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ORTHOMODULAR LATTICES CONTAINING MO2 AS A SUBALGEBRA

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Dedicated to Bernhard Banaschewski on the occasion of his 60th birthday

Introduction

In this paper we prove that 1 is a commutator in every irreducible commutator-finite orthomodular lattice in which the non-distinguished ($\neq 0, 1$) commutators are totally unordered. An example is presented to illustrate the limitations inherent in attempting to improve this result.

The commutator of two elements x and y of an orthomodular lattice L is the element $x*y$ given by the expression

$$x*y=(x\vee y)\wedge(x\vee y')\wedge(x'\vee y)\wedge(x'\vee y').$$

To say that $x*y=1$ is equivalent to saying that the subalgebra $\Gamma(\{x,y\})$ of L generated by x and y is $\{0,1,x,x',y,y'\}$, i.e. $\Gamma(\{x,y\})\cong$ MO2 the six element orthomodular lattice.

A part of the folklore of the subject is that MO2 is a homomorphic image of a subalgebra of every non-Boolean orthomodular lattice. In brief, any non-Boolean orthomodular lattice contains elements x and y which do not commute, so that $\alpha := x*y\neq 0$. Let $Z:=\{0,\alpha,x\wedge\alpha,x'\wedge\alpha,y\wedge\alpha,y'\wedge\alpha\}$. Then $Z\cup\{z' \mid z\in Z\}$ is a subalgebra of L isomorphic to either MO2 or $Z\times\{0,\alpha'\}\cong(\text{MO2})\times 2$, depending on whether $\alpha=1$ or not, so that MO2 is a homomorphic image of a subalgebra of L.

The question of when 1 is a commutator is equivalent to the question when is MO2 not only a homomorphic image of a subalgebra of L but when is it actually a subalgebra. The Remark below lists several statements equivalent to MO2 is a subalgebra of L. The verification, which is straightforward, is left to the reader. Readers unfamiliar with the rudiments of orthomodular lattice theory are referred to [4]. We write com L for the set of all commutators of L. Throughout the paper L is an orthomodular lattice.

Remark. These are equivalent:
(1) $1 \in \operatorname{com} L$.
(2) MO2 is a subalgebra of L.
(3) For some $x \in L$, x is (strongly) perspective to x'.
(4) $\alpha \in C(L) \cap \operatorname{com} L$ iff $\alpha' \in C(L) \cap \operatorname{com} L$.
(5) $C(L) \cap \operatorname{com} L$ is a subalgebra of L.
(6) $C(L) \subseteq \operatorname{com} L$.

Along the way to providing a non-trivial class of commutator-finite orthomodular lattices in which 1 is always a commutator, we isolate conditions which are equivalent in *any* orthomodular lattice to the subalgebra Π generated by the commutators being a projective plane with $\operatorname{com} L = \operatorname{com} \Pi$.

1. Condition Δ

The elements $0, 1$ of L are called *distinguished*. All other elements of L are called *non-distinguished*. Let $\operatorname{com}_0 L$ be the set of non-distinguished commutators of L. We say that L *satisfies condition* Δ in case
(1) $1 \notin \operatorname{com} L$ and
(2) for all distinct $\alpha, \beta \in \operatorname{com}_0 L$, $\alpha \vee \beta = 1$.
That is, L satisfies condition Δ if no non-zero commutator is distinguished and the join of two non-distinguished commutators is distinguished.

In this section, we shall show that an orthomodular lattice with at least two non-distinguished commutators satisfies condition Δ if and only if the subalgebra Π generated by the commutators of L is a projective plane and $\operatorname{com} L = \operatorname{com} \Pi$.

We begin with a lemma of frequently used results; the proofs may be found in [3].

Lemma 1.1. *Let* $x, y, z \in L$.
(1) $x * y = x * y' = x' * y = x' * y'$.
(2) *If* $xCyCz$, *then* $y \wedge (x * z) = (y \wedge x) * z$.
(3) *If* $\alpha = x * y$, *then* $\alpha = (x \wedge \alpha) * (y \wedge \alpha) = (x \wedge \alpha) \vee (y \wedge \alpha)$.
(4) xCy *iff* $x * y \leq x$ *iff* $x * y = 0$.
(5) *No atom is a commutator.*
(6) $\bigvee \operatorname{com} L$, *if it exists, is central in* L.
(7) *If* $\alpha = x * y$ *and* $z \leq \alpha$ *with* $xCzCy$, *then* $z \in \operatorname{com} L$.
(8) *If* $\alpha = x * y$ *and* $\beta \in \operatorname{com} L$ *with* $xC\beta Cy$, *then* $\alpha \vee \beta \in \operatorname{com} L$.

