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Journal of Algebra 264 (2003) 129–154

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

On commutative polarizations

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Received 20 December 2001

Communicated by Michel van den Bergh

Introduction

Let L be a finite-dimensional Lie algebra over a field k of characteristic zero and let $U(L)$ be its enveloping algebra with quotient division ring $D(L)$. Let P be a commutative Lie subalgebra of L. In [O2] the necessary and sufficient condition on P was given in order for $D(P)$ to be a maximal (commutative) subfield of $D(L)$. In particular, this condition is satisfied if P is a commutative polarization (CP) with respect to any regular $f \in L^*$ and the converse holds if L is ad-algebraic. The purpose of this paper is to study Lie algebras admitting these CP's and to demonstrate their widespread occurrence.

First we have the following characterisation if L is completely solvable: P is a CP of L if and only if there exists a descending chain of Lie subalgebras

$$L = L_n \supset \cdots \supset L_{j+1} \supset L_j \supset \cdots \supset L_p = P$$

such that $\dim L_j = j$ with increasing index, i.e., $i(L_j) = i(L_{j+1}) + 1$, j: $p, \ldots, n-1$ (Theorem 1.11). In low dimension this phenomenon appears frequently. In fact, in a case by case study of indecomposable nilpotent Lie algebras of dimension at most seven we discover that Lie algebras without CP's are rather exceptional: 1 (out of 9) in dimension at most 5; 3 (out of 22) in dimension 6 and 26 (out of 130) in dimension 7. These will be listed in Section 3, in which we also prove that non-abelian Lie algebras having a nondegenerate, invariant bilinear form do not admit any CP (Theorem 3.2).

Suppose k is algebraically closed. Then for a Lie algebra L to admit a CP P has the following advantage: in $U(L)$ the primitive ideals $I(f)$, with regular $f \in L^*$, can all be

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1 Partially supported by the CRDF grant RM1-2088.

doi:10.1016/S0021-8693(03)00146-7

constructed using the same polarization P, since $I(f)$ is the kernel of the (twisted) induced representation $\sigma = \operatorname{ind}^{\sim}(f|_P, L)$ [D, 10.3.4]. If in addition P is an ideal of L (a so called CP-ideal of L) then the representation σ is irreducible (in the completely solvable case P even turns out to be a Vergne polarization). Also, the semi-center $Sz(U(L))$ of $U(L)$ is contained in $U(P)$ (Corollary 4.4). Moreover, a standard technique using Grassmannians shows that if L is solvable with a CP, then it also has a CP-ideal (Theorem 4.1).

In Section 5, we look for CP-ideals in some Frobenius Lie algebras (i.e. Lie algebras of index zero [O1]). For instance, let $x \in L$ be a principal nilpotent element of a semi-simple Lie algebra L with centralizer P. Then the normalizer F of P is a Frobenius Lie algebra by a recent result of Panyushev [P2], in which P is a CP-ideal (Theorem 5.7). Next, let A be a finite dimensional associative algebra over k with a unit. A becomes a Lie algebra $\mathfrak{g}$ for the Lie bracket $[a,b] = ab - ba$, $a, b \in A$ and $V = A$ becomes a $\mathfrak{g}$-module by left multiplication. Consider the semi-direct product $L = \mathfrak{g} \oplus V$. Then the following are equivalent (Proposition 5.6):

(1) A is a Frobenius algebra.
(2) L is a Frobenius Lie algebra.
(3) V is a CP-ideal of L.
(4) $D(V)$ is a maximal subfield of $D(L)$.

A similar result can be obtained if A is a finite dimensional left symmetric algebra (Example 5.4) or if A is a finite dimensional simple Novikov algebra over k, $\operatorname{char}(k) = p > 2$.

CP-ideals also occur naturally in the nilradical N of any parabolic Lie subalgebra of a simple Lie algebra L of type A_r or C_r. As a bonus we obtain an explicit formula for the index $i(N)$ of N (Theorem 6.2).

Finally, Section 7 deals with some CP-preserving extensions.

1. Preliminaries and general results

Let L be a Lie algebra over a field k of characteristic zero with basis $x_1, \ldots, x_n$. Let $f \in L^*$ and consider the alternating bilinear form B_f on L sending (x, y) into $f([x, y])$. For any subset A of L we denote by $A^{\perp}$ or A^f the subspace

$$\{x \in L \mid f([x, a]) = 0 \text{ for all } a \in A\}.$$

We also put $L(f) = L^{\perp}$ and $i(L) = \min_{f \in L^*} \dim L(f)$, the index of L. Note that $L(f)$ is a Lie subalgebra of L containing the center $Z(L)$ of L. We recall from [D, 1.14.13] that

$$i(L) = \dim L - \operatorname{rank}_{R(L)}([x_i, x_j]),$$

where $R(L)$ is the quotient field of the symmetric algebra $S(L)$ of L. In particular, $\dim L - i(L)$ is an even number.

Furthermore, f is called regular if $\dim L(f) = i(L)$. It is well-known that the set L^*_{reg} of all regular elements of L^* is an open dense subset of L^* for the Zariski topology.

Definition 1.1 [D, 1.12.7]. A Lie subalgebra P of L is called a polarization w.r.t. $f \in L^*$ if $f([P, P]) = 0$ and $\dim P = \frac{1}{2}(\dim L + \dim L(f))$, in other words P is a maximal totally isotropic subspace of L (equipped with B_f). If in addition P is commutative then f is regular by the following observation.

Lemma 1.2 (see Theorem 14 of [O2]). *Let P be a commutative Lie subalgebra of L; $h_1, \dots, h_m$ a basis of P and $x_1, \dots, x_n$ a basis of L. Then the following conditions are equivalent*:

(a) $\dim P = \frac{1}{2}(\dim L + i(L))$, *i.e., P is a CP* (*commutative polarization*) *of L w.r.t. each $f \in L^*_{\mathrm{reg}}$*.
(b) $P = P^f$ *w.r.t. some $f \in L^*$* (*such an f is necessarily regular*).
(c) $\mathrm{rank}_{R(L)}([h_i, x_j]) = \dim L - \dim P$.

Lemma 1.3. *Let P and M be Lie subalgebras of L such that $P \subset M \subset L$. Then the following conditions are equivalent*:

(1) *P is a CP of L.*
(2) *P is a CP of M and $i(M) = i(L) + \dim L - \dim M$.*

Under these conditions the following hold:

$$f \in L^*_{\mathrm{reg}} \quad \Rightarrow \quad f|_M \in M^*_{\mathrm{reg}}.$$

Proof. (1) $\Rightarrow$ (2). Take any $f \in L^*_{\mathrm{reg}}$. Then $P = P^f$. Put $g = f|_M \in M^*$. W.r.t. B_g we have:

$$P^g = \{x \in M \mid g([x, P]) = 0\} = \{x \in M \mid f([x, P]) = 0\} = M \cap P^f = M \cap P = P.$$

Hence P is a CP of M and $g \in M^*_{\mathrm{reg}}$ by Lemma 1.2. In particular,

$$\tfrac{1}{2}\big(\dim M + i(M)\big) = \dim P = \tfrac{1}{2}\big(\dim L + i(L)\big).$$

Consequently, $i(M) = i(L) + \dim L - \dim M$.

(2) $\Rightarrow$ (1). P is commutative and

$$\begin{aligned} \dim P &= \tfrac{1}{2}\big(\dim M + i(M)\big) = \tfrac{1}{2}\big(\dim M + i(L) + \dim L - \dim M\big) \\ &= \tfrac{1}{2}\big(\dim L + i(L)\big). \end{aligned}$$

Hence, P is a CP of L. $\square$

The following is a direct application of [D, Lemma 1.12.2].

Lemma 1.4. *Let M be a Lie subalgebra of L of codimension one. Let $f \in L^*$ and put $g = f|_M \in M^*$. Then we distinguish two cases*:

(i) *If $L(f) \subset M$ then $L(f)$ is a hyperplane in $M(g)$.*
(ii) *If $L(f) \not\subset M$ then $M(g) = L(f) \cap M$ is a hyperplane in $L(f)$.*

Remark 1.5. In [O2] we introduced the notion of the Frobenius semiradical $F(L)$ of a Lie algebra L, namely

$$F(L) = \sum_{f \in L^*_{\mathrm{reg}}} L(f).$$

This is a characteristic ideal of L containing the center $Z(L)$ of L. It seems to play a natural role in the study of commutative polarizations. For instance if L admits a CP P, then $F(L) \subset P$ and hence is commutative [O2, p. 710].

Proposition 1.6. *Let M be a Lie subalgebra of L of codimension one, $f \in L^*$ and $g = f|_M \in M^*$. Then we have*:

(1) *either $i(M) = i(L) + 1$ or $i(M) = i(L) - 1$;*

(2) $$\begin{cases} f \in L^*_{\mathrm{reg}} \\ i(M) = i(L) + 1 \end{cases} \Leftrightarrow \begin{cases} g \in M^*_{\mathrm{reg}} \\ L(f) \subset M \end{cases};$$

(3) $$\begin{cases} f \in L^*_{\mathrm{reg}} \\ L(f) \not\subset M \end{cases} \Leftrightarrow \begin{cases} g \in M^*_{\mathrm{reg}} \\ i(M) = i(L) - 1 \end{cases};$$

(4) $i(M) = i(L) + 1 \Leftrightarrow F(L) \subset M$;

(5) *Suppose $i(M) = i(L) + 1$ and let P be a Lie subalgebra of M. Then*

$$P \text{ is a CP of } L \quad \Leftrightarrow \quad P \text{ is a CP of } M;$$

(6) *suppose $i(M) = i(L) - 1$. If H is a CP* (*respectively a CP-ideal*) *of L, then $H \cap M$ is a CP* (*resp. a CP-ideal*) *of M and* $\dim(H \cap M) = \dim H - 1$.

