Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Functional Analysis 200 (2003) 505–535
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Groups of hierarchomorphisms of trees and related Hilbert spaces

Yu.A. Neretin[1,*]

Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse, 9, Wien 1020, Austria

Received 16 November 2001; accepted 28 October 2002

Communicated by M. Vergne

To the memory of M.V. Smurov

0. Introduction

0.1. Hierarchomorphisms (spheromorphisms)

The Bruhat–Tits tree $\mathcal{T}_p$ is an infinite tree such that any vertex belongs to $(p+1)$ edges, $p \geqslant 2$. Cartier [3] observed that the groups $\mathrm{Aut}(\mathcal{T}_p)$ of automorphisms of the trees $\mathcal{T}_p$ are analogs of rank 1 real and p-adic groups (as $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{O}(1,n)$, $\mathrm{SL}_2(\mathbb{Q}_p)$, etc.). The representation theory of $\mathrm{Aut}(\mathcal{T}_p)$ was developed in Cartier's [3] and Olshansky's [31] papers.

The group $\mathrm{Aut}(\mathcal{T}_p)$ is essentially simpler than the rank 1 groups over locally compact fields, but many nontrivial phenomena related to rank 1 groups survive for the group of automorphisms of Bruhat–Tits trees.

The absolute of a Bruhat–Tits tree is an analogue of the boundaries of rank 1 symmetric spaces; in particular, the absolute is an analogue of the circle. A group of hierarchomorphisms[2] $\mathrm{Hier}(\mathcal{T}_p)$ (defined in [25]) is a tree analogue of the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle. The group $\mathrm{Hier}(\mathcal{T}_p)$ consists of homeomorphisms of the absolute of $\mathcal{T}_p$ that can be extended to the whole Bruhat–Tits tree except a finite subtree. It turns out [25,26], that the representation theory of $\mathrm{Diff}(S^1)$ partially survives for the groups $\mathrm{Hier}(\mathcal{T}_p)$.

*Permanent address: Math. Phys. Group, Institute of Theoretical and Experimental Physics, Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia.

E-mail address: neretin@main.mccme.rssi.ru, neretin@gate.itep.ru.

[1] Partially supported by the Grant NWO 047-008-009.

[2] In [26], there was proposed the term *ball-morphisms*, which is difficult for pronouncement. In English translation, it was replaced by *spheromorphism*. I want to propose the neologism *hierarchomorphism*, this is a map, which regards hierarchy of balls on the absolute; see below Section 5.1.

doi:10.1016/S0022-1236(02)00146-5

In fact, the group $\mathrm{Hier}(\mathcal{T}_p)$ contains the group of locally analytic diffeomorphisms of p-adic line (see [26]), and this partially explains the points of similarity of $\mathrm{Diff}(S^1)$ and $\mathrm{Hier}(\mathcal{T}_p)$.[3]

The following facts 1–4 are known about the groups $\mathrm{Hier}(\mathcal{T}_p)$. Phenomena 1–3 are a reflection of the representation theory of $\mathrm{Diff}(S^1)$, the last phenomenon now does not have a visible analogue over $\mathbb{R}$.

1. (Neretin [25,26]) Denote by $\mathrm{O}(\infty)$ the group of all orthogonal operators in a real Hilbert space H. Denote by $\mathrm{GLO}(\infty)$ the group of all invertible operators in H having the form $A = B + T$, where $B \in \mathrm{O}(\infty)$ and T has finite rank. Denote by $H_{\mathbb{C}}$ the complexification of H. Denote by $\mathrm{UO}(\infty)$ the group of all unitary operators in $H_{\mathbb{C}}$ having the form $A = B + T$, where $B \in \mathrm{O}(\infty)$ and T has finite rank. There exist some series of embeddings

$$\mathrm{Hier}(\mathcal{T}_p) \to \mathrm{GLO}(\infty), \quad \mathrm{Hier}(\mathcal{T}_p) \to \mathrm{UO}(\infty).$$

This allows to apply the second quantization machinery (see [27,29,33]) for obtaining unitary representations of $\mathrm{Hier}(\mathcal{T}_p)$.

2. Embeddings $\mathrm{Hier}(\mathcal{T}_p) \to \mathrm{GLO}(\infty)$ allow to develop a theory of fractional diffusions with a Cantor set time (the Cantor set appears as the absolute of the tree). I never wrote a text on this topic, but, on the whole, the picture here is quite parallel to fractional diffusions with real time (see [28]).

3. (Kapoudjian [16,19]) There exists a $\mathbb{Z}/2\mathbb{Z}$-central extension of $\mathrm{Hier}(\mathcal{T}_p)$.

4. (Kapoudjian, [18]) Consider the dyadic Bruhat–Tits tree $\mathcal{T}_2$. There exists a canonical action of the group $\mathrm{Hier}(\mathcal{T}_2)$ on the inductive limit of the Deligne–Mumford [5] moduli spaces $\lim_{n\to\infty} \mathcal{M}_{0,2^n}$ of 2^n point configurations on the Riemann sphere. This construction also has two versions over $\mathbb{R}$. The first variant is an action on the inductive limit of Stasheff associahedrons [42]. The second variant is an action on the inductive limit of the spaces constructed by Davis et al. [4]. Some group-theoretical properties of Hier $(\mathcal{T}_p)$ are discussed in [17].

0.2. The purposes of this paper

The paper has two purposes. The first aim is to construct a new series of embeddings of the groups of hierarchomorphisms to the group $\mathrm{GLO}(\infty)$. By the Feldman–Hajek theorem (see [41]), this gives constructions of unitary representations of groups of hierarchomorphisms, but we do not discuss this subject.

There exists the wide and nice theory of actions of groups on trees (see [20,38–40]). It is clear that a hierarchomorphism type extension can be constructed for any group Γ acting on a tree (and even on an $\mathbb{R}$-tree), it is sufficient to allow to cut finite collections of edges. The second purpose of this paper[4] is to understand, is this "hierarchomorphization" of an arbitrary group Γ a reasonable object?

[3] Another heuristic explanation can be obtain by the monstrous degeneration construction from [20, Chapter 9]; the Lobachevsky plane can be degenerated to the universal $\mathbb{R}$-tree.

[4] See also the recent preprint of Nekrashevich [24].

One example of such "hierarchomorphization" is quite known, this is the Richard Thompson group [23], which firstly appeared as an counterexample in the theory of discrete groups. Later it became clear, that this group is not a semi-pathological counterexample, but a rich and unusual object (see works of Greenberg, Ghys, Sergiescu, Penner, Freyd, Heller and others [1,10,12,13,34,35], see also [2]), relation of the hierarchomorphisms and the Thompson group was observed by Sergiescu).

If the group Γ acting on a tree is discrete, then the corresponding group of hierarchomorphisms is a discrete Thompson-like group. If the group Γ is locally compact, then the group of hierarchomorphisms (see some examples in [26]) is an "infinite-dimensional group" (or, it is better to say, "large group") similar to the group of diffeomorphisms of the circle or diffeomorfisms of p-adic line.

0.3. New results

We define two groups $\mathrm{Hier}^\circ(J)$, $\mathrm{Hier}(J)$ of hierarchomorphisms of a tree.

The first groups $\mathrm{Hier}^\circ(J)$ consists of transformations of set of the vertices. These transformations are almost compatible with the structure of the tree, we allow "breaking" of arbitrary finite collections of edges.

The second group $\mathrm{Hier}(J)$ consists of homeomorphisms of the absolute that admit a continuation to whole tree except a finite subtree. This group is a quotient group of $\mathrm{Hier}^\circ(J)$.

The second group is more similar to the group of diffeomorphisms of the circle, and apparently it is more important.

We construct embeddings of the both groups to $\mathrm{GLO}(\infty)$.

First, we introduce a family of Hilbert spaces $\mathscr{H}_\lambda(J)$, where $0<\lambda<1$, associated with a tree J. The space $\mathscr{H}_\lambda(J)$ contains the system of vectors (a "nonorthogonal basis") e_a enumerated by vertices a of the tree, and the inner products of the vectors e_a, e_b are given by

$$\langle e_a, e_b \rangle = \lambda^{\{\text{distance between } a \text{ and } b\}}$$

(these spaces are well-known, see [14,15,32]). We show that the group $\mathrm{Hier}^\circ(J)$ acts in $\mathscr{H}_\lambda(J)$ by operators of the class $\mathrm{GLO}(\infty)$.

Second, for sufficiently large values of λ, we construct an operator of the "restriction to the absolute" in the space $\mathscr{H}_\lambda(J)$. In other words, we construct an $\mathrm{Hier}^\circ(J)$-invariant subspace $\mathscr{E}_\lambda(J)$ in $\mathscr{H}_\lambda(J)$. The action of $\mathrm{Hier}^\circ(J)$ in $\mathscr{E}_\lambda(J)$ is not faithful, and it is reduced to the action of the quotient group $\mathrm{Hier}(J)$. Thus, we obtain the embedding of $\mathrm{Hier}(J)$ to the group GLO of the space $\mathscr{E}_\lambda(J)$.

0.4. The structure of the paper

Sections 1 and 2 contain preliminary definitions and examples. In Section 3, we define the groups of hierarchomorphisms of trees. In Section 4, we consider the Hilbert spaces $\mathscr{H}_\lambda(J)$ and the associated embeddings of the groups Hier° to

$\mathrm{GLO}(\infty)$ associated with a tree J. In Section 5, for sufficiently large λ, we construct an operator of the 'restriction to the absolute' in the space $\mathscr{H}_\lambda$. In Section 6, we discuss the action of the group of hierarchomorphisms in spaces of functions (distributions) on the absolute.

1. Notation and terminology

1.1. Simplicial trees

A simplicial tree J is a connected graph without circuits. By $\mathrm{Vert}(J)$ we denote the set of vertices of J. By $\mathrm{Edge}(J)$ we denote the set of edges of J. We say that two vertices $a, b \in \mathrm{Vert}(J)$ are *adjacent*, if they are connected by an edge. We denote this edge by $[a, b]$.

We assume that the sets $\mathrm{Vert}(J)$, $\mathrm{Edge}(J)$ are countable or finite. A simplicial tree is *locally finite* if any vertex a belongs to finitely many edges. We admit nonlocally finite trees.

A *way* in J is a sequence of *distinct* vertices

$$\ldots, a_1, a_2, a_3, \ldots$$

such that a_j, a_{j+1} are adjacent. A way can be finite, or infinite to one side, or infinite to the both sides.

For each vertices a, b, there exists a unique way $a_0 = a, a_1, \ldots, a_k = b$ connecting a and b. We say that k is the *simplicial distance* between a and b. We denote the simplicial metrics by

$$d_{\mathrm{simp}}(a, b).$$

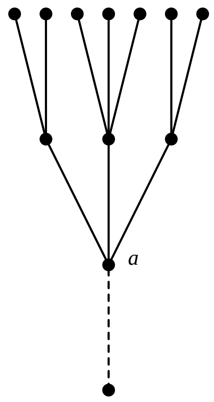
A *subtree* $I \subset J$ is a connected subgraph in the tree J. The *boundary* ∂I of a subtree $I \subset J$ is the set of all $a \in \mathrm{Vert}(I)$ such that there exists an edge $[a, b]$ with $b \notin \mathrm{Vert}(I)$. A subtree $I \subset J$ is a *thicket-subtree*, if the number of edges $[a, b] \in \mathrm{Edge}(J)$ such that $a \in I$, $b \notin I$ is finite. If we delete a finite collection of edges of J, then we obtain a finite collection of thicket subtrees.