Lemma 1.2. *Assume that* $\alpha, \beta \in \operatorname{com}_0 L$ *with* $\alpha \neq \beta$ *implies* $\alpha \vee \beta = 1$. *Let* $\alpha, \beta, \gamma \in \operatorname{com}_0 L$ *and let* $x \in L$.

(1) $\text{com}_0 L$ *is totally unordered.*
(2) *If* $\alpha C\beta$ *and* $\alpha \neq \beta$, *then* $\alpha' \leq \beta$.
(3) *If* $\alpha \not{C} x$, *then* $\alpha \vee x = 1$.
(4) *If* $\alpha \not{C} \beta$, *then* $\alpha * \beta = \alpha' \vee \beta'$.
(5) *If* $\alpha C\beta C\gamma$ *but* $\alpha \not{C} \gamma$, *then* $\beta = \alpha * \gamma$.

Proof. Claim (1) is obvious. To prove (2), we need observe only that $\alpha' \wedge \beta' = (\alpha \vee \beta)' = 1' = 0$; then $\alpha' C \beta'$ implies $\alpha' \perp \beta'$. To prove (3), we observe that $\alpha \not{C} x$ implies $\alpha \neq \alpha * x$ by Lemma 1.1(4). By hypothesis, $1 = \alpha \vee (\alpha * x) \leq \alpha \vee x$. Claim (4) follows from Claim (3) by expanding $\alpha * \beta$. Finally to prove (5), we observe that by (2) and (4), we have $0 < \alpha * \gamma = \alpha' \vee \gamma' \leq \beta$; thus $\alpha * \gamma = \beta$ by (1). □

It will assist the reader to appreciate the naturalness of our approach if he keeps in mind that ultimately we show that if L satisfies condition Δ and has more than one non-distinguished commutator, then the elements of $\text{com}\, L$ together with their orthocomplements form a projective plane in which the elements α in $\text{com}_0 L$ are the lines and the elements α' with α in $\text{com}_0 L$ are the points.

Lemma 1.3. *Assume that* L *satisfies condition* Δ. *If* α_0 *and* α_1 *are distinct non-distinguished commutators with* $\alpha_0 C \alpha_1$, *then there is a non-distinguished commutator* α_2 *with* $\alpha_0 \not{C} \alpha_2$ *and* $\alpha_1 C \alpha_2$.

Proof. We may assume $\alpha_i = x_i * y_i = x_i \vee y_i$ by Lemma 1.1(3). If $\alpha_1 \in C(x_0, y_0)$, then $1 = \alpha_0 \vee \alpha_1 \in \text{com}\, L$ by Lemma 1.1(8), which contradicts our hypothesis. Thus we may assume $\alpha_1 \not{C} x_0$, and similarly $\alpha_0 \not{C} x_1$. Since L satisfies condition Δ, we have $x_0 * \alpha_1 \leq x_0 \vee \alpha_1' \leq \alpha_0$ by lemma 1.2(2); thus $x_0 * \alpha_1 = \alpha_0$. By symmetry, we have also that $x_1 * \alpha_0 = \alpha_1$. Since $x_0 * \alpha_1 = \alpha_0 \not{C} x_1$, we have $x_0 \not{C} x_1$. Since $x_0 \not{C} \alpha_1$ and $x_1 \not{C} \alpha_0$, it follows that α_0, α_1 and $x_0 * x_1$ are distinct non-distinguished commutators. If $x_0 * x_1$ were to commute with α_1, we would find the contradiction $1 = \alpha_0 \vee (x_0 * x_1) = (x_0 * \alpha_1) \vee (x_0 * x_1) \in \text{com}\, L$ by Lemma 1.1(8). Thus $x_0 * x_1 \not{C} \alpha_1$ and similarly $x_0 * x_1 \not{C} \alpha_0$. Put $\alpha_2 = (x_0 * x_1) * \alpha_1$. Then clearly $\alpha_1 C \alpha_2$. Also $x_1 \in C(x_0 * x_1, \alpha_1)$ implies $x_1 C \alpha_2$. Then, however, if $\alpha_0 C \alpha_2$, we would have the contradiction $1 = \alpha_1 \vee \alpha_2 = (x_1 * \alpha_0) \vee \alpha_2 \in \text{com}\, L$. Thus $\alpha_0 \not{C} \alpha_2$. □

An *n-cycle*, $n \geq 4$, in L is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ of distinct non-distinguished commutators with $\alpha_i C \alpha_{i+1}$ for each index i (mod n). An *n-loop* is an n-cycle such that $\alpha_i \not{C} \alpha_j$ for $|i-j| \not\equiv 1$ (mod n).

