Multiple positive solutions to a singular boundary value problem for a superlinear Emden–Fowler equation

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Abstract

A multiplicity result for the singular ordinary differential equation
\[ y'' + \lambda x^{-2} y^\sigma = 0, \]
posed in the interval (0, 1), with the boundary conditions \( y(0) = 0 \) and \( y(1) = \gamma \), where \( \sigma > 1 \), \( \lambda > 0 \) and \( \gamma \geq 0 \) are real parameters, is presented. Using a logarithmic transformation and an integral equation method, we show that there exists \( \Sigma^* \in (0, \sigma/2] \) such that a solution to the above problem is possible if and only if
\[ \lambda \gamma^\sigma - 1 \leq \Sigma^*. \]
For \( 0 < \lambda \gamma^\sigma - 1 < \Sigma^* \), there are multiple positive solutions, while if \( \gamma = (\lambda^{-1} \Sigma^*)^{1/(\sigma-1)} \) the problem has a unique positive solution which is monotonic increasing. The asymptotic behavior of \( y(x) \) as \( x \to 0^+ \) is also given, which allows us to establish the absence of positive solution to the singular Dirichlet elliptic problem
\[ -\Delta u = d(x)^{-2} u^\sigma \]
in \( \Omega \), where \( \Omega \subset \mathbb{R}^N \), \( N \geq 2 \), is a smooth bounded domain and \( d(x) = \text{dist}(x, \partial \Omega) \).

Keywords: Ordinary differential equation; Second order; Singular boundary value problem; Multiple positive solutions; Asymptotic behavior

1. Introduction

The main purpose of this paper is to investigate the effect of the competing parameters \( \lambda > 0 \) and \( \gamma \geq 0 \) on the existence and non-existence of positive solutions to the following singular Emden–Fowler equation
\[ y'' + \lambda x^p y^\sigma = 0, \quad 0 < x < 1, \]
where \( p = -2 \) and \( \sigma > 1 \), subject to the boundary conditions
\[ y(0) = 0, \quad y(1) = \gamma. \]

By a positive solution to (1), (2) we mean a positive twice continuously differentiable function \( y \) on the interval (0, 1) that satisfies (1) and such that \( \lim_{x \to 0^+} y(x) = 0 \) and \( \lim_{x \to 1^-} y(x) = \gamma \) (see [28]).

The first motivation for studying the above problem stems from the singular non-standard equation [9,21]
\[ |x|^p u_t = (u^m)_{x,x}, \quad (x, t) \in (0, 1) \times (0, T). \]
Substituting a solution of the form

\[ u_s(x,t) = \mu(t)v(x) \]  

(4)

into (3) leads to

\[ (v^m)'' + \lambda |x|^p v = 0, \quad 0 < x < 1, \]

with

\[ \mu(t) = \left[ \mu_0^{1-m} + \lambda(m-1)t \right]^{1/(1-m)}, \]

if \( m \neq 1 \), and if \( m = 1 \)

\[ \mu(t) = \mu_0 e^{-\lambda t} \]

for some \( \mu_0 > 0 \). Setting \( \sigma = 1/m \) and \( y = v^m \) gives rise to Eq. (1).

Solutions to (3) of the form (4) may be viewed as special cases of a class of similarity solutions of Eq. (3). Since \( \partial u_s/\partial t \) has no jump for \( 0 < m < 1 \), \( u_s \) shows that the non-standard Eq. (3), may have solutions with total extinction in finite time for \( 0 < m < 1 \), whenever \( v \) or \( y \) exists.

The second interest in problem (1), (2) has been motivated by the singular Lane–Emden–Fowler problem

\[ \begin{cases} -\Delta u = k(x)u^\sigma & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]

(5)

recently considered, as an application, by Hernández and Mancebo [20]. The authors proved that if the function \( k \) satisfies \( 0 < k(x) \leq k_0 d(x)^p \), where \( d(x) = \text{dist}(x, \partial \Omega) \), \( k_0 > 0 \) is a constant, \( 1 + \sigma > -p > 0 \) and \( -1 < \sigma < 1 \), the problem has a unique solution.

The situation is different if \( p \leq -2 \) and \( \sigma < 0 \). In [33] it is shown that if \( k(x) \sim d(x)^p \), i.e.,

\[ c_1 d(x)^p \leq k(x) \leq c_2 d(x)^p, \]

(6)

where \( c_1, c_2 \) are positive constants, no regular solution can exist. In the present we shall, as application, consider problem (5) in which \( p = -2 \) and \( \sigma > 1 \). With the help of a solution to (1), (2) we shall see that problem (5), (6) has no regular solution.

Let us return to Eq. (1). This equation appears also in many different areas of investigation. For example, in boundary-layers, gas dynamics, chemical physics and physics of plasma. Excellent historical developments and bibliography are given in [30]. Depending on parameters \( p \) and \( \sigma \) this equation has been extensively studied for a long time.