A subtree $I \subset J$ is a *branch*, if there is a unique edge $[a, b] \in \mathrm{Edge}(J)$ such that $a \in \mathrm{Vert}(I)$, $b \notin \mathrm{Vert}(I)$, see Fig. 1. The vertex a is called a *root* of the branch. If we delete an edge of the tree J, then we obtain two branches.

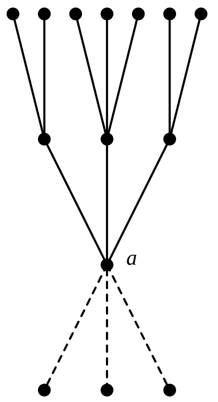
A subtree $I \subset J$ is a *bush*, if its boundary contains only one point a (a *root*) and the number of edges $[a, b] \in \mathrm{Edge}(J)$ such that $b \notin I$ is finite, see Fig. 1.

Lemma 1.1. (a) *The intersection of a finite family of thicket-subtrees is a thicket-subtree.*

(b) *For a thicket-subtree* $I \subset J$, *there exists a finite collection of edges* $\ell_1, \ldots, \ell_k \in \mathrm{Edge}(I)$ *such that* I *without* $\ell_1, \ldots, \ell_k$ *is a union of bushes and single point sets.*



A branch



A bush

Fig. 1.

Proof. Statement (a) is obvious.

(b) Let $a_1, \ldots, a_k$ be the boundary of I. Let $L \subset I$ be the minimal subtree containing the vertices $a_1, \ldots, a_k$. It is sufficient to delete all edges of L.

The rest is a (disconnected) subgraph M, satisfying the following properties:

(a) ∂M is finite.
(b) Let $a, b \in \partial M$, and an edge $[c, c^1]$ is on the way between a and b, we have $[c, c^1] \notin M$.

Each connected component of M satisfies the same properties. But a connected graph satisfying (a), and (b) is a bush. $\square$

1.2. Actions of groups on simplicial trees

A bijection $\mathrm{Vert}(J) \to \mathrm{Vert}(J)$ is an *automorphism* of a simplicial tree J, if the images of adjacent vertices are adjacent vertices. An action of a group Γ on a simplicial tree J is an embedding of Γ to the group of automorphisms of J.

1.3. Absolute

The absolute $\mathrm{Abs}(J)$ of a tree is the set of points of the tree at infinity. Let us give the formal definition.

We say that a *ray* is an infinite way $a_1, a_2, \ldots$. We say that rays $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are equivalent if there exist k and a sufficiently large N such that $b_j = a_{j+k}$ far all $j \geqslant N$.

A point of an absolute is a class of equivalent ways.

1.4. Metric trees

Let J be a simplicial tree. Let us assign a positive number $\rho(a,b)$ to each edge $[a,b]$. Let a, c be arbitrary vertices of J, let $a_0 = a, a_1, \dots, a_k = c$ be the way connecting a and c. We assume

$$\rho(a,c) = \sum_{j=1}^{k} \rho(a_{j-1}, a_j).$$

Obviously, ρ is a metric on $\mathrm{Vert}(J)$. We call by *metric trees* the *countable* spaces $\mathrm{Vert}(J)$ equipped with the metrics ρ.

Obviously, the edges of J can be restored from the metric ρ. By this reason, we prefer to think that the edges are present in a metric tree as an additional (combinatorial) structure.

Remark. Simplicial trees can be considered as partial cases of metric trees. Indeed, we can assume that lengths of all the edges are 1.

Remark. Sometimes the term *metric tree* is used in the quite different sense (for $\mathbb{R}$-trees, see below 4.7).

1.5. Actions of groups on metric trees

Let J be a metric tree. A bijection $\mathrm{Vert}(J) \to \mathrm{Vert}(J)$ is an *automorphism* of J, if it preserves the distance (hence it automatically preserves the structure of a simplicial tree).

An action of group Γ on a metric tree J is an embedding of Γ to the group of automorphisms of J.

2. Examples of actions of groups on trees

The purpose of this section is to give a collection of examples for abstract constructions given in Sections 3–6 (all these examples are standard). For algebraic and combinatorial theory of actions of groups on trees, see [38–40].

2.1. Bruhat–Tits trees

The Bruhat–Tits tree $\mathscr{T}_p$ is the tree, in which each vertex belongs to $(p+1)$ edges. The group $\mathrm{Aut}(\mathscr{T}_p)$ of automorphisms of $\mathscr{T}_p$ is a locally compact group. This group is similar to rank 1 groups over $\mathbb{R}$ and over p-adic fields. The representation theory of $\mathrm{Aut}(\mathscr{T}_p)$ and related harmonic analysis are well understood, see [3,7,8,31].

2.2. The tree $\mathcal{T}_\infty$

We denote by $\mathcal{T}_\infty$ the simplicial tree, in which each vertex belongs to a countable set of edges. At first sight, the group $\mathrm{Aut}(\mathcal{T}_\infty)$ seems pathological. Nevertheless, it is a useful object as one of the simplest examples of infinite-dimensional groups, see [29,32]. This group is an imitation of the group $\mathrm{O}(1,\infty)$.

2.3. The tree of free group

Denote by F_2 the free group with two generators α, β. Vertices of the tree $J(F_2)$ are numerated by elements of the group F_2. Vertices v_p, v_q are connected by an edge if

$$p = q\alpha^{\pm 1} \quad \text{or} \quad p = q\beta^{\pm 1}.$$

Obviously, $J(F_2)$ is the Bruhat–Tits tree $\mathcal{T}_3$. The group F_2 acts on the tree $J(F_2)$ by the transformations

$$r : v_p \mapsto v_{rp}, \quad \text{where } r \in F_2.$$

Fix $l_1, l_2 > 0$. Assign the length l_1 to any edge $[v_p, v_{p\alpha}]$, and the length l_2 to any edge $[v_p, v_{p\beta}]$. Thus we obtain the metric tree with the action of F_2.

2.4. Another tree of free group

Let us contract all the edges of the type $[v_p, v_{p\alpha}]$ of the tree $J(F_2)$ described in 2.3. Thus, we obtain the action of F_2 on $\mathcal{T}_\infty$.

2.5. Dyadic intervals

Vertices $V_{u;n}$ of the tree $J_2(\mathbb{R})$ are enumerated by segments in $\mathbb{R}$ having the form

$$S_{u;n} = \left[\frac{u}{2^n}, \frac{u+1}{2^n}\right], \quad \text{where } u \in \mathbb{Z}, n \in \mathbb{Z}.$$

We connect $V_{u;n}$ and $V_{w;n-1}$ by an edge if $S_{w;n-1} \supset S_{u;n}$.

Obviously, we obtain the simplicial Bruhat–Tits tree $\mathcal{T}_2$.

2.6. Balls on p-adic line

Denote by $\mathbb{Q}_p$ the field of p-adic numbers, denote by $\mathbb{Z}_p$ the p-adic integers. Denote by $B_{a,k}$ the ball

$$|z - a| \leqslant p^{-k}.$$

Remark. The radius p^{-k} is determined by the ball. But $B_{a,k} = B_{c,k}$ for any $c \in B_{a,k}$.

The set of vertices of the tree $J(\mathbb{Q}_p)$ is in one-to-one correspondence with the set of balls $B_{a,k}$. Vertices $v_{a,k}$ and $v_{b,k+1}$ are connected with an edge, if $B_{b,k+1} \supset B_{a,k}$. Obviously, the group of affine transformations

$$z \mapsto \alpha z + \beta$$

of the p-adic line $\mathbb{Q}_p^1$ acts on the tree $J(\mathbb{Q}_p)$.

2.7. Tree of lattices

Consider the p-adic plane $\mathbb{Q}_p^2$ equipped with a skew symmetric bilinear form

$$A((v_1, v_2), (w_1, w_2)) = v_1 w_2 - v_2 w_1.$$

Denote by $\mathrm{Sp}_2(\mathbb{Q}_p)$ the group of linear transformations preserving the form $A(v, w)$.

A *lattice* in $\mathbb{Q}_p^2$ is a compact subset $R \subset \mathbb{Q}_p^2$ having the form

$$\mathbb{Z}_p u \oplus \mathbb{Z}_p v, \quad \text{where } v, w \text{ are not proportional.}$$

We say that a lattice R is *self-dual* if

1. $A(v, w) \in \mathbb{Z}_p$ for all v, w in R.
2. if $h \in \mathbb{Q}_p^2$ satisfies $A(h, v) \in \mathbb{Z}_p$ for all $v \in R$, then $h \in R$.

Vertices of the tree $\mathscr{T}(\mathbb{Q}_p^2)$ are self-dual lattices. Two vertices R, S are connected by an edge if

$$\text{volume of } R \cap S = \frac{1}{p}(\text{volume of } R).$$

It can be shown that $\mathscr{T}(\mathbb{Q}_p^2)$ is the Bruhat–Tits tree $\mathscr{T}_p$. Obviously, the symplectic group $\mathrm{Sp}_2(\mathbb{Q}_p) \simeq \mathrm{SL}_2(\mathbb{Q}_p)$ acts on our tree by automorphisms.

Let us show that the absolute of the tree $\mathscr{T}(\mathbb{Q}_p^2)$ can be identified with the p-adic projective line $\mathbb{P}\mathbb{Q}_p^1$. Any self-dual lattice admits a representation in the form $p^{-n}\mathbb{Z}_p u \oplus p^n \mathbb{Z}_p v$, where $u, v \in \mathbb{Z}_p^2$ and $A(u, v) \in \mathbb{Z}_p$. We say that a sequence of the lattices converges to a line L if the lattices R_j can be represented in the form

$$p^{-n_j}\mathbb{Z}_p u_j \oplus p^{n_j}\mathbb{Z}_p v_j,$$

where $n_j \to +\infty$ and the $\lim u_j$ exists and belongs to L.

2.8. Modular tree

Consider the following standard picture from an arbitrary textbook on complex analysis. Consider the Lobachevsky plane $L : \operatorname{Im} z > 0$ and the triangle Δ with three vertices $0, 1, \infty$ on the absolute $\operatorname{Im} z = 0$. Consider the reflections of Δ with respect to the sides of Δ. This gives us 3 new triangles $\Delta_1, \Delta_2, \Delta_3$. Then we consider the reflections of Δ_j with respect to their sides, etc. We obtain a filling of L by infinite triangles (with vertices in rational points of the absolute $\operatorname{Im} z = 0$).

Vertices of the modular tree are enumerated by the triangles of the filling. Two vertices are connected by an edge, if the corresponding triangles have a common side.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the modular tree in the obvious way.