In other words, an n-cycle is a sequence of distinct non-distinguished commutators in which immediately adjacent commutators commute; in an n-loop, no other pairs of commutators commute. By Lemma 1.2, we see that $\alpha_{i+1} = \alpha_i * \alpha_{i+2} = \alpha_i' \vee \alpha_{i+2}'$ (mod n) in an n-loop. It will assist the reader to follow our arguments by picturing a 5-loop as a graph with respect to the commutativity relation, as in Fig. 1.

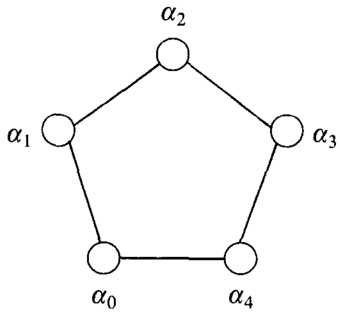

Fig. 1.

Lemma 1.4. *Assume that $\alpha, \beta \in \mathrm{com}_0 L$ with $\alpha \neq \beta$ implies $\alpha \vee \beta = 1$. If $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a 4-cycle, then $\alpha_0 C \alpha_2$ and $\alpha_1 C \alpha_3$. In particular, L has no 4-loops.*

Proof. If $\alpha_0 \not{C} \alpha_2$, then $\alpha_1 = \alpha_0 * \alpha_2 = \alpha_3$ by Lemma 1.2(5), which is a contradiction. Thus $\alpha_0 C \alpha_2$. Similarly $\alpha_1 C \alpha_3$.

Lemma 1.5. *Assume that L satisfies condition Δ. Then any two distinct non-distinguished commutators belong to a 5-loop. Thus, if $|\mathrm{com}_0 L| > 1$, then any non-distinguished commutator is the commutator of two commutators.*

Proof. Let $\alpha, \beta \in \mathrm{com}_0 L$ with $\alpha \neq \beta$. If $\alpha C \beta$, put $\alpha_0 = \alpha$, $\alpha_1 = \beta$ and construct α_2 as in the proof of Lemma 1.3. If $\alpha \not{C} \beta$, put $\alpha_0 = \alpha$, $\alpha_2 = \beta$ and $\alpha_1 = \alpha_0 * \alpha_2$. In either case α and β are contained in a sequence $\alpha_0, \alpha_1, \alpha_2$ with $\alpha_1 \in C(\alpha_0, \alpha_2)$ but $\alpha_0 \not{C} \alpha_2$. Now by Lemma 1.3, there is a non-distinguished commutator α_3 with $\alpha_1 \not{C} \alpha_3$ but $\alpha_2 C \alpha_3$. In particular, $\alpha_0 \neq \alpha_3$. Moreover as $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ could not be a 4-loop by Lemma 1.4, we have $\alpha_0 \not{C} \alpha_3$. Put $\alpha_4 = \alpha_0 * \alpha_3$. It follows easily from Lemma 1.4, that $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a 5-loop.

The second statement of this lemma follows from the remark preceding Lemma 1.4. □

Lemma 1.6. *Assume that L satisfies condition Δ. In a 5-loop $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$,*

$$\alpha_i' \vee \alpha_{i+1}' = \alpha_i * [(\alpha_i \wedge \alpha_{i+1}) * \alpha_{i+3}] \quad (\mathrm{mod}\ n).$$

Proof. For concreteness, we shall show that $\alpha_0' \vee \alpha_1' = \alpha_0 * [(\alpha_0 \wedge \alpha_1) * \alpha_3]$. Let $\gamma = (\alpha_0 \wedge \alpha_1) * \alpha_3$. The commutator γ is a simpler lattice polynomial than is apparent. By Lemma 1.2(3), $\alpha_0' \vee \alpha_1' \vee \alpha_3 = 1$; by Lemma 1.2(4) and (5), $\alpha_0' \vee \alpha_1' \vee \alpha_3' = (\alpha_1' \vee \alpha_3') \vee (\alpha_3' \vee \alpha_0') = \alpha_2 \vee \alpha_4 = 1$; by Lemma 1.2(2) and (5) and the Foulis–Holland Theorem,

$(\alpha_0\wedge\alpha_1)\vee\alpha_3=[(\alpha_0\wedge\alpha_1)\vee\alpha_4']\vee\alpha_3=[(\alpha_0\vee\alpha_4')\wedge(\alpha_1\vee\alpha_4')]\vee\alpha_3=(\alpha_0\wedge 1)\vee\alpha_3=\alpha_0\vee\alpha_3=1$. Then by expanding γ according to the definition of a commutator, we find that $\gamma=(\alpha_0\wedge\alpha_1)\vee\alpha_3'$.