Proof. (1) Choose $\varphi \in L^*_{\mathrm{reg}}$ such that $\gamma = \varphi|_M \in M^*_{\mathrm{reg}}$. Suppose $L(\varphi) \subset M$ then

$$i(M) = \dim M(\gamma) = \dim L(\varphi) + 1 = i(L) + 1$$

by (i) of Lemma 1.4. On the other hand, if $L(\varphi) \not\subset M$ then

$$i(M) = \dim M(\gamma) = \dim L(\varphi) - 1 = i(L) - 1$$

by (ii) of Lemma 1.4.

(2) (⇒) Suppose $L(f) \not\subset M$. By (ii) of Lemma 1.4

$$i(M) \leqslant \dim M(g) = \dim L(f) - 1 = i(L) - 1.$$

Contradiction. Therefore $L(f) \subset M$. Hence,

$$i(M) - 1 = i(L) = \dim L(f) = \dim M(g) - 1$$

by (i) of Lemma 1.4. Hence $i(M) = \dim M(g)$, i.e., $g \in M^*_{\mathrm{reg}}$.

(⇐) By (i) of Lemma 1.4 $L(f) \subset M$ implies that

$$i(L) \leqslant \dim L(f) = \dim M(g) - 1 = i(M) - 1.$$

So, $i(M) \geqslant i(L) + 1$. By (1), $i(M) = i(L) + 1$ and therefore $i(L) = \dim L(f)$, i.e., $f \in L^*_{\mathrm{reg}}$.

(3) (⇒) $L(f) \not\subset M$ implies that

$$i(M) \leqslant \dim M(g) = \dim L(f) - 1 = i(L) - 1$$

by (ii) of Lemma 1.4. Hence, by (1), $i(M) = i(L) - 1$ which forces $i(M) = \dim M(g)$, i.e., $g \in M^*_{\mathrm{reg}}$.

(⇐) Since $i(M) \neq i(L) + 1$ it follows from (2) that $L(f) \not\subset M$. Hence,

$$i(L) - 1 = i(M) = \dim M(g) = \dim L(f) - 1$$

by (ii) of Lemma 1.4. Consequently, $\dim L(f) = i(L)$, i.e., $f \in L^*_{\mathrm{reg}}$.

(4) (⇒) Follows from (2).

(⇐) Choose $f \in L^*_{\mathrm{reg}}$ such that $g = f|_M \in M^*_{\mathrm{reg}}$. Then $L(f) \subset F(L) \subset M$. Using (2) it follows that $i(M) = i(L) + 1$.

(5) Clearly, $i(M) = i(L) + \dim L - \dim M$. Now use Lemma 1.3.

(6) Suppose $i(M) = i(L) - 1$. Hence, by Lemma 1.3 $H \not\subset M$. Then $\dim(H \cap M) = \dim H - 1$. $H \cap M$ is abelian and

$$\dim(H \cap M) = \tfrac{1}{2}\bigl(\dim L + i(L)\bigr) - 1 = \tfrac{1}{2}\bigl(\dim M + i(M)\bigr).$$

Consequently, $H \cap M$ is a CP (resp. a CP-ideal) of M. □

Examples 1.7.

(1) Let E be a nonzero endomorphism of an n-dimensional vector space V over k. Consider the Lie algebra $L = kE \oplus V$ with Lie brackets $[E, v] = Ev$ and in which V is a commutative ideal. L is solvable and $i(L) = n - 1$. Clearly, $i(V) = n = i(L) + 1$ and V is a CP-ideal of L by (5) of Proposition 1.6.

(2) Let L be a Frobenius Lie algebra (i.e., $i(L) = 0$) and M a Lie subalgebra of L of codimension one. Then $i(M) = 1$ $(= i(L) + 1)$.

(3) Let M be a Lie subalgebra of codimension one in a non-abelian Lie algebra L with $F(L) = L$. Then, $i(M) = i(L) - 1$ and L does not have any CP's (by Proposition 1.6

and Remark 1.5). For instance, let L be the diamond Lie algebra with basis t, x, y, z and nonvanishing brackets $[t, x] = -x[t, y] = y$ and $[x, y] = z$. Clearly, $i(L) = 2$ and $M = [L, L] = \langle x, y, z\rangle$ is an ideal of codimension one in L with $i(M) = 1$. Put $f = x^* \in L^*_{\mathrm{reg}}$ and $g = f|_M \in M^*$. Then, $L(f) = \langle y, z\rangle \subset M$, $i(M) = i(L) - 1$ and $g \notin M^*_{\mathrm{reg}}$. Also, $P_1 = \langle y, z\rangle$ is a CP of M. But there is no CP P of L such that $P \cap M = P_1$ (in fact L does not admit any CP since $F(L) = L$). See also Theorem 3.2 and (2) of Examples 3.3.

Definition 1.8. A Lie algebra L is called square integrable if $L(f) = Z(L)$ for some $f \in L^*$, i.e., $i(L) = \dim Z(L)$.

In the nilpotent case these Lie algebras are precisely the Lie algebras of simply connected Lie groups admitting square integrable representations [MW, pp. 450–453].

Proposition 1.9. *Let L be a Lie algebra having an element $u \in L$ such that its centralizer $M = C(u)$ has codimension one in L. Then we have*

(i) $i(M) = i(L) + 1$.
(ii) *L has a CP if and only if M has a CP.*
(iii) *If L is square integrable then so is M.*

Remark 1.10. Note that $C(u)$ is an ideal of codimension one of L if either u is a noncentral semi-invariant of L (i.e., for a suitable $\lambda \in L^*\backslash\{0\}$: $[x, u] = \lambda(x)u$, $x \in L$) or $[u, L]$ is a one dimensional subspace of the center $Z(L)$ (such an u always exists if L is nilpotent and $\dim Z(L) = 1 < \dim L$). In that situation, if L has a CP-ideal then the same holds for $C(u)$.

Proof of the proposition. (i) Take $x \in L\backslash C(u)$ and choose $f \in L^*$ such that $f|_M$ is regular and such that $f([x, u]) \neq 0$. Then $C(u) = u^f$ (since both have the same dimension and $C(u) \subset u^f$). Then $L(f) = L^f \subset u^f = M$. It follows by (2) of Proposition 1.6 that $i(M) = i(L) + 1$ and $f \in L^*_{\mathrm{reg}}$.

(ii) First, let P be a commutative Lie subalgebra of M. Then,

$$P \text{ is a CP of } L \quad \text{if and only if} \quad P \text{ is a CP of } M$$

by (5) of Proposition 1.6. Next, let P be a CP of L such that $P \not\subset M$. Then $\dim(P \cap M) = \dim P - 1$ and $u \notin P \cap M$ (otherwise $[u, P] = 0$ and thus $P \subset C(u) = M$).

Finally, $P_1 = (P \cap M) \oplus ku$ is a CP of M since it is commutative and

$$\dim P_1 = \dim P = \tfrac{1}{2}\bigl(\dim L + i(L)\bigr) = \tfrac{1}{2}\bigl(\dim M + i(M)\bigr).$$

(iii) Clearly, $Z(L) \subset C(u) = M$ and $u \in Z(M)\backslash Z(L)$. Hence, $Z(L) \oplus ku \subset Z(M)$. Therefore,

$$i(M) \geqslant \dim Z(M) \geqslant \dim Z(L) + 1 = i(L) + 1.$$

As $i(M) = i(L) + 1$ we may conclude that $i(M) = \dim Z(M)$, i.e. M is square integrable. $\square$

Theorem 1.11. *Let P be a commutative Lie subalgebra of a completely solvable Lie algebra L. Then the following conditions are equivalent*:

(1) *P is a CP* (*resp. CP-ideal*) *of L.*
(2) *There exists a descending series of Lie subalgebras* (*resp. ideals*) *of L.*

$$L = L_n \supset \cdots \supset L_{j+1} \supset L_j \supset \cdots \supset L_p = P$$

$\dim L_j = j$, *with increasing index* (*i.e.*, $i(L_j) = i(L_{j+1}) + 1$).

Proof. Let P be a Lie subalgebra (resp. ideal) of L. P (resp. L) acts on the quotient space L/P. Application of Lie's theorem to this action shows the existence of Lie subalgebras (resp. ideals) L_j of L such that $L = L_n \supset \cdots \supset L_p = P$ with $\dim L_j = j$.

(1) $\Rightarrow$ (2). Now suppose P is a CP of L. Then, by Lemma 1.3, P is also a CP for each L_j and

$$i(L_j) = i(L) + (n - j) = i(L) + \big(n - (j+1)\big) + 1 = i(L_{j+1}) + 1.$$

(2) $\Rightarrow$ (1) By induction on j we show that P is a CP of L_j. This is trivial for $j = p$. Next, let $j \geqslant p + 1$. Then P is a CP of L_{j-1} and also of L_j since $i(L_{j-1}) = i(L_j) + 1$ by (5) of Proposition 1.6. $\square$

Corollary 1.12. *Let L be a completely solvable Frobenius Lie algebra of dimension* $2n$ *having a CP P. Then L can be obtained from the n-dimensional abelian Lie algebra P with n successive extensions as described in Theorem* 1.11.

Lemma 1.13. *Let P be a CP* (*resp. a CP-ideal*) *of a Lie algebra L, A an ideal of L contained in P and $f \in L^*_{\mathrm{reg}}$ such that $f(A) = 0$. Then P/A is a CP* (*resp. a CP-ideal*) *of the Lie algebra L/A and*

$$i(L/A) = i(L) - \dim A.$$

Proof. Let $\varphi : L \to L/A$ be the quotient homomorphism. As $f(A) = 0$ there is a $g \in (L/A)^*$ such that $g \circ \varphi = f$. Clearly, P/A is an abelian Lie subalgebra (resp. ideal) of L/A. It suffices to show that $(P/A)^g = P/A$.

$$\begin{aligned}(P/A)^g &= \big\{\varphi(x) \in L/A \mid g\big([\varphi(x), \varphi(P)]\big) = 0, x \in L\big\} \\ &= \varphi\big(\big\{x \in L \mid f\big([x, P]\big) = 0\big\}\big) = \varphi\big(P^f\big) = \varphi(P) = P/A\end{aligned}$$

as $P^f = P$. So, by Lemma 1.2 P/A is a CP (resp. CP-ideal) of L/A and $g \in (L/A)^*_{\mathrm{reg}}$. Therefore, $\dim P/A = \frac{1}{2}(\dim L/A + i(L/A))$ and

$$\begin{aligned}i(L/A) &= 2\dim P/A - \dim L/A = 2(\dim P - \dim A) - (\dim L - \dim A) \\ &= (2\dim P - \dim L) - \dim A = i(L) - \dim A. \quad \square\end{aligned}$$

2. CP's in square integrable nilpotent Lie algebras

The following lemma is easy to verify.