2.9. Tree of pants

Let R be a compact Riemann surface. Fix a collection $C_1, \dots, C_k$ of closed mutually disjoint geodesics on R. The universal covering of R is the Lobachevsky plane.

The coverings of the cycles C_j are geodesics on L. Thus we obtain the countable family of mutually disjoint geodesics on L. They divide L into a countable collection of domains.

Now we construct a tree. Vertices of the tree are enumerated by the domains on L obtained above. Two vertices are connected by an edge, if the corresponding domains have a common side.

The fundamental group $\pi_1(R)$ of the surface R acts on this tree in the obvious way.

3. Hierarchomorphisms

3.1. Large group of hierarchomorphisms

Consider a group Γ acting on a simplicial (or metric) tree J. Consider a partition of J into a finite collection of thicket-subtrees $S_1, \dots, S_k$; i.e., the subtrees S_j are mutually disjoint, and $\operatorname{Vert}(J) = \bigcup \operatorname{Vert}(S_j)$. Let

$$g_1 : S_1 \to J, \dots, g_k : S_k \to J, \quad g_j \in \Gamma$$

be a collection of embeddings such that

(1) the subtrees $g_j(S_j)$ are mutually disjoint;
(2) $\bigcup \operatorname{Vert}(g(S_j)) = \operatorname{Vert}(J)$.

Thus we obtain the bijection

$$g = \{g_j, S_j\} : \operatorname{Vert}(J) \to \operatorname{Vert}(J)$$

given by

$$g(a) = g_j(a) \quad \text{if } a \in \mathrm{Vert}(S_j).$$

We call such maps $\circ$-*hierarchomorphisms*, see Fig. 2.

Lemma 3.1. *The product of two $\circ$-hierarchomorphisms is a $\circ$-hierarchomorphism.*

Proof. Consider $\circ$-hierarchomorphisms $g = \{g_j, S_j)\}$ and $h = \{h_k, T_k\}$. Their product hg has the form

$$\{h_k g_j, g_j^{-1}(T_k) \cap S_j\}.$$

By Lemma 1.1, all the sets $g_j^{-1}(T_k) \cap S_j$ are thicket-subtrees. $\square$

Denote the group of all such $\circ$-hierarchomorphisms by $\mathrm{Hier}^{\circ}(J, \Gamma)$.

3.2. *Action of $\circ$-hierarchomorphisms on absolute*

Consider a $\circ$-hierarchomorphism $g = \{g_j, S_j\}$. Let $\omega \in \mathrm{Abs}(J)$. Let $a_1, a_2, \ldots$ be a way leading to ω. For a sufficiently large N and for some S_j, we have $a_N, a_{N+1}, \ldots \in S_j$. Hence $g_j(a_N), g_j(a_{N+1}), \ldots \in g_j(S_j)$ is a way leading to some point

$$v \in \mathrm{Abs}(g_j(S_j)) \subset \mathrm{Abs}(J).$$

We assume

$$v = g(\omega).$$

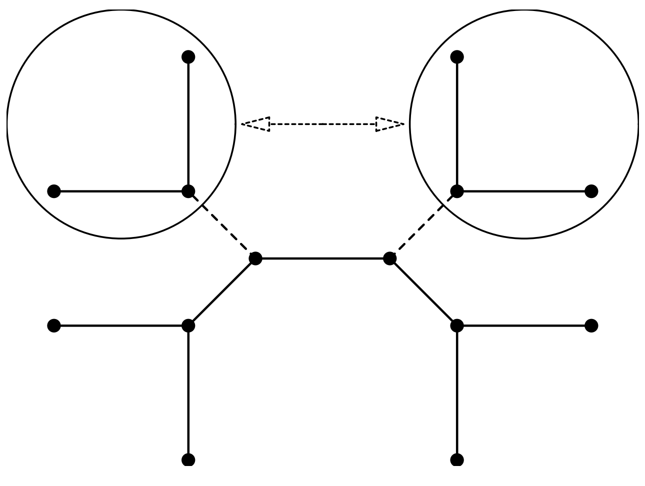


Fig. 2. An example of hierarchomorphism: a re-glueing of two branches.

3.3. Pseudo-derivative

Fix a point $\xi \in \mathrm{Vert}(J)$. Under the previous notation, consider the sequence

$$n_M = \rho(\xi, a_M) - \rho(\xi, g_j(a_M)).$$

This sequence becomes a constant after a sufficiently large M. We denote this constant (the *pseudo-derivative* of the $\circ$-hierarchomorphism g at the point $\omega \in \mathrm{Abs}(J)$) by

$$n(g, \omega) = n^{[\xi]}(g, \omega).$$

The following statement is obvious.

Proposition 3.1. *For* $g, h \in \mathrm{Hier}^{\circ}(J, \Gamma)$, $\omega \in \mathrm{Abs}(J)$,

$$n(gh, \omega) = n(h, \omega) + n(g, h\omega). \tag{3.1}$$

3.4. Small group of hierarchomorphisms

Denote by $\mathrm{Hier}(J, \Gamma)$ the group of transformations of the absolute induced by elements $g \in \mathrm{Hier}^{\circ}(J, \Gamma)$.

We have the obvious canonical map

$$\mathrm{Hier}^{\circ}(J, \Gamma) \to \mathrm{Hier}(J, \Gamma).$$

Its kernel Δ is a countable subgroup, and each element of Δ is a finite permutation of the set $\mathrm{Vert}(J)$.

Example. (a) Let $J = \mathscr{T}_p$ be the Bruhat–Tits tree and $\Gamma \simeq \mathrm{Aut}(\mathscr{T}_p)$ be the whole group of its automorphisms. In this case, the group Δ is the group of all the finite permutations of the set $\mathrm{Vert}(\mathscr{T}_p)$. The same is valid for the tree $J(F_2)$ of the free group (see 2.3).

(b) Let $J = \mathscr{T}_\infty$ be the Bruhat–Tits tree of the infinite order and $\Gamma \simeq \mathrm{Aut}(\mathscr{T}_\infty)$ be the whole group of its automorphisms. Then Δ consists of one element and the groups Hier° and Hier coincide. The same is valid for the action of the free group on $\mathscr{T}_\infty$ defined in 3.4.

(c) Consider a locally finite tree J and some group Γ of its automorphisms. Let $\Omega_1, \Omega_2, \dots$ be the orbits of Γ on the set $\mathrm{Vert}(J)$. Denote by $S(\Omega_i)$ the group of finite permutations of the set Ω_i. Then

$$\Delta = \prod S(\Omega_j).$$

3.5. *Shorter definition of the group* Hier(J, Γ)

A homeomorphism q of Abs(J) is an element of the group Hier(J, Γ) if

(1) there exists a finite collection of bushes $I_1, \dots, I_k$ such that the sets Abs$(I_1), \dots,$ Abs(I_k) are mutually disjoint and Abs$(J) = \cup$Abs(I_α);
(2) there exist elements of $g_1, \dots, g_k \in \Gamma$ such that

$$q\omega = g_\alpha \omega \quad \text{for } \omega \in \mathrm{Abs}(I_\alpha).$$

3.6. *Example: p-adic diffeomorphisms*

Consider the tree $J(\mathbb{Q}_p^2)$ of p-adic lattices described in 2.7 and its boundary $\mathbb{P}\mathbb{Q}_p^1$. We say that a bijection $q : \mathbb{P}\mathbb{Q}_p^1 \to \mathbb{P}\mathbb{Q}_p^1$ is locally analytic diffeomoirphism if it can be expanded into a Taylor series in a small neighborhood of each point.

Proposition 3.2 (Neretin [26]). *Any locally analytic diffeomorphism* $q : \mathbb{P}\mathbb{Q}_p^1 \to \mathbb{P}\mathbb{Q}_p^1$ *is a hierarchomorphism of the tree* $J(\mathbb{Q}_p)$.

3.7. *Pseudo-derivative on* Hier(J, Γ)

Obviously, the pseudo-derivative $n(g, \omega)$ is well defined for $g \in$ Hier(J, Γ).

Consider the Bruhat–Tits tree $\mathscr{T}_p$. Fix a vertex ξ of the Bruhat–Tits tree $\mathscr{T}_p$. Let us introduce the canonical measure v on Abs$(\mathscr{T}_p)$. Consider a branch S of $\mathscr{T}_p$, such, that $\xi \notin S$. Let v be the root of S. We assume

$$v(B[S]) = \frac{1}{p+1} \frac{1}{p^{\rho(\xi, v) - 1}},$$

The Radon–Nykodim derivative of the $g \in$ Hier$(\mathscr{T}_p)$ at a point ω is given by

$$g'(\omega) = p^{n(g, \omega)}.$$

Thus, for the Bruhat–Tits trees, the pseudo-derivative is reduced to the Radon–Nykodim derivative.

But for general metric trees there is no canonical measure an the absolute, and hence no the Radon–Nykodym derivative.

3.8. *A variant: planar hierarchomorphisms*

Assume a simplicial tree J be planar (this means, that for each vertex a we fix the cyclic order on the set of edges containing a; it is the case in some of our examples. Then also we have a canonical cyclic order on the absolute.

Now we can consider the group of hierarchomorphisms that preserves the cyclic order on the absolute.

4. Hilbert spaces $\mathscr{H}_\lambda(J)$

4.1. Definition

Let J be a metric tree, let $0<\lambda<1$. Denote by $\mathscr{H}_\lambda(J)$ the real Hilbert space spanned by the formal vectors e_a, where a ranges in $\mathrm{Vert}(J)$, with inner products given by

$$\langle e_a, e_b \rangle = \lambda^{\rho(a,b)}, \quad \forall a, b \in \mathrm{Vert}(J). \tag{4.1}$$

We must show that a system of vectors with the inner products (4.1) can be realized in a Hilbert space.

4.2. Existence of $\mathscr{H}_\lambda(J)$

Let a be a vertex of J. Let $b_1, b_2, \ldots$ be the vertices adjacent to a. Consider an arbitrary unit vector e_a in a real infinite-dimensional Hilbert space $\mathscr{H}$. Consider a collection $L_{b_1}, L_{b_2}, \ldots$ of pairwise perpendicular two-dimensional planes[5] containing e_a. For each plane L_{b_k}, we draw a vector $e_{b_k} \in L_{b_k}$ such that

$$\langle e_{b_k}, e_a \rangle = \lambda^{\rho(a,b_k)},$$

see Fig. 3.

By the perpendicularity,

$$\langle e_{b_k}, e_{b_l} \rangle = \langle e_{b_k}, e_a \rangle \cdot \langle e_a, e_{b_l} \rangle = \lambda^{\rho(b_k,b_l)}.$$

Then we apply the following inductive process. Assume that for a subtree S the required embedding $\mathrm{Vert}(S) \to \mathscr{H}$ is constructed, i.e., we have a subspace $\mathscr{H}_\lambda(S) \subset \mathscr{H}$. Let $b \in \mathrm{Vert}(S)$, and $c \notin \mathrm{Vert}(J)$ be adjacent to b. Consider the two-dimensional plane $L_c \subset \mathscr{H}$ that contains e_b and is perpendicular to $\mathscr{H}_\lambda(S)$. Let us draw a unit vector $e_c \in L_c$ such that

$$\langle e_c, e_b \rangle = \lambda^{\rho(b,c)}.$$

Thus we obtained the required embedding $\mathrm{Vert}(S) \bigcup \{b\} \to \mathscr{H}$.