It is now easy to verify that γ is not in our loop. Obviously $\alpha_3\vee\gamma=1$. By Lemma 1.2(3), we have immediately that $\alpha_0\vee\gamma=1=\alpha_1\vee\gamma$. Using the Foulis–Holland Theorem, we have also $\alpha_2\vee\gamma=[\alpha_2\vee(\alpha_0\wedge\alpha_1)]\vee\alpha_3'=[(\alpha_2\vee\alpha_0)\wedge(\alpha_2\vee\alpha_1)]\vee\alpha_3'=1\vee\alpha_3'=1$. Similarly $\alpha_4\vee\gamma=1$. It follows that $\gamma\neq\alpha_0,\alpha_1,\alpha_2,\alpha_3$, or α_4.

Now since $(\alpha_0,\gamma,\alpha_3,\alpha_4)$ is not a 4-loop by Lemma 1.4, we see that $\alpha_0 \not{C}\gamma$. Similarly, since $(\alpha_1,\alpha_2,\alpha_3,\gamma)$ is not a 4-loop, we see that $\alpha_1\not{C}\gamma$. In particular $\alpha_0\neq\alpha_0*\gamma$ and $\alpha_1\neq\alpha_1*\gamma$. By Lemma 1.2(2), we have $0<\alpha_0'\le\alpha_0*\gamma\le\alpha_0'\vee\gamma'=\alpha_0'\vee[(\alpha_0'\vee\alpha_1')\wedge\alpha_3]\le\alpha_0'\vee\alpha_1'$. By symmetry, we have also $0<\alpha_1'\le\alpha_1*\gamma\le\alpha_0'\vee\alpha_1'$. If $\alpha_0*\gamma\neq\alpha_1*\gamma$, then $1=(\alpha_0*\gamma)\vee(\alpha_1*\gamma)\le\alpha_0'\vee\alpha_1'$ and so $\alpha_0'=\alpha_1\in\text{com}_0 L$. Then, however, $1=\alpha_0'\vee a_0\in\text{com}\, L$, which is a contradiction. Thus $\alpha_0',\alpha_1'\le\alpha_0*\gamma=\alpha_1*\gamma\le\alpha_0'\vee\alpha_1'$, and our argument is completed. □

Lemma 1.7. *Assume that L satisfies condition Δ. If* $\alpha,\beta\in\text{com}_0 L$ *with* $\alpha\neq\beta$, *then* $\alpha'\vee\beta'\in\text{com}_0 L$.

Proof. If $\alpha\not{C}\beta$, then $\alpha'\vee\beta'=\alpha*\beta$ by Lemma 1.2(4). Thus we may assume that $\alpha C\beta$. By the proof of Lemma 1.5, there is a 5-loop $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ with $\alpha_0=\alpha$ and $\alpha_1=\beta$. Then by Lemma 1.6, $\alpha'\vee\beta'=\alpha_0'\vee\alpha_1'=\alpha_0*[(\alpha_0\wedge\alpha_1)*\alpha_3]$. □

Lemma 1.8. *Let L be an orthomodular lattice of height* 3. *These conditions are equivalent*:

(1) $1\in\text{com}\, L$.

(2) *There are atoms a and b with* $a*b=1$.

(3) *L is non-modular.*

Proof. Assume $1\in\text{com}\, L$. Then there are elements $x,y\in L$ with $x*y=x'*y=x*y'=x'*y'=1$. Either x or x' is an atom and either y or y' is an atom. Thus we have established (1) implies (2). Clearly (2) implies (1). Assume (3) holds. Then we have the pentagon sublattice of L given in Fig. 2 (cf. [4, p. 33]).

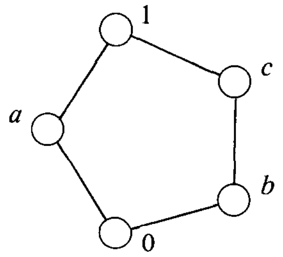

Fig. 2.