Lemma 2.1. *Suppose L is a direct product of Lie algebras*; $L = L_1 \times L_2$. *Then we have the following*:

(1) $i(L) = i(L_1) + i(L_2)$ *and* $Z(L) = Z(L_1) \times Z(L_2)$.
(2) L *is square integrable if and only if the same holds for* L_1 *and* L_2.
(3) L *has a CP* (*resp. CP-ideal*) *if and only if the same holds for* L_1 *and* L_2.

Proposition 2.2. *Let L be a square integrable nilpotent Lie algebra over $\mathbb{C}$, of dimension n at most seven. Then L admits a CP-ideal.*

Proof. By Lemma 2.1 we may assume that L is indecomposable. In particular, $1 \leqslant \dim Z(L) = i(L) < \dim L$.

We now distinguish the following cases:

(1) $i(L) = 1$. Then n is 3, 5, or 7. Let m be the maximum dimension of all abelian ideals of L. Then by [Mo, p. 161] and [O2, p. 706] we have the following inequalities:

$$\tfrac{1}{2}\big(\sqrt{8n+1} - 1\big) \leqslant m \leqslant \tfrac{1}{2}\big(\dim L + i(L)\big) = \tfrac{1}{2}(n+1).$$

This implies that $m = \frac{1}{2}(\dim L + i(L))$ in case $n = 3, 5$, or 7, showing the existence of a CP-ideal in L.

(2) $i(L) = 2$. Then $n = 6$ (the case $n = 4$ does not occur since L is indecomposable). We select from Morozov's classification of 6-dimensional nilpotent Lie algebras those that are indecomposable, square integrable and of index 2; in each $\{e_1, \ldots, e_6\}$ is a basis of L. The numbering is Morozov's [Mo, p. 168].

4. $[e_1, e_2] = e_5, \quad [e_1, e_3] = e_6, \quad [e_2, e_4] = e_6,$
5. $[e_1, e_3] = e_5, \quad [e_1, e_4] = e_6, \quad [e_2, e_4] = e_5, \quad [e_2, e_3] = \gamma e_6, \quad \gamma \neq 0,$
6. $[e_1, e_2] = e_6, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = e_5, \quad [e_2, e_3] = e_5,$
7. $[e_1, e_3] = e_4, \quad [e_1, e_4] = e_5, \quad [e_2, e_3] = e_6,$
8. $[e_1, e_2] = e_3 + e_5, \quad [e_1, e_3] = e_4, \quad [e_2, e_5] = e_6,$
9. $[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_1, e_5] = e_6, \quad [e_2, e_3] = e_6,$
10. $[e_1, e_2] = e_3, \quad [e_1, e_3] = e_5, \quad [e_1, e_4] = e_6, \quad [e_2, e_4] = e_5,$
    $[e_2, e_3] = \gamma e_6, \quad \gamma \neq 0,$
11. $[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = e_5, \quad [e_2, e_3] = e_6.$

In each one of these, $P = \langle e_3, e_4, e_5, e_6 \rangle$ is a CP-ideal, since P is an abelian ideal and $\dim P = 4 = \frac{1}{2}(\dim L + i(L))$.

(3) $i(L) = 3$. Then $n = 7$ (the case $n = 5$ does not occur since L is indecomposable).

We have the following possibilities according to Seeley's classification of 7-dimensional nilpotent Lie algebras. We maintain the same notation as in [See]. In particular $\{a, b, c, d, e, f, g\}$ is a basis of L. In each case we exhibit a commutative ideal P of dimension 5 $(= \frac{1}{2}(\dim L + i(L)))$.

In the following 3 Lie algebras we take $P = \langle a, d, e, f, g\rangle$.

$$3\,7_B:\quad [a,b]=e,\quad [b,c]=f,\quad [c,d]=g,$$
$$3\,7_C:\quad [a,b]=e,\quad [b,c]=f,\quad [c,d]=e,\quad [b,d]=g,$$
$$3\,7_D:\quad [a,b]=e,\quad [b,d]=g,\quad [c,d]=e,\quad [a,c]=f.$$

In the following 3 we take $P = \langle c, d, e, f, g\rangle$.

$$3,5,7_A:\quad [a,b]=c,\quad [a,c]=e,\quad [a,d]=g,\quad [b,d]=f,$$
$$3,5,7_B:\quad [a,b]=c,\quad [a,c]=e,\quad [a,d]=g,\quad [b,c]=f,$$
$$3,5,7_C:\quad [a,b]=c,\quad [a,c]=e,\quad [a,d]=g,\quad [b,c]=f,\quad [b,d]=e.\quad \square$$

Remark 2.3. Among the Lie algebras described in Proposition 2.2 there is one which is characteristically nilpotent, namely $1, 2, 4, 5, 7_N$ with basis $\{a, b, c, d, e, f, g\}$ and nonzero brackets: $[a,b]=c$, $[a,c]=d$, $[a,d]=g$, $[a,e]=f$, $[a,f]=g$, $[b,c]=e$, $[b,d]=f$, $[b,e]=\xi g$, $[b,f]=g$, $[c,d]=g$, $[c,e]=-g$ with $\xi \neq 0, 1$ [See, p. 493]. In this case take $P = \langle d, e, f, g\rangle$.

3. Lie algebras without CP'S

First we want to show that the restriction on the dimension in Proposition 2.2 cannot be removed.

Examples 3.1.

(i) Let L be the 8-dimensional Lie algebra over k with basis $\{e_1, \dots, e_8\}$ and nonvanishing brackets: $[e_1,e_2]=e_5$, $[e_1,e_3]=e_6$, $[e_1,e_4]=e_7$, $[e_1,e_5]=-e_8$, $[e_2,e_3]=e_8$, $[e_2,e_4]=e_6$, $[e_2,e_6]=-e_7$, $[e_3,e_4]=-e_5$, $[e_3,e_5]=-e_7$, $[e_4,e_6]=-e_8$.
L is characteristically nilpotent [DL]. L is also square integrable of index 2, but it does not admit a CP-ideal (and not any CP's either, see Section 4).

Proof. Suppose L has a CP-ideal P. So, P is a 5-dimensional abelian ideal of L. Now take the linear functional $f = e_7^* \in L^*$, which is regular. Put $A = ke_8 \subset Z(L)$. This is a 1-dimensional ideal of L contained in P and $f(A) = 0$. By Lemma 1.13 $Q = P/A$ is a CP-ideal of L/A. Clearly, L/A is a 7-dimensional nilpotent Lie algebra of index 1, with basis $x_1 = e_1 + A, \dots, x_7 = e_7 + A$. So, Q is a 4-dimensional abelian ideal of L/A. One verifies that there are $\lambda, \mu \in k$, not both zero such that Q is generated by $\lambda x_1 + \mu x_4, x_5, x_6, x_7$. Then P is generated by $\lambda e_1 + \mu e_4, e_5, e_6, e_7, e_8$. But this contradicts the fact that P is commutative, since

$$[\lambda e_1 + \mu e_4, e_5] = -\lambda e_8 \quad \text{and} \quad [\lambda e_1 + \mu e_4, e_6] = -\mu e_8.$$

(ii) Let V be a vector space over k with basis $e_1, \ldots, e_n$; $n \geqslant 2$. Take the vector space $\bigwedge^2 V$ with basis $e_{ij} = e_i \wedge e_j$, $i < j$. Next, consider the Lie algebra

$$L = V \oplus \bigwedge^2 V$$

with nonvanishing brackets $[e_i, e_j] = e_{ij}$, $i < j$. Clearly, $[L, L] = \bigwedge^2 V = Z(L)$. So, L is 2-step nilpotent of dimension $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$. Let $x, y \in V$. Then it is easy to see that

$$[x, y] = 0 \quad \Leftrightarrow \quad x, y \text{ are linearly dependent over } k. \tag{$*$}$$

Next, we take n to be even. Then, $\operatorname{rank}_{R(L)}([e_i, e_j]) = n$. This implies that

$$i(L) = \dim L - n = \tfrac{1}{2}n(n-1) = \dim Z(L),$$

i.e., L is square integrable.

Finally, take $n = 4$. Then $\dim L = 10$, $\dim Z(L) = i(L) = 6$ and $\frac{1}{2}(\dim L + i(L)) = 8$. But, because of $(*)$, L has no 8-dimensional abelian Lie subalgebra containing $Z(L)$, i.e., L has no CP's. The same holds for all even $n \geqslant 4$, using a similar argument.

Theorem 3.2. *Let L be a Lie algebra having a nondegenerate, invariant bilinear form b. Then $F(L) = L$. In particular, L does not admit a CP unless L is abelian.*

Proof. Take $y \in L$ and consider the map φ_y sending each $x \in L$ into $b(x, y)$. Clearly, $\varphi_y \in L^*$ and the map $\varphi : L \to L^*$ sending y into φ_y is an isomorphism of L-modules. Consequently, y and φ_y have the same stabilizer in L, i.e., $C(y) = L(\varphi_y)$.

Next, put $\Omega = \varphi^{-1}(L^*_{\text{reg}})$. Then,

$$F(L) = \sum_{f \in L^*_{\text{reg}}} L(f) = \sum_{y \in \Omega} L(\varphi_y) = \sum_{y \in \Omega} C(y)$$

Clearly, $F(L)$ contains Ω, which is an open dense subset of L for the Zariski topology since φ is a linear isomorphism. Consequently, $F(L) = L$. $\square$

Examples 3.3.

(1) L semi-simple (take b to be the Killing form of L).