"There is lot of rooms left in Hilbert space", and hence we obtain the embedding $\mathrm{Vert}(J) \to \mathscr{H}$.

Remark. This geometric picture is especially pleasant, if lengths of all the edges are equal.

[5] Subspaces M_1, M_2 in a Hilbert space are *perpendicular* iff there is an orthogonal system of vectors $u_1, u_2, \ldots, v_1, v_2, \ldots, w_1, w_2, \ldots$ such that M_1 is spanned by the vectors u_i, v_j, and M_2 is spanned by the vectors w_n, v_j.

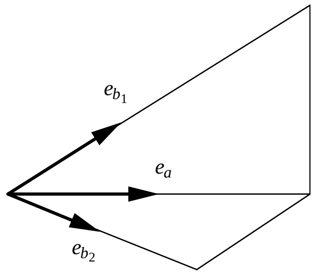


Fig. 3.

Remark. Each finite collection of vectors $e_{u_1}, \ldots, e_{u_k}$ is linear independent. This is quite clear from our geometrical description.

4.3. *More formal description of $\mathscr{H}_\lambda(J)$*

Consider an *affine* real infinite-dimensional Hilbert space $\mathscr{K}$, i.e., a Hilbert space, where the origin of coordinates is not fixed. Denote by $||\cdot||$ the length in $\mathscr{K}$. Consider a collection of points $N_a \in \mathscr{K}$, where $a \in \mathrm{Vert}(J)$, such that

(1) if $[a,b]$, $[c,d]$ are different edges of J, then $N_a N_b \perp N_c N_d$;

(2) for $[a,b] \in \mathrm{Edge}(J)$,

$$||N_a N_b||^2 = \sigma\rho(a,b),$$

where $\sigma > 0$ is fixed.

The existence of such embedding is obvious (see Fig. 4).

By the Pythagoras theorem,

$$||N_b N_c||^2 = \sigma\rho(b,c) \quad \forall b,c \in \mathrm{Vert}(J).$$

Now let us apply the following standard Fock–Schoenberg construction [9,36]. For an affine Hilbert space $\mathscr{K}$, there exists a linear Hilbert space $\mathrm{Exp}(\mathscr{K})$ and an embedding $\phi : \mathscr{K} \to \mathrm{Exp}(\mathscr{K})$ such that for all $X, Y \in \mathscr{K}$

$$\langle \phi(X), \phi(Y) \rangle = \exp(-||XY||^2).$$

Fix any origin of the coordinates in $\mathscr{K}$ and fix an orthonormal basis $e_1, e_2, \ldots$ in $\mathscr{K}$. Denote by $S^k(\mathscr{K})$ the symmetric powers of $(\mathscr{K})$. The vectors

$$e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_N^{\alpha_N}, \quad \text{where } N = 1, 2, \ldots \text{ and } \alpha_1 + \cdots + \alpha_N = k$$

form an orthogonal basis in $S^k(\mathscr{K})$, and

$$||e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_N^{\alpha_N}||^2 = \alpha_1! \cdots \alpha_N!$$

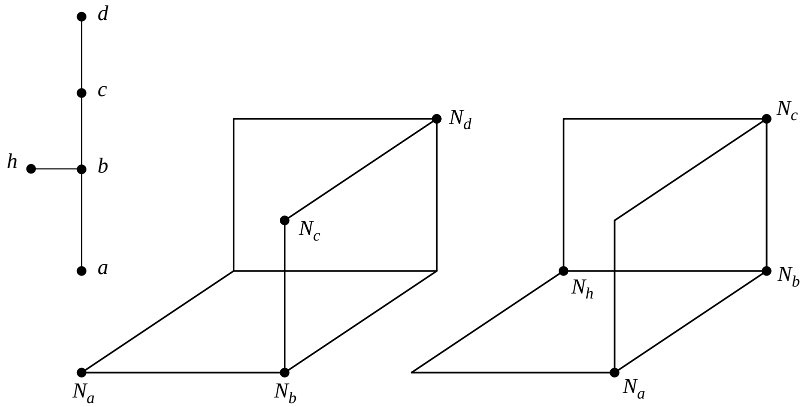


Fig. 4. Five points N_a, N_b, N_c, N_d, N_h span a four-dimensional subspace in the affine Hilbert space K. We portray the relative positions of N_a, N_b, N_c, N_d in the corresponding three-dimensional space, and also the relative positions N_a, N_b, N_c, N_h in (another) three-dimensional space.

We can assume that $\mathrm{Exp}(\mathscr{K})$ is the direct sum of all the symmetric powers of $\mathscr{K}$

$$\mathrm{Exp}(\mathscr{K}) = \mathbb{R} \oplus \mathscr{K} \oplus S^2\mathscr{K} \oplus S^3\mathscr{K} \oplus \cdots$$

and

$$\phi(X) = e^{-\|X\|^2}\left[1 \oplus \frac{X}{1!} \oplus \frac{X^{\otimes 2}}{2!} \oplus \frac{X^{\otimes 3}}{3!} \oplus \cdots\right].$$

It remains to apply the Fock–Schoenberg construction to the space $\mathscr{K}$ constructed above. The vectors $\phi(N_a)$ satisfy relations (4.1) for $\lambda = e^{-\sigma}$.

Remark. The spaces $\mathscr{H}_\lambda$ associated with a tree are well known; they are present in Haagerup's paper [14] and Olshansky's paper [32]. In an implicit form, they are present in [15] (Hilbert space itself without the underlying tree).

Remark. The trees are analogues of rank 1 noncompact Riemannian symmetric spaces. Consider the Lobachevsky plane $L = U(1,1)/U(1) \times U(1)$, i.e., the disc $|z| < 1$ in $\mathbb{C}$. Denote by $\mathscr{D}'$ the space of compactly supported distributions on L. Fix $\lambda > 0$. Consider the scalar product in $\mathscr{D}'$ given by

$$\langle \chi_1, \chi_2 \rangle := \left\{ \frac{(1 - z\bar{z})^\lambda (1 - u\bar{u})^\lambda}{(1 - u\bar{z})^{2\lambda}}, \chi_1(z)\overline{\chi_2(u)} \right\},$$

where $\{\cdot,\cdot\}$ denotes the pairing of smooth functions and distributions. Denote by $\mathscr{H}_\lambda(L)$ the Hilbert space associated with the pre-Hilbert space $\mathscr{D}'$. Our spaces $\mathscr{H}_\lambda(J)$ are an imitation of the spaces $\mathscr{H}_\lambda(L)$. The latter spaces were defined by Vershik et al. [11,43].

4.4. Action of the group of $\circ$-hierarchomorphisms in $\mathscr{H}_\lambda(J)$

Let a group Γ acts on J by isometries. Then Γ acts in $\mathscr{H}_\lambda(J)$ by the orthogonal operators[6] of the Hilbert space $\mathscr{H}_\lambda(J)$ by the formula

$$U_\lambda(g)e_a = e_{ga}. \tag{4.2}$$

Let now $g \in \mathrm{Hier}^\circ(J,\Gamma)$ be a $\circ$-hierarchomorphism. Define the operators $U_\lambda(g)$ by the same formula (4.2).

Theorem 4.1. (a) *The operators $U_\lambda(g)$ are well defined and bounded.*

(b) *Each operator $U_\lambda(g)$ can be represented in the form $U_\lambda(g) = A(1+R)$, where A is an orthogonal operator and R is an operator of finite rank.*

The theorem is proved below in 4.6.

4.5. The subspaces $\mathscr{H}_\lambda(S)$

Let S be a subtree in J. Denote by $\mathscr{H}_\lambda(S)$ the subspace in $\mathscr{H}_\lambda(J)$ generated by the vectors e_c, where $c \in \mathrm{Vert}(S)$. Denote by P_S the operator of projection $\mathscr{H}_\lambda(J) \to \mathscr{H}_\lambda(S)$.

Lemma 4.2. *Let A, B be two disjoint subtrees in J. Let $a \in \mathrm{Vert}(A)$, $b \in \mathrm{Vert}(B)$ be the nearest vertices of the subtrees A, B.*

(a) *The sum $\mathscr{H}_\lambda(A) + \mathscr{H}_\lambda(B)$ is a topological direct sum in $\mathscr{H}_\lambda(J)$.*

(b) *Let $Q : \mathscr{H}_\lambda(A) \to \mathscr{H}_\lambda(B)$ be the projection operator P_B restricted to $\mathscr{H}_\lambda(A)$. Then the image of Q is the line spanned by e_b, and the kernel of Q is the orthocomplement in $\mathscr{H}_\lambda(A)$ to e_a.*

Remark. Let V, W be subspaces in a Hilbert space H. We say that the sum $V + W$ is a *topological direct sum* in H, if the operator $\Omega : V \oplus W \to H$ given by $\Omega(v \oplus w) = v + w$ is injective and its image is closed. In particular, the operator $\Omega^{-1} : V + W \to V \oplus W$ is bounded.

Denote by P the projector operator $V \to W$. The sum $V + W$ is direct if and only if $||P|| < 1$.

Proof. Let u be a vertex of A. Let v ranges in $\mathrm{Vert}(B)$. We have equality

$$\langle e_u, e_v \rangle = \lambda^{\rho(e_u,e_v)} = \lambda^{\rho(e_u,e_b)} \lambda^{\rho(e_b,e_v)},$$

hence

$$\langle e_u, e_v \rangle = \langle e_u, e_b \rangle \cdot \langle e_b, e_v \rangle.$$

[6] An orthogonal operator is an invertible operator in a real Hilbert space preserving the inner product.

Thus, for each $h \in \mathscr{H}_\lambda(B)$,

$$\langle e_u, h \rangle = \langle e_u, e_b \rangle \cdot \langle e_b, h \rangle.$$

Hence the projection of e_u onto $\mathscr{H}_\lambda(B)$ is e_b for all the vertices $u \in \mathrm{Vert}(A)$. This proves (b).

The norm of the projector $P_B : \mathscr{H}_\lambda(A) \to \mathscr{H}_\lambda(B)$ is $\lambda^{\rho(b,c)} < 1$. This implies (a).

4.6. Proof of Theorem 4.1

Denote by $\mathscr{H}_\lambda^{\mathrm{fin}}(J)$ the space of all linear combinations of the vectors e_v, where v ranges in J. Our operator $U_\lambda(g)$ was defined on the space $\mathscr{H}_\lambda^{\mathrm{fin}}(J)$.

For a given $\circ$-hierarchomorphism g, consider the Hermitian form

$$Q(h_1, h_2) = \langle U_\lambda(g)h_1, U_\lambda(g)h_2 \rangle - \langle h_1, h_2 \rangle \tag{4.3}$$

on $\mathscr{H}_\lambda^{\mathrm{fin}}(J) \times \mathscr{H}_\lambda^{\mathrm{fin}}(J)$. It is sufficient to prove that Q is a bounded Hermitian form on $\mathscr{H}_\lambda^{\mathrm{fin}}(J) \times \mathscr{H}_\lambda^{\mathrm{fin}}(J)$ and the rank of Q is finite.

