alpha(0), \dots, \alpha(m)\rangle$. Then player I wins this game iff $\exists m\varphi((\alpha{\restriction}m))$.

The complexity of these games are all in $<\omega$-$\mathbf{\Sigma^0_2}$. In the case that the final break is never played, we have to check which of the players did not play a "break" at level k although the other player played infinitely many breaks at level $k-1$. The statement "one player played infinitely (only finitely) many breaks of a certain level" is $\mathbf{\Pi^0_2}$ (respectively $\mathbf{\Sigma^0_2}$), i.e., the complexity stays in $<\omega$-$\mathbf{\Sigma^0_2}$.

To talk about the set of winning positions in these games we need the following lemma.

Lemma 3.1. *Let $\varphi(x, \alpha)$ be a $<\omega$-$\mathbf{\Sigma^0_2}$-game. Then $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 proves that*

$$\exists X_\varphi \forall x((X_\varphi)_x \text{ is a winning strategy for one player in } \varphi(x, \alpha)).$$

Proof. Consider the following game. First player I plays a natural number x, and then player II decides which of the games $\varphi(x, \alpha)$ and $\neg\varphi(x, \alpha)$ they play. The winner of this game is the winner of the whole game. Player I obviously has no winning strategy in this game; therefore by determinacy player II has a winning strategy. From that strategy X_φ can be defined. □

We say that s is a winning position in the game for $\exists^n x\varphi(x)$ iff s is a finite sequence of courses of games played according to $rule_{n-1}$ and player I wins the game for $\exists^n x\varphi(s^\frown x)$. Using the previous lemma, we define this set formally by

$$W^n_\varphi := \{s \mid (X_\psi)_s \text{ is a strategy for player I}\},$$

where ψ is the $<\omega$-$\mathbf{\Sigma^0_2}$-formula for the game of $\exists^n x\varphi(s^\frown x)$. Since the existence of W^n_φ is proved in $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0, we can treat W^n_φ as a constant in our language from now on. We now have to show that W^n_φ is the least fixed point of the formula $\varphi^{\exists^{n-1}}(x, \vec{y}, X, \vec{Y})$ (for the definition of $\varphi^{\exists^{n-1}}$ see 1.8). Then we have shown that we can map $\exists^n x\varphi(x)$ to $\langle\rangle \in W^n_\varphi$ and our embedding of ⅁ame into $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 is done.

By Lemma 1.5, it suffices to prove that $\mathrm{IGF}(\varphi^{\exists^{n-1}}, W^n_\varphi, \preceq^n_\varphi, \prec^n_\varphi)$ for appropriate $\preceq^n_\varphi, \prec^n_\varphi$. We will first have a look at the game for $s \preceq^2_\varphi t$ for two appropriate sequences s and t. In the notation introduced above, this game looks like this:

$$\begin{array}{cccccccccc}
\mathrm{I} & f^s_1 & & & n^t_1 & f^s_2 & & & n^t_2 & \\
 & & & & & & & & & \cdots \\
\mathrm{II} & & n^s_1 & f^t_1 & & & n^s_2 & f^t_2 & &
\end{array}$$

Let $\alpha^s := (f^s_1 \restriction n^s_1)(f^s_2 \restriction n^s_2) \cdots$ and $\alpha^t := (f^t_1 \restriction n^t_1)(f^t_2 \restriction n^t_2) \cdots$. Then the winning condition for player I is

$$\exists n(\varphi(s^\frown(\alpha^s \restriction n)) \wedge \forall n' < n \neg\varphi(t^\frown(\alpha^t \restriction n'))).$$

In this game, the two games for $\exists^2 x\varphi(s^\frown x)$ and $\exists^2 x\varphi(t^\frown x)$ are played simultaneously: player I tries to prove that he can win the game to show that $\exists^2 x\varphi(s^\frown x)$ is true, and that he can do this at least as quickly as player II can win the game in which he shows that $\exists^2 x\varphi(t^\frown x)$ is true. The diagram for the general case would be

$$\begin{array}{ccccccc}
\mathrm{I} & & & & & & \\
 & \overline{rule}_{n-1} & rule_{n-1} & \overline{rule}_{n-1} & rule_{n-1} & \overline{rule}_{n-1} & \cdots \\
\mathrm{II} & & & & & &
\end{array}$$