In [28] Taliaferro studied the more general problem

\[ y'' + \phi(x)y^q = 0, \quad 0 < x < 1, \]

(7)

with positive continuous function \( \phi \). The author showed that (7), where \( q < 0 \), has a (unique) positive solution, such that \( y(0) = y(1) = 0 \) (\( y = 0 \)), if and only if

\[ \int_0^1 x(1-x)\phi(x) \, dx < \infty. \]

(8)

Recently, Lima and Oliveira [23] studied numerical positive solutions to Eq. (7), where \( \phi(x) = x^p g(x) \), in which \( g \) is a continuous and positive function on \([0, 1]\). Exponents \( p \) and \( q \) satisfy \( p > -2 \), \( q < -1 \) and \( p + q < -1 \). Based on [28], the authors concluded that any local solution satisfies

\[ y(x) \sim \left( \frac{(1-q)^2 g(0)}{(p+2)(-p-q-1)} \right)^{1/(1-q)} x^{\tau}, \]

as \( x \to 0^+ \), where \( \tau = \frac{p+2}{1-q} \). Note that the above estimate breaks down if \( p = -2 \).

In [8] Fowler studied the question of the oscillatory behavior of solutions to

\[ y'' + x^p |y|^{\sigma-1} y = 0, \quad x > 0, \]

(9)
where \( p \) is a real parameter and \( \sigma > 1 \). Fowler established, among other results, that all solutions to (9) are oscillatory for \( p \geq -2 \), i.e., for any \( \varepsilon > 0 \), there exists \( x_0 \geq \varepsilon \) such that \( y(x_0) = 0 \). Since Eq. (9) is invariant by dilatation if (and only if) \( p = -2 \), we may deduce that problem (1), (2) has a positive solution for \( \gamma = 0 \) (see also [30]). In [12] Gilding discussed the nonnegative solutions of the equation (\( p = 0 \))

\[
y'' + \lambda y^{\sigma-1} = 0,
\]

with boundary conditions (2). It is shown that for \( \sigma > 1 \) and \( \gamma > 0 \), there exists \( \lambda_0 > 0 \) such that the problem has a unique nonnegative solution for \( \lambda = \lambda_0 \), precisely two nonnegative solutions for all \( 0 < \lambda < \lambda_0 \) and no nonnegative solutions for all \( \lambda > \lambda_0 \).

In [2] Berestycki and Esteban investigated the problem

\[
\begin{aligned}
&x^2 y'' + \lambda y + |y|^{\sigma-1} y = 0, \\
&y(0) = y(1) = 0,
\end{aligned}
\]

where \( \sigma > 1 \) and \( \lambda \) is a real parameter. It was shown that for all \( 0 < \lambda < \frac{1}{4} \) there exists an uncountable infinity of positive solutions to (11). In [2,25–27,29,31] we shall use, in the first part of the present section, an admissible functional transformation to reduce problem (1), (2) to an Abel equation, with one parameter, and give some connections between the present problem and some problems in boundary-layer equations.

2. The main results

The aim of this section is to analyze the effect of \( \lambda > 0 \) and \( \gamma > 0 \) on the existence, multiplicity and non-existence of positive solutions to (1), (2), where \( \sigma > 1 \) and \( p = -2 \). From the boundary-layer view point, the main interest lies in the existence of some (positive) values of \( \gamma \) such that the problem has positive solutions for any fixed \( \lambda > 0 \).

Let us note that for the linear case \( \sigma = 1 \) the problem is solved explicitly

\[
y(x) = \gamma \sqrt{x} \cos(\sqrt{\lambda - 1/4} \log \sqrt{x}) + a \sqrt{x} \sin(\sqrt{\lambda - 1/4} \log \sqrt{x}),
\]

if \( \lambda > \frac{1}{4} \), where \( a \) is an arbitrary constant. If \( \lambda = \frac{1}{4} \), we have

\[
y(x) = \gamma \sqrt{x} (1 + a \log(x)), \quad a \in \mathbb{R},
\]

and for \( \lambda < \frac{1}{4} \)

\[
y(x) = \gamma \sqrt{x} \left[ ax^{1/4} \sqrt{1-4\lambda} + bx^{-1/4} \sqrt{1-4\lambda} \right],
\]

where \( a, b \in \mathbb{R} \) satisfying \( a + b = 1 \).

Note also that for \( \sigma \neq 1 \) and \( p < -2 \) problem (1), (2) has the explicit solution

\[
y(x) = \gamma x^{-\frac{p+2}{\sigma-1}},
\]

where

\[
\lambda y^{\sigma-1} = -\frac{(p+2)(\sigma+p+1)}{(\sigma-1)^2},
\]

provided \( (p+2)(\sigma-1) < -(p+2)^2 \).

2.1. Reduction to an Abel equation

In the same vein as in [2,25–27,29,31] we shall use, in the first part of the present section, an admissible functional transformation to reduce problem (1), (2) to an Abel equation, with one parameter, and give some connections between the present problem and some problems in boundary-layer equations.
The starting point is the logarithmic $x$ scale by setting ($\lambda > 0$, $\gamma > 0$)

\[
 w(t) = \frac{1}{y(x), \quad t = -\Gamma^{-1}\log(x), \quad \Gamma = \lambda^{-1/2}\gamma^{-(\sigma-1)/2}.
\]

Substitution into Eq. (1) gives the following generalized Duffing equation without external forcing [6,16]

\[
 w'' + \Gamma w' + w^\sigma = 0, \quad 0 < t < \infty,
\]

which has to be solved subject to the boundary condition

\[
 w(0) = 1, \quad \lim_{t \to \infty} w(t) = 0.
\]

With $\sigma = 3$ Eq. (12) reads

\[
 w'' + \Gamma w' + w^3 = 0,
\]

which is the classical Duffing equation without external forcing.