We may assume that a is an atom by applying the orthocomplementation if necessary. Now $a \vee b = 1$ and $a' \vee b' = (a \wedge b)' = 1$. We claim that $a \vee b' = 1 = a' \vee b$ as well. Since a' and b' are coatoms, this is clear unless $a = b'$. However, if $a = b'$, then we would have the nonsense: $b < c$ but $b' \wedge c = a \wedge c = 0$. So $a \vee b' = 1 = a' \vee b$. Then by expanding $a * b$ according to its definition, we find $a * b = 1$. Thus (3) implies (2).

Finally, assume that (2) holds. Since L has height 3, if it is a horizontal sum, then it is clearly non-modular. So we may assume that L is not a horizontal sum; in particular, then, no atom of L is also a coatom. Consider any coatom $c > b$. Then $1 = a * b \leq a \vee b$. Moreover, $a \not\leq c$; for otherwise $1 = a \vee b \leq c$. It follows that $a \wedge c = 0$ since a is an atom. We have produced the sublattice of Fig. 2, which shows that L is non-modular. Thus (2) implies (3). □

This lemma permits us to identify the commutators in an orthocomplemented projective plane Π. Since Π is a modular orthomodular lattice of height 3, $1 \notin \operatorname{com} \Pi$. No atom of Π is a commutator by Lemma 1.1(5). The coatoms of Π, however, are clearly commutators. Thus the commutators of Π are precisely the coatoms. Moreover, it is easy to see that for two lines α and β of Π, $\alpha * \beta = 0$ if the point α' lies on the line β, and $\alpha * \beta$ is the line determined by the points α' and β' otherwise.

Theorem 1.9. *Let L be an orthomodular lattice. These conditions are equivalent:*

(1) *L satisfies condition Δ and* $|\operatorname{com}_0 L| > 1$.

(2) *Let* $\Pi = \{\alpha, \alpha' : \alpha \in \operatorname{com} L\}$; *then* $1 \notin \operatorname{com} L$ *and Π is a subalgebra of L which is a projective plane.*

(3) *The subalgebra generated by the commutators of L is a projective plane, and every commutator of L is a commutator of this subalgebra.*

Proof. Assume (1) holds. By (1), $|\Pi| \geq 6$. By lemma 1.7, Π is a subalgebra of L. Condition Δ implies that Π contains no chains of length greater than 3. Let $\alpha \in \operatorname{com}_0 L$. Since each non-distinguished commutator belongs to a 5-loop, Π contains commutators β, γ with $\alpha C \beta$ and $\alpha \not{C} \gamma$. This remark has two consequences. First, $\alpha' < \beta$ implies that Π contains a chain of length 3 and hence Π has height 3. Second $\alpha \not{C} \gamma$ implies $\alpha \notin C(\Pi)$. Thus $C(\Pi)$ contains no non-distinguished elements of Π and hence Π is irreducible. Finally, by Lemma 1.8, $1 \notin \operatorname{com} L$ implies that Π is modular.

Now assume (2) holds. Clearly Π is the subalgebra generated by $\operatorname{com} L$. Thus the subalgebra generated by $\operatorname{com} L$ is a projective plane. Let $\alpha \in \operatorname{com}_0 L$. If α were an atom of Π, then α' would be a coatom of Π. But then $1 = \alpha \vee \alpha' \in \operatorname{com} L$. Thus α is a coatom of the projective plane Π and hence $\alpha \in \operatorname{com} \Pi$. Thus $\operatorname{com} \Pi = \operatorname{com} L$.

Now assume (3) holds. Let $\Gamma(\operatorname{com} L)$ be the subalgebra generated by $\operatorname{com} L$. Then by Lemma 1.8, $\Gamma(\operatorname{com} L)$ being modular implies $1 \notin \operatorname{com} \Gamma(\operatorname{com} L) = \operatorname{com} L$. Since an ortho-complemented projective plane contains infinitely many commutators (all the coatoms), we see that $|\operatorname{com}_0 L| > 1$. Now let $\alpha, \beta \in \operatorname{com}_0 L$ with $\alpha \neq \beta$. Then α and β are coatoms of $\Gamma(\operatorname{com} L)$, and so $\alpha \vee \beta = 1$. Thus L satisfies condition Δ. □

2. Commutator-finite orthomodular lattices

In this section we focus on commutator-finite orthomodular lattices in the light of Theorem 1.9. In Proposition 2.1 we show that, because of Baer's Theorem [1] to the effect that there are no finite orthocomplemented projective planes, there is a paucity of commutator-finite orthomodular lattices satisfying condition Δ. This proposition essentially says that a commutator-finite orthomodular lattice satisfies condition Δ if and ony if it has no more than one non-distinguished commutator.