Let $g = \{g_j, S_j\} \in \mathrm{Hier}^\circ(J)$ be a $\circ$-hierarchomorphism. Without loss of generality (see Lemma 1.1), we can assume that S_j are bushes or single-point sets.

By Lemma 4.2, the decomposition

$$\mathscr{H}_\lambda(J) = \bigoplus_j \mathscr{H}_\lambda(S_j)$$

is a topological direct sum, i.e. the equality is an isomorphism of topological vector spaces.

The matrix of Q in the basis e_a is

$$Q(e_a, e_b) = \langle e_{ga}, e_{gb} \rangle - \langle e_a, e_b \rangle = \lambda^{\rho(ga,gb)} - \lambda^{\rho(a,b)}.$$

The matrix $Q(e_a, e_b)$ has the natural block decomposition

$$Q = \{Q_{ij}\}$$

corresponding to the partition

$$\mathrm{Vert}(J) = \bigcup \mathrm{Vert}(S_j).$$

It is sufficient to prove that each block Q_{ij} has finite rank.

Thus, let a ranges in $\mathrm{Vert}(S_i)$, b ranges in $\mathrm{Vert}(S_j)$. If S_i is an one-point space, then the required statement is obvious. Hence, we assume that S_i, S_j are bushes. Let u_i, u_j be their roots. If $S_i = S_j$, then $Q_{ii}(e_a, e_b)$ is the identical zero. Thus, assume $S_i \neq S_j$. Then

$$\rho(a, b) = \rho(a, u_i) + \rho(u_i, u_j) + \rho(u_j, b),$$

$$\begin{aligned}\rho(ga, gb) &= \rho(ga, gu_i) + \rho(gu_i, gu_j) + \rho(gu_j, gb)\\ &= \rho(a, u_i) + \rho(gu_i, gu_j) + \rho(u_j, b).\end{aligned}$$

Thus,

$$\begin{aligned}Q(e_a, e_b) &= \left[\lambda^{\rho(gu_i, gu_j)} - \lambda^{\rho(u_i, u_j)}\right] \cdot \lambda^{\rho(a, u_i)} \cdot \lambda^{\rho(b, u_j)}\\ &= \text{const} \cdot \langle e_{u_i}, e_a \rangle \cdot \langle e_{u_j}, e_b \rangle.\end{aligned}$$

Hence we obtain that the Hermitian form Q_{ij} on $\mathscr{H}_\lambda^{\text{fin}}(S_i) \times \mathscr{H}_\lambda^{\text{fin}}(S_j)$ is given by the formula

$$Q_{ij}(h_1, h_2) = \text{const} \cdot \langle e_{u_i}, h_1 \rangle \cdot \langle e_{u_j}, h_2 \rangle.$$

Thus the form Q_{ij} on $\mathscr{H}_\lambda(S_i) \times \mathscr{H}_\lambda(S_j)$ is bounded and its rank is $\leqslant 1$. This finishes the proof.

4.7. Remark. Spaces $\mathscr{H}_\lambda$ associated with $\mathbb{R}$-trees

Let we have a countable family $J_1, J_2, \ldots$ of metric trees and let we have isometric embeddings $\iota_k : J_k \to J_{k+1}$:

$$\cdots \overset{\iota_{k-1}}{\to} J_k \overset{\iota_k}{\to} J_{k+1} \overset{\iota_{k+1}}{\to} J_{k+2} \overset{\iota_{k+2}}{\to} \cdots .$$

Let $\mathbf{J}$ be the direct limit (the union) of J_k. Such spaces are called $\mathbb{R}$-*trees*.[7]

Obviously, we have the chain of inclusions

$$\cdots \subset \mathscr{H}_\lambda(J_k) \subset \mathscr{H}_\lambda(J_{k+1}) \subset \mathscr{H}_\lambda(J_{k+2}) \subset \cdots .$$

Denote the inductive limit of this chain by $\mathscr{H}_\lambda(\mathbf{J})$. Thus the Hilbert spaces $\mathscr{H}_\lambda$ survive for $\mathbb{R}$-trees. Nevertheless, the analogue of Theorem 4.1 is wrong.

First, Lemma 4.1(a) becomes noncorrect, since the norm of the projector $\mathscr{H}_\lambda(A) \to \mathscr{H}_\lambda(B)$ can be 1 (if for each $\varepsilon > 0$ there exists a pair of vertices $u \in A$, $v \in B$ such that $\rho(u, v) < \varepsilon$).

In fact, for an $\mathbb{R}$-tree we have also the form Q given by (4.3) on the space $\mathscr{H}_\lambda^{\text{fin}}$. This form has a finite rank, but Q is unbounded.

5. Boundary spaces

In Sections 5 and 6, we construct a canonical Hier°-invariant subspace $\mathscr{E}_\lambda \subset \mathscr{H}_\lambda$. In fact, this gives a triangular representation of the operators $U_\lambda(g)$ defined in the

[7] Up to a possible minor variation of terminology.

previous section

$$U_\lambda(g) = \begin{pmatrix} P(g) & Q(g) \\ 0 & T(g) \end{pmatrix} : \mathscr{E}_\lambda^\perp \oplus \mathscr{E}_\lambda \to \mathscr{E}_\lambda^\perp \oplus \mathscr{E}_\lambda. \tag{5.0}$$

Thus we obtain two representations of Hier°, the first is the representation $T(g)$ in $\mathscr{E}_\lambda$, the second is the representation $P(g)$ in the quotient space $\mathscr{H}_\lambda/\mathscr{E}_\lambda$ (the operators $U_\lambda(g)$ are not unitary and hence the subspace $\mathscr{E}_\lambda^\perp$ is not Hier°-invariant).

The object of our interest is the representation $T(g)$ and the subspace $\mathscr{E}_\lambda$. We show, that elements of the space $\mathscr{E}_\lambda$ are some kind of "distributions" on the absolute of a metric tree. These spaces can be considered as analogues of the Sobolev spaces on the spheres.

The representation $T(g)$ is trivial on the subgroup Δ and thus it is a representation of the quotient group $\mathrm{Hier} = \mathrm{Hier}^\circ/\Delta$.

This construction is valid under some conditions on the tree and on λ. Conditions on the tree are not restrictive, conditions on the parameter λ are essential. It turns out to be, that there is a critical value σ such that $0<\sigma<1$; for $\lambda>\sigma$ the boundary space $\mathscr{E}_\lambda$ and the triangular representation (5.0) exist, and the space $\mathscr{E}_\lambda$ is trivial for $\lambda<\sigma$.

We obtain upper and lower estimates for σ.

In this section, the term tree means a locally finite tree such that each vertex is contained in $\geqslant 3$ edges, the both restrictions are not really important, we only trying simplify the text.

Constructions of this section are an imitation of the work [30] on the level of trees.

5.1. Balls in absolute

Let S be a branch of J. A ball $B[S] \subset \mathrm{Abs}(J)$ is the absolute of the branch S. If we delete the root of the S and all the edges containing the root, then S will be disintegrated into a finite collection of branches $S^{(1)}, S^{(2)}, \dots, S^{(k)}$. Hence the ball $B[S]$ admits the canonical partition

$$B[S] = B[S^{(1)}] \cup \cdots \cup B[S^{(k)}] \tag{5.1}$$

into the balls $B[S^{(k)}]$.

We define the topology on $\mathrm{Abs}(J)$ by the assumption that all the balls $B[S]$ are open-and-closed subsets in $\mathrm{Abs}[J]$. Obviously, $\mathrm{Abs}(J)$ is a completely discontinuous compact set.

Remark. Hierarchomorphisms locally preserve hierarchy of balls on the absolute.[8] Obviously, hiearchomorphisms are homeomorphisms of the absolute. But preserving of the hierarchy of balls is a very rigid condition on a homeomorphism.

[8] Firstly, this hierarchy structure on p-adic manifolds was mentioned in Addendum in Serre's book [37].

5.2. *New notation in the space* $\mathscr{H}_\lambda(J)$

Let us fix a vertex $\xi \in \mathrm{Vert}(J)$. Let $a, b \in \mathrm{Vert}(J)$. Consider the way $a_0 = a, a_1, \dots, a_l = b$ connecting a, b. Assume

$$\theta(a, b) = 2 \min \rho(\xi, a_j).$$

We emphasis that this function has sense also if a or b are points of the absolute, and the value $\theta(a, b)$ is finite except the case $a = b \in \mathrm{Abs}(J)$.

For $a \in \mathrm{Vert}(J)$, consider the vector $f_a \in \mathscr{H}_\lambda(J)$ given by

$$f_a = \lambda^{-\rho(\xi,a)} e_a.$$

Then

$$\langle f_a, f_b, \rangle = \lambda^{-\theta(a,b)}.$$

Remark. Let S be a subtree in J containing ξ. For $c \in \mathrm{Vert}(J)$, consider the nearest vertex $b \in \mathrm{Vert}(S)$. Then the projection of f_c to $\mathscr{H}_\lambda(S)$ is f_b.

5.3. *Measures on* Abs(J) *and compatible systems of measures on* Vert(J)

Let $R \subset J$ be a finite subtree. We say that R is *complete* if any $a \in \mathrm{Vert}(R)$ satisfies one of two following conditions (see Fig. 5):

1. Any vertex b of J adjacent to a is contained in R.
2. Only one vertex of J adjacent to a is contained in R.

Let ∂R denote the boundary of R, i.e., the set of all vertices of the second type. We also assume

$$\xi \in \mathrm{Vert}(R) \setminus \partial R.$$

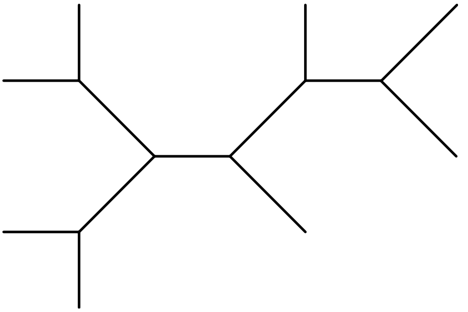


Fig. 5. A complete subtree in the dyadic Bruhat–Tits tree $\mathscr{T}_2$.

Consider a real-valued measure (charge) μ of a *finite variation* on $\mathrm{Abs}(J)$. Recall that any measure μ of a finite variation admits the canonical representation

$$\mu = \mu^+ - \mu^-,$$

where $\mu^\pm$ are nonnegative finite measures, and for some (noncanonical) Borel subset $U \subset \mathrm{Abs}$,

$$\mu^-(U) = 0, \quad \mu^+(\mathrm{Abs} \setminus U) = 0.$$

The *variation* of the measure μ is

$$\mathrm{var}(\mu) = \mu^+(U) + \mu^-(\mathrm{Abs} \setminus U).$$

For a complete subtree R, denote by $u_1, u_2, \ldots$ the points of ∂R. For any u_k, there exists a unique branch $S_{u_k} \subset J$ such that u_k is the root of S_{u_k} and $\xi \notin S_{u_k}$.