The definition of $s \prec^n_\varphi t$ is analogous. We give the diagram for the general case:

$$\begin{array}{ccccccc}
\mathrm{I} & & & & & & \\
 & rule_{n-1} & \overline{rule}_{n-1} & rule_{n-1} & \overline{rule}_{n-1} & rule_{n-1} & \cdots \\
\mathrm{II} & & & & & &
\end{array}$$

Let α^t be the sequence whose elements are the courses of the $rule_{n-1}$-parts of the game, and α^s respectively for the $\overline{rule}_{n-1}$-parts. Then the winning condition for player I is

$$\exists n[\varphi(s^\frown(\alpha^s \restriction n)) \wedge \forall n' \leq n \neg\varphi(t^\frown(\alpha^t \restriction n'))].$$

The complexity of these games is again in $<\omega$-$\mathbf{\Sigma^0_2}$.

Analogously to W^n_φ, we define $\preceq^n_\varphi$ and $\prec^n_\varphi$ using the games described above. For $\preceq^n_\varphi$, let

$$\psi(\langle s, t\rangle, \alpha) :\Leftrightarrow \exists n(\varphi(s^\frown(\alpha^s \restriction n)) \wedge \forall n' < n \neg\varphi(t^\frown(\alpha^t \restriction n'))),$$

where α^s is the sequence of the $\overline{rule}_{n-1}$-parts and α^t is the sequence of the $rule_{n-1}$-parts of α. We now apply 3.1 to ψ and define

$$s \preceq^n_\varphi t :\Leftrightarrow (X_\psi)_{\langle s,t\rangle} \text{ is a strategy for player I.}$$

The definition of $\prec^n_\varphi$ is analogous.

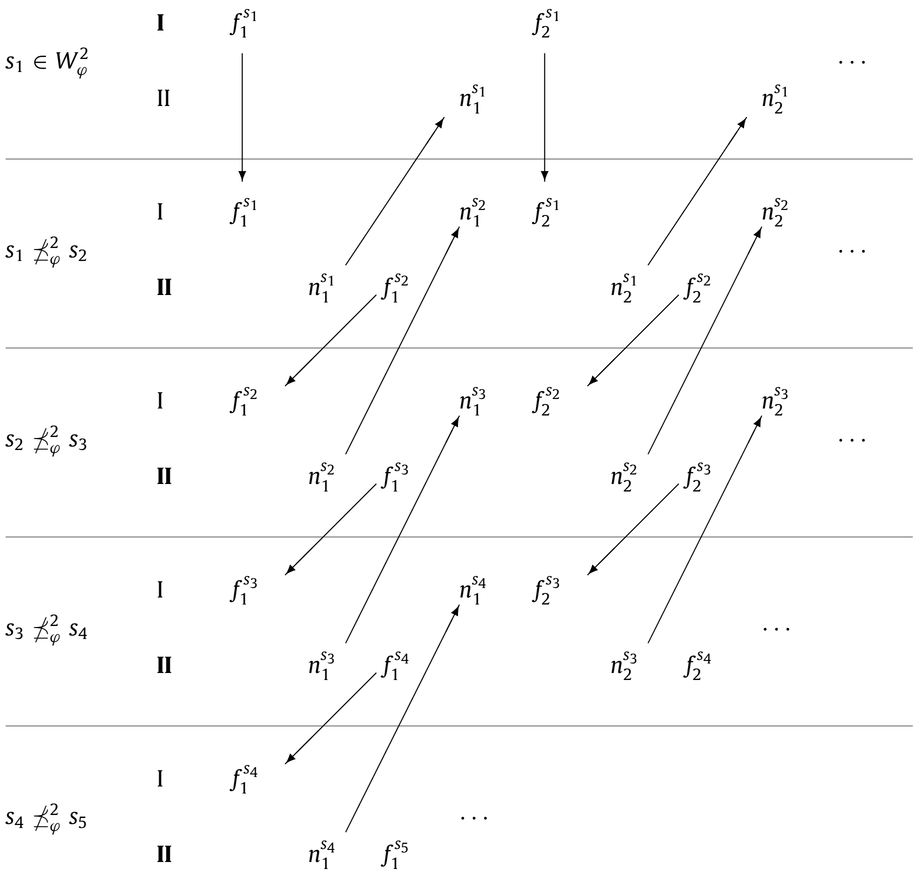
From now on, we will talk about $\exists^n$ and $\forall^n$ in the theory $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0. That should be read as an abbreviation for the corresponding second-order formula (see 1.8). We will prove the following theorem.

