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STABLE PARALLELIZABILITY OF LENS SPACES

John EWING*, Suresh MOOLGAVKAR*, Larry SMITH†
Indiana University, Bloomington, Ind. 47401, U.S.A.

and

R.E. STONG†
University of Virginia, Charlottesville, Va. 22903, U.S.A.

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Let p be an odd prime and n a non-negative integer. Define $T: \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ by

$$T(z_0, z_1, \ldots, z_n) = (\lambda^{b_0} z_0, \lambda^{b_1} z_1, \ldots, \lambda^{b_n} z_n)$$

where $\lambda = \exp(2\pi i/p)$ and $1 \leq b_i \leq p-1$. The map T induces a free action of $\mathbf{Z}/p$ on the sphere S^{2n+1} and the orbit space is the lens space which we denote by $L(p; b_0, b_1, \ldots, b_n)$.

We are concerned here with determining when these lens spaces are framable, that is have stably trivial tangent bundles. Our first step is to reduce this problem to algebra.

Proposition. *The lens space $L(p; b_0, \ldots, b_n)$ is framable iff $n < p$ and*

$$b_0^{2j} + b_1^{2j} + \cdots + b_n^{2j} \equiv 0 \bmod p$$

for $j = 1, 2, \ldots, [n/2]$.

By applying a key lemma of Deligne (from his positive solution to the Weil conjectures) we obtain the following global result.

Theorem. *Let n be a positive integer. Then for all sufficiently large primes p there exists a $2n+1$ dimensional* mod p *lens space $L(p; b_0, b_1, \ldots, b_n)$ that frames.*

Much of the rest of the paper is concerned with the solvability of the system

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$$\sum_{i=0}^{n} x_i^{2j} \equiv 0 \bmod p, \qquad j = 1, 2, \ldots, [n/2]$$

for fixed n or p from a more elementary viewpoint.

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1. Reduction to algebra

Associated with the principal $\mathbf{Z}/p$ bundle $\pi: S^{2n+1} \downarrow L(p; b_0, b_1, \ldots, b_n)$ one may form a complex line bundle γ over $L(p; b_0, \ldots, b_n)$ by dividing out the diagonal action of $\mathbf{Z}/p$ on $S^{2n+1} \times \mathbf{C}$ where the generator of $\mathbf{Z}/p$ acts on $\mathbf{C}$ by multiplication by λ. There are also the similarly defined line bundles where $\mathbf{Z}/p$ acts on $\mathbf{C}$ by multiplication by λ^b which are just the complex tensor powers γ^b.

Lemma 1.1. *The tangent bundle of* $L(p; b_0, b_1, \ldots, b_n)$ *is stably isomorphic to* $\mathrm{re}(\gamma^{b_0} \oplus \gamma^{b_1} \oplus \cdots \oplus \gamma^{b_n})$.

Proof. Clearly $\tau(S^{2n+1}) \oplus \mathbf{R}$ with $\mathbf{Z}/p$ action given by $dT \oplus 1$ is equivariantly isomorphic to $S^{2n+1} \times \mathbf{C}^{n+1}$ with $\mathbf{Z}/p$ action given by $T \times T$. Dividing out the action gives the result. □

Now let $L^n(p) = L(p; 1, 1, \ldots, 1)$ ($n+1$ ones).

Proposition 1.2. $L(p; b_0, b_1, \ldots, b_n)$ *is framable iff* $\mathrm{re}(\gamma^{b_0} \oplus \gamma^{b_1} \oplus \cdots \oplus \gamma^{b_n})$ *is stably trivial over* $L^n(p)$.

Proof. For any $(b_0, b_1, \ldots, b_n)$ the principal $\mathbf{Z}/p$ bundle $\pi: S^{2n+1} \downarrow L(p; b_0, b_1, \ldots, b_n)$ is $2n+1$ universal. Thus there are maps (**N.B.** Not unique up to homotopy nor homotopy equivalences in general.)

$$f: L(p; b_0, \ldots, b_n) \to L^n(p)$$

$$g: L^n(p) \to L(p; b_0, \ldots, b_n)$$

such that (abusing notations)

$$f^*\gamma = \gamma, \qquad g^*\gamma = \gamma$$

and the result follows from Lemma 1.1. □

In [5] $\widetilde{\mathrm{KO}}(L^n(p))$ is computed. Setting $\bar{\sigma} = \mathrm{re}(\gamma) - 2$ the p torsion part of $\widetilde{\mathrm{KO}}(L^n(p))$ is a direct sum of cyclic groups generated by $\bar{\sigma}^i$, $1 \le i \le \frac{1}{2}(p-1)$, where if $n = s(p-1)+r$, $0 \leq r < p-1$, the order of $\tilde{\sigma}^i$ is p^{s+1} for $i \leq [\frac{1}{2}r]$ and p^s for $i > [\frac{1}{2}r]$.

Realification is not a ring homomorphism. But one has (the $*$ denotes complex conjugation)

$$\begin{aligned}\gamma \otimes_{\mathbf{R}} \gamma^a &= (\gamma \otimes_{\mathbf{R}} \mathbf{C}) \otimes_{\mathbf{C}} \gamma^a \\ &= (\gamma \oplus \gamma^*) \otimes_{\mathbf{C}} \gamma^a \\ &= (\gamma \oplus \gamma^{-1}) \otimes_{\mathbf{C}} \gamma^a = \gamma^{a+1} \oplus \gamma^{a-1}\end{aligned}$$

(*) $\mathrm{re}(\gamma^b) - 2 = \tilde{\sigma}^b +$ terms of lower degree in $\tilde{\sigma}$ provided $1 \leq b \leq \frac{1}{2}(p-1)$.