(2) The diamond Lie algebra with basis t, x, y, z and nonvanishing brackets $[t, x] = -x$, $[t, y] = y$ and $[x, y] = z$. Let b be the symmetric bilinear form with nonzero entries $b(t, z) = 1$ and $b(x, y) = -1$.
(3) Let $\mathfrak{g}_5$ be the 5-dimensional nilpotent Lie algebra over k with basis $x_1, \dots, x_5$ and nonvanishing brackets $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$, $[x_2, x_3] = x_5$.
Let b be the symmetric bilinear form with nonzero entries:

$$b(x_1, x_5) = b(x_3, x_3) = 1 \quad \text{and} \quad b(x_2, x_4) = -1.$$

(4) Let $\mathfrak{g}_6$ be the 6-dimensional 2-step nilpotent Lie algebra with basis $x_1, \dots, x_6$ and nonvanishing brackets $[x_1, x_2] = x_6$, $[x_1, x_3] = x_4$, $[x_2, x_3] = x_5$.
Let b be the symmetric bilinear form with nonzero entries:

$$b(x_1, x_5) = b(x_3, x_6) = 1 \quad \text{and} \quad b(x_2, x_4) = -1$$

(see [B1, p. 133]).
(5) Consider the semi-direct product $L = sl(2, k) \oplus W_2$, where W_2 is the 3-dimensional irreducible $sl(2, k)$-module. L also admits a nondegenerate, invariant, symmetric bilinear form.

Proposition 3.4. *Among all the different types of indecomposable nilpotent Lie algebras over $\mathbb{C}$ of dimension $n \leqslant 7$, only the following* 30 *Lie algebras do not have a CP*:

(1) $n = 5$: $\mathfrak{g}_5$ (*see* (3) *of Examples* 3.3).
2) $n = 6$: *From Morozov's classification* [Mo, p. 168] *the Lie algebras* $3 (\cong \mathfrak{g}_6)$, 21 *and* 22.
(3) $n = 7$: *From Seeley's classification* [See]: 2, 5, 7_K; 2, 5, 7_L; 2, 4, 7_D; 2, 4, 7_E; 2, 4, 7_G; 2, 4, 7_H; 2, 4, 7_J; 2, 4, 7_K; 2, 4, 7_Q; 2, 4, 7_R; 2, 3, 5, 7_C; 2, 3, 5, 7_D; 2, 3, 4, 5, 7_B; 2, 3, 4, 5, 7_C; 2, 3, 4, 5, 7_D; 2, 3, 4, 5, 7_F; 2, 3, 4, 5, 7_G; 1, 3, 5, 7_S, $\xi = 1$; 1, 3, 4, 5, 7_H; 1, 2, 4, 5, 7_C; 1, 2, 4, 5, 7_F; 1, 2, 4, 5, 7_H; 1, 2, 4, 5, 7_K; 1, 2, 4, 5, 7_L; 1, 2, 4, 5, 7_N, $\xi = 1$; 1, 2, 3, 4, 5, 7_I, $\xi = 0$.
Note that the infinite families fail to have a CP only for exceptional values of the parameter ξ.

Proof. This is done case by case, considering only the ones that are not square integrable (Proposition 2.2). Usually, CP's are easy to spot by looking at the multiplication table. To prove that a Lie algebra L has no CP's is more difficult however. This can be achieved by using Proposition 1.9 or by showing that $F(L)$ is not commutative. For instance, take $L = 1, 2, 4, 5, 7_N$, $\xi = 1$. See Remark 2.3 for its Lie brackets. One verifies that $F(L) = \langle a - b, c, d, e, f, g \rangle$, which is not commutative. □

Remark 3.5. Having a CP is not preserved under degeneration (for a definition we refer to [GO1] or [GO2]). Indeed, g_5, which has no CP's (see 3 of Examples 3.3), is a degeneration of the Lie algebra h_5 with basis $x_1, \dots, x_5$ over $\mathbb{C}$ and nonzero brackets $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$, $[x_1, x_4] = x_5$ and $[x_2, x_3] = x_5$ for which $\langle x_3, x_4, x_5 \rangle$ is a CP. On the other hand, the Lie algebra j_5 with the same basis and nonzero brackets $[x_1, x_2] = x_3$ and $[x_1, x_3] = x_4$ admits a CP (namely $\langle x_2, x_3, x_4, x_5\rangle$) and is a degeneration of $\mathfrak{g}_5$ [GO1, p. 323].

4. CP-ideals

These are by far the most interesting CP's. The following shows that they occur as often as ordinary CP's, at least in the solvable case.

Theorem 4.1. *Let L be solvable and k algebraically closed. Let m be the maximum dimension of all abelian ideals of L. Clearly, $m \leqslant \frac{1}{2}(\dim L + i(L))$* [O2, p. 706]. *Then the following are equivalent*:

(1) *L admits a CP.*
(2) *L admits a CP-ideal.*
(3) $m = \frac{1}{2}(\dim L + i(L))$.

Proof. It suffices to show that (1) $\Rightarrow$ (2), since (2) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (3) are clear. Let G be the adjoint algebraic group of L, i.e., the smallest algebraic subgroup of $\operatorname{Aut} L$ such that $L(G)$ contains $\operatorname{ad} L$ [D, 1.1.14]. Clearly, $\operatorname{ad} L$ and hence its algebraic hull $L(G)$ are solvable (since they have the same derived algebra [Ch, p. 173], which is nilpotent). Therefore G is a solvable connected group. Next put $p = \frac{1}{2}(\dim L + i(L))$. Then the set C of all CP's is a nonempty (by assumption) closed subset of the Grassmannian $\operatorname{Gr}(L, p)$, which is an irreducible and complete algebraic variety [D, 1.11.8-9]. Hence C is also complete. Now G acts morphically on C, mapping each CP H on $g(H)$, $g \in G$. By Borel's theorem, G has a fixed point P in C [Bo, p. 242]. So, $g(P) = P$ for all $g \in G$. In particular, $\operatorname{ad} x(P) \subset P$ for all $x \in L$. Consequently, P is a CP-ideal of L. $\square$

Remark 4.2. (a) The number m is an important characteristic of a Lie algebra, often used in classifications.

(b) It is now easy to see that the 8-dimensional Lie algebra (i) of 3.1 has no CP's (go over to the algebraic closure of k and use Theorem 4.1).

Theorem 4.3. *Let P be an ideal of a Lie algebra L and let P be a polarization of L with respect to some $f \in L^*$. Then we have*

(1) *If $f \in L^*_{\mathrm{reg}}$ then P is solvable* (*in fact $P'' = 0$*). *If in addition L is Frobenius or nilpotent of index one, then P is a CP-ideal of L.*
(2) *If k is algebraically closed and $f \in L^*_{\mathrm{reg}}$, then the induced representation* $\operatorname{ind}(f|_P, L)$ *is simple.*
(3) *If L is completely solvable then P is a Vergne polarization. In particular,* $\operatorname{ind}(f|_P, L)$ *is absolutely simple.*

Proof.

(1) Take $x \in L$ and $y, y' \in P$, then

$$f\big([x,[y,y']]\big) = f\big([[x,y],y']\big) + f\big([y,[x,y']]\big) = 0 \quad \text{since } P \text{ is an ideal}$$

and $f([P,P]) = 0$.

Hence, $[y,y'] \in L(f)$. Therefore, $P' = [P,P] \subset L(f)$. This implies that $P'' = 0$ since $L(f)$ is abelian by [D, 1.11.7]. Now, suppose L is Frobenius, i.e., $i(L) = 0$. Then $L(f) = 0$ which forces $[P,P] = 0$. On the other hand, if L is nilpotent of index 1, then $\dim L(f) = 1$. We may assume that $f \neq 0$. Clearly, $[P,P] \neq L(f)$ since $f([P,P]) = 0$ and $f(L(f)) \neq 0$ [BC, p. 89]. So, we conclude that $[P,P] = 0$.

(2) By [RV, p. 395] or [D, 10.5.7] there exists a solvable polarization H of L w.r.t. f such that $H \cap P$ is a solvable polarization of P w.r.t. $f|_P$ and such that the twisted induced representation $\mathrm{ind}^\sim(f|_H, L)$ is simple. First we observe that

$$\dim H = \tfrac{1}{2}\big(\dim L + \dim L(f)\big) = \dim P. \qquad (\bullet)$$

Similarly,

$$\dim(H \cap P) = \tfrac{1}{2}\big(\dim P + \dim P(f|_P)\big) = \dim P$$

since $P(f|_P) = \{x \in P \mid f([x,P]) = 0\} = P$. It follows that $H \cap P = P$, i.e., $P \subset H$. Hence, by $(\bullet)$, we see that $P = H$.

Consequently, $\mathrm{ind}^\sim(f|_P, L)$ is simple. Finally, $\mathrm{ind}^\sim(f|_P, L) = \mathrm{ind}(f|_P, L)$ because P is an ideal of L [D, 5.2.1].