In fluid dynamics, Eq. (14) appears when looking for pseudosimilarity solutions to nonlinear convection over a vertical flat plate embedded in a porous medium at a temperature of a maximum density and where the wall temperature distribution varies as $x^{-1/2}$, where $x$ is the distance along the plate [19,22]. The case $\sigma = 2$ i.e.

\[
 w'' + \Gamma w' + w^2 = 0,
\]

appears when studying pseudosimilarity solutions to boundary-layer flows induced by a continuous permeable plane surface where the stretching velocity varies inversely-linear with the distance [24] (see also [35]).

To obtain a global information on the phase portrait for problem (12), (13), we can use the modified Liapunov function [16]

\[
 V = \frac{1}{2}w'^2 + \frac{1}{\sigma+1}w^{\sigma+1} + \beta\left(ww' + \frac{\Gamma}{2}w^2\right),
\]

where $\beta$ is a positive parameter. The function $V$ satisfies

\[
 V' = (\beta - \Gamma)w'^2 - \beta|w|^{\sigma+1}.
\]

Since

\[
 V = \frac{1}{2}(w' + \beta w)^2 + \frac{\beta}{2}(\Gamma - \beta)w^2 + \frac{1}{\sigma+1}|w|^{\sigma+1},
\]

we deduce, for $\beta < \Gamma$, that the (unique) equilibrium point $(0, 0)$ is globally asymptotically stable. All local solutions to (12) remain bounded and tend to 0 at infinity. However, this approach is not sufficient to provide the existence of a positive (monotonic decreasing) solution.

Let us note that, since the equilibrium point $(0, 0)$ is a center for the ODE

\[
 w'' + |w|^{\sigma-1}w = 0,
\]

problem (12), (13) may have no positive solution if $\Gamma > 0$ is small.

To establish the existence of monotonic decreasing (positive) solutions we use, as in [4,5,18] and [26], a Crocco variables approach. The solution $w$ will be considered as an independent variable. At first, we assume that $w$ is monotonic decreasing on $(0, \infty)$. Let $t(w)$ denote the inverse function of $w(t)$ defined on $[0, 1]$. Under the transformation

\[
 s = w, \quad \varphi(s) = -w'(t(s))
\]

problem (12), (13) is reduced into the following Abel equation

\[
 \varphi\varphi' = \Gamma\varphi - s^\sigma, \quad 0 < s < 1,
\]

or into the following integral equation [15]

\[
 \varphi(s) = \Gamma s - \int_0^s \frac{r^\sigma}{\varphi(r)} \, dr, \quad 0 < s < 1,
\]
supplemented by the initial condition
\[ \varphi(0) = 0. \]  
(18)

Let us note that the first derivative of \( y \) at \( x = 1 \) is given by
\[ \frac{dy}{dx}(1) = \lambda^2 \gamma^{\frac{\sigma+1}{\tau}} \varphi(1). \]
It may be also noted that the integral equation (17) was used to determine travelling wave solutions for the equation
\[ u_t = u_{xx} + u^\sigma. \]
A detailed discussion is given in the book by Gilding and Kersner [15].

The next subsection is devoted to study the question of existence and uniqueness of positive solutions to (16), (18), with the help of the theory of the integral equation developed in [15].

2.2. Multiple solutions

As mentioned before we shall use the integral equation method [15], which is particularly useful for Abel equations. The unknown solution \( \varphi \) is assumed to be continuous on \([0, 1]\) and positive on \((0, 1)\). From (17) one sees immediately that \( \varphi(s) \leq \Gamma s \), for all \( 0 \leq s \leq 1 \).

In the following result, we will see, by a simple argument, that problem (17), (18) has at most one solution \( \varphi \) satisfying
\[ \lim_{s \to 0^+} \frac{\varphi(s)}{s} > 0. \]  
(19)

Suppose that there exist two solutions \( \varphi_1 \) and \( \varphi_2 \) satisfying (19). Let \( \beta_1, \beta_2 \) be real positive numbers such that
\[ \varphi_1(s) \geq \beta_1 s, \quad \varphi_2(s) \geq \beta_2 s, \]
for all \( 0 \leq s \leq \varepsilon \leq 1, \varepsilon \) small.
Using (17) we deduce
\[ \left| \varphi_1(s) - \varphi_2(s) \right| \leq \int_{0}^{s} \frac{\left| \varphi_1(r) - \varphi_2(r) \right|r^\sigma}{\varphi_1(r)\varphi_2(r)} dr, \]
for all \( 0 < s \leq \varepsilon \). Thereafter
\[ \left| \varphi_1(s) - \varphi_2(s) \right| \leq \frac{\varepsilon^{\sigma-1}}{(\sigma-1)\beta_1\beta_2} \max_{0 \leq r \leq \varepsilon} \left| \varphi_1(r) - \varphi_2(r) \right|, \]
for all \( 0 < s \leq \varepsilon \). Hence, if \( \varepsilon \) is sufficiently small, \( \varphi_1 \equiv \varphi_2 \) on \([0, \varepsilon]\). Consequently, problem (16), (18) has at most one solution satisfying (19). This finding allows us to examine the behavior of \( \varphi(s) \) in the right neighbourhood of \( s = 0 \).