Lemma 1.3. *If $L(p; b_0, b_1, \ldots, b_n)$ is framable then $n < p$.*

Proof. Since $\mathrm{re}(\gamma^b) = \mathrm{re}((\gamma^*)^b) = \mathrm{re}(\gamma^{p-b})$ and complex conjugation in the $(i+1)$st coordinate induces a diffeomorphism

$$L(p; b_0, \ldots, b_n) \to L(p; b_0, \ldots, p-b_i, \ldots, b_n)$$

we may as well suppose $1 \leq b_i \leq \frac{1}{2}(p-1)$.

In order that $(\mathrm{re}(\gamma^{b_0}) - 2) + \cdots + (\mathrm{re}(\gamma^{b_n}) - 2)$ be zero in $\widetilde{\mathrm{KO}}(L^n(p))$ it is necessary that $n + 1 \equiv 0 \bmod p^s$, since from (*) it follows that for each b with $1 \leq b \leq \frac{1}{2}(p-1)$ the number of copies of $\mathrm{re}(\gamma^b) - 2$ must be divisible by p^s. Since $p^s > s(p-1) + r + 1$ if $s > 1$, and $(p-1) + r + 1 \equiv 0 \bmod p$ implies $r = 0$, one must have $n \leq p - 1$. □

Remark. When $n = p-1$, then one sees that $\gamma \oplus \gamma \oplus \cdots \oplus \gamma = p\gamma$ is stably trivial, so that $L^{p-1}(p)$ is framable. This was pointed out to us by Idar Hansen and provided one of the starting points of our inquiry.

Lemma 1.4. *Let ξ be an oriented vector bundle over a finite complex X, and suppose that:*

(a) *$\dim X < 2p + 2$, p an odd prime, and*

(b) *$\tilde{H}^{4*}(X; \mathbf{Z})$ has no q torsion for any odd prime $q < p$.*

If $\mathcal{P}(\xi) = 0$ then $\xi - \dim \xi \in \widetilde{\mathrm{KO}}(X)$ is a 2-torsion element.

Proof. Let

$$\pi : \mathrm{BSO}(m, \ldots, \infty) \to \mathrm{BSO}$$

be the $m-1$ connective fibring over BSO and write

$$\mathcal{P}_{i,m} = \pi^* \mathcal{P}_i \in H^{4i}(\mathrm{BSO}(m, \ldots, \infty); \mathbf{Z}).$$

Suppose inductively we have constructed a lifting as in the diagram

$$\begin{array}{ccc} & & \mathrm{BSO}(m,\ldots,\infty) \\ & \overset{\tilde f_m}{\nearrow} & \downarrow \pi \\ X & \xrightarrow[2^k f]{} & \mathrm{BSO} \end{array}$$

where $m < 2p+2$, $k \geq 0$ and f is a stable classifying map for ξ. Consider the lifting problem posed by the diagram

$$\begin{array}{ccc} & & \mathrm{BSO}(m+1,\ldots,\infty) \\ & & \downarrow \pi_m \\ X & \xrightarrow{\tilde f_m} & \mathrm{BSO}(m,\ldots,\infty). \end{array}$$

If $m \not\equiv 0,1,2,4 \bmod 8$ then π_m is a homotopy equivalence so $\tilde f_m$ lifts over π_m. If $m \equiv 1,2 \bmod 8$ then π_m is classified by a map

$$\mathrm{BSO}(m,\ldots,\infty) \to K(\mathbf{Z}/2, m)$$

so that $2\tilde f_m$ lifts over π_m, say to $\tilde f_{m+1}$, whence we get a lift as in the diagram

$$\begin{array}{ccc} & & \mathrm{BSO}(m+1,\ldots,\infty) \\ & \overset{\tilde f_{m+1}}{\nearrow} & \downarrow \pi \\ X & \xrightarrow[2^{k+1} f]{} & \mathrm{BSO}. \end{array}$$

If $m \equiv 0(4)$ then π_m is classified by a map

$$g_m : \mathrm{BSO}(m,\ldots,\infty) \to K(\mathbf{Z}, m)$$

such that

$$a_m g_m^*(i) = \mathcal{P}_{m/4,m}$$

where $a_m \neq 0 \in \mathbf{Z}$ has a prime factorization involving only primes $q < p$ [7]. Therefore

$$a_m \tilde f_m^* g_m^*(i) = 2^b \mathcal{P}_{m/4}(\xi) = 0$$

whence $\tilde f_m^* g_m^*(i) = 0$ since a_m is not a zero divisor in $H^m(X;\mathbf{Z})$ by condition (b). Therefore $\tilde f_m \tilde g_m$ is null homotopic, so $\tilde f_m$ lifts over π_m. Therefore inductively it follows that there is a lift as in the diagram

$$\begin{array}{ccc} & & \mathrm{BSO}(2p+2,\ldots,\infty) \\ & \overset{\tilde f}{\nearrow} & \downarrow \pi \\ X & \xrightarrow[2^k f]{} & \mathrm{BSO} \end{array}$$

where $k = \mathrm{card}\{n \mid 0 < n \leq 2p+2 \text{ and } n \equiv 1,2(8)\}$. But by condition (a) $\dim X \leq$