(3) L being completely solvable, we can find a flag of ideals of L:

$$L = L_n \supset \cdots \supset L_p \supset \cdots \supset L_1 \supset L_0 = (0)$$

such that $L_p = P$ where $p = \dim P$. Put $f_i = f|_{L_i}$ and $P_j = \sum_{i \leqslant j} L_i(f_i)$. Then P_n is the so called Vergne polarization w.r.t. this flag and $f \in L^*$ [BGR, 9.4]. We claim that $P = P_n$. Clearly,

$$L_i(f_i) = \big\{x \in L_i \mid f\big([x,L_i]\big) = 0\big\} = L_i \cap L_i^{\perp}.$$

In particular, $L_p(f_p) = L_p \cap L_p^{\perp} = P \cap P^{\perp} = P$ since $P = P^{\perp}$ w.r.t. $f \in L^*$. This implies that $P \subset P_n$. On the other hand consider $L_j(f_j)$. If $j \leqslant p$, then $L_j(f_j) \subset L_j \subset L_p = P$. If $j > p$, then $P = L_p \subset L_j$ implies that $L_j(f_j) = L_j \cap L_j^{\perp} \subset L_j^{\perp} \subset P^{\perp} = P$. Consequently, $P_n = \sum_{j=1}^{n} L_j(f_j) \subset P$. □

Corollary 4.4. *Let P be a CP-ideal of a Lie algebra L and take any $f \in L^*_{\mathrm{reg}}$. Then,*

(1) *If k is algebraically closed, then* $\mathrm{ind}(f|_P, L)$ *is simple.*
(2) *If L is completely solvable, then P is a Vergne polarization w.r.t. f and any flag of ideals containing P. In particular,* $\mathrm{ind}(f|_P, L)$ *is absolutely simple.*
(3) (a) *$Sz(U(L)) \subset U(P)$ and $Sz(D(L)) \subset D(P)$ where $Sz(U(L)) = \bigoplus_\lambda U(L)_\lambda$ is the semi-center of $U(L)$. Similarly for $Sz(D(L))$. This generalizes* [D, 6.1.6].
 (b) *Put $\wedge(L) = \{\lambda \in L^* \mid U(L)_\lambda \neq 0\}$ and $L_\wedge = \bigcap_{\lambda \in \wedge(L)} \ker\lambda$. Then, $P \subset L_\wedge$.*

Proof. (1) and (2) follow directly from Theorem 4.3.

(3) Let $u \in U(L)_\lambda$ be any semi-invariant with weight $\lambda \in \wedge(L)$, i.e., $[x, u] = \lambda(x)u$ for all $x \in L$.

Now, take $x \in P$. Then $adx(L) \subset P$ and $(adx)^2 = 0$ since P is a commutative ideal of L. So, adx is nilpotent. This implies that $\lambda(x) = 0$ and $[x, u] = 0$. Consequently, $x \in L_\wedge$ which shows (b) and also $u \in C(U(P)) = U(P)$. Therefore, $Sz(U(L)) \subset U(P)$. Similarly for $Sz(D(L)) \subset D(P)$ (since $C(D(P)) = D(P)$). □

Remark 4.5. The previous corollary does not hold for arbitrary CP's of L. For example, let L be the 2-dimensional Lie algebra over an algebraically closed field k with basis x, y and nonzero bracket $[x, y] = y$. L is Frobenius and $f \in L^*$ with $f(x) = 0$ and $f(y) = 1$ is regular. Clearly, $P = kx$ is a CP of L w.r.t. $f \in L^*$. But $\operatorname{ind}^{\sim}(f|_P, L)$ is not simple [BGR, p. 95]. Also, y is a semi-invariant of L but $y \notin U(P)$.

The following, which we recall from [O2, p. 708], describes how CP-ideals naturally arise in certain semi-direct products.

Proposition 4.6. *Let $\mathfrak{g}$ be a Lie algebra with basis $\{x_1, \dots, x_m\}$ and let V be a $\mathfrak{g}$-module with basis $\{v_1, \dots, v_n\}$ with $\dim \mathfrak{g} \leqslant \dim V$. For each $f \in V^*$ we put*

$$\mathfrak{g}(f) = \bigl\{x \in \mathfrak{g} \mid f(xv) = 0 \text{ for all } v \in V\bigr\}$$

the stabilizer of f. Consider the semi-direct product $L = \mathfrak{g} \oplus V$ in which $[x, v] = xv$, $x \in \mathfrak{g}$, $v \in V$ and in which V is an abelian ideal. Then the following are equivalent:

(1) *$D(V)$ $(= R(V))$ is a maximal subfield of $D(L)$.*
(2) *V is a CP-ideal of L.*
(3) $i(L) = \dim V - \dim \mathfrak{g}$.
(4) $\operatorname{rank}_{R(V)}(e_i v_j) = \dim \mathfrak{g}$.
(5) *$\mathfrak{g}(f) = 0$ for some $f \in V^*$.*

Remark 4.7. If k is algebraically closed, $\mathfrak{g}$ a simple Lie algebra, acting irreducibly on V, then the conditions of the proposition are satisfied if and only if $\dim \mathfrak{g} < \dim V$ [AVE, p. 196].

The following shows that if a Lie algebra L admits a CP-ideal then its structure comes close to that of the semi-direct product considered in Proposition 4.6.

Corollary 4.8. *Let V be a commutative ideal of L. Clearly, the Lie algebra $\mathfrak{g} = L/V$ acts on V. Consider the semi-direct product $L_1 = \mathfrak{g} \oplus V$. Then,*

$$V \text{ is a CP of } L \quad \Leftrightarrow \quad V \text{ is a CP of } L_1.$$

In that case, $i(L_1) = i(L)$.

Proof. Let $g \in L^*$ and put $f = g|_V \in V^*$. Then, we claim that $\mathfrak{g}(f) = V^g/V$. Indeed,

$$\begin{aligned}
\bar{x} = x + V \in \mathfrak{g}(f) \quad &\Leftrightarrow \quad f([\bar{x}, V]) = 0 \\
&\Leftrightarrow \quad f([x, V]) = 0 \\
&\Leftrightarrow \quad g([x, V]) = 0 \\
&\Leftrightarrow \quad x \in V^g \quad \Leftrightarrow \quad \bar{x} \in V^g/V.
\end{aligned}$$

We now proceed with the proof

($\Rightarrow$) $\dim V = \frac{1}{2}(\dim L + i(L))$. Also, $V^g = V$ for some $g \in L^*$ by Lemma 1.2. Hence, $\mathfrak{g}(f) = 0$. By Proposition 4.6 V is a CP of L_1 and

$$i(L_1) = \dim V - \dim \mathfrak{g} = \dim V - (\dim L - \dim V) = 2\dim V - \dim L = i(L).$$

($\Leftarrow$) By Proposition 4.6, $\mathfrak{g}(f) = 0$ for some $f \in V^*$. Next, choose $g \in L^*$ such that $f = g|_V$. Then, $V^g/V = \mathfrak{g}(f) = 0$. So, $V^g = V$ which by Lemma 1.2 implies that V is a CP of L. □

5. CP-ideals in certain Frobenius Lie algebras

Let L be a Frobenius Lie algebra with a CP-ideal P. Take any $f \in L^*_{\mathrm{reg}}$ and assume that k is algebraically closed. Then $I(f) = 0$ by [O1, p. 42]. So, by Corollary 4.4 $\mathrm{ind}(f|_P, L)$ is a faithful irreducible representation of $U(L)$. Next, let $x_1, \dots, x_m, y_1, \dots, y_m$ be a basis of L such that $y_1, \dots, y_m$ is a basis of P. Then $\det([x_i, y_j]) \in S(P)$ is a nonzero semi-invariant under the action of $\mathrm{Aut}\, L$ [O1, p. 28]. It is also known that Frobenius Lie algebras give rise to constant solutions for the classical Yang–Baxter equation [BD].

The following is a special case of Proposition 4.6.

Corollary 5.1. *Let* $\mathfrak{g}$ *be a Lie algebra and* V *a* $\mathfrak{g}$*-module such that* $\dim \mathfrak{g} = \dim V$. *Consider the semi-direct product* $L = \mathfrak{g} \oplus V$. *Then the following are equivalent*:

(1) $R(V)$ *is a maximal subfield of* $D(L)$.
(2) V *is a CP-ideal of* L.
(3) L *is Frobenius.*
(4) $\mathfrak{g}(f) = 0$ *for some* $f \in V^*$.

Example 5.2. Let $\mathfrak{g}$ be Frobenius and let $V = \mathfrak{g}$ be the adjoint representation.

Example 5.3. The above condition is satisfied if $\mathfrak{g}$ is reductive over an algebraically closed field k and V^* is a prehomogeneous $\mathfrak{g}$-module (i.e. V^* has an open $\mathfrak{g}$-orbit) with $\dim \mathfrak{g} = \dim V$. These modules have been studied extensively by the Japanese school since 1977 [SK,KKTI].

Example 5.4. Let A be a left-symmetric algebra (LSA), i.e., a finite dimensional vector space provided with a bilinear product $A \times A \to A$, $(a,b) \to ab$ which satisfies

$$a(bc) - (ab)c = b(ac) - (ba)c \tag{$*$}$$

for all $a, b, c \in A$. There is an extensive literature on LSA's, see for example [H,Seg]. Vinberg used LSA's to classify convex homogeneous cones [V]. A left-symmetric algebra is Lie-admissable. This means that A becomes a Lie algebra, which we denote by $\mathfrak{g}$, for the Lie bracket $[a,b] = ab - ba$, $a, b \in A$. Using $(*)$ we observe that

$$[a,b]c = (ab)c - (ba)c = a(bc) - b(ac).$$

Therefore, A becomes a $\mathfrak{g}$-module, which we denote by V, for the bilinear map

$$\mathfrak{g} \times V \to V, \quad (x,v) \to xv.$$

Now, suppose A contains a nonzero element $f \in A$ which is not a right zero divisor of A. Let V^* be the dual module of V. Identifying the module V^{**} with V, we may consider f to be an element of $(V^*)^*$. Clearly, the stabilizer $\mathfrak{g}(f) = \{x \in \mathfrak{g} \mid xf = 0\} = 0$ by assumption.

Finally, using Corollary 5.1 we may conclude that the semi-direct product $L = \mathfrak{g} \oplus V^*$ is a Frobenius Lie algebra in which V^* is a CP-ideal.

Remark 5.5. In characteristic $p > 2$ a similar result can be obtained if A is a finite dimensional simple Novikov algebra and where V is a certain irreducible A-module. We recall that a nonassociative k-algebra is said to be a left Novikov algebra if A is left symmetric, satisfying the identity $(ab)c = (ac)b$ for all $a, b, c \in A$. In characteristic zero E. Zelmanov showed that finite dimensional simple Novikov algebras are all one-dimensional [Z]. Recently simple Novikov algebras and their irreducible modules have been determined by Osborn and Xu [Os,X].

We now focus on a special case, which provides an interesting link between Frobenius algebras and Frobenius Lie algebras.