If we look for a solution to (16) under the “approximate” form [18]
\[ \varphi_{app}(s) = \beta s^\alpha + ds^\tau, \]
as \( s \to 0^+ \), where \( \beta, d, 0 < \alpha < \tau \) are real parameters, then we get
\[ \alpha \beta^2 s^{2\alpha-1} + \beta d(\alpha + \tau)s^{\alpha+\tau-1} + \tau d^2 s^{2\tau-1} = \beta \Gamma s^\alpha + \Gamma ds^\tau - s^\sigma, \]
which is true in the limit \( s \to 0^+ \) provided that either
\[ \alpha = 1, \quad \tau = \sigma, \quad \beta = \Gamma \quad \text{and} \quad d = -\frac{1}{\sigma \Gamma}, \]
or
\[ \alpha = \sigma, \quad \tau = 2\sigma - 1, \quad \beta = \frac{1}{\Gamma} \quad \text{and} \quad d = -\frac{\sigma}{\Gamma^3}. \]
For a more exact result, we intuitively suppose
\[ \varphi(s) = s^\alpha \left[ \beta + h(s) \right], \]
where \( h \in C^1(0, 1) \), \( h(s) \) tends to 0 with \( s \), and \( \beta > 0 \) and \( \alpha \) are real parameters to be determined. Using (17), we get
\[ s^{\alpha-1} \left[ \beta + h(s) \right] + \frac{1}{s} \int_0^s \frac{r^{\sigma-\alpha}}{\beta + h(r)} dr = \Gamma, \]
for all \( 0 < s \leq 1 \). A simple analysis of the above, as \( s \to 0 \), shows that either \( \alpha = 1 \) and \( \beta = \Gamma \), or \( \alpha = \sigma \) and \( \beta = \frac{1}{\Gamma} \). The foundation of those results is that problem (17), (18) may have two sets of solutions. In the one any solution satisfies
\[ \lim_{s \to 0} \frac{\varphi(s)}{s^\alpha} = \frac{1}{\Gamma}. \]  
\[ \text{(20)} \]

The other set contains at most one solution which satisfies
\[ \lim_{s \to 0} \frac{\varphi(s)}{s} = \Gamma. \]  
\[ \text{(21)} \]

In the theory of the integral equation [13–15] it is known that either problem (17), (18) has no solution, or it has one parameter family of solutions which contains a (unique) solution called “maximal” solution satisfying (21).

We recall briefly some relevant results of [13–15] concerning problem (17), (18). It is shown that there exists a minimal \( \Gamma^* = \Gamma^*(\sigma) > 0 \) such that (17), (18) has a positive solution if and only if \( \Gamma \geq \Gamma^* \). Moreover, for \( \Gamma = \Gamma^* \) the solution is unique and vanishes at \( s = 1 \). For any \( \Gamma > \Gamma^* \) there exist multiple ordering solutions with respect to the parameter \( \varphi(1) \). These solutions intersect only at \( s = 0 \) and there always exists a unique (positive) solution satisfying \( \varphi(1) = 0 \). Moreover, for \( \Gamma^* > \Gamma \) the unique solution satisfies (21), while if \( \Gamma > \Gamma^* \) any solution satisfies (20).

Consequently, the Emden–Fowler problem (1), (2) has a unique positive monotonic increasing solution if \( \gamma = \gamma_c \), where \( \gamma_c = (\lambda \Gamma^* \sigma)^{-1/(\sigma-1)} \), multiple monotonic increasing solutions for any \( 0 < \gamma < \gamma_c \) and no monotonic increasing solution for \( \gamma > \gamma_c \). In fact, we shall see that problem (1), (2) has no solution for any \( \gamma > \gamma_c \). Suppose that the Emden–Fowler problem has a solution \( y \), for some \( \gamma > \gamma_c \). Hence, there exists \( 0 < b < 1 \) such that \( y \) is monotonic increasing on \( (0, b) \) and monotonic decreasing on \( (b, 1) \). Since Eq. (1) is invariant by dilatation the new function \( y_b(x) = y(bx) \) satisfies (1), (2). Because \( y_b \) is monotonic increasing, \( y_b(1) \leq (\lambda^{-1} \Sigma^*)^{\frac{1}{r+1}} \), or \( y(b) \leq (\lambda^{-1} \Sigma^*)^{\frac{1}{r+1}} \), which is a contradiction. The above argument indicates also that any positive solution to (1), (2) satisfies the following estimate
\[ y \leq (\lambda^{-1} \Sigma^*)^{\frac{1}{r+1}} \equiv \gamma_c, \]  
\[ \text{(22)} \]
on \([0, 1]\). In the remainder of this section we shall obtain some estimates for \( \varphi(1) \) and \( \Sigma^* \), or \( \Gamma^* \). It turns out that the real number \( \varphi(1) = -w'(0) \) plays the role of the shooting parameter for problem (12), (13). So, one advantage gained from the integral equation (17) (or the Abel equation) is that the estimate of \( \varphi(1) \) will be easily deduced. This can be used for approximate numerical solutions. To exhibit an estimate to \( w'(0) \) or \( \varphi(1) \), which may depend on \( \Gamma \), we use an idea of [1,15]. Multiplying (16) by \( 2/s \) and integrating from 0 to \( s \) yield (see [15, p. 49])
\[ \frac{\varphi(s)^2}{s} + \int_0^s \frac{1}{r^2} (\varphi(r) - \Gamma r)^2 dr + \frac{2}{\sigma} s^{\sigma} = \Gamma^2 s, \]  
\[ \text{(23)} \]
for all \( 0 < s < 1 \). Thereby
\[ \frac{\varphi(s)^2}{s^2} + \frac{2}{\sigma} s^{\sigma-1} \leq \Gamma^2, \]  
\[ \text{(24)} \]
for all \( 0 < s < 1 \). In particular,
\[ \Gamma^2 \geq \frac{2}{\sigma}, \quad \varphi(1) \leq \sqrt{\Gamma^2 - \frac{2}{\sigma}}, \]  
\[ \text{(25)} \]
Therefore $\Gamma^* \geq \sqrt{\frac{\gamma}{2}}$, or