Theorem 3.2. *$<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 proves that* $\mathrm{IGF}(\varphi^{\exists^{n-1}}, W^n_\varphi, \preceq^n_\varphi, \prec^n_\varphi)$ *and therefore also* $\mathrm{LFP}(\varphi^{\exists^{n-1}}, W^n_\varphi)$ *for each first-order formula φ.*

The proof is by meta-induction on n (i.e., induction outside of $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0). To make clear the idea of the proof we will first consider the case $n = 2$.

Lemma 3.3. *$<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 proves that there exists no $(s_i)_{i\in\omega}$ such that $s_1 \in W^2_\varphi \wedge s_1 \not\preceq^2_\varphi s_2 \not\preceq^2_\varphi s_3 \cdots$.*

Proof. We assume that $s_1 \in W^2_\varphi \wedge s_1 \not\preceq^2_\varphi s_2 \not\preceq^2_\varphi s_3 \cdots$. Then by determinacy of the corresponding game we have winning strategies for one player. We link these strategies in the following way (the player with the winning strategy is printed in boldface):



This diagram is to be read as follows. The winning strategy for player I in the game for $s_1 \in W^2_\varphi$ delivers an $f_1^{s_1}$. We play that function in the game for $s_1 \not\preceq^2_\varphi s_2$ and get by the winning strategy for player II an $n_1^{s_1}$ and $f_1^{s_2}$, and so on. Let

$$\alpha_i := (f_1^{s_i}{\restriction}n_1^{s_i})(f_2^{s_i}{\restriction}n_2^{s_i})(f_3^{s_i}{\restriction}n_3^{s_i})\cdots.$$

Player I wins the game for $s_1 \in W^2_\varphi$, i.e.

$$\exists n_1 \varphi(s_1{}^\frown(\alpha_1{\restriction}n_1)).$$

Because player II wins the game for $s_i \preceq^2_\varphi s_{i+1}$, we get for each i

$$\forall n_i[\neg\varphi(s_i{}^\frown(\alpha_i{\restriction}n_i)) \vee \exists n_{i+1} < n_i \varphi(s_{i+1}{}^\frown(\alpha_{i+1}{\restriction}n_{i+1}))].$$

Therefore we have constructed an infinite descending sequence of natural numbers $n_1 > n_2 > n_3 \cdots$, a contradiction. □

Corollary 3.4. *$<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 proves that*

1. *$s \in W^2_\varphi$ implies that $s \preceq^2_\varphi s$;*
2. *$\prec^2_\varphi$ is wellfounded.*

Proof. The first statement follows directly from Lemma 3.3. For 2, we prove that

$$s \prec^2_\varphi t \Rightarrow t \not\preceq^2_\varphi s \quad \text{and} \quad s \prec^2_\varphi t \Rightarrow s \in W^2_\varphi$$

by linking the strategies of the corresponding games like in the proof of the last lemma. Then wellfoundedness follows immediately from the last lemma. □

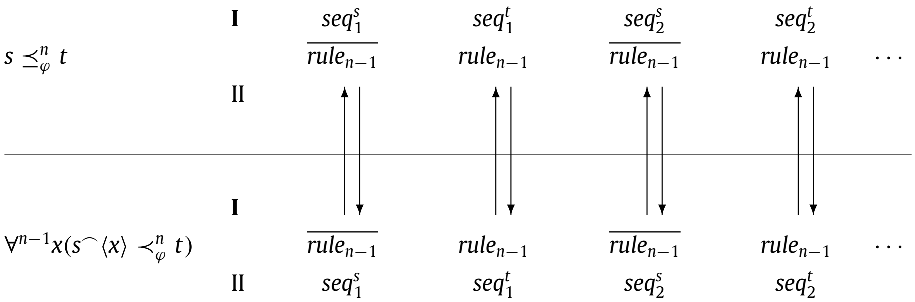
In the next lemma, we prove the general case.

Lemma 3.5. $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 *proves that*

$$s \preceq^n_\varphi t \leftrightarrow \varphi^{\exists^{n-1}}(s, \{u \mid u \prec^n_\varphi t\}).$$

Proof. Assume that $s \preceq^n_\varphi t$. If $\varphi(s)$, then we are done. So we may assume that $\neg\varphi(s)$.