$2p+1$ = connectivity of $BSO(2p+2,\ldots,\infty)$, where $\bar{f}$ is null homotopic, so $2^k f = \pi \circ f$ is null homotopic and hence $2^k(\xi - \dim \xi) = 0 \in \tilde{KO}(X)$. □

Proposition 1.5. *The lens space $L(p; b_0, b_1, \ldots, b_n)$ is framable iff $n < p$ and the Pontrjagin classes of the bundle $\gamma^{b_0} \oplus \gamma^{b_1} \oplus \cdots \oplus \gamma^{b_n} \downarrow L^n(p)$ are trivial.*

Proof. If $L(p; b_0, \ldots, b_n)$ is framable then $\gamma^{b_0} \oplus \cdots \oplus \gamma^{b_n} \downarrow L^n(p)$ is trivial so has vanishing Pontrjagin classes. Conversely if $\gamma^{b_0} \oplus \cdots \oplus \gamma^{b_n}$ has vanishing Pontrjagin classes and $n < p$ then by (1.4) $\mathrm{re}(\gamma^{b_0} \oplus \cdots \oplus \gamma^{b_n}) \downarrow L^n(p)$ is 2 torsion in $\tilde{KO}(L^n(p))$. But

$$\mathrm{re}(\gamma^{b_0} \oplus \cdots \oplus \gamma^{b_n}) - 2(n+1) \in \mathrm{Im}\{\tilde{K}(L^n(p) \to \tilde{KO}(L^n(p))\}$$

and as $\tilde{K}(L^n(p))$ has no 2-torsion [5] it follows $\gamma^{b_0} \oplus \cdots \oplus \gamma^{b_n}$ is stably trivial and thus $L(p; b_0, \ldots, b_n)$ frames by Proposition 1.2. □

Recall that

$$H^*(L^n(p); Z/p) \simeq E[u] \otimes \frac{Z/p[v]}{(v^{n+1})}$$

where $\deg u = 1$, $\deg v = 2$, $\beta u = v$. Moreover

$$\tilde{H}^{\text{even}}(L^n(p); Z) \to \tilde{H}^{\text{even}}(L^n(p); Z/p)$$

is an isomorphism.

The total mod p Pontrjagin class of the bundle

$$\gamma^{b_0} \oplus \gamma^{b_1} \oplus \cdots \oplus \gamma^{b_n} \downarrow L^n(p)$$

is given by

$$\prod_{i=0}^{n} (1 + b_i^2 v^2) \in H^*(L^n(p); Z/p)$$

and thus we obtain

Corollary 1.6. *The lens space $L(p; b_0, \ldots, b_n)$ is framable iff $n < p$ and one of the following equivalent conditions holds:*

(1) $(1 + b_0^2 v^2)(1 + b_1^2 v^2) \cdots (1 + b_n^2 v^2) = 1$ *in* $Z/p[v^2]/v^{n+1}$

(2) $b_0^{2j} + b_1^{2j} + \cdots + b_n^{2j} = 0 \bmod p$, $j = 1, 2, \ldots, [\frac{1}{2}n]$.

Proof. From the preceding discussion (1) is equivalent to the vanishing of the Pontrjagin class of $\mathrm{re}(\gamma^{b_0} \oplus \cdots \oplus \gamma^{b_n})$ so applying Proposition 1.5 yields the result.

The equivalence of (1) and (2) follows from Newton's identity

$$0 = m\sigma_m - Q_1\sigma_{m-1} + \cdots + (-1)^{m-1} Q_{m-1}\sigma_1 + (-1)^m Q_m$$

where

$$\sigma_i = i\text{th elementary symmetric function of } b_0^2, \dots, b_n^2$$

$$Q_i = b_0^{2i} + b_1^{2i} + \cdots + b_n^{2i},$$

and the fact that the coefficient of v^{2i} in the polynomial of (1) is σ_i. □

2. Elementary consequences

By the results of the preceding section, it is clear that given an odd prime p, a non-negative integer $n < p$, and integers $b_0, \dots, b_n$ with $1 \leq b_i \leq p-1$, one can determine with no *theoretical* difficulty whether or not the lens space $L(p; b_0, \dots, b_n)$ is framable. The interesting questions arise when one fixes n or p or both and asks whether a framable lens space exists for that n and p.

Proposition 2.1. *Let p be an odd prime and ε a primitive root* mod p. *If $a \mid (p-1)/2$ and $b = (p-1)/2a$, then the lens spaces*

$$L(p; \varepsilon^b, \varepsilon^{2b}, \dots, \varepsilon^{(a-1)b})$$

and

$$L(p; 1, \varepsilon^b, \varepsilon^{2b}, \dots, \varepsilon^{(2a-1)b}),$$

of dimension $2a-1$ and $4a-1$ respectively are framable.