$$\Sigma^* \leq \frac{\sigma}{2}.$$

This indicates, in particular that the Emden–Fowler problem (1), (2) has no solution if $\lambda \gamma^{\sigma-1} > \frac{\gamma}{2}$.

To obtain an estimate from above for $\varphi(1)$ or to obtain increasing–decreasing solutions, we look for a solution to (12), (13) such that $\zeta = w'(0) > 0$, $\zeta$ being the shooting parameter. In Section 2 it is argued that for any $\zeta \neq 0$ the (unique) local solution to (12) with the initial condition $w(0) = 1$, $w'(0) = \zeta$, is global and tends to 0 at infinity. Let us denote this solution by $w_\zeta$. The real number $\zeta$ has to be selected in order to have $w_\zeta > 0$ on $(0, \infty)$. Since $\zeta$ is positive there exists $t_0 = t_0(\zeta)$ such that $w_\zeta$ is monotonic increasing on $(0, t_0)$ and $w_\zeta'(t_0) = 0$. Define

$$w(t) = w_\zeta(t + t_0)$$

for all $t \geq 0$. The function $w$ is a solution to (12) with the initial condition

$$w(0) = w_\zeta(t_0) > 1, \quad w'(0) = 0.$$

Using the scaling transformations

$$t \to d^{-\frac{\sigma-1}{2}} t, \quad w \to \frac{1}{d} w,$$

where $d = w_\zeta(t_0)$, the new function solves the problem

$$w'' + \Gamma d w' + w^\sigma = 0,$$

and

$$w(0) = 1, \quad w'(0) = 0, \quad w(\infty) = 0,$$

where $\Gamma_d = d^{-\frac{\sigma-1}{2}} \Gamma$.

According to the above analysis problem (26), (27) has a decreasing solution if and only if

$$\Gamma \geq d^{-\frac{\sigma-1}{2}} \Gamma^*.$$

On the other hand, it follows from (12)

$$\Gamma_d + \int_0^{t_0} w_\zeta(s)\sigma ds = \zeta + \Gamma'.$$

Thus $\Gamma_d < \zeta + \Gamma$. So, inequality (28) is satisfied for

$$0 < \zeta \leq \Gamma \left[ \left( \frac{\Gamma}{\Gamma^*} \right)^{\frac{2}{\sigma-1}} - 1 \right] \leq \Gamma \left[ \left( \frac{\Gamma}{\Gamma^*} \right)^{\frac{2}{\sigma-1}} - 1 \right].$$

Consequently, for this range of values of $\zeta$, the solutions $w_\zeta$ gives a family of monotonic increasing–decreasing solutions to (12), (13) for $\Gamma > \Gamma^*$.

In summary, the main result is that problem (1), (2) has multiple monotonic increasing and increasing–decreasing solutions such that $\frac{dy}{dx}(1)$ takes place in the interval

$$-\lambda^{-1/(\sigma-1)}[\Sigma^{1/(\sigma-1)} - \gamma^{1/(\sigma-1)}] \leq \frac{dy}{dx}(1) \leq \gamma \sqrt{1 - 2\lambda \gamma^{\sigma-1}/\sigma},$$

for all $0 < \gamma < \gamma_\zeta$, where $\gamma_\zeta$ is given by (22). If $\gamma = \gamma_\zeta$ the solution is unique, monotonic increasing and satisfies $\frac{dy}{dx}(1) = 0$.