We have to show $\forall^{n-1}x(s^\frown\langle x\rangle \prec^n_\varphi t)$, which is equivalent to $\neg\exists^{n-1}x\neg(s^\frown\langle x\rangle \prec^n_\varphi t)$ by the last clause of Definition 1.8. By induction hypothesis of the induction in the proof of Theorem 3.2 together with the definition of $\exists^{n-1}$ (see Definition 1.8), this is equivalent to $\neg\langle\rangle \in W^{n-1}_{\neg\psi}$ with $\psi(x) := s^\frown\langle x\rangle \prec^n_\varphi t$. By definition of W, this means "Player II has a winning strategy in the game for $\exists^{n-1}x\neg\psi(x)$", which is the $rule_{n-1}$-game which player I wins if the position x after the final break satisfies $\neg\psi(x)$. Considering this game with changed roles of player I and II, this is equivalent to "Player I has a winning strategy in the $\overline{rule_{n-1}}$-game which player II wins if the position x after the final break satisfies $\neg\psi(x)$". But this is the same as "Player I has a winning strategy in the $\overline{rule_{n-1}}$-game which player I wins if the position x after the final break satisfies $s^\frown\langle x\rangle \prec^n_\varphi t$". Hence we have to give a strategy for player I to play according to $\overline{rule_{n-1}}$, such that if we substitute the course of the game for x, player I has a winning strategy in the game for $s^\frown\langle x\rangle \prec^n_\varphi t$. The strategy for player I is described in the following diagram:



This diagram is to be read as follows. Each pair of arrows indicates that we use the part of player I's winning strategy which belongs to the corresponding part of the upper game to define the strategy for player I in the corresponding part of the lower game. The first $\overline{rule_{n-1}}$-part of the lower game is to determine the x, and the rest of the game is to determine if for this x it holds that $s^\frown\langle x\rangle \prec^n_\varphi t$. As strategy for player I in the first $\overline{rule_{n-1}}$-part of the lower game we take the strategy in the first $\overline{rule_{n-1}}$-part from the winning strategy for player I in the $s \preceq^n_\varphi t$-game. The result of the first part of the games is a sequence seq^s_1. The elements of seq^s_1 are sequences which are played according to $rule_{n-2}$, and so on; hence the depth of these nested sequences increases with the induction on n. For this seq^s_1 we have to prove that $s^\frown\langle seq^s_1\rangle \prec^n_\varphi t$, which means that we have to give a winning strategy for player I in the corresponding game. This winning strategy is defined by the next parts of the winning strategy for player I in the $s \preceq^n_\varphi t$-game, as the diagram above indicates. We name the sequences played in each part as denoted in the diagram and define

$$\alpha^s := (seq^s_1)(seq^s_2)(seq^s_3)\cdots,$$
$$\alpha^t := (seq^t_1)(seq^t_2)(seq^t_3)\cdots$$

and

$$\alpha^{s^\frown\langle x\rangle} := (seq^s_2)(seq^s_3)(seq^s_4)\cdots.$$

We have to show that

$$\exists m[\varphi(s^\frown\langle seq^s_1\rangle^\frown(\alpha^{s^\frown\langle x\rangle}\restriction m)) \wedge \forall m' \leq m\neg\varphi(t^\frown(\alpha^t\restriction m'))].$$

Player I wins the $s \preceq^n_\varphi t$-game; hence

$$\exists k[\varphi(s^\frown(\alpha^s\restriction k)) \wedge \forall k' < k\neg\varphi(t^\frown(\alpha^t\restriction k'))].$$

Because of $\neg\varphi(s)$ we have $k \geq 1$, so we choose $m := k - 1$ and are done. The other direction is analogous. □

The other properties that are needed for $\mathrm{PWO}(W^n_\varphi, \preceq^n_\varphi, \prec^n_\varphi)$ are proved analogously; therefore we have proved Theorem 3.2.

Corollary 3.6. $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0 *proves that if* $\varphi(x)$ *is equivalent to a first-order formula then also* $\exists^n x\varphi(x)$ *and* $\forall^n x\varphi(x)$*. Therefore we have comprehension for all* $\mathcal{L}_\Game$*-formulas and we have an embedding of* $\Game$ame *into* $<\omega$-$\mathbf{\Sigma^0_2}$-Det_0.

Proof. By 3.2 we have $\exists^n x\varphi(x) \leftrightarrow \langle\rangle \in W^n_\varphi$ and $\forall^n x\varphi(x) \leftrightarrow \langle\rangle \notin W^n_{\neg\varphi}$. $\square$

This, together with Theorem 1.9 and Lemma 2.2, finishes the proof of Theorem 1.10.

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