Consider the measure μ_R defined on the finite set ∂R by

$$\mu_R(u_j) = \mu(B[S_{u_j}]).$$

Consider also the vector

$$\Psi[\mu|R] = \sum_{u_j \in \partial R} \mu(B[S_{u_j}]) f_{u_j}.$$

Let $R_2 \supset R_1$ be complete subtrees. Then we have the obvious retraction

$$\eta_{R_1}^{R_2} : \mathrm{Vert}(R_2) \to \mathrm{Vert}(R_1)$$

defined by the condition: if $a \in \mathrm{Vert}(R_2)$, then $\eta_{R_1}^{R_2}(a)$ is the nearest vertex of R_1 (see Fig. 6).

Lemma 5.1. (a) *μ_{R_1} is the image of μ_{R_2} under the retraction $\eta_{R_1}^{R_2}(a)$.*

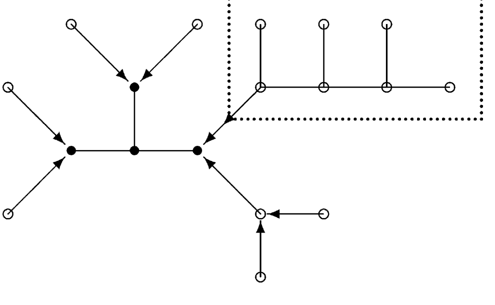


Fig. 6. The tree R_2 and the retraction $\eta_{R_1}^{R_2}$. Vertices of R_1 are black.

(b) *The vector* $\Psi[\mu|R_1]$ *is the projection of* $\Psi[\mu|R_2]$ *to the subspace* $\mathscr{H}_\lambda(R_1)$. *In particular*,

$$||\Psi[\mu|R_1]|| \leqslant ||\Psi[\mu|R_2]||.$$

Proof. Assertion (a) is obvious, and assertion (b) follows from the last remark from 5.2. □

Conversely, consider a family of complete subtrees

$$R_1 \subset R_2 \subset R_3 \subset \cdots,$$

such that $\bigcup R_j = J$. Let for each j we have a measure ν_j on ∂R_j, and $\eta_{R_j}^{R_{j+1}} \nu_{j+1} = \nu_j$ for all j. If $\sup \mathrm{var}(\nu_j) < \infty$, then there exists a unique measure ν on Abs such that $\nu_j = \nu_{R_j}$.

5.4. *Boundary spaces* $\mathscr{E}_\lambda \subset \mathscr{H}_\lambda$

Let $R_1 \subset R_2 \subset \cdots$ be a sequence of complete subtrees in J, and $\bigcup R_k = J$ (the construction below do not depend on choice of the sequence).

Let μ be a measure of finite variation on $\mathrm{Abs}(J)$. We say that μ belongs to the class $\mathscr{E}_\lambda = \mathscr{E}_\lambda(J)$ if

$$\lim_{j\to\infty} ||\Psi[\mu|R_j]||_{\mathscr{H}_\lambda} < \infty.$$

Proposition 5.2. *For* $\mu, \mu' \in \mathscr{E}_\lambda$, *the following statements hold*:

(a) *There exists the following limit in the space* $\mathscr{H}_\lambda(J)$

$$\Psi[\mu] := \lim_{j\to\infty} \Psi[\mu|R_j]. \tag{5.2}$$

(b)
$$||\Psi[\mu]||_{\mathscr{H}_\lambda} = \lim_{j\to\infty} ||\Psi[\mu|R_j]||_{\mathscr{H}_\lambda}. \tag{5.3}$$

(c)
$$\langle \Psi[\mu], \Psi[\mu'] \rangle_{\mathscr{H}_\lambda} = \lim_{j\to\infty} \langle \Psi[\mu|R_j], \Psi[\mu'|R_j] \rangle_{\mathscr{H}_\lambda}. \tag{5.4}$$

Proof. All statements follow from Lemma 5.1. □

Thus we obtain the embedding $\mathscr{E}_\lambda(J) \to \mathscr{H}_\lambda(J)$ given by $\Psi : \mu \mapsto \Psi[\mu]$. We define the inner product in $\mathscr{E}_\lambda(J)$ by

$$\langle \mu_1, \mu_2 \rangle_{\mathscr{E}_\lambda(J)} := \langle \Psi[\mu_1], \Psi[\mu_2] \rangle_{\mathscr{H}_\lambda(J)}.$$

Denote by $\mathfrak{E}_\lambda \subset \mathscr{H}_\lambda$ the image of the embedding Ψ. Denote by $\overline{\mathfrak{E}_\lambda}$ the closure of $\mathfrak{E}_\lambda$ in $\mathscr{H}_\lambda$, and also denote by $\overline{\mathscr{E}_\lambda}$ the completion of the space $\mathscr{E}_\lambda$ with respect to the norm (5.3).

5.5. *More direct description of* $\mathscr{E}_\lambda$

We can write formally

$$||\mu||^2_{\mathscr{E}_\lambda} = \int\int_{\mathrm{Abs}\times\mathrm{Abs}} \lambda^{-\theta(\omega_1,\omega_2)}\, d\mu(\omega_1)\, d\mu(\omega_2), \tag{5.5}$$

$$\langle \mu_1, \mu_2 \rangle_{\mathscr{E}_\lambda} = \int\int_{\mathrm{Abs}\times\mathrm{Abs}} \lambda^{-\theta(\omega_1,\omega_2)}\, d\mu_1(\omega_1)\, d\mu_2(\omega_2). \tag{5.6}$$

These integrals are very simple, since the integrand $\lambda^{-\theta(\omega_1,\omega_2)}$ has only countable set of values. Nevertheless, generally (even for the Bruhat–Tits tree $\mathscr{T}_2$) for $\mu_1, \mu_2 \in \mathscr{E}_\lambda$, these integrals diverge as Lebesgue integrals, i.e., the integral

$$\int\int_{\mathrm{Abs}\times\mathrm{Abs}} \lambda^{-\theta(\omega_1,\omega_2)}(d\mu^+(\omega_1) + d\mu^-(\omega_1))(d\mu^+(\omega_2) + d\mu^-(\omega_2))$$

can be infinite.

Our limit procedure is equivalent to the Riemann improper integration in the following sense. Consider a complete subtree $R \subset J$ such that $\xi \in R$. Then $J \backslash R$ is a union of disjoint branches $S_1, \dots, S_k$. Thus

$$\mathrm{Abs}(J) = B[S_1] \cup \cdots \cup B[S_k].$$

Let us define the integral sum

$$\mathscr{S}_R(\mu_1, \mu_2) = \sum_{i,j} \left\{ \min_{\omega_1 \in B[S_i], \omega_2 \in B[S_j]} \lambda^{-\theta(\omega_1,\omega_2)} \right\} \mu_1(B[S_i])\mu_2(B[S_j]).$$

Obviously,

$$\mathscr{S}_R(\mu_1, \mu_2) = \langle \Psi[\mu_1|R], \Psi[\mu_2|R] \rangle_{\mathscr{H}_\lambda}.$$

Remark. If $i \neq j$, then the value $\lambda^{-\theta(\omega_1,\omega_2)}$ is a constant on $B[S_i] \times B[S_j]$.

By Lemma 5.1, we have

$$R_2 \supset R_1 \quad \Rightarrow \quad \mathscr{S}_{R_1}(\mu, \mu) \leqslant \mathscr{S}_{R_2}(\mu, \mu). \tag{5.7}$$

Now we can define integral (5.5) as the limit of these integral sums under refinement of the partition. A measure μ is contained in $\mathscr{E}_\lambda$ iff the Riemann integral (5.5) is finite.

After this, we can define the inner product in $\langle \mu_1, \mu_2 \rangle_{\mathscr{E}_\lambda}$ as the Riemann improper integral (5.6).

This definition coincides with formula (5.4) for $\langle \Psi[\mu_1], \Psi[\mu_2] \rangle$, but it is given independently on the space $\mathscr{H}_\lambda$.

Nevertheless, the space $\mathscr{H}_\lambda$ was essentially used in the justification of this construction, since the convergence of integral sums and positiveness of integral (5.5) are not obvious.

5.6. Nontriviality of $\mathscr{E}_\lambda$

Theorem 5.3. (a) *There exists* σ, *which belongs to* $0 \leqslant \sigma \leqslant 1$, *such that the space* $\mathscr{E}_\lambda$ *is zero for* $\lambda < \sigma$ *and the space* $\mathscr{E}_\lambda$ *is not zero for* $\lambda > \sigma$.

(b) *If lengths of edges of* J *are bounded, then* $\sigma < 1$.

(c) *Let* J *contain a subtree* I *that is isomorphic to the Bruhat–Tits tree* $\mathscr{T}_p$ *as a simplicial tree, and lengths of all edges of* I *are* $\leqslant \tau$. *Then* $\sigma \leqslant 1/\sqrt[2\tau]{p}$.

(d) *Assume lengths of edges of* J *are bounded away from zero. Denote by* $s(N)$ *the number of* $a \in \mathrm{Vert}(J)$, *satisfying* $d_{\mathrm{simp}}(\xi, a) \leqslant N$. *Assume that* $s(N)$ *has exponential growth, i.e.*, $s(N) \leqslant \exp(\alpha N)$ *for some constant* α. *Then* $\sigma > 0$.

The proof of the theorem is contained below in 5.7–5.11

5.7. Expansion of $||\Psi||^2$ into series with positive terms

Let $R_0 \subset R_1 \subset R_2 \subset \cdots$ be a sequence of complete subtrees in J, and $\bigcup R_m = J$. We say that the sequence R_j is *incompressible* if

1. R_0 consists of the vertex ξ;

2. for each m, there exists $u \in \partial R_m$ such that $\mathrm{Vert}(R_{m+1}) \backslash \mathrm{Vert}(R_m)$ consists of vertices adjacent to u.

Fix a measure (charge) μ on Abs.

Obviously, $\Psi[\mu|R_0] = \mu(\mathrm{Abs}) f_\xi$, and hence

$$||\Psi[\mu|R_0]||^2 = \mu(\mathrm{Abs})^2.$$

Let us evaluate

$$z^{(m)}(\lambda) = ||\Psi[\mu|R_{m+1}]||^2_{\mathscr{H}_\lambda} - ||\Psi[\mu|R_m]||^2_{\mathscr{H}_\lambda}.$$

Let u be the vertex defined in 2. Let $v_1, \ldots, v_n \in \partial R_{m+1}$ be the vertices adjacent to u, see Fig. 7.