Proof. Since ε^{2b} has order a mod p, one has the identities in $\mathbf{Z}/p[X]$:

$$\prod_{i=0}^{a-1} (X - (\varepsilon^{ib})^2) = X^a - 1$$

and

$$\prod_{i=0}^{2a-1} (X - \varepsilon^{ib})^2) = (X^a - 1)^2.$$

It follows that

$$\prod_{i=0}^{a-1} (1 + \varepsilon^{ib})^2 V^2) = 1 + (-1)^{a+1} V^{2a} \equiv 1 \text{ in } \mathbf{Z}/p[V^2]/(V^a)$$

and

$$\prod_{i=0}^{2a-1} (1 + (\varepsilon^{ib})^2 v^2) = 1 + (-1)^{a+1} 2V^{2a} + V^{4a}$$

$$\equiv 1 \text{ in } \mathbf{Z}/p[V^2]/(V^{2a})$$

and the result follows from Corollary 1.6. □

Remark. Taking $a = \frac{1}{2}(p-1)$ in the above $\{1, \varepsilon, \dots, \varepsilon^{2a-1}\} = \{1, 2, \dots, p-1\}$. Hence the lens space $L(p; 1, 2, \dots, p-1)$ of dimension $2p-3$ is framable.

Corollary 2.2. *For each integer n, there is a framable lens space $L(p; b_0, \dots, b_n)$ if*

$p \equiv 1 \bmod n+1$. *Thus there are framable lens spaces of dimension* $2n+1$ *for an infinite number of primes.*

Proof. Write $p-1 = M(n+1)$, take $a = n+1$ if $n+1$ is odd and $a = \frac{1}{2}(n+1)$ if $n+1$ is even. In the first case $2a-1 = 2n+1$, while in the second $4a-1 = 2n+1$. □

From Proposition 1.5 (2) we easily deduce the following nonexistence result.

Corollary 2.3. *Suppose* $p \equiv 1 \bmod 4$, $\frac{1}{2}(p-1) \leq n < p-1$, *and* n *is even. Then no lens space* $L(p; b_0, \ldots, b_n)$ *of dimension* $2n+1$ *can frame.*

Proof. By a suitable choice of generator of $\mathbf{Z}/p$ we may of course suppose $b_0 = 1$. By Proposition 1.5 (2) we then have for $j = \frac{1}{4}(p-1)$

$$1 + b_1^{(p-1)/2} + \cdots + b_n^{(p-1)/2} \equiv 0 \bmod p$$

if $L(p; 1, b_1, \ldots, b_n)$ frames. Since $b^{p-1} = 1$ for any $0 \neq b \in \mathbf{Z}/p$ we must have $b^{(p-1)/2} = \pm 1$ and therefore

$$1 \pm 1 \pm 1 \cdots \pm 1 \equiv 0 \bmod p,$$

which is impossible, as there are an odd number, $n+1$, of ± 1 in the sum and $n+1 < p$. □

3. A non elementary consequence

In this section we apply a lemma of Deligne [1], which follows from his proof of the Weil conjectures, to obtain the following result.

Theorem 3.1. *Let* n *be a positive integer, then for all sufficiently large primes* p *there exists a* $2n+1$ *dimensional lens space* $L(p; b_0, b_1, \ldots, b_n)$ *that frames.*

This theorem follows immediately from the following arithmetical results by applying Propsoition 1.5 (2).

Theorem 3.2. *Let* p *be a prime and denote by* N *the number of solutions to the system of congruences*

$$x_0^{2j} + x_1^{2j} + \cdots + x_n^{2j} \equiv 0 \bmod p, \qquad j = 1, \ldots, m$$

satisfying

$$x_0, x_1, \ldots, x_n \not\equiv 0 \bmod p.$$

Then there exists a positive constant A, *depending only on* n *and* m *such that*

$$|p^m N - (p-1)^{n+1}| \leq A(p^m - 1)p^{(n+1)/2}.$$

Corollary 3.3. *In the notations of Theorem 3.2 if $m \leq \frac{1}{2}n$ then $N > 0$ for sufficiently large p.*

Proof. We have

$$\lim_{p\to\infty}\left|\frac{p^m}{(p-1)^{n+1}}N-1\right| \leq \lim_{p\to\infty} A\frac{p^m-1}{(p-1)^{n+1}}p^{(n+1)/2}$$
$$\leq A\lim_{p\to\infty}\frac{(p^{n/2}-1)(p^{(n+1)/2})}{(p-1)^{n+1}}=0,$$

whence $\lim_{p\to\infty} N = \infty$. □

The proof of Theorem 3.2 requires two lemmas: the first is due to Deligne [1, 8.4–8.13], [6; 6.2] and the second is an elementary consequence of the first.

Lemma 3.4 (Deligne). *Let p be a prime and $g(x_0,\ldots,x_n)$ a polynomial with coefficients in $\mathbf{Z}/p$ of degree d satisfying*

(a) $(d,p)=1$,

(b) *the homogenous component of degree d of $g(x_0,\ldots,x_n)$ defines a non-singular hypersurface of projective n-space.*

Then

$$\left|\sum_{0\leq x_0,\ldots,x_n\leq p-1}\lambda^{g(x_0,\ldots,x_n)}\right| \leq (d-1)^{n+1}p^{(n+1)/2}$$

where $\lambda=\exp 2\pi i/p$. □

Lemma 3.5. *Let p be a prime and $g(x_0,\ldots,x_n)$ a polynomial with coefficients in $\mathbf{Z}/p$ of degree d satisfying*

(a) $(d,p)=1$,

(b) *the polynomial obtained from $g(x_0,\ldots,x_n)$ by setting some proper subset of the variables equal to zero is of degree d and defines a non-singular hypersurface of the appropriate projective space.*

Then there exists a constant A, depending only on d and n such that

$$\left|\sum_{1\leq x_0,\ldots,x_n\leq p-1}\lambda^{g(x_0,\ldots,x_n)}\right| \leq Ap^{(n+1)/2}.$$

(**N.B.** This time the sum ranges over those $n+1$ tuples satisfying $x_0,\ldots,x_n \neq 0$.)