Proposition 2.2 states that any element dominating all the non-distinguished commutators is central, provided L has no Boolean horizontal summand. Given the proviso, this generalizes [3, Corollary 2]. We then characterize orthomodular lattices not excluded by the proviso in terms of the commutants of non-distinguished commutators.

Theorem 2.4 gives equivalent ways of saying part (2) of condition Δ in a commutator-finite orthomodular lattice in which the join of the commutators is 1. One of these equivalent conditions is that $\mathrm{com}_0 L$ be totally unordered. These are precisely the conditions under which we can prove that 1 is a commutator. Finally, an example shows that these conditions appear to be best possible.

Proposition 2.1. *Those commutator-finite orthomodular lattices satisfying condition Δ consist of Boolean algebras and of nontrivial direct products of Boolean algebras with nontrivial horizontal sums of Boolean algebras.*

Proof. By [1], there are no finite orthocomplemented projective planes. So by Theorem 1.9, $|\mathrm{com}_0 L| \leq 1$. If $\mathrm{com}\, L = \{0\}$, then L is Boolean [3]. Otherwise $\mathrm{com}\, L = \{0, \alpha\}$ with $0 < \alpha < 1$. Then $\alpha \in C(L)$ by [3, Corollary 2] and so $L \cong [0, \alpha'] \times [0, \alpha]$. Since α is a maximal commutator, $[0, \alpha']$ is a Boolean lattice by [3, Corollary 6], since α is a minimal nonzero commutator, $[0, \alpha]$ is a nontrivial horizontal sum of Boolean lattices by [3, Proposition 7]. □

A *comparability chain* from $\{z, z'\}$ to $\{x, x'\}$ is a sequence $y_0, y_1, \ldots, y_n \in L \setminus \{0, 1\}$ with $y_0 = z$ or z' and $y_n = x$ or x', and $y_{i-1} \leq y_i$ or $y_{i-1} \geq y_i$ for each $i = 1, \ldots, n$. The integer n is called the *length* of the chain. If $y_0 = z$, we say that the path is *based* at z. Note that if there is a chain of length n based at z' to $\{x, x'\}$, then there is a chain of length n based at z to $\{x, x'\}$. Eric Schreiner [6, Theorem 1.5.6] has proved essentially the following: L cannot be written as a horizontal sum if and only if for all $x, z \in L \setminus \{0, 1\}$ with $x \neq z$, there exists a comparability chain from $\{z, z'\}$ to $\{x, x'\}$.

Proposition 2.2. *If L is an orthomodular lattice with no Boolean horizontal summand, and if $y \leq z$ for all $y \in \mathrm{com}_0 L$, then $z \in C(L)$ and $[0, z']$ is a Boolean factor of L.*

Proof. If L is a horizontal sum, say $L = L_1 \circ L_2$ then there exists $\alpha_i \in L_i \cap \mathrm{com}_0 L$,

$i=1,2$ (else some L_i is a Boolean algebra) and so $1=\alpha_1\vee\alpha_2\leq z$, i.e. $z=1\in C(L)$.

Thus we may assume that L is not a horizontal sum. Let $x\in L$. Suppose $x\not{C}z$. Let $y_0,y_1,\ldots,y_n$ be a comparability chain based at z to $\{x,x'\}$ of minimal length. Then $n>1$ and $z=y_0\leq y_1\geq y_2$ or $z=y_0\geq y_1\leq y_2$. In either case $z*y_2<1$; so $z*y_2\leq z$, i.e. zCy_2 so $y_2\neq x,x'$. Hence $n>2$.

We claim that $z*y_3=1$. Suppose not, say $z\wedge y_3>0$; then $z\geq z\wedge y_3\leq y_3$ initiates a shorter path, contradicting the minimality of the given path. Hence $z*y_3=1$. Since zCy_2Cy_3, we have $(y_2\wedge z)*y_3=y_2\wedge(z*y_3)=y_2\wedge 1=y_2$; so $y_2\leq z$, and $z=y_0,y_2,y_3,\ldots,y_n$ is a shorter path than the original, which is a contradiction. Hence xCz. Thus $z\in C(L)$.