Let $\mu_{R_{m+1}}(v_k) = t_k$ (these numbers can be negative), respectively, $\mu_{R_m}(u) = t_1 + \cdots + t_n$. It is readily seen that

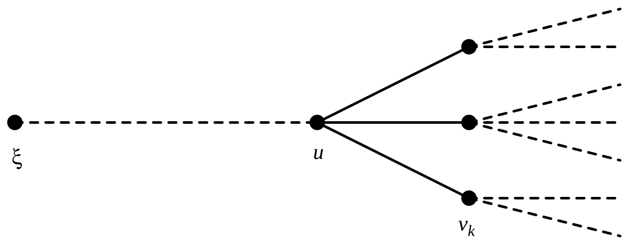


Fig. 7.

$$z^{(m)}(\lambda) = \left(\lambda^{-2\rho(\xi,u)} \sum_{k\neq l} t_k t_l + \lambda^{-2\rho(\xi,u)} \sum_k \lambda^{-2\rho(u,v_k)} t_k^2\right) - \lambda^{-2\rho(\xi,u)} \left(\sum t_k\right)^2$$
$$= \lambda^{-2\rho(\xi,u)} \sum_k \left(\lambda^{-2\rho(u,v_k)} - 1\right) t_k^2. \tag{5.8}$$

First, we observe that this expression is completely determined by the measure μ and the vertex u. The subtrees R_m, R_{m+1} are nonessential. Hence it is natural to denote $z^{(m)}(\lambda)$ by $z_u(\lambda)$.

Thus,

$$||\Psi[\mu]||^2 = \mu(\mathrm{Abs})^2 + \sum_{m=1}^{\infty} z^{(m)}(\lambda) \tag{5.9}$$

$$= \mu(\mathrm{Abs})^2 + \sum_{u\in \mathrm{Vert}(J), u\neq\xi} z_u(\lambda). \tag{5.10}$$

We emphasis that

(a) all summands of these series are nonnegative;

(b) all summands $z_u(\lambda)$ are decreasing functions on λ for $0 \leqslant \lambda \leqslant 1$.

5.8. Existence of σ

The Statement (a) of Theorem 5.4 follows from the last observation of the previous subsection.

5.9. Existence of $\mathscr{E}_\lambda$

It is sufficient to prove (c), since (b) is a corollary of (c). Furthermore, it is sufficient to prove nontriviality of $\mathscr{E}_\lambda(I)$ for the subtree I. Denote by R_k the subtree of I, consisting of all vertices $a \in I$ such that the simplicial distance $d_{\mathrm{simp}}(\xi, a) \leqslant k$.

Consider the uniform measure μ_{R_k} on ∂R_k, i.e., the measure of each point is $1/(p^{k-1}(p+1))$. Obviously, the measures μ_k form a compatible system of the measures, denote by μ the inverse limit of the measures μ_{R_k}.

Let us estimate

$$\Psi[\mu|R_k]||^2 = \frac{1}{(p+1)^2 p^{2(k-1)}} \left\| \sum_{a \in \partial R_k} f_a \right\|^2$$

$$= \frac{1}{(p+1)^2 p^{2(k-1)}} \sum_{a,b \in \partial R_k} \lambda^{-2\rho(a,b)}$$

$$\leqslant \frac{1}{(p+1)^2 p^{2(k-1)}} \sum_{j=0}^{k} \lambda^{-2\tau j} \cdot \left\{ \begin{array}{c} \text{number of pairs } (a,b) \in \partial R_k \\ \text{such that } d_{\mathrm{simp}}(a,b) = 2(k-j) \end{array} \right\}$$

$$= \frac{1}{(p+1)^2 p^{2(k-1)}} \Bigg[(p+1)p^{2k-1} + \sum_{j=1}^{k-1} \lambda^{-2\tau j}(p+1)p^{k-1}(p-1)p^{k-j-1}$$

$$+ \lambda^{-2k\tau}(p+1)p^{k-1} \Bigg]$$

$$\leqslant \sum_{j=0}^{k} \lambda^{-2\tau j} p^{-j}.$$

If $\lambda^{2\tau} p > 1$, then these sums are uniformly bounded in k; hence $\mu \in \mathscr{E}_\lambda(I) \subset \mathscr{E}_\lambda(J)$ (Fig. 7).

5.10. Localization

Lemma 5.4. *Let $\mu \in \mathscr{E}_\lambda$, and let $B[S] \subset \mathrm{Abs}$ be a ball. Let ν be the restriction of μ to $B[S]$* (i.e., $\nu(A) = \mu(A \bigcap B([S])$ *for any Borel subset* $A \subset \mathrm{Abs}$). Then $\nu \in \mathscr{E}_\lambda$.

Proof. We can assume $\xi \notin S$ (otherwise we divide the ball $B[S]$ into a *finite* collection of smaller balls $B[S_j]$ satisfying the property $\xi \notin B[S_j]$). Denote by v the root of the branch S. The quantity $||\mu||^2_{\mathscr{E}_\lambda}$ is the sum of the series $\sum z_u(\lambda)$ given by (5.8) and (5.10). The series for $||\nu||^2_{\mathscr{E}_\lambda}$ is obtained from the series for $||\mu||^2_{\mathscr{E}_\lambda}$ by the following operations:

(1) For u lying between ξ and v, the summands $z_u(\lambda)$ are changed in a nonpredictable way.

(2) For any $u \in S$, the summand $z_u(\lambda)$ does not change.

(3) All other summands become zero.
Obviously, the new series $\sum z_u(\lambda)$ is convergent. $\square$

Remark. Consider a Borel subset U in the absolute. Let ν be the restriction of $\mu \in \mathscr{E}_\lambda$ to U. Generally, $\nu \notin \mathscr{E}_\lambda$. Also, generally, $\mu^\pm \notin \mathscr{E}_\lambda$.

5.11. *Lower estimate of σ*

By Lemma 5.4, if $\mathscr{E}_\lambda \neq 0$, then there exists a measure $\mu \in \mathscr{E}_\lambda$ such that $\mu(\mathrm{Abs}) \neq 0$. For definiteness, assume $\mu(\mathrm{Abs}) = 1$.

Let σ be a lower bound for lengths of edges. Consider a complete subtree $R \subset J$ defined by the condition $d_{\mathrm{simp}}(\xi, a) \leqslant N$. Consider the measure μ_R on ∂R. In Section 5.7,

$$\begin{aligned} \|\Psi[\mu]\|^2 &= 1 + \sum_{u \in \mathrm{Vert} J, u \neq \xi} z_u \geqslant \sum_{u \in \partial R} z_u \\ &\geqslant \lambda^{-2N\sigma}(\lambda^{-2\sigma} - 1) \sum_{u \in \partial R} \mu_R(u)^2 \quad \text{(by formula (5.8))}. \end{aligned} \tag{5.11}$$

The number of points of ∂R is less than $\exp\{\alpha N\}$, where α is a constant. Furthermore, $\sum_{u \in \partial R} \mu_R(u) = 1$, hence the last expression is larger than

$$\lambda^{-2N\sigma}(\lambda^{-2\sigma} - 1)\exp\{-\alpha N\}.$$

For a sufficiently small $\lambda > 0$, the last expression tends to ∞ as $N \to \infty$, and thus $\mathscr{E}_\lambda = 0$.

5.12. *Bruhat–Tits trees*

In this case, the space $\mathscr{E}_\lambda$ coincides with the following well-known construction (see [3]).

Consider the scalar product in the space of real functions on $\mathrm{Abs}(\mathscr{T}_p)$ given by

$$\langle f_1, f_2 \rangle = \int\int_{\mathrm{Abs} \times \mathrm{Abs}} \lambda^{-\theta(\omega_1, \omega_2)} f_1(\omega_1) f_2(\omega_2)\, d\nu(\omega_1)\, d\nu(\omega_2). \tag{5.12}$$

This space is the space of the representation of $\mathrm{Aut}(\mathscr{T}_p)$ of the complementary series (see [3,7,31]).

If we assume in (5.6)

$$d\mu_1 = f_1(\omega)\, d\mu(\omega), \quad d\mu_2 = f_2(\omega)\, d\mu(\omega)$$

then we obtain exactly expression (5.12).

Nevertheless, this can not be repeated for a general metric tree, since the construction uses a *canonical* measure on the boundary of tree, and there is no visible way invent such canonical measures for arbitrary metric tree. But our construction of the spaces $\mathscr{E}_\lambda$ is canonical.

6. Action of groups of hierarchomorphisms in $\mathscr{E}_\lambda$

Let J satisfy the same conditions as in Section 5, i.e., J be locally finite and any vertex of J belong to $\geqslant 3$ edges.

Let $\mathscr{H}_\lambda(J) \supset \overline{\mathfrak{E}_\lambda}(J) \simeq \overline{\mathscr{E}_\lambda}(J)$ be the same spaces as above. Let a group Γ act on J by isometries. Let $\mathrm{Hier}^\circ(J,\Gamma)$, $\mathrm{Hier}(J,\Gamma)$ be the corresponding hierarchomorphisms groups. The group $\mathrm{Hier}^\circ(J,\Gamma)$ acts in $\mathscr{H}_\lambda(J)$ by the operators $U_\lambda(g)$ given by (4.1).

6.1. Action of hierarchomorphisms in $\mathscr{E}_\lambda$

Proposition 6.1. (a) *The space $\overline{\mathfrak{E}_\lambda}(J) \subset \mathscr{H}_\lambda(J)$ is invariant with respect to* $\mathrm{Hier}^\circ(J,\Gamma)$.

(b) *For $g \in \mathrm{Hier}^\circ(J,\Gamma)$, the restriction of the operator $U_\lambda(g)$ to $\overline{\mathfrak{E}_\lambda}$ depends only on the corresponding element $\tilde{g} \in \mathrm{Hier}(J,\Gamma)$.*

(c) *The action of* $\mathrm{Hier}(J,\Gamma)$ *in $\mathscr{E}_\lambda(J) \simeq \mathfrak{E}_\lambda(J)$ is given by*

$$T_\lambda(\tilde{g})\mu(\omega) = \lambda^{n(g,\omega)} \cdot \mu(g\omega), \quad \textit{where } g \in \mathrm{Hier}(J,\Gamma), \tag{6.1}$$

where the pseudo-derivative $n(g,\omega) = n(\tilde{g},\omega)$ of a hierarchomorphism on the absolute was defined in 3.2, 3.7, *and $\mu(g\omega)$ is the image of the measure μ under the transformation $\omega \mapsto g\omega$.*

Proof. Fix $g \in \mathrm{Hier}^\circ(J,\Gamma)$. Let $R_0 \subset R_1 \subset \cdots$ be an incompressible sequence of complete subtrees as in 5.7, $\bigcup R_k = J$. Consider the sequence $g \cdot \partial R_1$, $g \cdot \partial R_2, \ldots$. There exists l such that for all $k \geqslant l$

$$g \cdot \partial R_k = \partial T_k, \quad \text{where } T_k \text{ is a complete subtree.}$$

Hence,

$$U_\lambda(g)\Psi[\mu|R_k] = \Psi[\nu|T_k],$$

where ν is some measure on $\mathrm{Abs}(J)$.

We must show that the numbers $||\Psi[\nu|T_k]||$ are bounded. Consider the expansion of $||\Psi[\mu]||^2$ and $||\Psi[\nu]||^2$ into the series $\sum z^k(\lambda)$, see (5.9) and (5.10). The summands with numbers $<l$ are essentially different, but this do not influence on the convergence. Other summands are rearranged and multiplied by the factors $\lambda^{n(g,\omega)}$.