Proof. By induction on n. If $n=0$ then

$$\sum_{0\leq x_0\leq p-1}\lambda^{g(x_0)}=\lambda^{g(0)}+\sum_{1\leq x_0\leq p-1}\lambda^{g(x_0)}$$

and hence by Lemma 3.4

$$\left|\sum_{1\le x_0\le p-1} \lambda^{g(x_0)}\right| \le \left|\sum_{0\le x_0\le p-1} \lambda^{g(x_0)}\right| + |\lambda^{g(0)}|$$
$$\le (d-1)p^{\frac{1}{2}} + 1 \le dp^{\frac{1}{2}}.$$

Assume the result when the number of variables is less than $n+1$ and consider a polynomial as in the hypothesis. Let $\mathcal{S}$ denote the set of all subsets of $\{x_0, \ldots, x_n\}$. For each $\pi \in \mathcal{S}$ let

$$B(\pi) = \sum_{\substack{0\le x_0,\ldots,x_n\le p-1\\ x_i=0\Leftrightarrow x_i\in\pi}} \lambda^{g(x_0,\ldots,x_n)}.$$

Then

$$\sum_{0\le x_0,\ldots,x_n\le p-1} \lambda^{g(x_0,\ldots,x_n)} = \sum_{\pi\in\mathcal{S}} B(\pi).$$

Clearly

$$B(\emptyset) = \sum_{1\le x_0,\ldots,x_n\le p-1} \lambda^{g(x_0,\ldots,x_n)}.$$

If card $\pi = k > 0$, then by the induction hypothesis there exists a constant C_π depending only on k and d such that

$$|B(\pi)| \le C_\pi p^{(n+1-k)/2} < C_\pi p^{(n+1)/2}.$$

Let $C = \max_{\pi\in\mathcal{S}, \pi=0} C_\pi$. Then

$$\left|\sum_{1\le x_0,\ldots,x_n\le p-1} \lambda^{g(x_0,\ldots,x_n)}\right| \le \left|\sum_{0\le x_0,\ldots,x_n\le p-1} \lambda^{g(x_0,\ldots,x_n)}\right|$$
$$+ \sum_{\substack{\pi\neq\emptyset\\ \pi\in\mathcal{S}}} |B(\pi)| \le (d-1)^{n+1} p^{(n+1)/2} + (2^k-1)Cp^{(n+1)/2}$$

completing the induction. □

Proof of Theorem 3.2. Let

$$f_j(x_0, \ldots, x_n) = x_0^{2j} + x_1^{2j} + \cdots + x_n^{2j}$$

and for a *fixed* $n+1$ tuple $(x_0, \ldots, x_n)$ and a *fixed* $j > 0$ notice that

$$\sum_{t_j=0}^{p-1} \lambda^{t_j f_j(x_0,\ldots,x_n)} = \begin{cases} 0: & \text{if } f_j(x_0,\ldots,x_n) \not\equiv 0 \bmod p \\ p: & \text{if } f_j(x_0,\ldots,x_n) \equiv 0 \bmod p \end{cases}$$

where as usual $\lambda = \exp 2\pi i/p$. Hence

$$\prod_{j=1}^{m} \sum_{t_j=0}^{p-1} \lambda^{t_j f_j(x_0,\ldots,x_n)} = \begin{cases} p^m: & \text{if } f_j(x_0,\ldots,x_n) \equiv 0(p) \\ & \quad \text{for all } j = 1, \ldots, m \\ 0: & \text{otherwise.} \end{cases}$$

Thus we have

$$p^m N = \sum_{1\le x_0,\dots,x_n\le p-1} \prod_{j=1}^{m} \sum_{t_j=0}^{p-1} \lambda^{t_j f_j(x_0,\dots,x_n)}$$

$$= \sum_{1\le x_0,\dots,x_n\le p-1} \left(\sum_{t_1,t_2,\dots,t_m=0}^{p-1} \lambda^{t_1 f_1(x_0,\dots,x_n)+\dots+t_m f_m(x_0,\dots,x_n)} \right).$$

Now the inner sum consists of p^m terms. The term corresponding to $t_1 = \dots = t_m = 0$ is simply 1. The $p^m - 1$ remaining terms are all of the form

$$\lambda^{g(x_0,\dots,x_n)}$$

where $g(x_0,\dots,x_n)$ is a polynomial satisfying the conditions of Lemma 3.4. Since

$$\sum_{1\le x_0,\dots,x_n\le p-1} 1 = (p-1)^{n+1}$$

we obtain

$$|p^m N - (p-1)^{n+1}| \le (p^m - 1)Ap^{(n+1)/2}$$

as required. □

4. Low dimensional examples

Noting that all 1 and 3 dimensional lens spaces are framable one first considers dimension 5.

Proposition 4.1. *There is a framable lens space $L(p; b_0, b_1, b_2)$ of dimension 5 iff $p \ne 5$.*

Proof. For $p = 5$ and $n = 2$ Corollary 2.3 shows that no framable $L(5; b_0, b_1, b_2)$ exists.