Proposition 5.6. *Let A be a finite dimensional associative algebra over k with a unit element. A becomes a Lie algebra $\mathfrak{g}$ for the Lie bracket $[a,b] = ab - ba$, $a, b \in A$, and $V = A$ becomes a $\mathfrak{g}$-module by left multiplication. Consider the semi-direct product $L = \mathfrak{g} \oplus V$. Then the following conditions are equivalent*:

(1) *A is a Frobenius algebra.*
(2) *L is a Frobenius Lie algebra.*
(3) *V is a CP-ideal of L.*
(4) *$R(V)$ is a maximal subfield of $D(L)$.*

Proof. In view of Corollary 5.1 it suffices to show that (1) is equivalent with $\mathfrak{g}(f) = 0$ for some $f \in V^*$. So, take $f \in V^*$. Then

$$\mathfrak{g}(f) = \{a \in A \mid f(ab) = 0 \text{ for all } b \in A\}.$$

Clearly, $\mathfrak{g}(f) = 0$ if and only if the bilinear map $A \times A \to k$, $(a, b) \to f(ab)$ is nondegenerate, i.e., A is a Frobenius algebra [CR, Theorem 61.3]. $\square$

Finally, we devote our attention to certain Frobenius Lie subalgebras of a semi-simple Lie algebra.

Theorem 5.7. *Let L be a semi-simple Lie algebra of rank r over k, k algebraically closed, and let x be a principal nilpotent element of L (i.e. the centralizer $C(x)$ of x in L has dimension r). Then the normalizer F of $C(x)$ in L is a solvable Frobenius Lie subalgebra of L in which $C(x)$ is a CP-ideal.*

Proof. It is well known that $C(x)$ is abelian [K]. Clearly, $C(x)$ is an ideal of F. In 1991 R. Brylinski and B. Kostant showed that $\dim F = 2r$ and that $F/C(x)$, and hence also F, is solvable [BK]. Recently, D. Panyushev proved that F is Frobenius [P2, Theorem 5.5]. $\square$

6. CP-ideals in the nilradical of parabolic Lie subalgebras of a simple Lie algebra

Theorem 6.1. *Let B be a Borel subalgebra of a simple Lie algebra L over k, k algebraically closed, of rank r and let N be the nilradical of B. Then,*

(1) *N admits a CP $\Leftrightarrow$ L is of type A_r or C_r. In these 2 cases N has a CP-ideal P, which is an ideal of B.*
(2) *P is also a CP-ideal of B in case L is of type C_r, $r \geqslant 1$.*

Proof. The information on $i(N)$, $i(B)$ in Table 1 is obtained from [E1,E2]. Also, we know that $i(N) + i(B) = r$ [P2, 1.5].

The idea is to compare the maximum dimension m of abelian Lie subalgebras of N, computed by Malcev [Ma, p. 216] with the number $\frac{1}{2}(\dim N + i(N))$. Then N contains a CP if and only if these numbers coincide. According to the table this occurs precisely if L is of type A_r or C_r.

Furthermore, we know from [PR, Table 1] that in both types (A_r or C_r) B has a maximal abelian ideal P of dimension $\frac{1}{2}(\dim N + i(N))$. Clearly $P \subset N$. Therefore P is a CP-ideal of N. This can also be deduced from Theorem 4.1.

(2) Using Lemma 1.3 we see that P is also a CP-ideal of B if and only if

$$\begin{aligned} i(N) = i(B) + \dim B - \dim N \quad &\Leftrightarrow \quad i(N) - i(B) = r \\ &\Leftrightarrow \quad i(B) = 0 \quad (\text{since } i(N) + i(B) = r). \end{aligned}$$

and this happens when L is of type $A_1 (= C_1)$ or C_r, $r \geqslant 2$. $\square$

Table 1

		$\dim N$	$i(N)$	$i(B)$	$\frac{1}{2}(\dim N + i(N))$	m
A_{2t}	$t \geqslant 1$	$t(2t+1)$	t	t	$t(t+1)$	$t(t+1)$
A_{2t+1}	$t \geqslant 0$	$(t+1)(2t+1)$	$t+1$	t	$(t+1)^2$	$(t+1)^2$
B_3		9	3	0	6	5
B_r	$r \geqslant 4$	r^2	r	0	$\frac{1}{2}r(r+1)$	$\frac{1}{2}r(r-1)+1$
C_r	$r \geqslant 2$	r^2	r	0	$\frac{1}{2}r(r+1)$	$\frac{1}{2}r(r+1)$
D_{2t}	$t \geqslant 2$	$2t(2t-1)$	$2t$	0	$2t^2$	$t(2t-1)$
D_{2t+1}	$t \geqslant 2$	$2t(2t+1)$	$2t$	1	$2t(t+1)$	$t(2t+1)$
E_6		36	4	2	20	16
E_7		63	7	0	35	27
E_8		120	8	0	64	36
F_4		24	4	0	14	9
G_2		6	2	0	4	3

Theorem 6.2. *Let L be a simple Lie algebra over k, k algebraically closed, of type A_r or C_r, π a parabolic Lie subalgebra of L. Then the nilradical N of π admits a CP-ideal P. Furthermore,*

(1) *suppose L is of type A_r and π of type $(p_1, \ldots, p_m)$. Put $n = r + 1$ and $p = p_1 + \cdots + p_\ell$, $1 \leqslant \ell \leqslant m$, such that $|\sum_{i=1}^{\ell} p_i - \frac{n}{2}|$ is as small as possible. Then,*

$$i(N) = 2p(n-p) - \frac{1}{2}\left(n^2 - \sum_{i=1}^{m} p_i^2\right);$$

(2) *suppose L is of type C_r, $r \geqslant 2$, and π of type $(p_1, \ldots, p_m)$. Put $\ell = [m/2]$, then*

$$i(N) = \frac{1}{2}\sum_{i=1}^{\ell} p_i(p_i + 1).$$

Remark 6.3.

(a) The first formula is new. A recursive formula for $i(N)$ was already established in [E1]. A different proof for the second formula can also be found in [E1].

(b) (Made by the referee) A. Joseph already gave a formula for $i(N)$ in an arbitrary simple Lie algebra, using a maximal subset of strongly orthogonal positive roots [J, (ii) of Proposition 2.6]. Being applied to A_r or C_r, Joseph's formula gives the above explicit expressions.

Proof. (1) Let $L = sl(V)$ where V is an n-dimensional vector space over k. By [B2, p. 187] we can find a flag F of subspaces of V:

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_m = V, \quad F_{i-1} \subsetneqq F_i$$

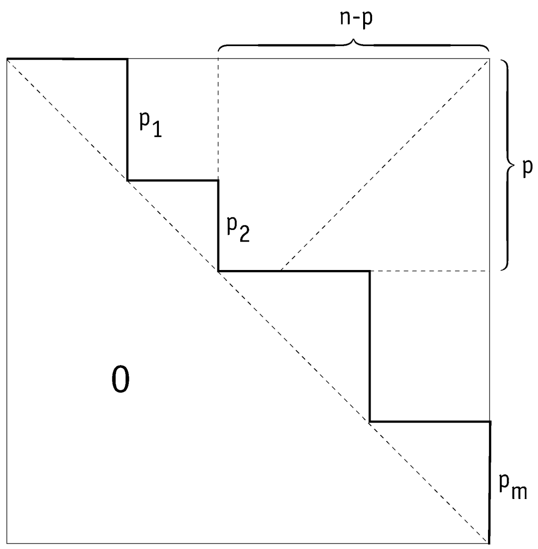

Fig. 1.

such that π (respectively its nilradical N) consists of all endomorphisms $x \in L$ such that $xF_i \subset F_i$ (resp. $xF_i \subset F_{i-1}$) for $1 \leqslant i \leqslant m$. Put $p_i = \dim(F_i/F_{i-1})$ then π is said to be of type $(p_1, \ldots, p_m)$. Next, choose a basis $e_1, \ldots, e_n$ of V compatible with the flag F (i.e., $e_1, \ldots, e_{p_1} \in F_1 \backslash F_0$, etc.). Then, N can be considered to be the Lie algebra of matrices of the form as shown in Fig. 1.

We may assume, as is the case in Fig. 1, that $p \leqslant n/2$ $(*)$. In particular, $p + p_{\ell+1} > n/2$. As usual we denote by E_{ij} the $n \times n$ matrix whose ijth entry is 1 and other entries are zero. Let P be the subspace of N generated by all E_{ij} with $1 \leqslant i \leqslant p$; $p+1 \leqslant j \leqslant n$. So, P consists of matrices of the form $\left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)$ where M is any $p \times (n-p)$ matrix. It is easy to see that P is an abelian ideal of N. We claim that P is a CP of N. Let $f \in N^*$ be defined by $f(E_{p,n-p+1}) = \cdots = f(E_{1n}) = 1$ and zero on all other E_{ij}. We want to show that $P^f = P$. Therefore we take $x \in P^f$. We write

$$x = \sum_{i<j} \lambda_{ij} E_{ij} + y$$

where $E_{ij} \in N \backslash P$, $\lambda_{ij} \in k$ and $y \in P$. We need to demonstrate that each $\lambda_{i_0 j_0} = 0$. There are two cases to distinguish:

(i) $j_0 \leqslant p$. Then $i_0 < j_0 \leqslant p$ and $s = (n+1) - i_0 > (n+1) - p > p$. Hence, $E_{j_0 s} \in P$ and

$$0 = f\big([x, E_{j_0 s}]\big) = \sum_{i<j} \lambda_{ij} f\big([E_{ij}, E_{j_0 s}]\big) + f\big([y, E_{j_0 s}]\big)$$

$$= \sum_{i<j} \lambda_{ij} f(\delta_{jj_0} E_{is} - \delta_{si} E_{j_0 j})$$

$$= \sum_{i<j_0} \lambda_{ij_0} f(E_{is}) - \sum_{j>s} \lambda_{sj} f(E_{j_0 j}) = \lambda_{i_0 j_0}$$

$(f(E_{j_0 j}) = 0$ since $j_0 + j > i_0 + s = n + 1)$.