From a different perspective, Magyari et al. [24] considered the case $\sigma = 2$. Using a natural point-mechanical analogy they conjectured the existence of $\Gamma_{\text{min}} (= \Gamma^*) > 0$ such that problem (12), (13), with $\sigma = 2$, has multiple solutions for $\Gamma > \Gamma_{\text{min}}$, a unique solution for $\Gamma = \Gamma_{\text{min}}$ and no solution for $\Gamma < \Gamma_{\text{min}}$. Moreover, for any $\Gamma > \Gamma_{\text{min}}$ there exist a smallest negative initial (velocity) $w_{0,\text{min}}'(\Gamma)$ and a maximum positive initial (velocity) $w_{0,\text{max}}'(\Gamma)$, such that the multiple solutions are parameterized by $w'(0) = \zeta$ in the interval

$$w_{0,\text{min}}'(\Gamma) \leq \zeta \leq w_{0,\text{max}}'(\Gamma).$$
The above predictions were supported by numerical calculations. Magyari et al. obtained that \( \Gamma_{\text{min}} = 1.079151 \) (for \( \sigma = 2 \)) and that (for \( \Gamma = 2 \))
\[
u_{0,\text{min}}(2) = -1.725126, \quad \nu_{0,\text{max}}(2) = +9.212868.
\]
The authors also conjectured that the width of the interval in (30) increases with \( \Gamma \) rapidly, and there exists an overlinear increase of \( \nu_{0,\text{max}}(\Gamma) \) and a linear decrease of \( \nu_{0,\text{min}}(\Gamma) \) for \( \Gamma \gg \Gamma_{\text{min}} \), which agree very well with (25) and (29).

In the present analysis we find that, for \( \sigma = 2 \), \( \Gamma^* = \Gamma_{\text{min}} \geq 1 \) and that, for \( \Gamma = 2 \), \( \nu_{0,\text{min}} \geq -\sqrt{3} = -1.732050808 \).

### 3. Asymptotic behavior of solutions at the origin

To complete our analysis, we discuss the asymptotic behavior of \( y(x) \) as \( x \) tends to 0. As in Section 2 it is convenient to study the asymptotic behavior of solutions to the Abel problem (16), (18). Using the same approach as in [3,17] we will obtain a more complete asymptotic expansion for solutions to (16), (18) \((\Gamma \geq \Gamma^*)\).

First, assume that \( \varphi(s)/s \to \Gamma^* \) as \( s \) goes to 0. From (17) it follows
\[
s^{\sigma+1} \left( \frac{\varphi(s)}{s} - \Gamma^* \right) = -\int_0^s r^{\sigma} \varphi(r)^{-1}(r) \, dr,
\]
for any \( 0 < s < 1 \). Then by L’Hôpital’s rule, we obtain
\[
\lim_{s \to 0} s^{\sigma+1} \left( \frac{\varphi(s)}{s} - \Gamma^* \right) = -\frac{1}{\sigma} \lim_{s \to 0} \frac{s}{\varphi(s)}.
\]
Thus
\[
\lim_{s \to 0} s^{\sigma+1} \left( \frac{\varphi(s)}{s} - \Gamma^* \right) = -\frac{1}{\sigma} \Gamma^*.
\]

Next, using again (17)
\[
s^{\sigma+2} \left\{ s^{\sigma+1} \left( \frac{\varphi(s)}{s} - \Gamma^* \right) + \frac{1}{\sigma} \frac{1}{\Gamma^*} \left( \frac{1}{\varphi(s)} - \Gamma^* \right) \right\} = \frac{1}{s^{2\sigma-1} \Gamma^*} \int_0^s r^{\sigma} \left( \frac{\varphi(r)}{r} - \Gamma^* \right) \, dr.
\]
Together with (32) and the L’Hôpital’s rule
\[
s^{\sigma+1} \left\{ s^{\sigma+1} \left( \frac{\varphi(s)}{s} - \Gamma^* \right) + \frac{1}{\sigma} \frac{1}{\Gamma^*} \right\} = -\frac{1}{\sigma} \frac{1}{(2\sigma - 1) \Gamma^{*3}}.
\]
Consequently,
\[
\varphi(s) = \Gamma^* s - \frac{s^{\sigma}}{\sigma \Gamma^*} - \frac{s^{2\sigma-1}}{\sigma(2\sigma - 1) \Gamma^{*3}} + o(s^{2\sigma-1}),
\]
as \( s \to 0 \).

Now, assume that \( \Gamma > \Gamma^* \), which leads to (20); that is
\[
\lim_{s \to 0} \frac{\varphi(s)}{s^\sigma} = \frac{1}{\Gamma^*}.
\]

Set
\[
\varphi(s) = s^\sigma \left[ \frac{1}{\Gamma} + h(s) \right],
\]
where the function \( h \in C^1(0, 1) \) and \( h(0) = 0 \). Inserting (35) into (16) yields
\[
\Gamma \frac{h}{s^{\sigma+1}} = \sigma \left( \frac{1}{\Gamma} + h \right)^2 + s \left( \frac{1}{\Gamma} + h \right) h'.
\]
Define
\[ H(s) = \frac{h(s)}{s^{\sigma-1}}. \]