For $p \equiv 1 \bmod 4$, $p \ne 5$, -1 is a quadratic residue and letting $a^2 = -1 \bmod p$ one has

$$0 = 25 + 25a^2 = 25 + (3a) + (4a)^2$$

whence $L(p; 5, 3a, 4a)$ is framable.

For $p \equiv 3 \bmod 4$, -1 is not a quadratic residue $\bmod p$. Consider the set $S = \{u^2 + v^2 \mid u, v \in \mathbf{Z}/p^\times\}$. If $0 \in S$ then

$$0 = u^2 + v^2$$

gives

$$-1 = (v/u)^2$$

which is not possible, so $0 \notin S$. If S consists entirely of quadratic residues, then for any quadratic residue a^2, $a^2 + 1^2 \in S$ so $a^2 + 1$ is also a quadratic residue, which implies every element of $\mathbf{Z}/p^\times$ is a quadratic residue, which is impossible. Therefore S contains a non-quadratic residue, say $-w^2 = u^2 + v^2$, so $L(p; u, v, w)$ frames. □

Proposition 4.2. *There is a framable lens space $L(p; b_0, b_1, b_2, b_3)$ of dimension 7 iff $p \neq 3$.*

Proof. Since $n = 3$ we must have $p > 3$ by Lemma 1.3. For $p = 5$, $p \equiv 1 \bmod n + 1$ so a framable mod p lens space exists by Corollary 2.2. For $p > 5$, Proposition 4.1 shows that there are $u, v, w \neq \bmod p$ such that

$$0 = u^2 + v^2 + w^2$$

and so

$$0 = 25u^2 + 25v^2 + 25w^2$$
$$= (3u)^2 + (4u)^2 + (5v)^2 + (5w)^2$$

and so $L(p; 3u, 4u, 5v, 5w)$ frames. □

Note. We are indebted to Charles Giffen for the very useful observation that $3^2 + 4^2 = 5^2$ even mod p.

In dimension 9 the situation becomes a great deal more complicated. In order to have $L(p; b_0, b_1, b_2, b_3, b_4)$ framable one must have $p > 4$. For $p = 5$, $L^{p-1}(p) = L^4(5)$ is framable. By Corollary 2.2 there is a framable 9 dimensional lens space for $p \equiv 1 \bmod 5$.

Proposition 4.3. *If $p \equiv 1 \bmod 8$ and there is a $v \neq 0 \in \mathbf{Z}/p$ satisfying the conditions*
(a) *$v^2 + 1 \neq 0$ is a quadratic residue,*
(b) *$v^4 + v^2 + 1 \neq 0$ is a fourth power,*
then there is a framable lens space of dimension 9 for p.

Proof. Since $p \equiv 1 \bmod 8$ there is a ν with $\nu^4 \equiv -1 \bmod p$. Let

$$u^2 = v^2 + 1$$
$$w^4 = v^4 + v^2 + 1.$$

Then

$$(\nu w)^2 + (\nu^3 w)^2 + (\nu^2 u)^2 + v^2 + 1^2 = (\nu^2 w^2 - \nu^2 w^2 - u^2 + v^2 + 1 = 0$$

and

$$(\nu w)^4 + (\nu^3 w)^4 + (\nu^2 u)^4 + v^4 + 1^4 =$$
$$= -w^4 - w^4 + u^4 + v^4 + 1^4$$
$$= -2(v^4 + v^2 + 1) + (v^2 + 1)^2 + v^4 + 1$$
$$= 0$$

so $L(p; 1, \nu w, \nu^3 w, \nu^2 u, v)$ is framable. □

For example, taking $v = 1$, $v^2 + 1 = 2$ is a square, and $v^4 + v^2 + 1 = 3$. Thus if 3 is a biquadratic residue, there is a framable lens space. Gauss has characterized these primes [3]. For $p \equiv 1 \bmod 8$, p can be uniquely expressed in the form $p = a^2 + b^2$ with a odd and b even, and 3 is a biquadratic residue iff $b \equiv 0 \bmod 3$. The smallest such primes are 193, 313, 433, 577, 601, 673, 769, 937, 1201, 1297, 1321.

One can use other values of v also. For example when $p = 89$, $30^2 = 3^2 + 1^2$ and $5^4 = 3^4 + 3^2 + 1$.

Using a slightly sharper estimate than that of Lemma 3.4 [6; 6.1] one obtains that a framable 9 dimensional lens space exists for $p \cdot 73$. Ad hoc calculations verify that for primes $p \geq 5$ such examples exist except for

$$p = 7, 13, 17 \text{ or } 23,$$

where no such examples are possible.

In dimension 11 a framable lens space $L(p; b_0, \ldots, b_5)$ exists if $p \equiv 1 \bmod 6 = 5 + 1$. We also have

Proposition 4.4. *If $p \equiv 1 \bmod 4$, $p > 5$, then there is a framable 11 dimensional lens space for p.*

Proof. Consider first the case $p \equiv 1(8)$. Then there is a ν with $\nu^4 = -1$, so

$$3^2 + (3\nu)^2 + (4\nu)^2 + (5\nu^3)^2 + (5\nu^2)^2 =$$
$$= (3^2 + 4^2 - 5^2)(1 + \nu^2) \equiv 0 \bmod p$$

and

$$3^4 + (3\nu)^4 + 4^4 + (4\nu)^4 + (5\nu^3)^4 + 5\nu^2)^4 =$$
$$= 3^4 - 3^4 + 4^4 - 4^4 + 5^4 - 5^4 \equiv 0 \bmod p$$

so that $L(p; 3, 3\nu, 4, 4\nu, 5\nu^3, 5\nu^2)$ frames.