(ii) $i_0 > p$ and $j_0 > p_1 + \cdots + p_\ell + p_{\ell+1} > n/2$. By definition of p:

$$(p_1 + \cdots + p_\ell + p_{\ell+1}) - \frac{n}{2} \geqslant \frac{n}{2} - p.$$

Hence

$$j_0 \geqslant (p_1 + \cdots + p_\ell + p_{\ell+1}) + 1 \geqslant n - p + 1.$$

So, $t = (n+1) - j_0 \leqslant p < i_0$ and $E_{ti_0} \in P$. Therefore

$$0 = f\bigl([E_{ti_0}, x]\bigr) = \sum_{i<j} \lambda_{ij} f\bigl([E_{ti_0}, E_{ij}]\bigr) + f\bigl([E_{ti_0}, y]\bigr)$$

$$= \sum_{i<j} \lambda_{ij} f(\delta_{i_0 i} E_{tj} - \delta_{jt} E_{ii_0})$$

$$= \sum_{j>i_0} \lambda_{i_0 j} f(E_{tj}) - \sum_{i<t} \lambda_{it} f(E_{ii_0}) = \lambda_{i_0 j_0}$$

$(f(E_{ii_0}) = 0$ since $i + i_0 < t + j_0 = n + 1)$.

In both cases: $x = y \in P$. So, $P^f \subset P$. Consequently, $P^f = P$ as the other inclusion is obvious by the commutativity of P.

By Lemma 1.2 we may conclude that P is a CP of N and $f \in N^*_{\mathrm{reg}}$. Finally, from $\dim P = \frac{1}{2}(\dim N + i(N))$ we obtain:

$$i(N) = 2\dim P - \dim N = 2p(n-p) - \frac{1}{2}\left(n^2 - \sum_{i=1}^{m} p_i^2\right).$$

(2) Let $L = \mathrm{sp}(V)$ where V is a vector space over k of dimension $n = 2r$ provided with a nondegenerate alternating bilinear form $\varphi : V \times V \to k$. There exists an isotropic flag

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_m = V,$$

i.e., $F_i^\perp = F_{m-i}$ for $0 \leqslant i \leqslant m$ such that π (respectively its nilradical N) consists of all $x \in L$ such that $xF_i \subset F_i$ (resp. $xF_i \subset F_{i-1}$) for $1 \leqslant i \leqslant m$. Put $p_i = \dim(F_i/F_{i-1})$ then

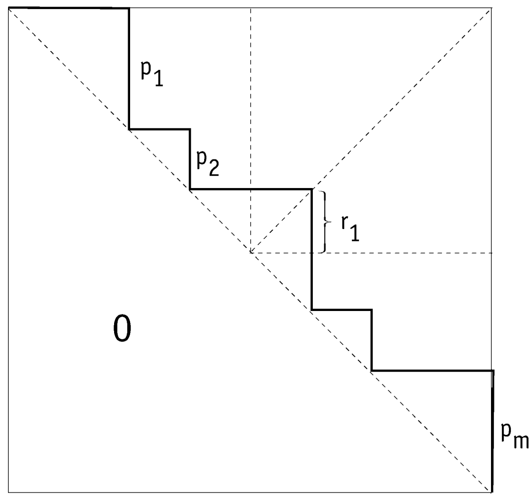

Fig. 2.

it follows that $p_i = p_{m+1-i}$ for $1 \leqslant i \leqslant m$. Following [B2, p. 200] we can find a Witt basis of V:

$$e_1, \dots, e_r,\ e_{-r}, \dots, e_{-1}$$

compatible with the given flag and such that $\varphi(e_i, e_{-j}) = \delta_{ij}$.

We now identify each $x \in L$ with its matrix with respect to this basis, i.e.,

$$x = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $r \times r$ matrices such that $B = \widehat{B}$, $C = \widehat{C}$, $D = -\widehat{A}$, where the transformation $\widehat{\ }$ is the transpose relative to the second diagonal. If $x \in N$ then x is of the form as shown in Fig. 2.

If $m = 2\ell + 1$ then we put $r_1 = \frac{1}{2} p_{\ell+1}$ ($p_{\ell+1}$ is even since $\sum_{i=1}^{m} p_i = n = 2r$ and $p_i = p_{m+1-i}$). If $m = 2\ell$ then we put $r_1 = 0$. π is determined by the sequence $(p_1, \dots, p_\ell; r_1)$. Note that $r = \sum_{i=1}^{\ell} p_i + r_1$. Next, let P be the subspace of N of matrices of the form $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ where B is an $r \times r$ matrix such that $B = \widehat{B}$ and with zero $r_1 \times r_1$ submatrix in the bottom left corner. Clearly,

$$X_{\varepsilon_i+\varepsilon_j} = E_{i,-j} + E_{j,-i}, \quad 1 \leqslant i \leqslant r - r_1;\ i \leqslant j \leqslant r.$$

form a basis of P which is an abelian ideal of N and $\dim P = \frac{1}{2}[(r^2 - r_1^2) + (r - r_1)]$. We enlarge this basis to a basis of N by adjoining some vectors of the type

$$X_{\varepsilon_i-\varepsilon_j} = E_{ij} - E_{-j,-i}, \quad i < j.$$

From Fig. 2 we see that

$$\dim N = \frac{1}{2}\left(r^2 - \sum_{i=1}^{\ell} p_i^2 - r_1^2\right) + \dim P = \left(r^2 - r_1^2\right) - \frac{1}{2}\sum_{i=1}^{\ell} p_i^2 + \frac{1}{2}(r - r_1).$$

Next, let $f \in N^*$ be defined by $f(X_{2\varepsilon_i}) = 1$ for $1 \leqslant i \leqslant r - r_1$ and zero on all other basis vectors of N. We want to show that $P^f = P$. For this purpose we take $x \in P^f$ which we can write as

$$x = \sum_{i<j} \lambda_{ij} X_{\varepsilon_i - \varepsilon_j} + y,$$

where $X_{\varepsilon_i - \varepsilon_j} \in N$, $\lambda_{ij} \in k$ and $y \in P$. Fix any λ_{st}, $s < t$ with $X_{\varepsilon_s - \varepsilon_t} \in N$. This implies that $s \leqslant r - r_1$, $t \leqslant r$. Hence $X_{\varepsilon_s + \varepsilon_t} \in P$. Therefore,

$$\begin{aligned}
0 &= f\big([x, X_{\varepsilon_s+\varepsilon_t}]\big) = \sum_{i<j} \lambda_{ij} f\big([X_{\varepsilon_i-\varepsilon_j}, X_{\varepsilon_s+\varepsilon_t}]\big) + f\big([y, X_{\varepsilon_s+\varepsilon_t}]\big) \\
&= \sum_{i<j} \lambda_{ij} f(\delta_{js} X_{\varepsilon_i+\varepsilon_t} + \delta_{jt} X_{\varepsilon_i+\varepsilon_s}) \\
&= \sum_{i<s} \lambda_{is} f(X_{\varepsilon_i+\varepsilon_t}) + \sum_{i<t} \lambda_{it} f(X_{\varepsilon_i+\varepsilon_s}) = 0 + \lambda_{st}
\end{aligned}$$

($f(X_{\varepsilon_i+\varepsilon_t}) = 0$ since $i < s < t$).

It follows that $x = y \in P$. So, $P^f \subset P$. Consequently, $P^f = P$ as the other inclusion is obvious. By Lemma 1.2 we may conclude that P is a CP of N and $f \in N^*_{\mathrm{reg}}$. Finally,

$$\begin{aligned}
i(N) &= 2\dim P - \dim N \\
&= \left(r^2 - r_1^2\right) + (r - r_1) - \left(r^2 - r_1^2\right) + \frac{1}{2}\sum_{i=1}^{\ell} p_i^2 - \frac{1}{2}(r - r_1) \\
&= \frac{1}{2}\left(\sum_{i=1}^{\ell} p_i^2 + (r - r_1)\right) = \frac{1}{2}\sum_{i=1}^{\ell} p_i(p_i + 1). \quad \square
\end{aligned}$$

7. CP-preserving extensions

Proposition 7.1. *Let M be a finite dimensional Lie algebra over k and let $d \in \operatorname{Der} M$ be a derivation such that $d(Z(M)) \neq 0$. Consider the extension $L = M \oplus kd$ in which $[d, x] = d(x)$, $x \in M$.*

Then we have

(i) $i(M) = i(L) + 1$.

(ii) *L has a CP if and only if M has a CP.*
(iii) *If L is square integrable, then so is M.*

Remark 7.2. Example (3) of 1.7 shows that the condition on d cannot be removed.

Proof. Take $u \in Z(M)$ such that $d(u) \neq 0$. Clearly $M = C(u)$. Now the assertions follow directly from Proposition 1.9. □

Proposition 7.3. *Let M be a finite dimensional Lie algebra over k and fix z, a non-zero central element of M. Let S be a $2r$-dimensional vector space, provided with a nondegenerate alternating bilinear form $\varphi : S \times S \to k$. Consider the Lie algebra $L = M \oplus S$ containing M as an ideal and in which $[x, s] = 0$ and $[s, t] = \varphi(s, t)z$ for $x \in M$; $s, t \in S$. Then we have*

(i) *$H = S \oplus kz$ is a Heisenberg Lie algebra.*
(ii) *$i(L) = i(M)$ and $Z(L) = Z(M)$.*
(iii) *M is square integrable if and only if L is square integrable.*
(iv) *If M allows a CP (resp. a CP-ideal) then the same holds for L.*

Proof. (i) It is easy to verify that L is a Lie algebra. There exists a $f \in L^*_{\mathrm{reg}}$ such that $f|_M \in M^*_{\mathrm{reg}}$ and $f(z) \neq 0$. We may assume that $f(z) = 1$ (by replacing f by $\frac{1}{f(z)} f$). Then for all $s, t \in S$

$$B_f(s, t) = f\big([s, t]\big) = \varphi(s, t).$$

From the assumption on φ, $S \cap S^\perp = 0$ and we can find a basis $s_1, \dots, s_r; t_1, \dots, t_r$ of S such that for all i, j:

$$\varphi(s_i, s_j) = 0 = \varphi(t_i, t_j) \quad \text{and} \quad \varphi(s_i, t_j) = \delta_{ij}.$$

This implies $[s_i, s_j] = 0 = [t_i, t_j]$ and $[s_i, t_j] = \delta_{ij} z$ for all i, j. Consequently, H is a Heisenberg Lie algebra.