Assume that \( H \) is not monotonic on any interval \((0, s_0), 0 < s_0 < 1\). Let \( s_k^{\pm} \) be a sequence converging to 0 as \( k \to \infty \) such that
\[ H'(s_k^{\pm}) = 0, \quad \liminf_{s \to 0} H(s) = \lim_{k \to \infty} H(s_k^{\pm}), \quad \limsup_{s \to 0} H(s) = \lim_{k \to \infty} H(s_k^{\pm}). \]
Since
\[ s_k^{\pm} h'(s_k^{\pm}) = \sigma h(s_k^{\pm}), \]
the LHS goes to 0 as \( k \to \infty \). Together with Eq. (36) we infer
\[ \lim_{k \to \infty} H(s_k^{-}) = \lim_{k \to \infty} H(s_k^{+}) = \sigma \frac{\Gamma}{\Gamma_3}. \]
Thus
\[ \lim_{s \to 0} \frac{h(s)}{s^{\sigma}} = \sigma \frac{\Gamma}{\Gamma_3}. \tag{37} \]

Now, assume that there exists \( 0 < s_0 < 1 \) such that the function \( H \) is monotonic on \((0, s_0)\). Without lost of generality we may assume that \( H \) is monotonic increasing; that is
\[ s h'(s) \geq h(s) \]
on \((0, s_0)\). Thus
\[ \Gamma H \geq \sigma \left( \frac{1}{\Gamma} + h \right)^2 + (\sigma - 1) \left( \frac{1}{\Gamma} + h \right) h \]
on \((0, s_1)\) for some \( s_1 < s_0 \). Therefore \( H \) is bounded and there exists \( \frac{\sigma}{\Gamma_3} \leq L < \infty \) such that \( \lim_{s \to 0} H(s) = L \) and this limit satisfies
\[ \Gamma L = \frac{\sigma}{L^2} + \frac{1}{\Gamma} \lim_{s \to 0} s h'(s). \]
Assume, on the contrary that \( L \neq \frac{\sigma}{\Gamma_3} \). We deduce from the above that \( h(s) \simeq l \log(s) \), as \( s \to 0 \) for some constant \( l \neq 0 \). A contradiction. Then \( L = \frac{\sigma}{\Gamma_3} \) which leads to (37). Therefore
\[ \varphi(s) = \frac{s^\sigma}{\Gamma} + \frac{\sigma s^{2\sigma-1}}{\Gamma_3} + o(s^{2\sigma-1}), \tag{38} \]
as \( s \to 0 \).

Returning to the original variables, we deduce that if \( y \) is a local positive solution to (1) such that \( y(0) = 0 \), either
\[ \lim_{x \to 0^+} \frac{y'(x)x}{y(x)} = 1 \tag{39} \]
or
\[ \lim_{x \to 0^+} \frac{y'(x)x}{y(x)^\sigma} = \lambda. \tag{40} \]
From this we deduce, by a simple integration, the following results.

(a) If \( \gamma = \gamma_c \) problem (1), (2) has a unique solution. Moreover this solution satisfies
\[ y(x) \simeq C \lambda^{-1/(\sigma-1)} x, \tag{41} \]
as \( x \to 0^+ \), where \( C = C(\sigma) \) is a positive constant.
(b) For all $0 < \gamma < \gamma_c$ there are multiple monotonic increasing solutions and multiple increasing–decreasing solutions which satisfy
\[ y(x) \simeq C \left( \lambda(\sigma - 1) \log(x) \right)^{-1/(\sigma - 1)}, \] as $x \to 0^+$, where $C = C(\sigma)$ is a positive constant.

(c) For all $\gamma > \gamma_c$ there is no positive solution.

As was pointed out by the referee, it should be noticed here that for $0 < \gamma < \gamma_c$, any solution is not Dini continuous at zero for $\sigma \geq 2$.

4. Absence of solution for the singular elliptic problem

As mentioned in the introduction, we are also interested in the non-solvability of the following singular Lane–Emden–Fowler problem
\[ \begin{cases} -\Delta u = \phi(d(x))u^\sigma & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \tag{43} \]
where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded domain with smooth boundary, $\sigma > 1, d(x) = \text{dist}(x, \partial \Omega)$, and $\phi$ is a continuous function satisfying
\[ c_1 t^{-2} \leq \phi(t), \]
for any $t$ small, where $c_1 > 0$ is a constant.

Recently, Ghergu and Rădulescu [10] considered the following singular Lane–Emden–Fowler equation
\[ \begin{cases} -\Delta u = \lambda f(u) + \phi(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \tag{44} \]
where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N, N \geq 2$, $\lambda$ is a positive parameter, $f$ is a positive continuous function and the function $\phi \neq 0$ is supposed to be nonnegative, Hölder continuous and satisfies $\lim_{d(x) \to 0} \frac{\phi(x)}{h(d(x))} = c_0$, where $h$ is a positive continuous nondecreasing function and $c_0$ is a positive constant. The function $g \in C^1(0, \infty)$ is assumed to be positive, nonincreasing and satisfies $\lim_{t \to 0^+} g(t) = \infty$. Under suitable conditions on $f$ the authors showed that there exists $\lambda_c$ such that problem (44) has a solution if and only if $\lambda < \lambda_c$. This solution is unique and satisfies the following boundary estimate
\[ \lim_{d(x) \to 0} \frac{u(x)}{y(d(x))} = c_1, \]
where $c_1$ is a positive constant and $y$ is a local positive solution to
\[ y'' + \phi(t)g(y(t)) = 0, \quad y(0) = 0. \]