For $p \equiv 1 \bmod 4$ Gegenbauer [4] proves

$$x^4 + y^4 + z^4 \equiv 0 \bmod p$$

is solvable with $xyz \not\equiv 0 \bmod p$ if $p \not\equiv 5, 17, 29$ or 41. So for $p \equiv 5 \bmod 8$ $p \not\equiv 29$, $\exists u, v, w \in \mathbf{Z}/p^\times$ such that

$$u^4 + v^4 + w^4 \equiv 0 \bmod p.$$

Let $\eta^2 = -1 \bmod p$. Then one finds $L(p; 1, u, v, \eta, \eta u, \eta v)$ frames. For $p = 29$ $L(29; 1, 6, 11, 14, 14)$ frames. □

By Lemma 1.3 no framable lens space of dimension 11 can exist for $p = 2, 3$ or 5. Using the methods of Deligne one finds that a framable lens space of dimension 11 exists for all primes $p > 41$, which leaves only $p = 11, 23$ to complete the determination of which primes admit framable 11 dimensional lens spaces. For $p = 11$ none can exist and $L(23; 11, 5, 8, 8, 11)$ is framable.

5. Composite moduli

One can obviously, generalize the problem by letting $\mathbf{Z}/m$ act on S^{2n+1} by means of

$$T(z_0,\dots,z_n)=(\lambda^{b_0}z_0,\dots,\lambda^{b_n}z_n)$$

where $\lambda=\exp(2\pi i/m)$ and $(b_i,m)=1$ for each i. The resulting lens space will be denoted $L(m;b_0,b_1,\dots,b_n)$. Throughout, we write $m=p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$; $p_1<p_2<\cdots<p_s$, $r_i\geq 1$, in it's prime factorization.

Proposition 5.1. *$L(m;b_0,\dots,b_n)$ is framable if and only if*
a) $b_0^{2j}+b_1^{2j}+\cdots+b_n^{2j}\equiv 0\ (\mathrm{mod}\ m)$ *for* $1\leq j\leq[n/2]$ *and,*
b) *if* $p_1=2$, *then* $n=0,1$ *or* $n=3$ *and* $r_1=1$, *or if* $p_1>2$, *then* $n<p_1$.

Proof. If $L(m;b_0,\dots,b_n)$ is framable, then it's covering space $L(p_1;b_0,\dots,b_n)$ is also framable. Thus, if $p_1=2$, $n=0,1$ or 3, and if $p_1>2$, $n<p_1$. Since condition a) follows from the vanishing of the Pontrjagin classes, necessity is reduced to showing that $L(4;b_0,b_1,b_2,b_3)$ cannot be framed, since for $p_1=2$, $r_1>1$ it is a covering space.

Assuming m is even, framability requires $n=0,1$ or 3, and for $n=0$ or 1 all are framable. For $n=3$, the mod 2 cohomology of $L(m;b_0,\dots,b_3)$ is the same as that of the standard lens space, with Steenrod operations, and so $L=L(m;b_0,\dots,b_3)$ is a Spin manifold. Letting

$$\tau=L\to \mathrm{BSpin}$$

be a classifying map for the tangent bundle, the first and only obstruction to lifting to $\mathrm{BO}(8,\dots,\infty)$, hence framing L, is $\tau^*(x)$, where $x\in H^4(\mathrm{BSpin};\mathbf{Z})=\mathbf{Z}$ is a generator. Since the mod 2 reduction of the first Pontrjagin class $\mathcal{P}_1$ is w_2^2, which vanishes for Spin bundles, and since $\mathcal{P}_1$ generates $H^4(\mathrm{BSpin};\mathbf{Z}/p)$ for all odd p, it follows that $\pm 2^s x=\mathcal{P}_1$ for some $s>0$. Letting γ be the Hopf bundle over $\mathbf{CP}(\infty)$, one sees that $\gamma\oplus\gamma$ is a Spin bundle and that $\mathcal{P}_1(\gamma\oplus\gamma)=2\alpha^2$, $\alpha\in H^2(\mathbf{CP}(\infty);\mathbf{Z})$ the generator. Thus $\pm 2x=\mathcal{P}_1$.

Now, if $L(m;b_0,\dots,b_3)$ frames and $p_1=2$, $r_1>1$, so does $L(4;b_0,\dots,b_3)$ which is diffeomorphic to $L(4;1,1,1,1)=L$ which has tangent bundle 4γ (where γ is induced by the map into $\mathbf{CP}(3)$). Now 2γ is a Spin bundle and the obstruction $x(2\gamma)$ generates $H^4(L;\mathbf{Z})\cong\mathbf{Z}_n$ and by additivity of the obstruction, $\tau^*(x)=2x(2\gamma)$ is twice the generator, so is nonzero. Thus if m is even and $n=3$, $r_1=1$.