Now let $y\leq z'$. Then $x*y\leq z\leq y'$ implies xCy. Thus $y\leq z'$ implies $y\in C(L)$ so that $[0,z']$ is a Boolean factor of L. □

It follows from [3, Corollary 2] that, whenever $z\geq\gamma$ for all $\gamma\in\operatorname{com} L$, $[0,z']$ is a Boolean factor of L. Proposition 2.2 can be regarded as a generalization of this result for those orthomodular lattices which have no Boolean algebra as a horizontal summand. The following remark characterizes these structures.

Proposition 2.3. *An orthomodular lattice L has no Boolean horizontal summand if and only if $L=\bigcup\{C(\alpha):\alpha\in\operatorname{com}_0 L\}$.*

Proof. Assume that L is the horizontal sum of a Boolean subalgebra B and a subalgebra L_1. Then $\operatorname{com}_0 L\subseteq L_1$, and so the non-distinguished elements of B fail to commute with any non-distinguished commutators. Thus $L\neq\bigcup\{C(\alpha):\alpha\in\operatorname{com}_0 L\}$.

Conversely, assume that L has no Boolean horizontal summand. Let us consider first the case in which L is non-Boolean and L is not itself a horizontal sum. Since L is non-Boolean, there must exist nonzero commutators. If 1 where the only nonzero commutator, then L would be a horizontal sum of Boolean subalgebras by [3, Proposition 7]. Thus $\operatorname{com}_0 L$ is non-empty. Let $x\in L$. If $x\in C(L)$, then $x\in C(\alpha)$ for each $\alpha\in\operatorname{com}_0 L$. Thus we may assume that $x\notin C(L)$. Let $x=x_0,x_1,\ldots,x_n$ be a comparability chain from $\{x,x'\}$ to $\{x_n,x_n'\}$ of minimal length with respect to the property: $x\not{C}x_n$. Clearly $n\geq 2$. If $n=2$, then $0<x*x_2\leq x_1$ or x_1' and we are finished since $xCx*x_2$. If $n>2$, then by the minimality of the chain, we see that xCx_{n-1}; hence, $x*(x_{n-1}\wedge x_n)=x_{n-1}\wedge(x*x_n)$. If $x*x_n<1$, then it is a non-distinguished commutator commuting with x. If $x*x_n=1$, then $x*(x_{n-1}\wedge x_n)=x_{n-1}$ has this property.

There is another case to consider. Suppose L is a horizontal sum of non-Boolean subalgebras L_i, which are not themselves horizontal sums. Then $x\in L$ implies $x\in L_i$ for some index i. Then the preceding paragraph shows that $x\in C(\alpha)$ for some $\alpha\in\operatorname{com}_0 L_i\subseteq\operatorname{com}_0 L$. □

Theorem 2.4. *Let L be a commutator-finite orthomodular lattice with $\bigvee\operatorname{com} L=1$. These are equivalent*:

(1) $\operatorname{com}_0 L$ *is totally unordered.*

(2) *The join of distinct non-distinguished commutators of* L *is the unit of* L.
(3) *If* $\alpha, \beta, \gamma, \delta \in \mathrm{com}_0 L$ *with* $\alpha \neq \beta$ *and* $\gamma \neq \delta$, *then* $\alpha \vee \beta = \gamma \vee \delta$.
(4) *If* $\alpha, \beta, \gamma \in \mathrm{com}_0 L$ *with* $\alpha \neq \beta$, *then* $\gamma \leq \alpha \vee \beta$.

Proof. We prove that (1) implies (2) by induction on $|\mathrm{com}_0 L|$. (2) is vacuously satisfied if $|\mathrm{com}_0 L| < 2$. Assume that the result holds for all such lattices having fewer than n non-distinguished commutators. Let $2 \leq |\mathrm{com}_0 L| = n$ with $\bigvee \mathrm{com}\, L = 1$ and $\mathrm{com}_0 L$ totally unordered. Let $\alpha \neq \beta$ be in $\mathrm{com}_0 L$. Suppose that $\alpha \vee \beta < 1$. Either L is not a horizontal sum or $\alpha \vee \beta$ is in a unique horizontal summand L_1 of L. Since the rest of this argument takes place in L_1 we may assume that $L_1 = L$, i.e. that L contains no horizontal summand. By Proposition 2.2 and the fact that $\bigvee \mathrm{com}\, L = 1$, $|\mathrm{com}_0 [0, \alpha \vee \beta]| < n$ (otherwise $[0, (\alpha \vee \beta)']$ would be a non-trivial Boolean factor). Since the other induction hypotheses clearly obtain, $[0, \alpha \vee \beta]$ satisfies (2). Since $\mathrm{com}_0 L$ is totally unordered and $\mathrm{com}[0, \alpha \vee \beta] \subseteq \mathrm{com}\, L$ we may conclude that $\alpha \vee \beta \notin \mathrm{com}[0, \alpha \vee \beta]$. By Theorem 1.9, $[0, \alpha \vee \beta]$ contains a finite orthocomplemented projective plane, the subalgebra generated by the commutators of $[0, \alpha \vee \beta]$, contradicting Baer's theorem. Thus $\alpha \vee \beta = 1$ and L satisfies (2). Hence (1) implies (2).