Very recently, Dupaigne et al. [7] discussed the problem
\[ \begin{cases} -\Delta u - \phi(d(x))g(u) = \lambda f(x, u) + \mu |\nabla u|^a & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \tag{45} \]
where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N, N \geq 2$, $\lambda > 0, \mu \in \mathbb{R}, 0 < a \leq 2$ and $g \in C^1(0, \infty)$ is a positive decreasing function and unbounded around the origin. Among other results, the authors showed that if $\int_0^1 t\phi(t) = \infty$, problem (45) has no regular solution. This result is established with the crucial help of a local solution to the problem
\[ \begin{cases} y'' + \phi(t)g(y(t)) = 0, \quad y' > 0, \quad 0 < t < b < 1, \\ y(0) = 0. \end{cases} \tag{46} \]
In particular, the authors showed that the problem
\[
\begin{cases}
  -\Delta u - d(x)^{-\alpha} u^{-\beta} = \lambda f(x, u) + \mu|\nabla u|^\alpha & \text{in } \Omega, \\
  u > 0 & \text{in } \Omega, \\
  u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( \beta > 0 \) and \( \alpha \geq -2 \), has no regular solution.

To prove the non-existence result for problem (43), we use a sub-supersolutions argument and the Emden–Fowler problem (1), (2) (see [7, 10, 11, 32–34]). Our non-existence result completes the results in [33] and [7].

Suppose that \( u \) is a regular solution to problem (43). Let \( 0 < \gamma < \gamma_c = (\Sigma^*)^{1/(\sigma-1)} \), where \( \Sigma^* \) is given in Section 3 and \( y \) be a solution to (1), (2), where \( \lambda = 1 \), such that \( y'(t) > 0 \) for all \( t \) in \((0, 1)\). Define
\[
u_t(x) = t^{-\tau} y(t \varphi_1),
\]
where \( \tau > 0 \) is a real parameter and \( \varphi_1 \) is a positive eigenfunction of \( -\Delta \) in \( H^1_0(\Omega) \). Recall that \( \varphi_1 \) satisfies
\[
\frac{1}{c_0} d(x) \leq \varphi_1(x) \leq c_0 d(x),
\]
for all \( x \in \Omega \), where \( c_0 \) is a positive constant. A simple computation yields to
\[
-\Delta \nu_t = \frac{u_{t}^\sigma}{\varphi_1^\sigma} t^{(\sigma-1)\tau} \left[ |\nabla \varphi_1|^2 + \lambda_1 t \varphi_1 y'(t \varphi_1) \frac{\tau}{y(t \varphi_1)^{1+\sigma}} \varphi_1^2 \right],
\]
where \( \lambda_1 \) is the first eigenvalue of \( -\Delta \). Together with (48) one sees
\[
-\Delta \nu_t \leq c_0^2 \frac{\lambda_1 t^{\sigma-1}}{d(x)^2} t^{\sigma \tau} \left[ |\nabla \varphi_1|^2 + \lambda_1 t \varphi_1 y'(t \varphi_1) \frac{\tau}{y(t \varphi_1)^{1+\sigma}} \varphi_1^2 \right].
\]
On the other hand, from (40), we deduce that there exists \( \xi_0 > 0 \) such that
\[
\frac{\xi y'(\xi)}{y(\xi)^\sigma} \leq 2
\]
for all \( 0 < \xi < \xi_0 \).

Clearly, if we choose \( t > 0 \) small enough such that \( t \varphi_1 < \xi_0 \) and \( c_0^2 t^{\sigma-1} t \varphi_1 y'(t \varphi_1) \frac{\tau}{y(t \varphi_1)^{1+\sigma}} \varphi_1^2 < 1 \) in \( \Omega \) we conclude that
\[
-\Delta \nu_t \leq c_1 d(x)^{-2} u_{t}^\sigma \leq \phi(d(x)) u_{t}^\sigma,
\]
in \( \Omega \). This shows that \( \nu_t \) is a sub-solution to (43). Consequently, \( \nu_t \leq u \) in \( \Omega \). Thus
\[
\frac{1}{C_1} \frac{1}{t^{\sigma-1}/(\lambda_1 t \varphi_1(x)^2)^{1/(\sigma-1)}} \leq u(x),
\]
for all \( x \in \Omega \), and all \( t > 0 \) small, where \( C_1 \) is a positive constant, thanks to (42). Passing to the limit as \( t \to 0^+ \), we see that \( u(x) = \infty \). Hence problem (43) has no regular solution.

The similar argument can be used for the class of non-standard equations of the type
\[
\begin{cases}
  \rho(x) u_t = \Delta u^m & \text{in } \Omega \times (0, T), \\
  u > 0 & \text{in } \Omega \times (0, T), \\
  u = 0 & \text{on } \partial \Omega \times (0, T), \tag{51}
\end{cases}
\]
where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N, N \geq 2, 0 < m < 1, 0 < T \leq \infty \) and \( \rho \geq 0 \).

It is simple to verify that, if \( \rho(x) \sim d(x)^{-2} \), problem (51) has no similarity solution in the form
\[
u(x, t) = [T_0 - t]_+^{1/(1-m)} v(x).
\]

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