But $\lambda^{n(g,\omega)}$ has only finite number of values and hence the series $\sum z^k(\lambda)$ for the measure ν is also convergent. Thus $\nu \in \mathscr{E}_\lambda(J)$.

Statement (a) is proved, statement (b) is obvious, and statement (c) follows from the same considerations.

6.2. Almost orthogonality

Theorem 6.2. *Let* $g \in \mathrm{Hier}(J, \Gamma)$. *The operators* $T_\lambda(g)$ *in* $\mathscr{E}_\lambda(J)$ *given by* (6.1) *admit the representation* $T_\lambda(g) = A(1+Q)$, *where* A *is an orthogonal operator and* Q *is a finite rank operator.*

This statement follows from Theorem 4.1. This can also be proved directly from the explicit formulas (5.6) and (6.1) in the same way as in [26].

6.3. Action of Γ

For $g \in \Gamma$, the operator $T_\lambda(g)$ is unitary in $\mathscr{E}_\lambda$. Thus we obtained a series T_λ of unitary representations of the group Γ.

If Γ is the group $\mathrm{Aut}(\mathscr{T}_p)$ of automorphisms of Bruhat–Tits tree, then this construction is nothing but the complementary series representations (see [3,7,31]).

Nevertheless, possibly our construction of representations of Γ in a general case is new and I'll try to describe its position with respect to known constructions.

There are many Γ-quasiinvariant measures on Abs, and for each quasiinvariant measure κ we have a series of unitary representations of Γ in $L^2(\mathrm{Abs}, \kappa)$ (“boundary representations”).

There arise two questions.

(1) Is it possible to realize a representation T_λ as a boundary representation?

(2) Is it possible to obtain T_λ as an analytic continuation of boundary representations (see [8])?

We will discuss the case of free groups, which is relatively well understood (see [21,22]).

The answer to the first question is negative. Indeed, the boundary representations are weakly contained in $L^2(F_2)$, see [21,22]. A representation ρ of F_2 is weakly contained in L^2 if for any vector v,

$$\sum_{g \in F_2} e^{-\varepsilon l(g)} \langle \rho(g)v, v \rangle < \infty \quad \text{for all } \varepsilon,$$

where $l(g)$ is the length of g.

It is easy to see that for sufficiently large λ our representations U_λ, T_λ are not weakly contained in L^2.

The second problem seems more complicated.

(α) In some cases answer is affirmative. For instance, this is correct for the tree $J(F_2)$ from 2.3 in the case $l_1 = l_2$, see [6].

(β) Definitely, answer is negative for the action of F_2 on nonlocally finite tree $\mathscr{T}_\infty$; in this case, $T_\lambda = U_\lambda$ (see also [32], where a “complementary series” that is not an analytic continuation of the “principal series” is constructed).

(γ) Consider the tree $\mathscr{T}(F_2)$ from 2.3 and the corresponding representations T_λ of F_2. Unfortunately, I do not understand the position of these “complementary series” T_λ with respect to the “principal series” of Kuhn–Steger [22].

Acknowledgments

I am grateful to V. Sergiescu and C. Kapoudjian for meaningfull discussions. I thank the administrations of the Erwin Schrödinger Institute (Wien) and Institute Fourier (Grenoble), where this work was done, for their hospitality. I thank the referee of this paper for constructive comments.

References

[1] K.S. Brown, R. Geoghegan, An infinite-dimensional torsion-free FP_∞ group, Invent. Math. 77 (2) (1984) 367–381.
[2] J.W. Cannon, W.J. Floyd, W.R. Parry, Introductory notes on Richard Thompson's groups, Enseign. Math. (2) 42 (3–4) (1996) 215–256.
[3] P. Cartier, Geómétrie et analyse sur les arbres, Lecture Notes in Math. 317 (1973) 123–140.
[4] M. Davis, T. Januszkiewicz, R. Scott, Nonpositive curvature of blow-ups, Selecta Math. (N.S.) 4 (4) (1998) 491–547.
[5] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, Inst. Hautes tudes Sci. Publ. Math. 36 (1969) 75–109.
[6] A. Figá-Talamanca, M.A. Picardello, Spherical functions and harmonic analysis on free groups, J. Funct. Anal. 47 (3) (1982) 281–304.
[7] A. Figá-Talamanca, C. Nebbia, Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees, Cambridge University Press, Cambridge, 1991.
[8] A. Figá-Talamanca, T. Steger, Harmonic analysis for anisotropic random walks on homogeneous trees, Mem. Amer. Math. Soc. 531 (1994).
[9] V.A. Fock, Konfugurationsraum und zweite Quantelug, Z. Phys. 75 (1932) 622–647.
[10] P. Freyd, A. Heller, Splitting homotopy idempotents. II, J. Pure Appl. Algebra 89 (1–2) (1993) 93–106.
[11] I.M. Gelfand, M.I. Graev, A.M. Vershik, Models of representations of current groups, Representations of Lie groups and Lie algebras (Budapest, 1971), Akad. Kiadó, Budapest, 1985, pp. 121–179.
[12] É. Ghys, V. Sergiescu, Sur un groupe remarquable de difféomorphismes du cercle, Comment. Math. Helv. 62 (1987) 185–239.
[13] P. Greenberg, V. Sergiescu, An acyclic extension of the braid group, Comment. Math. Helv. 66 (1991) 109–138.
[14] U. Haagerup, An example of a nonnuclear C^*-algebra, which has the metric approximation property, Invent. Math. 50 (3) (1978/79) 279–293.
[15] R.S. Ismagilov, Representations of $\mathrm{SL}(2, P)$, where P is not locally compact, Funktsional Anal. i Prilozhen. 7 (4) (1973) 85–86 (English transl. Funct. Anal. Appl. 7 (1974) 328–329).
[16] C. Kapoudjian, Sur des analogues p-adic du group des diféomorphisms du cercle, Thése de Doctorat, Univ. LYON-I, 1998.
[17] C. Kapoudjian, Simplicity of Neretin's group of spheromorphisms, Ann. Inst. Fourier (Grenoble) 49 (4) (1999) 1225–1240.
[18] C. Kapoudjian, From symmetries of the modular tower of genus zero real stable curves to an Euler class for the dyadic circle, math.GR/0006055.
[19] C. Kapoudjian, Homological aspects and the Virasoro-type extension of Higman–Thopson and Neretin groups, to appear.
[20] M. Kapovich, Hyperbolic Manifolds and Discrete Groups, Birkhäuser, Basel, 2001.
[21] G. Kuhn, T. Steger, More irreducible boundary representations of free groups, Duke Math. J. 82 (2) (1996) 381–436.

[22] G. Kuhn, T. Steger, Monotony of certain free group representations, J. Funct. Anal. 179 (1) (2001) 1–17.
[23] R. McKenzie, R.J. Thompson, An elementary construction of unsolvable word problems in group theory, in: Word problems: Decision Problems and the Burnside Problem in Group Theory, North-Holland, Amsterdam, 1973, pp. 457–478.
[24] V. Nekrashevich, Cuntz–Pimsner algebras of group actions, preprint, 2001.
[25] Yu.A. Neretin, Unitary representations of the groups of diffeomorphisms of the p-adic projective line, Funktsional Anal. i Prilozhen. 18 (4) (1984) 92–93 (English transl.: Functional Anal. Appl. 18 (1984) 345–346).
[26] Yu.A. Neretin, Combinatorial analogues of the group of diffeomorphisms of the circle, Izv. Ross. Akad. Nauk, Ser. Mat. 56 (5) (1992) 1072–1085 (English transl. Russian Acad. Sci. Izvestiya. Math. 41 (2) (1993) 337–349).
[27] Yu.A. Neretin, Representations of Virasoro and affine Lie algebras, in: Representation Theory and Noncommutative Harmonic Analysis, I, Encyclopaedia of Mathematical Science, Vol. 22, Springer, Berlin, 1994, pp. 157–234.
[28] Yu.A. Neretin, Fractional diffusions and quasi-invariant actions of infinite-dimensional groups, Truely Mat. Inst. Steklova 217 (1997) 135–181 (English transl. Proc. Steklov Inst. Math. 217 (1997) 126–173).
[29] Yu.A. Neretin, Categories of Symmetries and Infinite-dimensional Groups, London Mathematical Society Monographs, Clarendon Press, Oxford, 1996.
[30] Yu.A. Neretin, G.I. Olshanskii, Boundary values of holomorphic functions, singular unitary representations of the groups $\mathrm{O}(p,q)$ and their limits as $q \to \infty$, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 223 (1995) 9–91 (English transl. J. Math. Sci. (New York) 87 (6) (1997) 3983–4035).
[31] G.I. Olshanskii, Classification of the irreducible representations of the automorphism groups of Bruhat–Tits trees, Funktsional Anal. i Prilozhen. 11 (1) (1977) 32–42, 96 (English transl. Funct. Anal. Appl. 11(1) (1977), 26–34).
[32] G.I. Olshanskii, New "Large" Groups of Type I, Current problems in Mathematics, Vol. 16, VINITI, Moscow, 1980, pp. 31–52, 228 (English transl. J. Sov. Math. 18 (1982) 22–39).
[33] G.I. Olshanskii, Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe, in: Representation of Lie Groups and Related Topics, Gordon and Breach, NY, 1990, pp. 269–463.
[34] R.C. Penner, The universal Ptolemy group and its completions, in: Geometric Galois Actions, Vol. 2, Cambridge University Press, Cambridge, 1997, pp. 293–312.
[35] A. Reznikov, Analytic Topology of Groups, Actions, Strings and Varietes, Preprint, available via http://de.arXiv.org/math.DG/0001135.
[36] I.J. Schoenberg, Metric spaces and positive definite functions, Trans. Amer. Math. Soc. 44 (1938) 522–536.
[37] J.-P. Serre, Lie Algebras and Lie Groups, W.A. Benjamin, Inc., New York, Amsterdam (1965) (Second edition: Lecture Notes in Mathematics, Vol. 1500, 1992).
[38] J.-P. Serre, Arbres, amalgames, SL_2, Astërisque No. (46) (1977) (English transl. J.-P. Serre, Trees, Springer, Berlin, 1980).
[39] P. Shalen, Dendrology of groups: An introduction, in: Essays in Group Theory, Springer, Berlin, 1987, pp. 265–319.
[40] P. Shalen, Dendrology and its applications. in: Group Theory from a Geometrical Viewpoint (Trieste, 1990), World Scientific Publishing, Singapore, 1991, pp. 543–616.
[41] A.V. Skorohod, Integration in Hilbert Space, Springer, Berlin, 1974.
[42] J. Stasheff, The pre-history of operads, in: Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), Contemporary Mathematics, Vol. 202, American Mathematical Society, Providence, RI, 1997, pp. 9–14.
[43] A.M. Vershik, I.M. Gelfand, M.I. Graev, Representations of the group $\mathrm{SL}(2, R)$, where R is a ring of functions, Uspehi Mat. Nauk 28 (5) (173) (1973), 83–128. (English transl. Russian Math. Surveys 28 (5) (1973) 87–132).