To prove sufficiency, if m is even, $n=3$, note that $b_0^2+\cdots+b_3^2\equiv 0 \bmod m$ is equivalent to vanishing of $\mathcal{P}_1$. Hence, since $r_1=1$, $\tau^*(x)\in H^4(L;\mathbf{Z})=\mathbf{Z}_m$ has order 2. Since $L(2;b_0,\dots,b_3)=\mathrm{RP}(7)$ frames this obstruction dies in $\mathrm{RP}(7)$, but $H^4(L;\mathbf{Z})\to H^1(\mathrm{RP}(7);\mathbf{Z})=\mathbf{Z}_2$ annihilates only the odd torsion. Thus $\tau^*(x)=0$ and L frames.

Finally, supposing m is odd and $n < p_1$, the arguments in Section 1 carry over with only trivial modification, completing the proof of sufficiency. □

As in the prime case, there is no further theoretical difficulty in determining whether a given lens space frames. For fixed n and m, the existence of a framing reduces immediately to the case $\mathbf{Z}/p$ for p on odd prime.

Proposition 5.2. *There is a framable lens space $L(m; b_0, \ldots, b_n)$ of dimension $2n+1$ for $\mathbf{Z}/m$ if and only if:*

a) *there is a framable lens space $L(p_i; b'_0, \ldots, b'_n)$ for $\mathbf{Z}/p_i$ for each odd p_i dividing m, and*

b) *if $p_1 = 2$, then $n = 0,1$ or $n = 3$ and $r_1 = 1$, or if $p_1 > 2$ then $n < p_1 - 1$ or $n = p_1 - 1$ and $r_1 = 1$.*

Proof. Since $m = p_1^{r_1} \cdots p_s^{r_s}$, solvability of the system $b_0^{2j} + \cdots + b_n^{2j} \equiv 0 \pmod m$ for $1 \leq j \leq [n/2]$ is equivalent to solvability of $b_0^{2j} + \cdots + b_n^{2j} \equiv 0 \pmod{p_i^{r_i}}$ for $1 \leq j \leq [n/2]$ for each i by an easy application of the Chinese Remainder Theorem.

For $p_i = 2$, there is no difficulty in solving the system, so one need only consider odd primes. Thus, one supposes that the system $b_0^{2j} + \cdots + b_n^{2j} \equiv 0 \bmod p$, $1 \leq j \leq [n/2]$, has a solution ($b_i \not\equiv 0 \bmod p$, of course) and seeks a solution $\bmod p^r$.

Suppose then that one has a system $b_0^{2j} + \cdots + b_n^{2j} \equiv 0$ (p^s), $s \geq 1$, $1 \leq j \leq [n/2]$ with $b_i \not\equiv 0$ (p), and consider the system of equations obtained by replacing $b_0, \ldots, b_{[n/2]-1}$ by $b_i + t_i p^s$

$$(b_0 + t_0 p^s)^{2j} + (b_1 + t_1 p^s)^{2j} + \cdots + (b_{[n/2]-1} + t_{[n/2]-1} p^s)^{2j} + b_{[n/2]}^{2j} + \cdots + b_n^{2j} \equiv$$
$$\equiv b_0^{2j} + \cdots + b_n^{2j} + 2jp^s[t_0 b_0^{2j-1} + \cdots + t_{[n/2]-1} b_{[n/2]-1}^{2j-1}] \qquad \bmod p^{s+1},$$

and one then wishes to solve the system

$$2j[t_0 b_0^{2j-1} + \cdots + t_{[n/2]-1} b_{[n/2]-1}^{2j-1}] \equiv -\frac{[b_0^{2j} + \cdots + b_n^{2j}]}{p^s}$$

$\bmod p$, $1 \leq j \leq [n/2]$. The coefficients of t_i form a Vandermonde determinant, and hence a solution can be found provided the b's are distinct. By reordering the b's, if needed, and using the obvious induction on s, we then see that a solution will exist provided the system $b_0^{2j} + \cdots + b_n^{2j} \equiv 0$ (p) has a solution with $[n/2]$ distinct b's.

If one supposes $b_0, \ldots, b_i$ are distinct, and a_j is the number of b's equal to b_j, one has

$$a_0 + a_1 + \cdots + a_i = n + 1$$
$$a_0 b_0^2 + \cdots + a_i b_i^2 \equiv 0\ (p)$$
$$\vdots$$
$$a_0 b_0^{2[n/2]} + \cdots + a_i b_i^{2[n/2]} \equiv 0\ (p)$$

and if $i \leq [n/2] - 1$, the first $i+1$ congruences have coefficients a nonsingular Vandermonde determinant, giving $a_j \equiv 0 \bmod p$ for each j. Since $0 \leq a_j \leq n+1 \leq p$, each a_j is either 0 or p. Thus one a is p and the rest are zero, so the b's are all equal and $n+1=p$.

This shows that if there is a framable lens space of dimension $2n+1$ for $\mathbf{Z}/p$ there is one for $\mathbf{Z}/p^s$ except when $n=p-1$, where none exists for $\mathbf{Z}/p^2$. For the latter, some verification is required, but since all the b's are congruent mod p (or more precisely the b^2's) one may assume $b_i = 1(p)$. The system

$$\sum_{i=0}^{p-1} (1+t_i p)^{2j} \equiv 0\,(p^2)$$

becomes

$$2pj\sum_{i=0}^{p-1} t_i + p \equiv 0\,(p^2)$$

so

$$2j\sum_{i=0}^{p-1} t_i = -1\,(p)$$

for each j, which has no solutions. □

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