(ii) First, we notice that $M = S^\perp$. Indeed, $M \subset S^\perp$ since $f([M, S]) = 0$. For the other inclusion, take $x \in S^\perp$, which we decompose as $x = m + s$ with $m \in M$ and $s \in S$. Then, $s = x - m \in S \cap S^\perp = \{0\}$. Hence, $x = m \in M$. As $M = S^\perp$ we deduce from [D, 1.12.4] that

$$M(f|_M) = M \cap M^\perp = S \cap S^\perp + L^\perp = L(f).$$

Taking dimensions yields $i(M) = i(L)$. Clearly, the elements of $Z(M)$ commute with those of M and S. Hence, $Z(M) \subset Z(L)$. Conversely, take $x \in Z(L)$ which we can decompose as $x = m + s$ with $m \in M$ and $s \in S$. For all $s' \in S$:

$$[s, s'] = [x - m, s'] = [x, s'] - [m, s'] = 0$$

and hence also $\varphi(s,s') = f([s,s']) = 0$ which implies that $s = 0$ and so $x = m \in M \cap Z(L) \subset Z(M)$.

(iii) This follows at once from (ii).

(iv) Suppose P_1 is a CP of M. Put $P_2 = ks_1 + \cdots + ks_r$ and $P = P_1 \oplus P_2$. Then P is a CP of L since P is commutative and

$$\dim P = \dim P_1 + \dim P_2 = \tfrac{1}{2}\big(\dim M + i(M)\big) + \tfrac{1}{2}\dim S = \tfrac{1}{2}\big(\dim L + i(L)\big).$$

Finally, if P_1 is an ideal of M then P is an ideal of L since

$$[M,P] = [M,P_1] + [M,P_2] = [M,P_1] \subset P_1 \subset P$$

and

$$\begin{aligned}[t_j,P] &= [t_j,P_1] + [t_j,P_2] = [t_j,P_2] \\ &= \sum_i k[t_j,s_i] = kz \subset Z(M) \subset P_1 \subset P. \quad \square\end{aligned}$$

Proposition 7.4. *Let A be an n-dimensional commutative (associative) Frobenius algebra over k and M an m-dimensional Lie algebra over k.*

Consider the Lie algebra $L = A \otimes_k M$ for which $[a \otimes x, a' \otimes y] = aa' \otimes [x,y]$, $a, a' \in A$ and $x, y \in M$. Then we have

- (i) *M is square integrable if and only if L is square integrable.*
- (ii) *M is Frobenius if and only if L is Frobenius.*
- (iii) *If M allows a CP (resp. a CP-ideal) then the same holds for L.*

Proof. (i) From [F, pp. 241–243] we know that $i(L) = n.i(M)$. On the other hand, $Z(L) = A \otimes_k Z(M)$ and so $\dim Z(L) = n.\dim Z(M)$. Therefore, $i(L) = \dim Z(L)$ if and only if $i(M) = \dim Z(M)$.

(ii) This follows from (i) and its proof.

(iii) Let P be a CP (resp. a CP-ideal) of M. Then $Q = A \otimes_k P$ is a commutative Lie subalgebra (resp. ideal) of L and

$$\begin{aligned}\dim Q &= n.\dim P = n.\tfrac{1}{2}\big(\dim M + i(M)\big) \\ &= \tfrac{1}{2}\big(n.\dim M + n.i(M)\big) = \tfrac{1}{2}\big(\dim L + i(L)\big). \quad \square\end{aligned}$$

Acknowledgments

The authors thank Jacques Alev for his interest and his valuable comments on the subject. They also express their gratitude to Dmitri Panyushev for providing some useful preprints of his work. The first author is grateful for the generous hospitality he received during his visits at the University of Limburg and the University of Bochum, which greatly contributed to the completion of this paper.

Finally, the authors thank the referee for pointing out Remark 6.3(b) and for giving helpful suggestions in connection with the presentation of the paper.

References

[AVE] E.M. Andreev, E.B. Vinberg, A.G. Elashvili, Orbits of greatest dimension of semi-simple linear groups, Funkt. Anal. Prilozh. 1 (4) (1967) 3–7.

[BD] A.A. Belavin, V.G. Drinfeld, Triangle equations and simple Lie algebras, Sov. Sci. Rev. Sect. C (Math. Phys. Rev.) 4 (1984) 93–165.

[BC] P. Bernat, N. Conze, et al., Représentations des Groupes de Lie Résolubles, in: Monographies Soc. Math. France, Dunod, Paris, 1972.

[Bo] A. Borel, Linear Algebraic Groups, Benjamin, New York, 1969.

[BGR] W. Borho, P. Gabriel, R. Rentschler, Primideale in Einhüllenden auflösbarer Lie-Algebren, in: Lecture Notes in Math., Vol. 357, Springer-Verlag, Berlin, 1973.

[B1] N. Bourbaki, Groupes et algèbres de Lie, Chap. 1, Hermann, Paris, 1971.

[B2] N. Bourbaki, Groupes et algèbres de Lie, Chap. 7 et 8, Hermann, Paris, 1975.

[BK] R. Brylinski, B. Kostant, The variety of all invariant symplectic structures on a homogeneous space and normalizers of isotropy subgroups, in: P. Donato, et al. (Eds.), Symplectic Geometry and Mathematical Physics, in: Progr. Math., Vol. 99, Birkhäuser, Basel, 1991, pp. 80–113.

[Ch] C. Chevalley, Théorie des groupes de Lie, Vol. III, Hermann, Paris, 1968.

[CR] C.W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience Publishers/Wiley, New York, 1962.

[DL] J. Dixmier, W.G. Lister, Derivations of nilpotent Lie algebras, Proc. Amer. Math. Soc. 8 (1957) 155–158.

[D] J. Dixmier, Enveloping Algebras, in: Graduate Stud. Math., Vol. 11, American Mathematical Society, Providence, RI, 1996.

[E1] A.G. Elashvili, On the index of orispherical subalgebras of semisimple Lie algebras, Trudy Razmadze Math. Institute (Tbilisi) 77 (1985) 116–126.

[E2] A.G. Elashvili, On the index of parabolic subalgebras of semisimple Lie algebras, Preprint, 1990.

[F] A.T. Fomenko, Integrability and Nonintegrability in Geometry and Mechanics, Kluwer Academic, Dordrecht, 1988.

[GO1] F. Grunewald, J. O'Halloran, Varieties of nilpotent Lie algebras of dimension less than six, J. Algebra 112 (1988) 315–325.

[GO2] F. Grunewald, J. O'Halloran, Deformations of Lie algebras, J. Algebra 162 (1993) 210–224.

[H] J. Helmstetter, Radical d'une algèbre symétrique à gauche, Ann. Inst. Fourier 29 (1979) 17–35.

[J] A. Joseph, A preparation theorem for the prime spectrum of a semisimple Lie algebra, J. Algebra 48 (1977) 241–289.

[K] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math. 81 (1959) 973–1032.

[KKTI] T. Kimura, S. Kasai, M. Taguchi, M. Inuzuka, Some P.V.-equivalences and a classification of 2-simple prehomogeneous vector spaces of type II, Trans. Amer. Math. Soc. 308 (1988) 433–494.

[Ma] A.I. Malcev, Commutative subalgebras of semisimple Lie algebras, Izvestiya Akademii Nauk. SSSR, Seriya Matematičeskaya 9 (1945) 291–300; English translation: AMS 40 (1951) 214–227.

[Mo] V. Morozov, Classification of nilpotent Lie algebras of 6th order, Irv. Vyssh. Uchebn. Zaved. Mat. 4 (5) (1958) 161–171.

[MW] C.C. Moore, J.A. Wolf, Square integrable representations of nilpotent groups, Trans. Amer. Math. Soc. 185 (1973) 445–462.

[O1] A.I. Ooms, On Frobenius Lie algebras, Comm. Algebra 8 (1980) 13–52.

[O2] A.I. Ooms, On certain maximal subfields in the quotient division ring of an enveloping algebra, J. Algebra 230 (2000) 694–712.

[Os] J.M. Osborn, Novikov algebras, Nova J. Algebra Geom. 1 (1992) 1–14.
[P2] D. Panyushev, The index of a Lie algebra, the centralizer of a nilpotent element and the normalizer of the centralizer, to appear.
[PR] D. Panyushev, G. Röhrle, Spherical orbits and abelian ideals, Adv. in Math. 159 (2001) 229–246.
[RV] R. Rentschler, M. Vergne, Sur le semi-centre du corps enveloppant d'une algèbre de Lie, Ann. Sci. Ecole Norm. Sup. 6 (1973) 380–405.
[See] C. Seeley, 7-dimensional nilpotent Lie algebras, Trans. Amer. Math. Soc. 335 (1993) 479–496.
[Seg] D. Segal, The structure of complete left-symmetric algebras, Math. Ann. 293 (1992) 569–578.
[SK] M. Sato, T. Kimura, Classification of irreducible prehomogeneous vector spaces and their invariants, Nagoya Math. J. 65 (1977) 1–155.
[V] E.B. Vinberg, Convex homogeneous cones, Transl. Moscow Math. Soc. 12 (1963) 340–403.
[X] X. Xu, On simple Novikov algebras and their irreducible modules, J. Algebra 185 (1996) 905–934.
[Z] E.I. Zelmanov, On a class of local translation invariant Lie algebras, Soviet Math. Dokl. 35 (1987) 216–218.

Further reading

[DK] V. Dergachev, A. Kirillov, Index of Lie algebras of seaweed type, J. Lie Theory 10 (2000) 331–343.
[P1] D. Panyushev, Inductive formulas for the index of seaweed Lie algebras, Moscow Math. J. (2001), to appear.