Clearly (2) implies (3) and (3) implies (4). Thus we need prove only that (4) implies (1). We may assume that $|\mathrm{com}_0 L| \geq 2$ and that L is not a horizontal sum since $\mathrm{com}_0 L$ is totally unordered if and only if $\mathrm{com}_0(L_i)$ is totally unordered for each horizontal summand L_i of L. Let $\alpha, \beta \in \mathrm{com}_0 L$ with $\alpha \neq \beta$. By (4) and Proposition 2.2, $[0, (\alpha \vee \beta)'] \subseteq C(L)$. Since $\bigvee \mathrm{com}\, L = 1$, $(\alpha \vee \beta)' = 0$ so that $\alpha \vee \beta = 1$. Since $\beta < 1$, we have $\alpha \not\leq \beta$. Thus $\mathrm{com}_0 L$ is totally unordered. □

Corollary 2.5. *Assume that* L *is a commutator-finite orthomodular lattice with* $\bigvee \mathrm{com}\, L = 1$. *If* $\mathrm{com}_0 L$ *is totally unordered, then* $1 \in \mathrm{com}\, L$.

Proof. We may assume that L has at least 2 elements. Since $\bigvee \mathrm{com}\, L = 1$, L is not a Boolean algebra and L has no Boolean factor. Suppose that $1 \notin \mathrm{com}\, L$. Then L satisfies condition Δ by Theorem 2.4. Thus, by Proposition 2.1, L is a horizontal sum of Boolean algebras so that $1 \in \mathrm{com}\, L$, which is a contradiction. Therefore $1 \in \mathrm{com}\, L$. □

By [3, Theorem 14], Corollary 2.5 implies that $1 \in \mathrm{com}\, L$ for every irreducible commutator-finite orthomodular lattice in which $\mathrm{com}_0 L$ is totally unordered. Fig. 3 depicts the Greechie diagram [4] for an irreducible commutator-finite OML in which 1 is not a commutator. Dichtl [2] first observed in print that this structure which we shall call the Dichtl triangle D_1, corresponds to an orthomodular lattice. The Hasse diagram for the poset of its commutators is given in Fig. 4. These facts have been checked by us as well as by an extensive computer program developed by Miller [5].

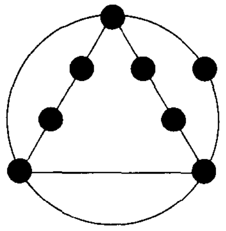

Fig. 3. The Dichtl Triangle D_1.

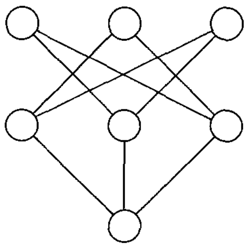

Fig. 4. com(D_1).

The maximal commutators of D_1 are of the form $x*y$ where x and y are atoms on distinct lines of the triangle and are not vertices of the triangle; whereas the minimal non-zero commutators are of the form $x*y$ where x is an atom of some line of the triangle but not a vertex and y is the (unique) atom on the circular block which is on no line of triangle. There appears to be no reasonable conjecture which generalizes Corollary 2.5 and is not refuted by the Dichtl triangle — unless it involves forbidden configurations.

References

[1] R. Baer, Polarities in finite projective planes, Bull. Amer. Math. Soc. 52 (1946) 77-93.

[2] M. Dichtl, There are loops of order three in orthomodular lattices, Arch. Math. (Basel) 37 (1981) 285-286.

[3] R. Greechie and L. Herman, Commutator-finite orthomodular lattices, Order 1 (1985) 277-284.

[4] G. Kalmbach, Orthomodular Lattices (Academic Press, London, 1983).

[5] W.D. Miller, Machine implemented computations for orthomodular lattices, Kansas State University Technical Report #125, July 1986.

[6] E. Schreiner, Ph.D. Dissertation, Wayne State University, 1964.